THE BIVARIATE HORADAM POLYNOMIALS AND HYBRINOMIALS

EMRE SEVGI

Abstract. In this paper, we define the Horadam polynomials and hybrinomials with two variables x and y, called the bivariate Horadam polynomials and hybrinomials, respectively. Also, we obtain Binet formula, generating function and some properties for the bivariate Horadam hybrinomials. Moreover, we get Catalan, Cassini and d'Ocagne identities for these hybrinomials. Finally, the matrix representations of the bivariate Horadam hybrinomials were introduced.

1. Introduction

Integer sequences have a great interest for researchers in many fields of science. Horadam defined the sequence W_n recursively by

$$W_n = pW_{n-1} + qW_{n-2}$$

for $n \geq 2$ and $a, b, p, q \in \mathbb{Z}$ with initial values $W_0 = a$, $W_1 = b$.

This sequence W_n is called Horadam sequence as a generalization of many integer sequences.

After that, Horzum and Kocer defined the Horadam polynomials recursively by

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x),\ n \geq 3$$

with $h_1(x) = a$ and $h_2(x) = bx$ [10].

Moreover, some authors worked on some properties of this sequence [9, 11, 12].

A new generalization of complex, hyperbolic and dual numbers called hybrid numbers were introduced by Özdemir [1]. The set of hybrid numbers is

$$\mathbb{K} = \{a + b\mathbf{i} + c\epsilon + d\mathbf{h} :\ a, b, c, d \in \mathbb{R}\}.$$

Let $Z_1 = a + b\mathbf{i} + c\epsilon + d\mathbf{h}$ and $Z_2 = e + f\mathbf{i} + g\epsilon + h\mathbf{h}$ be any two hybrid numbers. Then the main operations on hybrid numbers are defined as follows:

$Z_1 = Z_2$ if and only if $a = e, b = f, c = g, d = h$

$Z_1 + Z_2 = (a + e) + (b + f)\,\mathbf{i} + (c + g)\,\epsilon + (d + h)\,\mathbf{h}$

$Z_1 - Z_2 = (a - e) + (b - f)\,\mathbf{i} + (c - g)\,\epsilon + (d - h)\,\mathbf{h}$

$sZ_1 = sa + sb\mathbf{i} + sc\epsilon + sd\mathbf{h}$, where $s \in \mathbb{R}$.

Date: January 21, 2020.

2000 *Mathematics Subject Classification.* Primary 11B37, 11B39.

Key words and phrases. Hybrid numbers, Horadam polynomials, hybrinomials, recurrence relation.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Any two hybrid numbers can be multiplied by using the following multiplication table:

$\cdot$	1	$\mathbf{i}$	ϵ	$\mathbf{h}$
1	1	$\mathbf{i}$	ϵ	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\epsilon+\mathbf{i}$
ϵ	ϵ	$\mathbf{h}+1$	0	$-\epsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\epsilon-\mathbf{i}$	ϵ	1

Inspired by the definition of hybrid numbers, many authors worked on new kinds of hybrid numbers [2, 4, 5, 6, 7, 8]. As a generalization of these hybrid numbers, Horadam hybrid numbers were defined in [3] as

$$H_n = W_n + \mathbf{i}W_{n+1} + \epsilon W_{n+2} + \mathbf{h}W_{n+3}.$$

After that, for $n \geq 1$ Kızılateş defined the Horadam hybrinomials as

$$\mathbb{H}_n(x) = h_n(x) + \mathbf{i}h_{n+1}(x) + \epsilon h_{n+2}(x) + \mathbf{h}h_{n+3}(x)$$

where $h_n(x)$ is the nth Horadam polynomial [13].

2. Main Results

Definition 1. *The bivariate Horadam polynomials $w_n(x,y)$ are defined by the recurrence ralation*

$$w_n(x,y) = xpw_{n-1}(x,y) + yqw_{n-2}(x,y),\ n \geq 3.$$

with the initial values $w_1(x,y) = a$ and $w_2(x,y) = bx$.

The first few terms of this sequence are

n	$w_n(x,y)$
1	a
2	bx
3	$bpx^2 + aqy$
4	$bp^2x^3 + apqxy + bqxy$
5	$bp^3x^4 + ap^2qx^2y + 2bpqx^2y + aq^2y^2$
6	$bp^4x^5 + ap^3qx^3y + 3bp^2qx^3y + 2apq^2xy^2 + bq^2xy^2$

For simplicity, we will use w_n instead of $w_n(x,y)$.

The characteristic equation of this sequence is

$$v^2 - pxv - qy = 0$$

with the roots

$$\alpha = \frac{px + \sqrt{p^2x^2 + 4qy}}{2},\ \beta = \frac{px - \sqrt{p^2x^2 + 4qy}}{2}. \tag{2.1}$$

Lemma 1. *For $n \geq 0$ the Binet formula for the generalized Fibonacci polynomials is*

$$w_n = A\alpha^{n-1} + B\beta^{n-1}$$

where $A = \dfrac{bx - a\beta}{\sqrt{p^2x^2+4qy}}$ *and* $B = \dfrac{a\alpha - bx}{\sqrt{p^2x^2+4qy}}$.

Lemma 2. *The roots α and β defined in* (2.1) *satisfy the following properties*

- $\alpha + \beta = px$
- $\alpha - \beta = \sqrt{p^2x^2+4qy}$
- $\alpha\beta = -qy$
- $A(1-\beta) + B(1-\alpha) = a + bx - apx$

Definition 2. *For $n \geq 1$ the Horadam hybrinomials with two variables x and y, called the bivariate Horadam hybrinomials defined by*

$$\mathbb{BH}_n(x,y) = w_n + \mathbf{i}w_{n+1} + \epsilon w_{n+2} + \mathbf{h}w_{n+3}$$

where $w_n$is the nth bivariate Horadam polynomial.

For simplicity, we will use $\mathbb{BH}_n$ instead of $\mathbb{BH}_n(x,y)$.

Theorem 1. *The bivariate Horadam hybrinomials provides the recurrence relation*

$$\mathbb{BH}_n = px\mathbb{BH}_{n-1} + qy\mathbb{BH}_{n-2},\ n \geq 2$$

with the initial conditions

$$\begin{aligned}
\mathbb{BH}_1 &= a + \mathbf{i}bx + \epsilon(bpx^2 + aqy) + \mathbf{h}(bp^2x^3 + apqxy + bqxy) \\
\mathbb{BH}_2 &= bx + \mathbf{i}(bpx^2 + aqy) + \epsilon(bp^2x^3 + apqxy + bqxy) \\
&\quad + \mathbf{h}(bp^3x^4 + ap^2qx^2y + 2bpqx^2y + aq^2y^2).
\end{aligned}$$

Proof. For $n = 3$,we get

$$\begin{aligned}
\mathbb{BH}_3 &= px\mathbb{BH}_2 + qy\mathbb{BH}_1 \\
&= px\left[bx + \mathbf{i}(bpx^2 + aqy) + \epsilon(bp^2x^3 + apqxy + bqxy)\right. \\
&\quad \left. + \mathbf{h}(bp^3x^4 + ap^2qx^2y + 2bpqx^2y + aq^2y^2)\right] \\
&\quad\quad + qy\left[a + \mathbf{i}bx + \epsilon(bpx^2 + aqy) + \mathbf{h}(bp^2x^3 + apqxy + bqxy)\right] \\
&= bpx^2 + aqy + \mathbf{i}(bp^2x^3 + apqxy + bqxy) + \epsilon(bp^3x^4 + ap^2qx^2y + 2bpqx^2y + aq^2y^2) \\
&\quad + \mathbf{h}(bp^4x^5 + ap^3qx^3y + 3bp^2qx^3y + 2apq^2xy^2 + bq^2xy^2) \\
&= w_3 + \mathbf{i}w_4 + \epsilon w_5 + \mathbf{h}w_6.
\end{aligned}$$

For $n > 3$, using the definition of the bivariate Horadam polynomials, we obtain

$$\begin{aligned}
\mathbb{BH}_n &= w_n + \mathbf{i}w_{n+1} + \epsilon w_{n+2} + \mathbf{h}w_{n+3} \\
&= (pxw_{n-1} + qyw_{n-2}) + \mathbf{i}(pxw_n + qyw_{n-1}) \\
&\quad + \epsilon\,(pxw_{n+1} + qyw_n) + \mathbf{h}\,(pxw_{n+2} + qyw_{n+1}) \\
&= px\,(w_{n-1} + \mathbf{i}w_n + \epsilon w_{n+1} + \mathbf{h}w_{n+2}) + qy\,(w_{n-2} + \mathbf{i}w_{n-1} + \epsilon w_n + \mathbf{h}w_{n+1}) \\
&= px\mathbb{BH}_{n-1} + qy\mathbb{BH}_{n-2}.
\end{aligned}$$

So, the proof is completed. □

Theorem 2. *For any integer $n \geq 0$, the Binet formula for the bivariate Horadam hybrinomials is defined as*

$$\mathbb{BH}_n = A\alpha^{n-1}(1 + \mathbf{i}\alpha + \epsilon\alpha^2 + \mathbf{h}\alpha^3) + B\beta^{n-1}(1 + \mathbf{i}\beta + \epsilon\beta^2 + \mathbf{h}\beta^3)$$

where $\alpha = \dfrac{px + \sqrt{p^2x^2 + 4qy}}{2}$ *and* $\beta = \dfrac{px - \sqrt{p^2x^2 + 4qy}}{2}$.

Proof. Using the definition of the bivariate Horadam hybrinomials and the Binet formula for the bivariate Horadam polynomials, we get

$$\begin{aligned}
\mathbb{BH}_n &= w_n + \mathbf{i}w_{n+1} + \epsilon w_{n+2} + \mathbf{h}w_{n+3} \\
&= A\alpha^{n-1} + B\beta^{n-1} + \mathbf{i}\left(A\alpha^n + B\beta^n\right) + \epsilon\left(A\alpha^{n+1} + B\beta^{n+1}\right) \\
&\quad + \mathbf{h}\left(A\alpha^{n+2} + B\beta^{n+2}\right) \\
&= A\alpha^{n-1}(1 + \mathbf{i}\alpha + \epsilon\alpha^2 + \mathbf{h}\alpha^3) + B\beta^{n-1}(1 + \mathbf{i}\beta + \epsilon\beta^2 + \mathbf{h}\beta^3).
\end{aligned}$$

□

For expressing the notations simply, let

$$\begin{aligned}
\widehat{\alpha} &= 1 + \mathbf{i}\alpha + \epsilon\alpha^2 + \mathbf{h}\alpha^3 \\
\widehat{\beta} &= 1 + \mathbf{i}\beta + \epsilon\beta^2 + \mathbf{h}\beta^3.
\end{aligned}$$

Then, we can write the Binet formula for the bivariate Horadam hybrinomials as

$$\mathbb{BH}_n = A\alpha^{n-1}\widehat{\alpha} + B\beta^{n-1}\widehat{\beta}.$$

Theorem 3. *The generating function for the bivariate Horadam hybrinomials is*

$$\sum_{n=0}^{\infty} \mathbb{BH}_n t^n = \frac{\mathbb{BH}_0 + (\mathbb{BH}_1 - px\mathbb{BH}_0)t}{1 - pxt - qyt^2}.$$

Proof. Suppose that the formal power series representation of the generating function for the bivariate Horadam hybrinomials is

$$G(t) = \sum_{n=0}^{\infty} \mathbb{BH}_n t^n = \mathbb{BH}_0 + \mathbb{BH}_1 t + \mathbb{BH}_2 t^2 + \cdots \tag{2.2}$$

Then, multiplying the equation (2.2) by $-pxt$ and $-qyt^2$ respectively, we have
$-G(t)pxt = -\mathbb{BH}_0 pxt - \mathbb{BH}_1 pxt^2 - \mathbb{BH}_2 pxt^3 - \cdots$
and
$-G(t)qyt^2 = -\mathbb{BH}_0 qyt^2 - \mathbb{BH}_1 qyt^3 - \mathbb{BH}_2 qyt^4 - \cdots$
By using the above equations and the fact that for $n \geq 2$ the coefficients of t^n are zero by the recurrence relation of the bivariate Horadam hybrinomials, we obtain

$$G(t)\left(1 - pxt - qyt^2\right) = \mathbb{BH}_0 + (\mathbb{BH}_1 - px\mathbb{BH}_0)t. \tag{2.3}$$

Finally, by substituting $\mathbb{BH}_0$ and $\mathbb{BH}_1$ in the equation (2.3) , we get

$$G(t) = \frac{\mathbb{BH}_0 + (\mathbb{BH}_1 - px\mathbb{BH}_0)t}{1 - pxt - qyt^2}.$$

□

Lemma 3. *For any integer* $n \geq 2$, *the bivariate Horadam polynomials provides the summation formula*

$$\sum_{m=1}^{n-1} w_m = \frac{w_n + qyw_{n-1} - w_1 - qyw_0}{px + qy - 1}.$$

Proof. By using the Binet formula for the bivariate Horadam polynomials, we get

$$\begin{aligned}
\sum_{m=1}^{n-1} w_m &= \sum_{m=1}^{n-1} A\alpha^{m-1} + B\beta^{m-1} \\
&= A\left(\frac{1-\alpha^{n-1}}{1-\alpha}\right) + B\left(\frac{1-\beta^{n-1}}{1-\beta}\right) \\
&= \frac{1}{1-px-qy}\left[A(1-\beta)\left(1-\alpha^{n-1}\right) + B(1-\alpha)\left(1-\beta^{n-1}\right)\right] \\
&= \frac{1}{1-px-qy}\left[-\left(A\alpha^{n-1}+B\beta^{n-1}\right) + A\beta\alpha^{n-1} + B\alpha\beta^{n-1}\right. \\
&\qquad \left. + A(1-\beta) + B(1-\alpha)\right] \\
&= \frac{1}{1-px-qy}\left[-\left(A\alpha^{n-1}+B\beta^{n-1}\right) - qy\left(A\alpha^{n-2}+B\beta^{n-2}\right)\right. \\
&\qquad \left. + (A+B) - (A\beta + B\alpha)\right] \\
&= \frac{w_n + qyw_{n-1} - w_1 - qyw_0}{px+qy-1}.
\end{aligned}$$

□

Theorem 4. *For any integer $n \geq 2$, the bivariate Horadam hybrinomials provides the summation formula*

$$\sum_{m=1}^{n-1} \mathbb{BH}_m = \frac{\mathbb{BH}_n + qy\mathbb{BH}_{n-1} - \mathbb{BH}_1 - qy\mathbb{BH}_0}{px+qy-1}.$$

Proof. By using the definition of the bivariate Horadam hybrinomials, we have

$$\begin{aligned}
\sum_{m=1}^{n-1} \mathbb{BH}_m &= \sum_{m=1}^{n-1} A\widehat{\alpha}\alpha^{m-1} + B\widehat{\beta}\beta^{m-1} \\
&= A\widehat{\alpha}\left(\frac{1-\alpha^{n-1}}{1-\alpha}\right) + B\widehat{\beta}\left(\frac{1-\beta^{n-1}}{1-\beta}\right) \\
&= \frac{1}{1-px-qy}\left[A\widehat{\alpha}(1-\beta)\left(1-\alpha^{n-1}\right) + B\widehat{\beta}(1-\alpha)\left(1-\beta^{n-1}\right)\right] \\
&= \frac{1}{1-px-qy}\left[-\left(A\widehat{\alpha}\alpha^{n-1}+B\widehat{\beta}\beta^{n-1}\right) + A\widehat{\alpha}\beta\alpha^{n-1} + B\widehat{\beta}\alpha\beta^{n-1}\right. \\
&\qquad \left. + A\widehat{\alpha}(1-\beta) + B\widehat{\beta}(1-\alpha)\right] \\
&= \frac{1}{1-px-qy}\left[-\left(A\widehat{\alpha}\alpha^{n-1}+B\widehat{\beta}\beta^{n-1}\right) - qy\left(A\widehat{\alpha}\alpha^{n-2}+B\widehat{\beta}\beta^{n-2}\right)\right. \\
&\qquad \left. + \left(A\widehat{\alpha} + B\widehat{\beta}\right) - \left(A\widehat{\alpha}\beta + B\widehat{\beta}\alpha\right)\right] \\
&= \frac{\mathbb{BH}_n + qy\mathbb{BH}_{n-1} - \mathbb{BH}_1 - qy\mathbb{BH}_0}{px+qy-1}.
\end{aligned}$$

□

Theorem 5 (Catalan Identity)**.** *For the nonnegative integers n and r with $n \geq r$, we have*

$$\mathbb{BH}_{n-r}\mathbb{BH}_{n+r} - \left(\mathbb{BH}_n\right)^2 = (-qy)^{n-1} AB\left[\widehat{\alpha}\widehat{\beta}\left(\frac{\beta^r}{\alpha^r} - 1\right) + \widehat{\beta}\widehat{\alpha}\left(\frac{\alpha^r}{\beta^r} - 1\right)\right].$$

Proof. By using the Binet formula for the bivariate Horadam hybrinomials, we have

$$\begin{aligned}
&\mathbb{BH}_{n-r}\mathbb{BH}_{n+r}-(\mathbb{BH}_n)^2\\
&=\left(A\alpha^{n-r-1}\widehat{\alpha}+B\beta^{n-r-1}\widehat{\beta}\right)\left(A\alpha^{n+r-1}\widehat{\alpha}+B\beta^{n+r-1}\widehat{\beta}\right)\\
&\quad-\left(A\alpha^{n-1}\widehat{\alpha}+B\beta^{n-1}\widehat{\beta}\right)\left(A\alpha^{n-1}\widehat{\alpha}+B\beta^{n-1}\widehat{\beta}\right)\\
&=AB\alpha^{n-r-1}\beta^{n+r-1}\widehat{\alpha}\widehat{\beta}+AB\alpha^{n+r-1}\beta^{n-r-1}\widehat{\beta}\widehat{\alpha}\\
&\quad-AB\alpha^{n-1}\beta^{n-1}\widehat{\alpha}\widehat{\beta}-AB\alpha^{n-1}\beta^{n-1}\widehat{\beta}\widehat{\alpha}\\
&=AB\alpha^{n-1}\beta^{n-1}\widehat{\alpha}\widehat{\beta}\left(\frac{\beta^r}{\alpha^r}-1\right)+AB\alpha^{n-1}\beta^{n-1}\widehat{\beta}\widehat{\alpha}\left(\frac{\alpha^r}{\beta^r}-1\right)\\
&=(-qy)^{n-1}AB\left[\widehat{\alpha}\widehat{\beta}\left(\frac{\beta^r}{\alpha^r}-1\right)+\widehat{\beta}\widehat{\alpha}\left(\frac{\alpha^r}{\beta^r}-1\right)\right].
\end{aligned}$$

□

Theorem 6 (Cassini Identity)**.** *For any nonnegative integer n, we have*

$$\mathbb{BH}_{n-1}\mathbb{BH}_{n+1}-(\mathbb{BH}_n)^2=(-qy)^{n-1}\sqrt{p^2x^2+4qy}AB\left[\frac{\widehat{\beta}\widehat{\alpha}}{\beta}-\frac{\widehat{\alpha}\widehat{\beta}}{\alpha}\right].$$

Proof. Since the Cassini identity is a special case of the Catalan identity, by taking $r=1$ in the Catalan identity theorem can be proved easily. □

Theorem 7 (d'Ocagne Identity)**.** *For the nonnegative integers m and n with $m\geq n$, we have*

$$\mathbb{BH}_m\mathbb{BH}_{n+1}-\mathbb{BH}_{m+1}\mathbb{BH}_n=(-qy)^{n-1}\sqrt{p^2x^2+4qy}AB\left[\beta^{m-n}\widehat{\beta}\widehat{\alpha}-\alpha^{m-n}\widehat{\alpha}\widehat{\beta}\right].$$

Proof. By using the Binet formula for the bivariate Horadam hybrinomials, we have

$$\begin{aligned}
&\mathbb{BH}_m\mathbb{BH}_{n+1}-\mathbb{BH}_{m+1}\mathbb{BH}_n\\
&=\left(A\alpha^{m-1}\widehat{\alpha}+B\beta^{m-1}\widehat{\beta}\right)\left(A\alpha^{n}\widehat{\alpha}+B\beta^{n}\widehat{\beta}\right)\\
&\quad-\left(A\alpha^{m}\widehat{\alpha}+B\beta^{m}\widehat{\beta}\right)\left(A\alpha^{n-1}\widehat{\alpha}+B\beta^{n-1}\widehat{\beta}\right)\\
&=AB\alpha^{m-1}\beta^{n}\widehat{\alpha}\widehat{\beta}+AB\alpha^{n}\beta^{m-1}\widehat{\beta}\widehat{\alpha}\\
&\quad-AB\alpha^{m}\beta^{n-1}\widehat{\alpha}\widehat{\beta}-AB\alpha^{n-1}\beta^{m}\widehat{\beta}\widehat{\alpha}\\
&=AB\alpha^{m}\beta^{n}\widehat{\alpha}\widehat{\beta}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)+AB\alpha^{n}\beta^{m}\widehat{\beta}\widehat{\alpha}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\\
&=(-qy)^{n-1}\sqrt{p^2x^2+4qy}AB\left[\beta^{m-n}\widehat{\beta}\widehat{\alpha}-\alpha^{m-n}\widehat{\alpha}\widehat{\beta}\right].
\end{aligned}$$

□

Theorem 8. *For any nonnegative integer $n\geq 1$, we have*

$$\begin{bmatrix}\mathbb{BH}_{n+3} & \mathbb{BH}_{n+2}\\ \mathbb{BH}_{n+2} & \mathbb{BH}_{n+1}\end{bmatrix}=\begin{bmatrix}\mathbb{BH}_{3} & \mathbb{BH}_{2}\\ \mathbb{BH}_{2} & \mathbb{BH}_{1}\end{bmatrix}\begin{bmatrix}px & 1\\ qy & 0\end{bmatrix}^n.$$

Proof. We prove the theorem using induction method on n.

For $n=1$, the result is obvious.

Assume that for any $n\geq 2$ the theorem holds

$$\begin{bmatrix}\mathbb{BH}_{n+3} & \mathbb{BH}_{n+2}\\ \mathbb{BH}_{n+2} & \mathbb{BH}_{n+1}\end{bmatrix}=\begin{bmatrix}\mathbb{BH}_{3} & \mathbb{BH}_{2}\\ \mathbb{BH}_{2} & \mathbb{BH}_{1}\end{bmatrix}\begin{bmatrix}px & 1\\ qy & 0\end{bmatrix}^n.$$

We must show that for $n+1$ the theorem holds

$$\begin{bmatrix} \mathbb{BH}_{n+4} & \mathbb{BH}_{n+3} \\ \mathbb{BH}_{n+3} & \mathbb{BH}_{n+2} \end{bmatrix} = \begin{bmatrix} \mathbb{BH}_{3} & \mathbb{BH}_{2} \\ \mathbb{BH}_{2} & \mathbb{BH}_{1} \end{bmatrix} \begin{bmatrix} px & 1 \\ qy & 0 \end{bmatrix}^{n+1}.$$

By using the induction hypothesis, we have

$$\begin{aligned} &\begin{bmatrix} \mathbb{BH}_{3} & \mathbb{BH}_{2} \\ \mathbb{BH}_{2} & \mathbb{BH}_{1} \end{bmatrix} \begin{bmatrix} px & 1 \\ qy & 0 \end{bmatrix}^{n+1} \\ &\quad = \begin{bmatrix} \mathbb{BH}_{3} & \mathbb{BH}_{2} \\ \mathbb{BH}_{2} & \mathbb{BH}_{1} \end{bmatrix} \begin{bmatrix} px & 1 \\ qy & 0 \end{bmatrix}^{n} \begin{bmatrix} px & 1 \\ qy & 0 \end{bmatrix} \\ &\quad = \begin{bmatrix} \mathbb{BH}_{n+3} & \mathbb{BH}_{n+2} \\ \mathbb{BH}_{n+2} & \mathbb{BH}_{n+1} \end{bmatrix} \begin{bmatrix} px & 1 \\ qy & 0 \end{bmatrix} \\ &\quad = \begin{bmatrix} px\mathbb{BH}_{n+3} + qy\mathbb{BH}_{n+2} & \mathbb{BH}_{n+3} \\ px\mathbb{BH}_{n+2} + qy\mathbb{BH}_{n+1} & \mathbb{BH}_{n+2} \end{bmatrix} \\ &\quad = \begin{bmatrix} \mathbb{BH}_{n+4} & \mathbb{BH}_{n+3} \\ \mathbb{BH}_{n+3} & \mathbb{BH}_{n+2} \end{bmatrix} \end{aligned}$$

which completes the proof. □

3. Conclusion

In this research, we introduce the Horadam polynomial and hybrinomial with two variables x and y, called the bivariate Horadam polynomial and hybrinomia, respectively. Also, we obtain Binet formula, generating function and some properties for the bivariate Horadam hybrinomials. Moreover, we get Catalan, Cassini and d'Ocagne identities for these hybrinomials. Finally, the matrix representations of the bivariate Horadam hybrinomials were introduced.

4. Acknowledgments

The author would like to thank the anonymous reviewer for his/her valuable comments and suggestions to improve the quality of the paper

References

[1] Özdemir M. Introduction to Hybrid Numbers. *Advances in Applied Clifford Algebras* 2018; 28, 11.

[2] Szynal-Liana A., Wloch I. On Jacobsthal and Jacobsthal-Lucas Hybrid Numbers. Annales Mathematicae Silesianae 2019; 33:276-283.

[3] Szynal-Liana A. The Horadam Hybrid Numbers. *Discussiones Mathematicae-General Algebra and Applications* 2018; 38(1):91-98.

[4] Kızılateş C. A New Generalization of Fibonacci Hybrid and Lucas Hybrid Number. *Chaos, Solitons and Fractals* 2020; 130.

[5] Catarino P. On $k-$Pell Hybrid Numbers. *Journal of Discrete Mathematical Sciences and Cryptography 2019;* 22(1):83-89.

[6] Szynal-Liana A., Wloch I. On Pell and Pell-Lucas Hybrid Numbers. *Commentationes Mathematicae* 2018; 58:11-17.

[7] Szynal-Liana A., Wloch I. On Generalized Mersenne Hybrid Numbers. *Annales Universitatis Mariae Curie-Sklodowska Lublin-Polonia* 2020; 74(1):77-84.

[8] Szynal-Liana A., Włoch, I. The Fibonacci Hybrid Numbers. *Util. Math.* 2019; 110:3-10.

[9] Qi F., Kızılateş C., Du, W-S. A Closed Formula for the Horadam Polynomials in Terms of a Tridiagonal Determinant. *Symmetry* 2019; 11(6):782.

[10] Horzum T., Kocer EG. On Some Properties of Horadam Polynomials. *Int. Math. Forum* 2009; 4(25):1243-1252.

[11] Djordjevic SS., Djordjevic GG. Generalized Horadam Polynomials and Numbers. *Analele Stiintifice Ale Universitatii "Ovidius" Constanta - Seria Matematica* 2018; 26(1):91-101.

[12] Taşkara N., Uslu K., Güleç HH. The Algebraic Properties of Horadam Polynomials. *Selçuk Journal of Applied Mathematics* 2010; Special Issue:73-80.

[13] Kızılateş C. A Note on Horadam Hybrinomials, *Preprints* 2020; 2020010116 (doi:10.20944/preprints202001.0116.v1).

Email address: emresevgi@gazi.edu.tr

Department of Mathematics, Science Faculty, Gazi University, Ankara, Turkey