

A new fractional order chaotic dynamical system and its synchronization using optimal control

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Abstract

In this work, we introduce a new three-dimensional chaotic differential dynamical system. We find equilibrium points of this system and provide the stability conditions for various fractional orders. Numerical simulations will be used to investigate the chaos in the proposed system. A simple linear control will be used to control the chaotic oscillations. Further, we propose an optimal control which is based on the fractional order of the system and use it to synchronize the new chaotic system.

Keywords: Fractional derivative, Chaos, Synchronization, Optimal control.

1 Introduction

The differential equation is a prime tool used by Scientists in modeling various natural phenomena. To make it more realistic, the generalized operator viz. fractional derivative [1, 2] is introduced by the researchers. The order in fractional derivative can be any real or complex number, a function of time or may be distributed over some interval. This flexible order makes fractional derivative more suitable to model the intermediate processes and the memory properties. The fractional differential equations (FDE) have applications in Bioengineering [3], Viscoelasticity [4], Control theory [5], and so on. The analysis of FDEs is presented in [6, 7, 8, 9, 10, 11, 12]. The difficult task of numerical solutions of FDEs is handled in [13, 14].

The signals generated by a higher-order deterministic nonlinear system which are aperiodic for all the time and depend sensitively on initial conditions are termed as “chaotic” [15, 16]. The chaos can occur in a nonlinear autonomous differential dynamical system of order three or more, a delay differential equation and a discrete map. The most celebrated examples of chaos are the Lorenz system [17] and the logistic map [18]. Few other examples include the systems viz. Chen [19], Chua [20], Lu [21], Rossler [22], Bhalekar-Gejji(BG) [23], Modified BG [24], Proto BG [25], Pehlivan [26], and so on.

The fractional order counterparts of these classical systems also produce chaotic signals for certain values. For a commensurate order case, it is observed that the system remains chaotic up to some threshold value of fractional order and then becomes stable. Few examples of fractional order chaotic systems are described in [27, 28, 29, 30, 31, 32, 33].

Chaos in fractional ordered delayed systems is analyzed in [34, 35, 36, 37, 38].

The paper is organized as follows:

The basics are described in Section 2. We propose a new chaotic system and present the stability, bifurcation and chaos in Section 3. Sections 4 and 5 deal with the chaos control and synchronization, respectively. The comments on the incommensurate order case are presented in Section 6. The conclusions are summarized in Section 7.

2 Preliminaries

This section deals with basic definitions and results given in the literature [1, 2, 8, 39]. Throughout this section, we take $n \in \mathbb{N}$.

Definition 2.1. Let $\alpha \geq 0$ ($\alpha \in \mathbb{R}$). Then Riemann-Liouville (RL) fractional integral of function $f \in C[0, b]$, $t > 0$ of order ' α ' is defined as,

$${}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau. \quad (1)$$

Definition 2.2. The Riemann-Liouville (RL) fractional derivative of order $\alpha > 0$ of function $f \in C[0, b]$, $t > 0$ is defined as,

$${}^{RL}D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau, & \text{if } n-1 < \alpha < n \\ \frac{d^n}{dt^n} f(t), & \text{if } \alpha = n. \end{cases} \quad (2)$$

Definition 2.3. The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha \leq n$ is defined for $f \in C^n[0, b]$, $t > 0$ as,

$${}_0^CD_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t - \tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & \text{if } n-1 < \alpha < n \\ \frac{d^n}{dt^n} f(t), & \text{if } \alpha = n. \end{cases} \quad (3)$$

Definition 2.4. The one-parameter Mittag-Leffler function is defined as,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}, (\alpha > 0). \quad (4)$$

The two-parameter Mittag-Leffler function is defined as,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, (\alpha > 0, \beta > 0). \quad (5)$$

Properties

(i) Let $n-1 < \alpha \leq n$ and $\beta \geq 0$

$${}_0^CD_t^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(-\alpha+\beta+1)} t^{\beta-\alpha}, & \text{if } \beta > n-1, \beta \in \mathbb{R} \\ 0, & \text{if } \beta \in \{0, 1, 2, \dots, n-1\}. \end{cases}$$

$$(ii) {}_0^CD_t^\alpha {}_0I_t^\beta f(t) = \begin{cases} {}_0I_t^{\beta-\alpha} f(t), & \text{if } \beta > \alpha \\ f(t), & \text{if } \beta = \alpha \\ {}_0^CD_t^{\alpha-\beta} f(t), & \text{if } \alpha > \beta. \end{cases}$$

(iii) ${}_0^CD_t^\alpha c = 0$, where c is a constant.

(iv) ${}^{RL}D_t^\alpha c = \frac{c t^{-\alpha}}{\Gamma(1-\alpha)}$, where c is a constant.

Theorem 2.1. [40] Solution of homogeneous fractional order differential equation

$${}_0^CD_t^\alpha x(t) + \lambda x(t) = 0, \quad 0 < \alpha < 1 \quad (6)$$

is given by,

$$x(t) = x(0)E_\alpha(-\lambda t^\alpha). \quad (7)$$

Stability Analysis

Consider the following fractional order system,

$$\begin{aligned} {}_0^CD_t^{\alpha_1} x_1 &= f_1(x_1, x_2, \dots, x_n), \\ {}_0^CD_t^{\alpha_2} x_2 &= f_2(x_1, x_2, \dots, x_n), \\ &\vdots \\ {}_0^CD_t^{\alpha_n} x_n &= f_n(x_1, x_2, \dots, x_n) \end{aligned} \quad (8)$$

where $0 < \alpha_i < 1$ are fractional orders. If $\alpha_1 = \alpha_2 = \dots = \alpha_n$, then the system (8) is called commensurate order system, otherwise incommensurate order system.

A point $E = (x_1^*, x_2^*, \dots, x_n^*)$ is called an equilibrium point of the system (8) if

$$f_i(E) = f_i(x_1^*, x_2^*, \dots, x_n^*) = 0, \quad \text{for each } i = 1, 2, \dots, n.$$

(a) Commensurate order system:

Theorem 2.2. [41, 42] Consider $\alpha = \alpha_1 = \alpha_2 = \dots = \alpha_n$ in (8). An equilibrium point E of the system (8) is locally asymptotically stable if all the eigenvalues of the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (9)$$

evaluated at $E = (x_1^*, x_2^*, \dots, x_n^*)$ satisfy the following condition

$$|\arg(\text{Eig}(J|_E))| > \frac{\alpha\pi}{2}.$$

(b) Incommensurate order system:

Theorem 2.3. [7] Consider the incommensurate fractional ordered dynamical system given by (8). Let $\alpha_i = \frac{v_i}{u_i}$, $\gcd(u_i, v_i) = 1$, u_i, v_i be positive integers. Define M to be the least common multiple of u_i 's. Define,

$$\Delta(\lambda) = \text{diag}([\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \dots, \lambda^{M\alpha_n}]) - J \quad (10)$$

where, J is the Jacobian matrix as defined in (9) evaluated at point E . If all the roots λ 's of $\det(\Delta(\lambda)) = 0$ satisfy $|\arg(\lambda)| > \frac{\alpha\pi}{2}$, then E is locally asymptotically stable. This condition is equivalent to the following inequality

$$\frac{\pi}{2M} - \min_i |\arg(\lambda_i)| < 0. \quad (11)$$

The term $\frac{\pi}{2M} - \min_i |\arg(\lambda_i)|$ is called as the instability measure for equilibrium points in fractional order systems (IMFOS). Hence, a necessary condition [7] for fractional order system (8) to exhibit chaotic attractor is

$$\text{IMFOS} \geq 0. \quad (12)$$

Note that, the condition (12) is not sufficient [7, 29] for chaos to exist.

3 New chaotic system

We propose the following chaotic dynamical system,

$$\begin{aligned} \dot{x} &= ax - y^2, \\ \dot{y} &= by - z + dxz, \\ \dot{z} &= -gz + 4xy - hx^2, \end{aligned} \quad (13)$$

where $a, b, d, g, h \in \mathbb{R}$ are parameters. When $a = -1, b = 2.5, d = -5, g = 5.5$ and $h = -0.2$, system (13) shows chaotic behavior.

If $X_* = (x_*, y_*, z_*)$ is an equilibrium point of (13), then the Jacobian $J(X_*)$ of this system at X_* is given by

$$J(X_*) = \begin{bmatrix} a & -2y_* & 0 \\ dz_* & b & -1 + dx_* \\ -2hx_* + 4y_* & 4x_* & -g \end{bmatrix}.$$

3.1 Bifurcation Analysis

Out of five parameters a, b, d, g, h of the system (13), we hold any four parameters fixed and vary the remaining one to present the bifurcation analysis. This analysis is described in Table 1. The bifurcation diagrams and the trajectories/phase portraits of the corresponding cases are presented in Figure 1- Figure 4.

Fixed parameters	Changing parameter	Behavior of trajectories	Corresponding figures
$b = 2.5$ $d = -5$ $g = 5.5$ $h = -0.2$	$a = -4$	Stable orbit	Figure 1(a)
	$a = -3.5$	Limit cycle	Figure 1(b)
	$a = -2.8$	Limit cycle	Figure 1(c)
	$a = -2.6$	Limit cycle	Figure 1(d)
	$a = -2$	Chaos	Figure 1(e)
$a = -1$ $d = -5$ $g = 5.5$ $h = -0.2$	$b = 1.6$	Stable orbit	Figure 2(a)
	$b = 1.8$	Chaos	Figure 2(b)
	$b = 2.5$	Chaos	Figure 2(c)
	$b = 4.35$	Limit cycle	Figure 2(d)
	$b = 5.5$	Limit cycle	Figure 2(e)
$b = 2.5$ $d = -5$ $h = -0.2$ $a = -1$	$g = 2.7$	Limit cycle	Figure 3(b)
	$g = 3.4$	Limit cycle	Figure 3(c)
	$g = 3.8$	Limit cycle	Figure 3(d)
	$g = 4.5$	Chaos	Figure 3(a)
	$g = 5.5$	Chaos	Figure 3(e)
$a = -1$ $b = 2.5$ $d = -5$ $g = 5.5$	$h = -1.05$	Stable orbit	Figure 4(a)
	$h = -1$	Limit cycle	Figure 4(b)
	$h = -0.2$	Limit cycle	Figure 4(c)
	$h = 0.5$	Limit cycle	Figure 4(d)
	$h = 0.97$	Chaos	Figure 4(e)

Table 1: Observations for different values of parameters

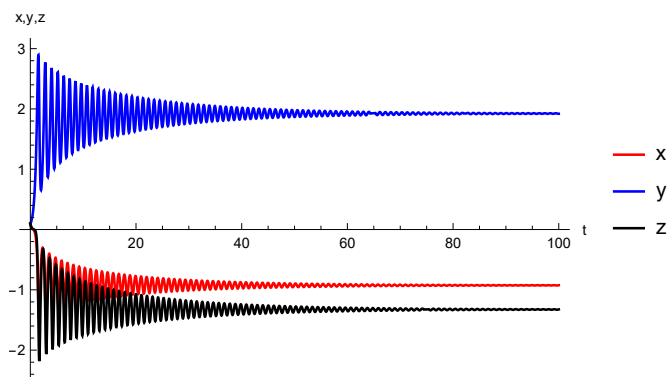
3.2 Equilibrium points and their nature

The equilibrium point of three dimensional dynamical system is called a saddle point of index 1 (respectively, index 2) if it generates one (respectively, two) unstable eigenvalue(s). The scrolls of chaotic attractors are usually around the saddle points of index 2, whereas the saddles of index 1 connect these scrolls [43, 44].

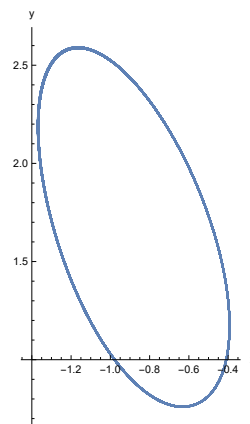
In Table 2, we classify equilibrium points of the proposed system (13) with parameter values $a = -1, b = 2.5, d = -5, g = 5.5$ and $h = -0.2$. It is observed that the chaotic attractor in this system is in the neighborhood of the equilibrium points O, E_1 and E_2 .

	Equilibrium points X_*	Eigenvalues of $J(X_*)$	Nature of X_*
O	$(0, 0, 0)$	$-5.5, 2.5, -1$	saddle point of index 1
E_1	$(-0.95606, -0.977783, 0.646631)$	$-4.67311, 0.336553 \pm 3.61409i$	saddle point of index 2
E_2	$(-0.91615, 0.957157, -0.668266)$	$-4.86932, 0.434662 \pm 3.57945i$	saddle point of index 2
E_3	$(-399.997, -19.9999, 0.0250126)$	$-399.997, -19.9999, 0.0250126$	saddle point of index 2

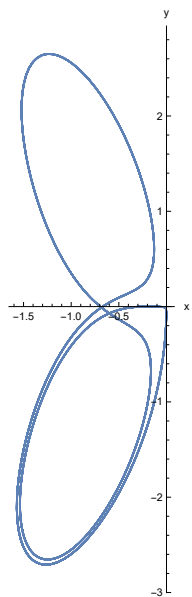
Table 2: Equilibrium points of (13), eigenvalues of Jacobian at these points and the nature of equilibriums



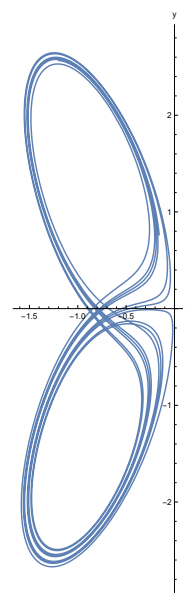
(a) $a = -4$



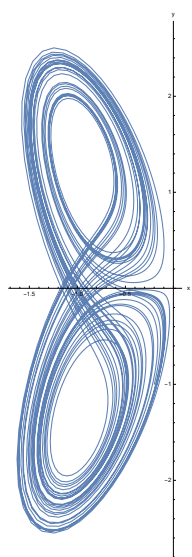
(b) $a = -3.5$



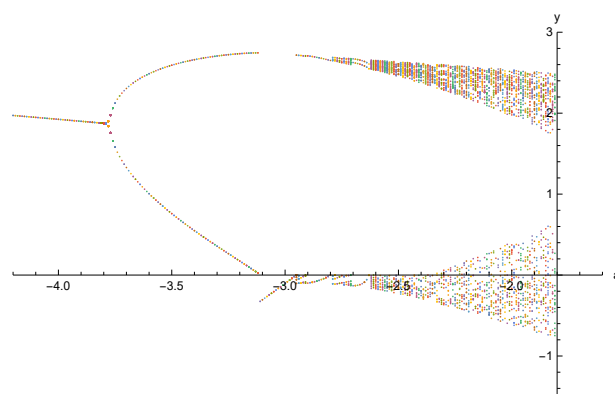
(c) $a = -2.8$



(d) $a = -2.6$

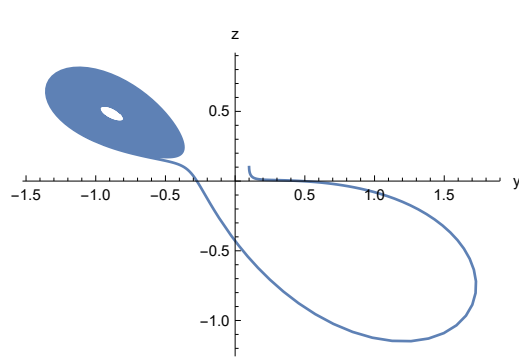


(e) $a = -2$

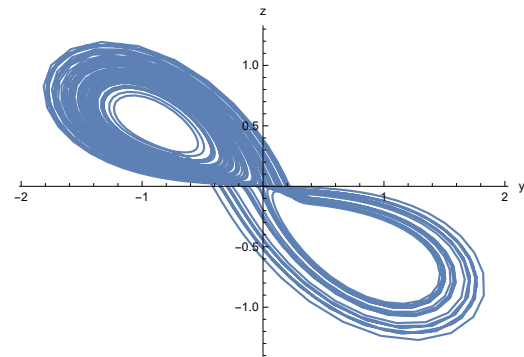


(f) Bifurcation with parameter a

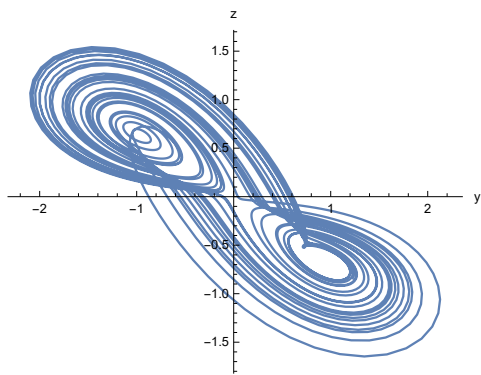
Figure 1: Bifurcation analysis for parameter a



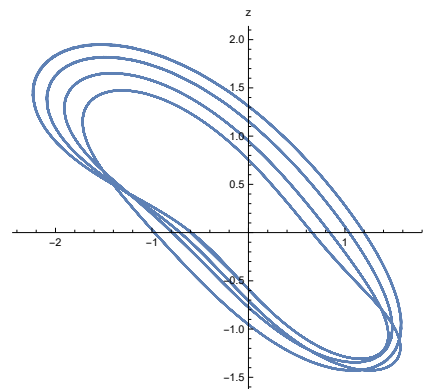
(a) $b = 1.6$



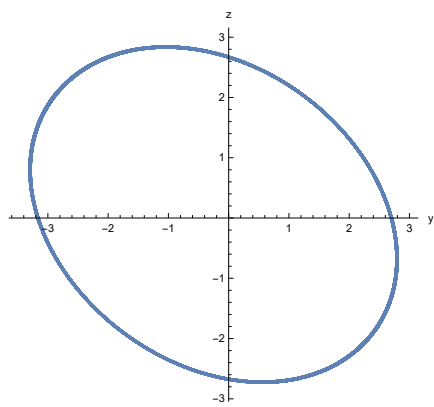
(b) $b = 1.8$



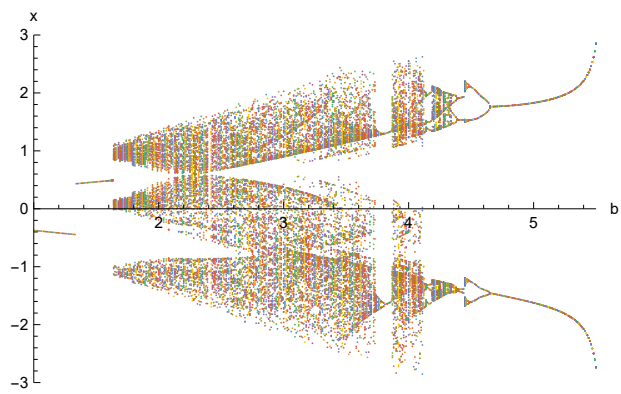
(c) $b = 2.5$



(d) $b = 4.35$



(e) $b = 5.5$



(f) Bifurcation with parameter b

Figure 2: Bifurcation analysis for parameter b

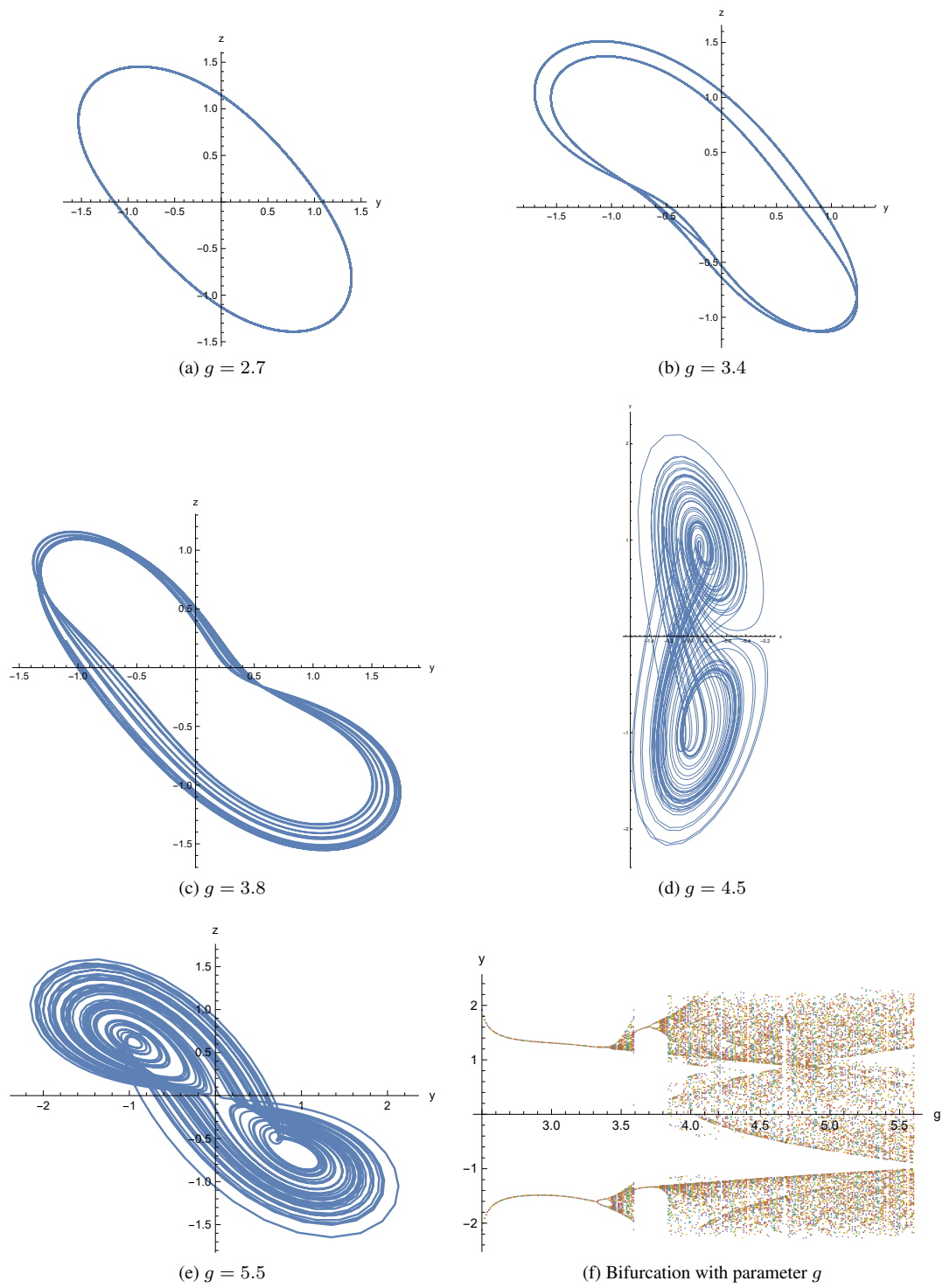
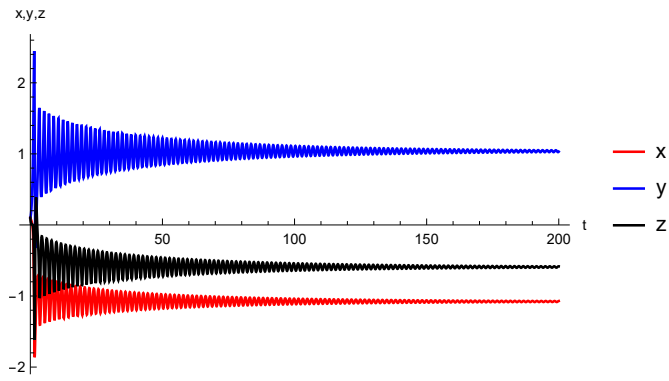
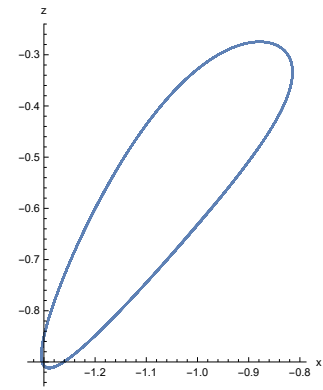


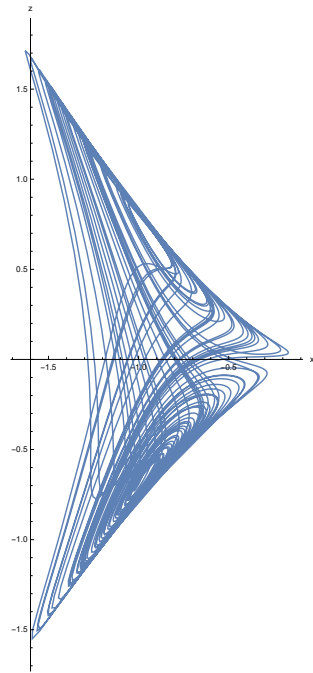
Figure 3: Bifurcation analysis for parameter g



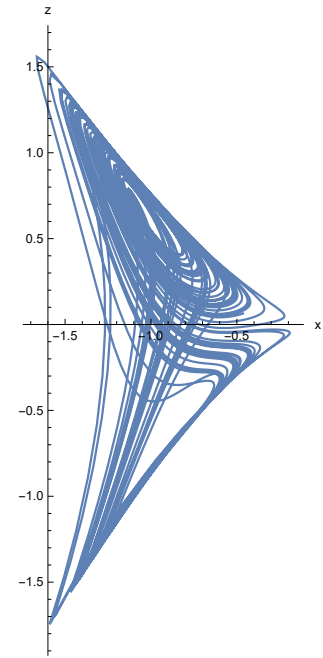
(a) $h = -1.05$



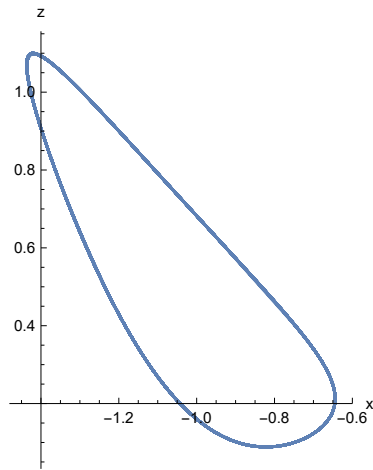
(b) $h = -1$



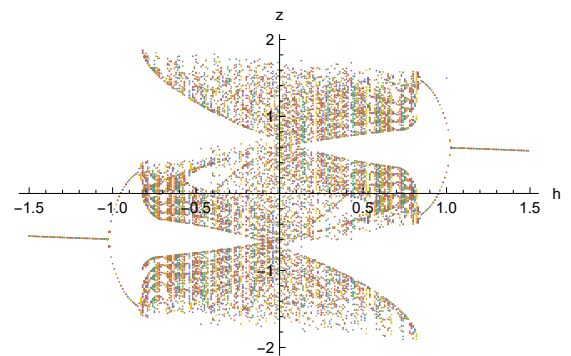
(c) $h = -0.2$



(d) $h = 0.5$



(e) $h = 0.97$



(f) Bifurcation with parameter h

Figure 4: Bifurcation analysis for parameter h

3.3 Chaos

Consider the chaotic system,

$$\begin{aligned}\dot{x} &= -x - y^2 \\ \dot{y} &= 2.5y - z - 5xz \\ \dot{z} &= -5.5z + 4xy - 0.2x^2.\end{aligned}\tag{14}$$

We know that, extreme sensitivity to initial conditions is one of the significant characteristic of chaos. In the Figure 5, we have plotted the trajectories $x(t)$ with initial conditions $(0.1, 0.1, 0.1)$ and $(0.10001, 0.1, 0.1)$ with small variation of 0.00001. We can see that, there is huge amount of difference between these two trajectories. This high sensitivity to initial conditions indicates chaos in (14).

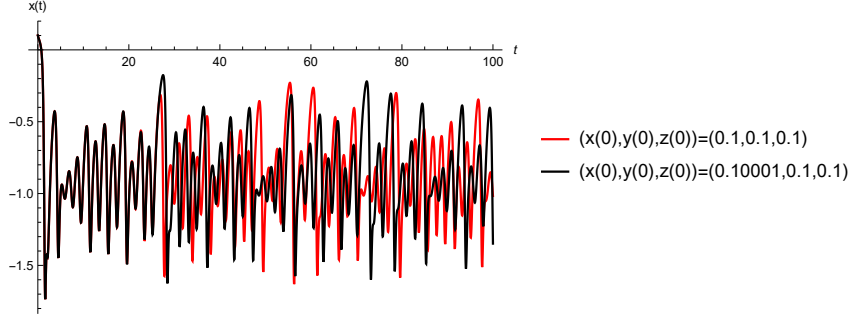


Figure 5: Solution trajectory $x(t)$ of the system (14) with slightly different initial conditions.

Chaotic attractor and wave forms are shown in Figure 6.

3.4 Stability analysis of proposed fractional order system

Consider the fractional order generalization of (14) as

$$\begin{aligned}{}_0^C D_t^\alpha x &= -x - y^2 \\ {}_0^C D_t^\beta y &= 2.5y - z - 5xz \\ {}_0^C D_t^\gamma z &= -5.5z + 4xy - 0.2x^2.\end{aligned}\tag{15}$$

Equilibrium points of fractional-order system (15) and the eigenvalues of the Jacobian evaluated at these equilibrium points are same as their classical counterparts, described in Table 2.

3.4.1 Commensurate fractional order system

Consider the fractional order system (15) with $\alpha = \beta = \gamma$.

Define

$$\alpha_*(E) = \frac{2}{\pi} \left[\min_{\lambda_E} |\arg(\lambda_E)| \right]$$

as stability bound for the equilibrium point E . Therefore, E is stable if $0 < \alpha < \alpha_*(E)$ and unstable for $\alpha_*(E) < \alpha \leq 1$. Note that, instability of all the equilibrium points is the necessary condition [45] to exists chaos in the commensurate order system (15).

In this case,

$$\begin{aligned}\alpha_*(O) &= 0, \\ \alpha_*(E_1) &= 0.92307 \\ \text{and } \alpha_*(E_2) &= 0.940887.\end{aligned}$$

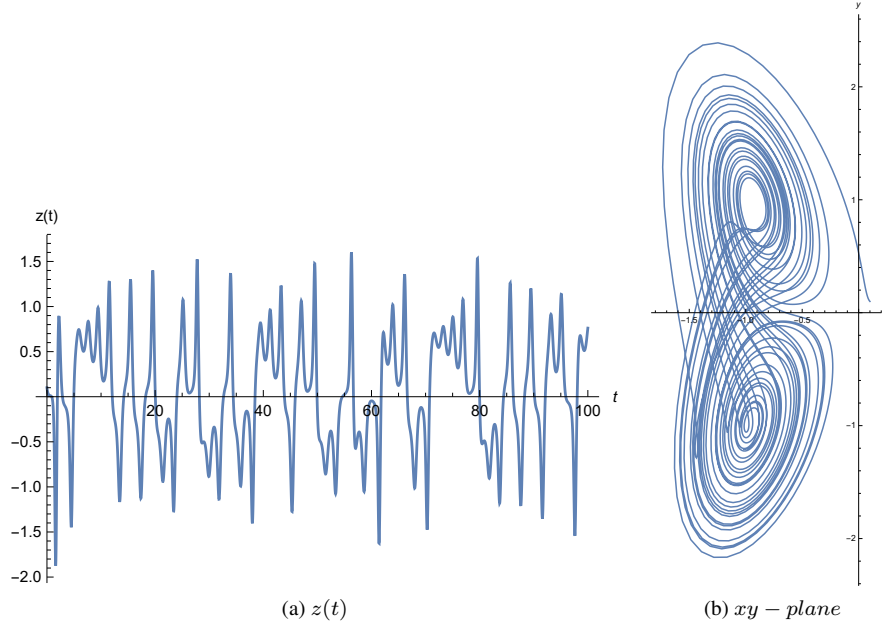


Figure 6: Chaos in system (14)

We define

$$\bar{\alpha} = \max_E \alpha_*(E). \quad (16)$$

In this case, $\bar{\alpha} = 0.940887$. If $\alpha < \bar{\alpha}$, then at least one of the equilibrium points O , E_1 or E_2 becomes stable and the system cannot be chaotic.

We further define the “threshold value” α_t such that the system is chaotic for $\alpha > \alpha_t$ and achaotic for $\alpha < \alpha_t$. For the commensurate order system (15) $\alpha_t = 0.940887$ (cf. Figures 7 (a-d)).

Note that, α_t is the value obtained using numerical observations and $\alpha_t \geq \bar{\alpha}$.

Based on the observations of various fractional order commensurate order chaotic systems we propose the following conjecture.

Conjecture 1. Suppose that the system

$${}_0^C D_t^\alpha X = f(X) \quad (17)$$

is chaotic with the threshold value α_t . If $\bar{\alpha}$ defined in (16) corresponds to the equilibrium points associated with the chaotic attractor then $\alpha_t = \bar{\alpha}$.

3.4.2 Incommensurate fractional order system

We consider the system (15) with two cases viz. $\alpha \geq 0.855$, $\beta = \gamma = 1$ and $\alpha = \beta = 1$, $\gamma \geq 0.68$. We show that $\text{IMFOS} \geq 0$ is not sufficient condition to exist chaos in the system.

Numerical simulations:

Case 1:

Let $\alpha = \frac{43}{50} = 0.86$ and $\beta = \gamma = 1$.

In this case $M = \text{LCM}(50, 1, 1) = 50$, $\Delta_1 = \text{diag}(\lambda^{43}, \lambda^{50}, \lambda^{50}) - J(O)$

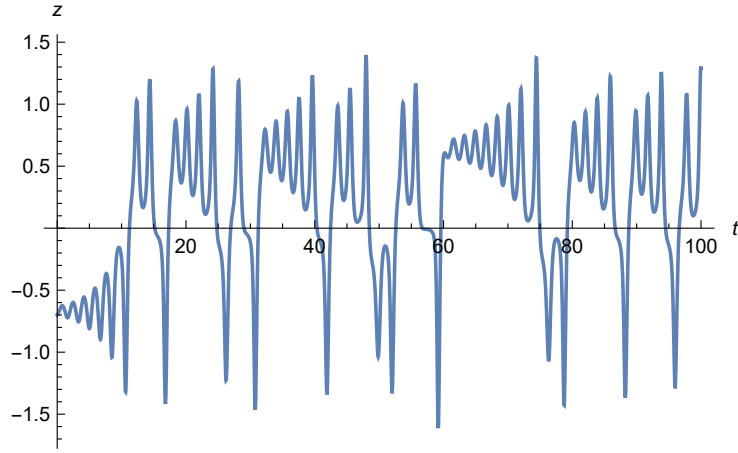
$\text{Det}(\Delta_1) = (1 + \lambda^{43})(-2.5 + \lambda^{50})(5.5 + \lambda^{50})$

$\text{IMFOS}(O) = \frac{\pi}{100} - 0 = 0.0314159 > 0$

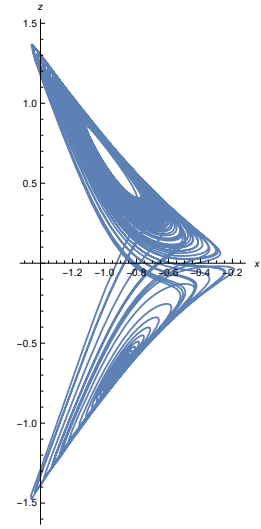
Similarly, we can find that, $\text{IMFOS}(E_1) = \frac{\pi}{100} - 0.0313462 = 0.0000696784 > 0$ and

$\text{IMFOS}(E_2) = \frac{\pi}{100} - 0.0308576 = 0.000558308 > 0$

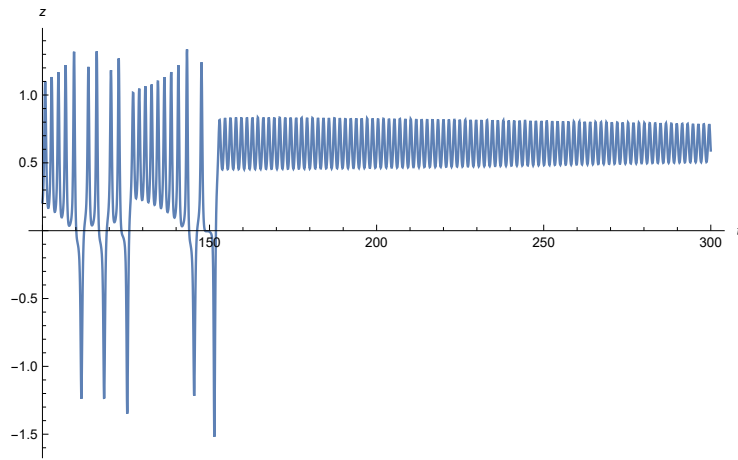
So, the system is unstable but there does not exist chaos (cf. Figure 8(a)).



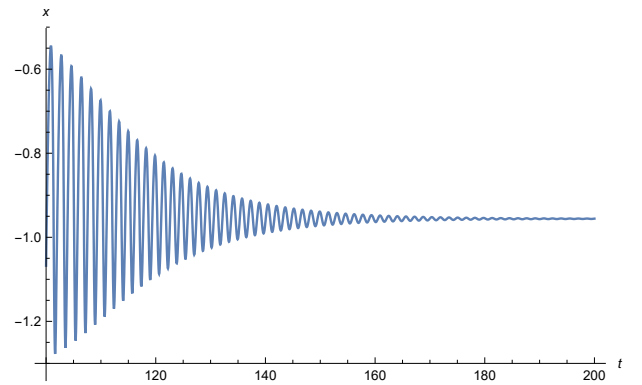
(a) $\alpha = 0.97$, Chaotic trajectory



(b) $\alpha = 0.940887$, Chaotic attractor



(c) $\alpha = 0.94$, stable



(d) $\alpha = 0.93$, stable

Figure 7: Stability of commensurate fractional order system (15)

Case 2:

Let $\alpha = \frac{9}{10} = 0.9$ and $\beta = \gamma = 1$.

In this case $M = LCM(10, 1, 1) = 10$, $\Delta_2 = \text{diag}(\lambda^9, \lambda^{10}, \lambda^{10}) - J(O)$

$\text{Det}(\Delta_2) = (1 + \lambda^9)(-2.5 + \lambda^{10})(5.5 + \lambda^{10})$

$\text{MFOS}(O) = \frac{\pi}{20} - 0 = 0.15708 > 0$

Similarly, we can find that, $\text{MFOS}(E_1) = \frac{\pi}{20} - 0.154105 = 0.00297418 > 0$ and

$\text{MFOS}(E_2) = \frac{\pi}{20} - 0.151553 = 0.00552665 > 0$

The system is chaotic (cf. Figure 8(b)).

Case 3:

Let $\gamma = \frac{85}{100} = \frac{17}{20}$ and $\alpha = \beta = 1$.

In this case $M = LCM(20, 1, 1) = 20$, $\Delta_3 = \text{diag}(\lambda^{20}, \lambda^{20}, \lambda^{17}) - J(O)$

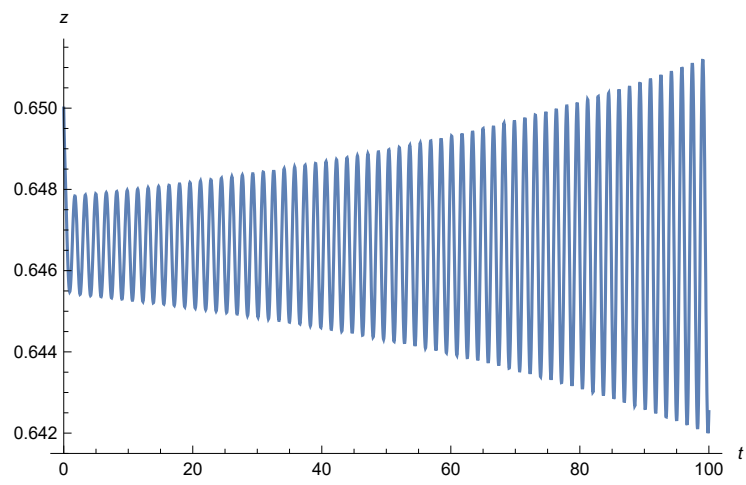
$\text{Det}(\Delta_3) = (5.5 + \lambda^{17})(-2.5 + \lambda^{20})(1 + \lambda^{20})$

$\text{MFOS}(O) = \frac{\pi}{40} - 0 = 0.0785398 > 0$

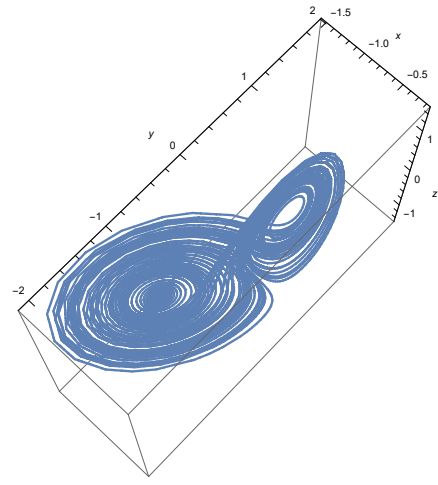
Similarly, we can find that, $\text{MFOS}(E_1) = \frac{\pi}{40} - 0.0763924 = 0.00214747 > 0$ and

$\text{MFOS}(E_2) = \frac{\pi}{40} - 0.0748748 = 0.00366499 > 0$

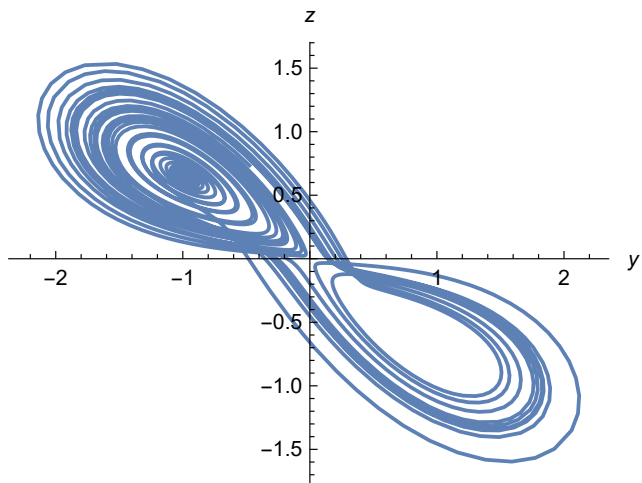
The system is chaotic (cf. Figure 8(c)).



(a) $\alpha = 0.86, \beta = \gamma = 1$



(b) $\alpha = 0.9, \beta = \gamma = 1$



(c) $\gamma = 0.85, \alpha = \beta = 1$

Figure 8: Stability of incommensurate fractional order system

4 Chaos control using linear control

Let us consider the controlled system,

$$\begin{aligned} {}^C_0D_t^\alpha x &= -x - y^2 \\ {}^C_0D_t^\alpha y &= 2.5y - z - 5xz + u \\ {}^C_0D_t^\alpha z &= -5.5z + 4xy - 0.2x^2, \end{aligned} \quad (18)$$

where u is the linear feedback control term. We set $u = ky$, where k is a parameter to be determined so that the system (18) is stable. Clearly $O = (0, 0, 0)$ is one of the equilibrium points of the system (18).

The Jacobian of system (18) evaluated at O is,

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2.5 + k & -1 \\ 0 & 0 & -5.5 \end{bmatrix}.$$

Eigenvalues of this Jacobian matrix are -1 , $2.5 + k$ and -5.5 . Therefore, if we choose $k < -2.5$ then the equilibrium point O becomes stable and the system (18) loose the chaotic nature. In Figure 9 we observe stable behavior of system (18) with $\alpha = 0.95$ and $k = -3$.

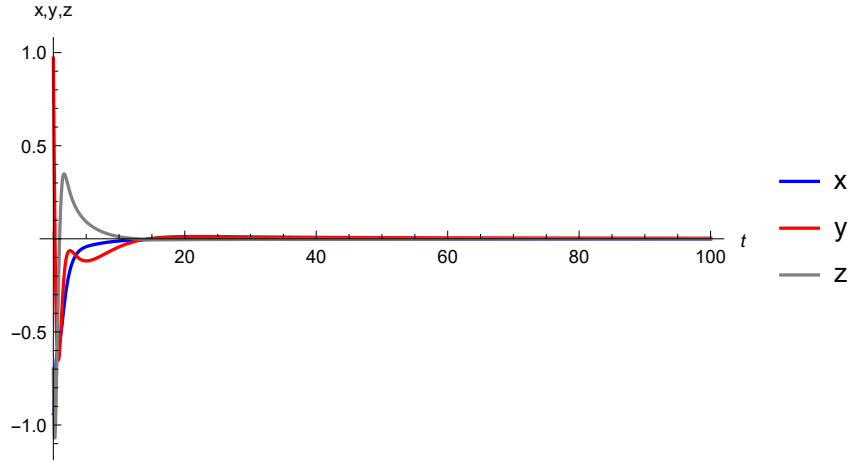


Figure 9: $\alpha = 0.95$, $k = -3$, $X(0) = (-0.94, 0.97, -0.699)^T$

5 Chaos synchronization using optimal control

We say that the chaotic systems get synchronized if the difference between their states tends to zero with increasing time. Synchronization between many pairs of fractional ordered chaotic systems are studied in the literature. Most of the fractional ordered chaotic systems are synchronized by using active control [46, 47], adaptive control [48, 49], sliding mode control [50, 51], impulsive control [52, 53], projective control [54, 55], etc. In this section, we propose a synchronization technique based on optimal control which depends on the fractional order of the system.

Here we have taken the system (15) as both drive and response system.

Let us consider a drive system as

$$\begin{aligned} {}^C_0D_t^\alpha x_1 &= -x_1 - y_1^2 \\ {}^C_0D_t^\alpha y_1 &= 2.5y_1 - z_1 - 5x_1z_1 \\ {}^C_0D_t^\alpha z_1 &= -5.5z_1 + 4x_1y_1 - 0.2x_1^2 \end{aligned} \quad (19)$$

and response system as

$$\begin{aligned} {}^C_0D_t^\alpha x_2 &= -x_2 - y_2^2 + u_1 \\ {}^C_0D_t^\alpha y_2 &= 2.5y_2 - z_2 - 5x_2z_2 + u_2 \\ {}^C_0D_t^\alpha z_2 &= -5.5z_2 + 4x_2y_2 - 0.2x_2^2 + u_3. \end{aligned} \quad (20)$$

We define the error functions as,

$$e_1 = x_1 - x_2, \quad e_2 = y_1 - y_2, \quad e_3 = z_1 - z_2 \quad (21)$$

The error system can be given by using equations (19), (20) and (21) as

$$\begin{aligned} {}^C_0D_t^\alpha e_1 &= -e_1 + y_2^2 - y_1^2 - u_1 \\ {}^C_0D_t^\alpha e_2 &= 2.5e_2 - e_3 - 5x_1z_1 + 5x_2z_2 - u_2 \\ {}^C_0D_t^\alpha e_3 &= -5.5e_3 + 4x_1y_1 - 4x_2y_2 - 2 + 0.2x_2^2 - 0.2x_1^2 - u_3. \end{aligned} \quad (22)$$

Let us choose the control terms $u_i(t)$ in system (20) as,

$$\begin{aligned} u_1 &= -e_1 + y_2^2 - y_1^2 - \cos\left(\frac{\alpha\pi}{2} + \epsilon\right)e_1 - \sin\left(\frac{\alpha\pi}{2} + \epsilon\right)e_2 \\ u_2 &= 2.5e_2 - e_3 - 5x_1z_1 + 5x_2z_2 + \sin\left(\frac{\alpha\pi}{2} + \epsilon\right)e_1 - \cos\left(\frac{\alpha\pi}{2} + \epsilon\right)e_2 \\ u_3 &= 4x_1y_1 - 4x_2y_2 - 2 + 0.2x_2^2 - 0.2x_1^2, \end{aligned} \quad (23)$$

where ϵ is any small positive real number.

Note that, the control terms u_i depend on fractional order α . Hence, for the given fractional order α the optimal strength of the controls is utilized in contrast with the conventional controls.

Now the error system (22) becomes,

$$\begin{aligned} {}^C_0D_t^\alpha e_1 &= \cos\left(\frac{\alpha\pi}{2} + \epsilon\right)e_1 + \sin\left(\frac{\alpha\pi}{2} + \epsilon\right)e_2 \\ {}^C_0D_t^\alpha e_2 &= -\sin\left(\frac{\alpha\pi}{2} + \epsilon\right)e_1 + \cos\left(\frac{\alpha\pi}{2} + \epsilon\right)e_2 \\ {}^C_0D_t^\alpha e_3 &= -5.5e_3. \end{aligned} \quad (24)$$

The eigenvalues of the coefficient matrix of linear system (24) are,

$$\lambda_{\pm} = e^{\pm i(\frac{\alpha\pi}{2} + \epsilon)} \quad \text{and} \quad \lambda_1 = -5.5. \quad (25)$$

Clearly, all the eigenvalues satisfy the condition, $|\arg(\lambda)| > \frac{\alpha\pi}{2}$.

So, the error system (24) is stable and hence the systems (19) and (20) get synchronized.

Figures 10(a)-(c) show the synchronization where the response system is shown by the dashed line. Here we have taken $\alpha = 0.95$ and the initial conditions as $x_1(0) = -0.94$, $y_1(0) = 0.97$, $z_1(0) = -0.699$, $x_2(0) = -0.9$, $y_2(0) = -0.9$ and $z_2(0) = 0.6$. In Figure 10(d), the error terms $e_i(t)$ of drive and response systems are plotted.

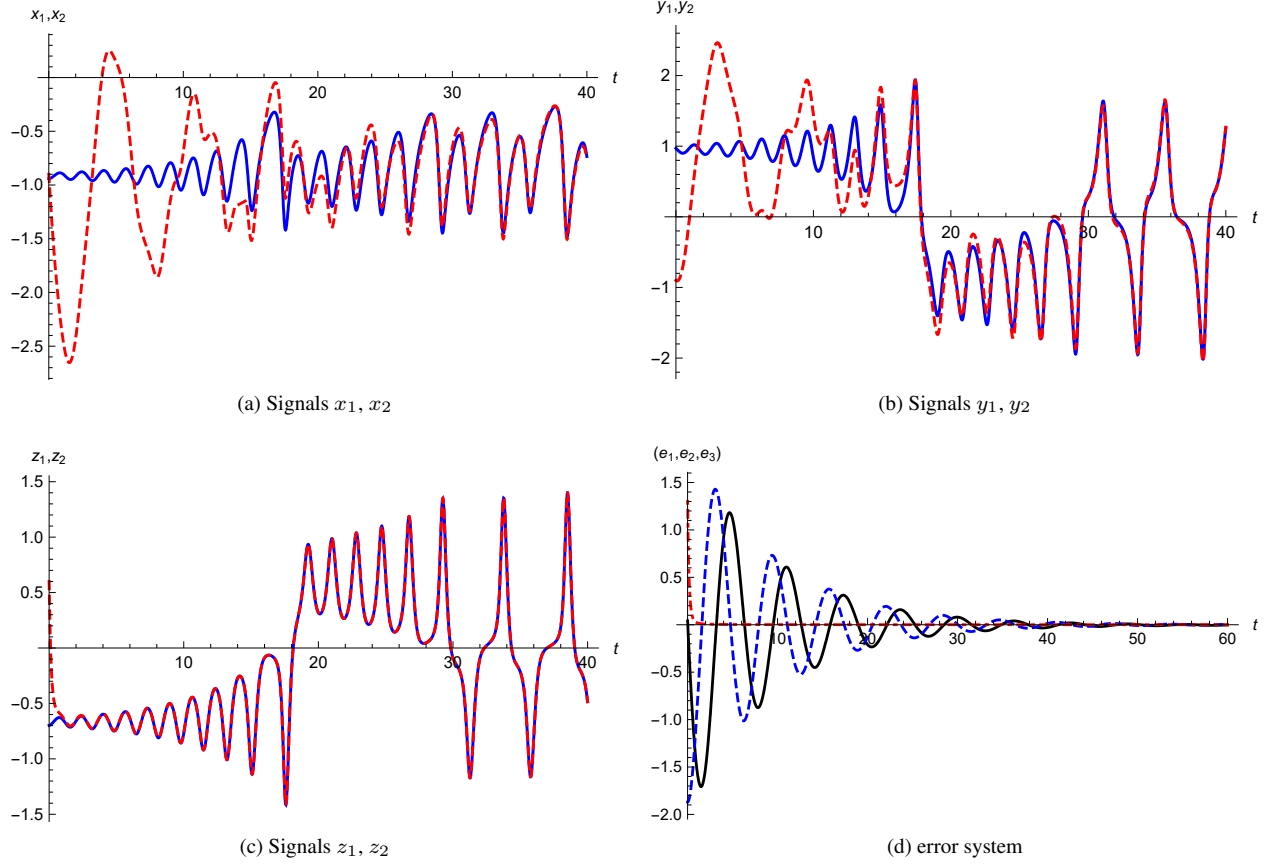


Figure 10: Synchronization of chaos in the proposed system

6 Comment on a case of incommensurate order

The behavior of the commensurate fractional order systems is relatively simpler than that of incommensurate order. Analyzing stability is a difficult task when the system is incommensurate.

If the commensurate order system is stable for some value α_* then it remains stable for all the orders $\alpha \in (0, \alpha_*)$. We cannot expect such nice behavior from the incommensurate order systems. We illustrate this with the following example.

Consider the system (15) with $\alpha = \gamma = 1$. The system is chaotic for $\beta = 0.92$ and is stable for $\beta = 0.4$. However, it does not remain stable for all $\beta \in (0, 0.4)$ e.g. $\beta = 0.3$ produces unstable oscillations (cf. Fig. 11). The Table 3 describes this weird behavior in details.

Order of derivative	IMFOS evaluated at equilibrium points			Stability	Corresponding figures
	O	E_1	E_3		
$\beta = 0.3, \alpha = \gamma = 1$	$0.15708 > 0$	$0.014928 > 0$	$0.0184283 > 0$	Unstable	Figure 11(a)
$\beta = 0.4, \alpha = \gamma = 1$	$0.314159 > 0$	$-0.00429731 < 0$	$0.00238612 > 0$	Stable	Figure 11(b)
$\beta = 0.92, \alpha = \gamma = 1$	$0.0628319 > 0$	$0.00217138 > 0$	$0.00338347 > 0$	Chaotic	Figure 11(c)

Table 3: Weird behavior of incommensurate order system

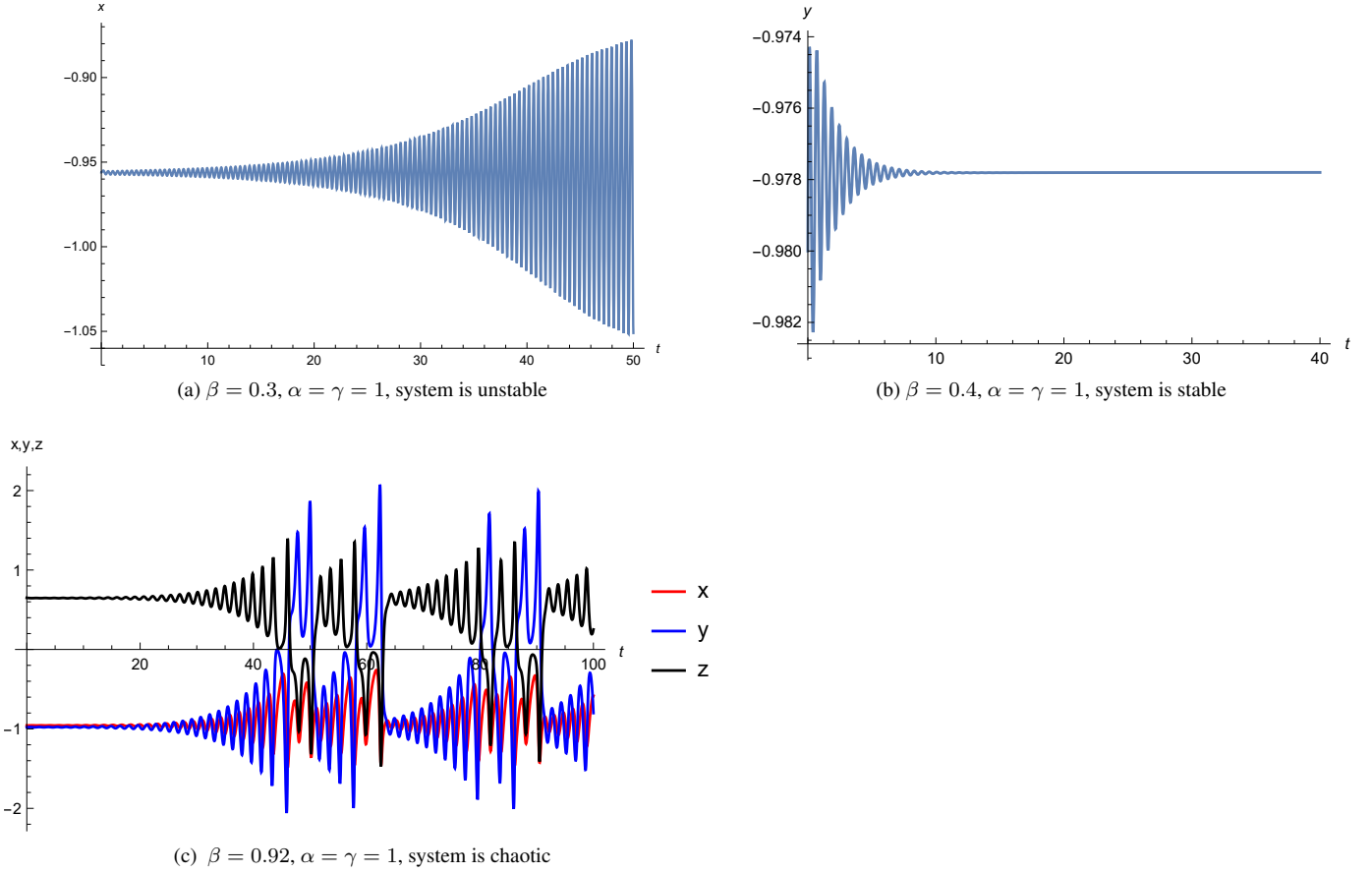


Figure 11

7 Conclusion

This article presents a new example of chaotic differential dynamical system. The bifurcation analysis with respect to various parameters is presented in the details. The stability analysis of fractional order generalization of the proposed system is described. Chaos in the commensurate as well as incommensurate order cases is investigated. The chaos in the system is controlled using a simple linear controller. The synchronization in the system is achieved using a nonlinear feedback controller. The peculiarity of this control is that it depends on fractional order. Unlike in other control strategies, the coupling strength is adjustable with the fractional order.

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