Nodal solutions of fourth-order Kirchhoff equations with critical growth in $\mathbb{R}^N$

Hongling Pu[a], Shiqi Li[a] and Sihua Liang[a*]

[a] College of Mathematics, Changchun Normal University, Changchun, 130032, P.R. China

Abstract

In this paper, we consider the following fourth-order elliptic equations of Kirchhoff type with critical growth in $\mathbb{R}^N$:

$$\Delta^2 u - \left(1 + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + V(x)u = \lambda f(u) + |u|^{2^{**}-2}u, \quad x \in \mathbb{R}^N,$$

where $\Delta^2 u$ is the biharmonic operator, $2^{**} = 2N/(N-4)$ is the critical Sobolev exponent with $N \geq 5$, b and λ are two positive parameters, and $V(x)$ is the potential. By using a main tool of constrained minimization in Nehari manifold, we establish sufficient conditions for the existence result of nodal (that is, sign-changing) solutions.

Keywords: Fourth-order elliptic equation; Kirchhoff problem; Critical exponent; Variational methods; Nodal solutions.

1 Introduction and main result

This paper concerns with the existence of nodal (that is, sign-changing) solutions for fourth-order elliptic equations of Kirchhoff type with critical growth in $\mathbb{R}^N$:

$$\Delta^2 u - \left(1 + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + V(x)u = \lambda f(u) + |u|^{2^{**}-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\Delta^2 u$ is the biharmonic operator, $2^{**} = 2N/(N-4)$ is the critical Sobolev exponent with $N \geq 5$, b and λ are two positive parameters. The continuous functions $V(x)$ and $f(u)$ satisfy the following conditions:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf\limits_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$, with V_0 being a positive constant. Moreover, for any $M > 0$, $\text{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty$, where $\text{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$.

(f_1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $f(u) = o(|u|)$ as $u \to 0$;

(f_2) There exists $p \in (4, 2^{**})$ such that $\lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = 0$;

*Corresponding author. E-mail address: pauline_phl@163.com (H. Pu), lishiqi59@126.com (S. Li), liangsihua@163.com (S. Liang)

(f_3) $\lim_{u\to\infty} \frac{F(u)}{u^4} = +\infty$, where $F(u) = \int_0^u f(t)dt$;

(f_4) The function $\frac{f(u)}{|u|^3}$ is a strictly increasing function of $u \in \mathbb{R} \setminus \{0\}$.

Problem (1.1) originates from the following Kirchhoff equations

$$-\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u) \quad \text{in } \Omega, \tag{1.2}$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain, $a > 0$, $b \geq 0$, and u satisfies some boundary conditions. The problem (1.2) stems from a typical model proposed by Kirchhoff [13]

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), \tag{1.3}$$

which serves as a generalization of the classic D'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = f(x,u)$$

by taking into account the effects of the changes in the length of the strings during the vibrations. The nonlocal term thus appears. See for example [7, 26] for more background on such problems. Thanks to the pioneering work of Lions [21] on problem (1.3), a lot of attention has been drawn to these nonlocal problems during the last decade. That was followed by some interesting results on the existence of various solutions to (1.2), including positive solutions, multiple solutions, bound state solutions, multibump solutions, and semiclassical state solutions, on both bounded domains and the whole space. More results on Kirchhoff-type equations, please refer to [8, 12, 15, 16, 20, 25, 29] and the references therein. Problem (1.2) with critical nonlinearity, as we notice, is seldom covered, mainly because of the challenge—lack of compactness—presented by the presence of the critical Sobolev exponent. We also refer the interested readers to [1, 10, 18, 27, 35, 36, 37, 38] about fractional Kirchhoff type problems.

Recently, various approaches have been adopted to consider the following fourth-order elliptic equations of Kirchhoff type

$$\begin{cases} \Delta^2 u - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Delta^2 u$ is the biharmonic operator, under different hypotheses on the nonlinearity. For instance, Ma [23] studied the existence and multiplicity of positive solutions for the fourth-order equation with the fixed point theorems in cones of ordered Banach spaces. Wang et al. [32] applied the mountain pass and the truncation method to get the existence of nontrivial solutions to fourth-order elliptic equations of Kirchhoff type with one parameter λ. Liang and Zhang [17] used the variational methods to obtain the existence and multiplicity of solutions to fourth-order elliptic equations of Kirchhoff type with critical growth in $\mathbb{R}^N$.

The motivation of this paper comes from [28, 30, 31, 40]. In [28], the author proved the existence of one least energy nodal solution u_b to problem (1.2), with its the energy strictly larger than the ground state energy. Meanwhile, the asymptotic behavior of u_b as the parameter $b \searrow 0$ was investigated as well. Later, under some more weak assumptions on f (especially, Nehari type monotonicity condition been removed), Tang and Cheng [30] improved and generalized some results obtained in [28] with some new analytical skills and Non-Nehari manifold method. In [31], the authors obtained the existence of least energy nodal solutions to Kirchhoff-type equation with critical growth in bounded domains by using the constraint variational method and the quantitative deformation lemma. In [40], the authors the studied following fourth-order elliptic equation with Kirchhoff-type

$$\begin{cases} \Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + V(x)u = f(u), & x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where the constants $a > 0$ and $b \geq 0$. By constraint variational method and quantitative deformation lemma, they obtain that the problem possesses one least energy nodal solution. For more results on nodal solutions to

Kirchhoff-type equations, please refer to [9, 14, 22, 39] and the references therein. However, to the best of our knowledge, there seems no such paper dealing with the existence of nodal solutions of the problem (1.1) involving critical nonlinearities in the whole space.

The purpose of the present paper is to study the existence, energy estimate and the convergence property of least energy nodal solutions for fourth-order elliptic equations of Kirchhoff type with critical growth in $\mathbb{R}^N$ (1.1). The novelty of this article is that the problem (1.1) concerns critical case on the whole space. Based on these facts, the problem turns out to be extremely complicated and more difficult than the one without critical nonlinearities in bounded domains. Since problem (1.1) involves critical exponent in the nonlinearity, it is rather difficult to show that the energy functional reaches a lower infimum on Nehari manifold because of the lack of compactness caused by the critical term. As we will see later, this problem prevents us from using the way as in [3, 28, 30, 40]. So we need some new ideas to overcome the above difficulties. Moreover, we will use the constraint variational method, topological degree theory and the quantitative deformation lemma to prove our main results. Thus, our main results generalize reference [3, 28, 30, 40] in several directions.

Before presenting our main results, first define

$$H^2(\mathbb{R}^N) := \left\{u \in L^2(\mathbb{R}^N) : |\nabla u|, \Delta u \in L^2(\mathbb{R}^N)\right\},$$

endowed the norm

$$\|u\|_{H^2(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left((\Delta u)^2 + (\nabla u)^2 + u^2\right) dx\right)^{\frac{1}{2}}.$$

In this article, we introduce the space

$$E := \left\{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 dx < \infty\right\}$$

with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \nabla v + V(x)uv)\, dx$$

and the norm

$$\|u\| = \int_{\mathbb{R}^N} \left(|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2\right) dx.$$

When condition (V) holds, we know that the embedding $E \hookrightarrow H^2(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $p \in (2, 2^{**})$ is compact and is continuous for $p \in [2, 2^{**}]$ (see [5]), then

$$S_p|u|_p \le \|u\|, \quad \forall u \in E. \tag{1.4}$$

In particular, if S is the best Sobolev constant for the embedding $E \hookrightarrow L^{2^{**}}(\mathbb{R}^N)$, then it is defined by

$$S = \inf\left\{\int_{\mathbb{R}^N} |\Delta u|^2 dx : \int_{\mathbb{R}^N} |u|^{2^{**}} dx = 1\right\}.$$

For the weak solution, we mean the one satisfying the given definition.

Definition 1.1. *We say that $u \in E$ is a weak solution to problem* (1.1), *if*

$$\int_{\mathbb{R}^N} (\Delta u \Delta \phi + \nabla u \nabla \phi + V(x)u\phi)\, dx + b\int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla \phi dx = \lambda \int_{\mathbb{R}^N} f(u)\phi dx + \int_{\mathbb{R}^N} |u|^{2^{**}-2}u\phi dx$$

for any $\phi \in E$.

For convenience, we will omit the terminology weak throughout this paper. The corresponding energy functional $I_b^\lambda : E \to \mathbb{R}$ to problem(1.1) is defined by

$$\begin{aligned} I_b^\lambda(u) =& \frac{1}{2}\int_{\mathbb{R}^N} \left(|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2\right) dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 \\ & - \lambda \int_{\mathbb{R}^N} F(u)dx - \frac{1}{2^{**}}\int_{\mathbb{R}^N} |u|^{2^{**}} dx. \end{aligned} \tag{1.5}$$

It is easily seen that I_b^λ belongs to $C^1(E,\mathbb{R})$ and the critical points of I_b^λ are the solutions to (1.1). Moreover, if every $u\in E$ can be written as

$$u^+(x)=\max\{u(x),0\} \quad \text{and} \quad u^-(x)=\min\{u(x),0\}$$

for $u\in E$, then every solution $u\in E$ to problem (1.1) with the property that $u^\pm\neq 0$ is a nodal solution to problem (1.1).

Our objective in this article is the least energy nodal solutions to problem (1.1). To our knowledge, there are some interesting studies on the following typical semilinear equation, which is related to problem (1.1)(see [4, 5])

$$-\Delta u+V(x)u=f(x,u) \quad \text{in} \quad \mathbb{R}^N. \tag{1.6}$$

These methods, however, depend heavily upon the following decompositions

$$J(u)=J(u^+)+J(u^-), \tag{1.7}$$

$$\langle J'(u),u^+\rangle=\langle J'(u^+),u^+\rangle \quad \text{and} \quad \langle J'(u),u^-\rangle=\langle J'(u^-),u^-\rangle, \tag{1.8}$$

where J is the energy functional of (1.6) given by

$$J(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)dx-\int_{\mathbb{R}^N}F(x,u)dx.$$

However, if $b>0$, the energy functional I_b^λ cannot be decomposed in the same way as it is done in (1.7) and (1.8). In fact, we have

$$I_b^\lambda(u)=I_b^\lambda(u^+)+I_b^\lambda(u^-)+\frac{b}{2}\int_{\mathbb{R}^N}|\nabla u^+|^2dx\int_{\mathbb{R}^N}|\nabla u^-|^2dx,$$

if $u^+\not\equiv 0$,

$$\langle (I_b^\lambda)'(u),u^+\rangle=\langle (I_b^\lambda)'(u^+),u^+\rangle+b\int_{\mathbb{R}^N}|\nabla u^+|^2dx\int_{\mathbb{R}^N}|\nabla u^-|^2dx>\langle (I_b^\lambda)'(u^+),u^+\rangle,$$

if $u^-\not\equiv 0$,

$$\langle (I_b^\lambda)'(u),u^-\rangle=\langle (I_b^\lambda)'(u^-),u^-\rangle+b\int_{\mathbb{R}^N}|\nabla u^-|^2dx\int_{\mathbb{R}^N}|\nabla u^+|^2dx>\langle (I_b^\lambda)'(u^-),u^-\rangle.$$

Therefore, the methods to obtain nodal solutions to the local problem (1.6) do not seem applicable to problem (1.1). In this paper, we follow the approach in [6] by defining the following constrained set

$$\mathcal{N}_b^\lambda=\{u\in E\,|\,u^\pm\neq 0,\langle (I_b^\lambda)'(u),u^\pm\rangle=0\} \tag{1.9}$$

and considering a minimization problem of I_b^λ on $\mathcal{N}_b^\lambda$. In fact, Shuai [28] proved $\mathcal{N}_b^\lambda\neq\emptyset$, in the absence of the nonlocal term, by applying the parametric method and implicit theorem. However, it is the nonlocal terms in problem (1.1), the biharmonic operator and the nonlocal term involved, that add to our difficulties. Roughly speaking, compared to the general Kirchhoff type problem (1.2), decompositions (1.7) and (1.8) corresponding to I_b^λ are much more complicated, which account for some technical difficulties during the proof of the nonempty of $\mathcal{N}_b^\lambda$. Moreover, the parametric method and implicit theorem are not applicable to problem (1.1) due to the complexity of the nonlocal problem there. Hence, inspired by [2], we follow a different path, specifically, resorting to a modified Miranda's theorem (see [24]). It is also feasible to prove that the minimizer of the constrained problem is also a nodal solution via the quantitative deformation lemma and degree theory. We can now present our first main result.

Theorem 1.1. *Assume that (V) and (f_1)-(f_4) hold. Then, there exists $\lambda^*>0$ such that for all $\lambda\geq\lambda^*$, problem (1.1) has a least energy nodal solution $u_b\in\mathcal{N}_b^\lambda$ such that $I_b^\lambda(u_b)=\inf_{u\in\mathcal{N}_b^\lambda}I_b^\lambda(u)$.*

Another goal of this article is to establish the so-called energy doubling property (cf. [33]), i.e., the energy of any nodal solution to problem (1.1) is strictly larger than twice the ground state energy. The conclusion is trivial for the semilinear equation problem (1.6). When $b > 0$, a similar result was obtained by Shuai [28] in a bounded domain Ω. We are also interested in whether energy doubling property still holds true for problem (1.1), to answer which question, we have the following result.

Theorem 1.2. *Assume that (V) and (f_1)-(f_4) hold. Then, there exists $\lambda^{**} > 0$ such that for all $\lambda \geq \lambda^{**}$, the $c^* := \inf_{u \in \mathcal{M}_b^\lambda} I_b^\lambda(u) > 0$ is achieved and $I_b^\lambda(u) > 2c^*$, where $\mathcal{M}_b^\lambda = \{u \in E \setminus \{0\} \mid \langle (I_b^\lambda)'(u), u\rangle = 0\}$ and u is the least energy nodal solution obtained in Theorem 1.1. In particular, $c^* > 0$ is achieved either by a positive or a negative function.*

It is obvious that the energy of the nodal solution u_b obtained in Theorem 1.1 depends on b. In the following, we give a convergence property of u_b as $b \to 0$, which reflects some relationship between $b > 0$ and $b = 0$ for problem (1.1).

Theorem 1.3. *Assume that $(V)$and (f_1)-(f_4) hold. For any sequence $\{b_n\}$ with $b_n \to 0$ as $n \to \infty$, there exists a subsequence, still denoted by $\{b_n\}$, such that $\{u_n\}$ converges to u_0 strongly in E as $n \to \infty$, where u_0 is a least energy nodal solution to the following problem*

$$\Delta^2 u - \Delta u + V(x) = \lambda f(u) + |u|^{2^{**}-2}u \quad \text{in } \mathbb{R}^N. \tag{1.10}$$

The plan of this paper is as follows: Section 2 covers the proof of the achievement of least energy for the constraint problem (1.1), and Section 3 is devoted to the proofs of our main theorems.

Throughout this paper, we use standard notations. For simplicity, we use "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in the related function space respectively. Denoted by C and C_i various positive constants, and by ":=" definitions. To simplify the notation, we denote a subsequence of a sequence $\{u_n\}_n$ as $\{u_n\}_n$ unless specified.

2 Some technical lemmas

As a start, fix $u \in E$ with $u^\pm \neq 0$. Consider function $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and mapping $W : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^2$ where

$$\varphi(\alpha, \beta) = I_b^\lambda(\alpha u^+ + \beta u^-) \tag{2.1}$$

and

$$W(\alpha, \beta) = \left(\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+\rangle, \langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \beta u^-\rangle\right). \tag{2.2}$$

Now for short, define the following quantity

$$A^+(u) = \int_{\mathbb{R}^N} |\nabla u^+|^2 dx, \quad A^-(u) = \int_{\mathbb{R}^N} |\nabla u^-|^2 dx, \quad B(u) = \int_{\mathbb{R}^N} \Delta u^+ \Delta u^- dx.$$

Lemma 2.1. *Assume that (V) and (f_1)-(f_4) hold. For any $u \in E$ with $u^\pm \neq 0$, there is the unique maximum point pair of positive numbers (α_u, β_u) such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_b^\lambda$.*

Proof. Our proof shall progress through three claims.

Claim 1. There exists a pair of positive numbers (α_u, β_u) such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_b^\lambda$ for any $u \in E$ with $u^\pm \neq 0$.

Note that

$$\begin{aligned}
\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+\rangle = & \int_{\mathbb{R}^N} \Delta(\alpha u^+ + \beta u^-)\Delta \alpha u^+ dx + \int_{\mathbb{R}^N} |\nabla \alpha u^+|^2 dx + \int_{\mathbb{R}^N} V(x)|\alpha u^+|^2 dx \\
& + b\int_{\mathbb{R}^N} |\nabla(\alpha u^+ + \beta u^-)|^2 dx \int_{\mathbb{R}^N} |\nabla \alpha u^+|^2 dx \\
& - \lambda \int_{\mathbb{R}^N} f(\alpha u^+)\alpha u^+ dx - \int_{\mathbb{R}^N} |\alpha u^+|^{2^{**}} dx
\end{aligned}$$

and

$$\begin{aligned}\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \beta u^- \rangle = &\int_{\mathbb{R}^N} \Delta(\alpha u^+ + \beta u^-)\Delta \beta u^- dx + \int_{\mathbb{R}^N} |\nabla \beta u^-|^2 dx + \int_{\mathbb{R}^N} V(x)|\beta u^-|^2 dx \\ &+ b\int_{\mathbb{R}^N} |\nabla(\alpha u^+ + \beta u^-)|^2 dx \int_{\mathbb{R}^N} |\nabla \beta u^-|^2 dx \\ &- \lambda \int_{\mathbb{R}^N} f(\beta u^-)\beta u^- dx - \int_{\mathbb{R}^N} |\beta u^-|^{2^{**}} dx.\end{aligned}$$

We obtain from a direct computation that

$$\begin{aligned}\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+ \rangle = &\alpha^2\|u^+\|^2 + \alpha\beta B(u) + \alpha^2\beta^2 b A^+(u)A^-(u) + \alpha^4 b \left(A^+(u)\right)^2 \\ &- \lambda \int_{\mathbb{R}^N} f(\alpha u^+)\alpha u^+ dx - \int_{\mathbb{R}^N} |\alpha u^+|^{2^{**}} dx\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \beta u^- \rangle = &\beta^2\|u^-\|^2 + \alpha\beta B(u) + \alpha^2\beta^2 b A^+(u)A^-(u) + \beta^4 b \left(A^-(u)\right)^2 \\ &- \lambda \int_{\mathbb{R}^N} f(\beta u^-)\beta u^- dx - \int_{\mathbb{R}^N} |\beta u^-|^{2^{**}} dx.\end{aligned} \tag{2.4}$$

On the assumption (f_1) and (f_2), we have

$$\int_{\mathbb{R}^N} f(\alpha u^+)\alpha u^+ dx \le \varepsilon \int_{\mathbb{R}^N} |\alpha u^+|^2 dx + C_\varepsilon \int_{\mathbb{R}^N} |\alpha u^+|^p dx. \tag{2.5}$$

Choose $\varepsilon > 0$ small enough such that $(1 - \lambda\varepsilon C_\varepsilon) > 0$, which along with (2.5) plus (2.3), enables

$$\begin{aligned}\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+ \rangle \ge &(1 - \lambda\varepsilon C_\varepsilon)\alpha^2\|u^+\|^2 + \alpha^2\beta^2 b A^+(u)A^-(u) + \alpha^4 b \left(A^+(u)\right)^2 \\ &- \lambda C_\varepsilon \int_{\mathbb{R}^N} |\alpha u^+|^p dx - \int_{\mathbb{R}^N} |\alpha u^+|^{2^{**}} dx.\end{aligned}$$

Since $2^{**} > 4$, we have $\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+ \rangle > 0$ for α small enough and all $\beta \ge 0$.

Similarly, according to (2.5) and (2.4), we get $\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \beta u^- \rangle > 0$ for β small enough and all $\alpha \ge 0$.

Hence, there exists $r > 0$ such that

$$\langle (I_b^\lambda)'(ru^+ + \beta u^-), ru^+ \rangle > 0 \quad \text{and} \quad \langle (I_b^\lambda)'(\alpha u^+ + ru^-), ru^- \rangle > 0 \tag{2.6}$$

for all $\alpha, \beta \ge 0$.

On the other hand, by (f_2) and (f_3), we have that

$$f(t)t > 0, t \neq 0;\ F(t) \ge 0,\ t \in \mathbb{R}. \tag{2.7}$$

Now, choose $R > r$, with R sufficiently large, by (2.3), (2.4) and by (2.7), we have

$$\langle (I_b^\lambda)'(Ru^+ + \beta u^-), Ru^+ \rangle < 0 \quad \text{and} \quad \langle (I_b^\lambda)'(\alpha u^+ + Ru^-), Ru^- \rangle < 0 \tag{2.8}$$

for all $\alpha, \beta \in [r, R]$.

In view of Miranda's theorem [24], together with (2.6) and (2.8), there exists $(\alpha_u, \beta_u) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $W(\alpha_u, \beta_u) = (0, 0)$, i.e., $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_b^\lambda$.

Claim 2. The pair (α_u, β_u) is unique.

• Case $u \in \mathcal{N}_b^\lambda$.

If $u \in \mathcal{N}_b^\lambda$, then, we have

$$\langle (I_b^\lambda)'(u), u^+ \rangle = 0 \quad \text{and} \quad \langle (I_b^\lambda)'(u), u^- \rangle = 0,$$

that is,

$$
\begin{aligned}
&\|u^+\|^2 + B(u) + bA^+(u)\left(A^+(u) + A^-(u)\right) \\
&= \lambda \int_{\mathbb{R}^N} f(u^+)u^+ dx + \int_{\mathbb{R}^N} |u^+|^{2^{**}} dx
\end{aligned} \tag{2.9}
$$

and

$$
\begin{aligned}
&\|u^-\|^2 + B(u) + bA^-(u)\left(A^+(u) + A^-(u)\right) \\
&= \lambda \int_{\mathbb{R}^N} f(u^-)u^- dx + \int_{\mathbb{R}^N} |u^-|^{2^{**}} dx.
\end{aligned} \tag{2.10}
$$

By Claim 1, we know that there exists at least one positive point pair (α_0, β_0) satisfying $\alpha_0 u^+ + \beta_0 u^- \in \mathcal{N}_b^\lambda$. Now, we state in the following that $(\alpha_0, \beta_0) = (1, 1)$ is the unique pair of numbers. Without loss of generality, let $\alpha_0 \leq \beta_0$, and it follows from (2.8) that

$$
\begin{aligned}
&\alpha_0^2 \left(\|u^+\|^2 + B(x)\right) + \alpha_0^4 bA^+(u)\left(A^+(u) + A^-(u)\right) \\
&= \lambda \int_{\mathbb{R}^N} f(\alpha_0 u^+)\alpha_0 u^+ dx + \int_{\mathbb{R}^N} |\alpha_0 u^+|^{2^{**}} dx.
\end{aligned} \tag{2.11}
$$

If $\alpha_0 < 1$, then due to (2.9), (2.11) and (f_4), we have that

$$
\begin{aligned}
0 &< [(\alpha_0)^{-2} - 1]\left(\|u^+\|^2 + B(u)\right) \\
&\leq \lambda \int_{\mathbb{R}^N} \left(\frac{f(x, \alpha_0 u^+)}{(\alpha_0 u^+)^3} - \frac{f(u^+)}{(u^+)^3}\right)(u^+)^4 dx + [(\alpha_0)^{2^{**}-4} - 1] \int_{\mathbb{R}^N} |u^+|^{2^{**}} dx < 0,
\end{aligned} \tag{2.12}
$$

which is a contradiction. Hence, $1 \leq \alpha_0 \leq \beta_0$.

Adopting a similar approach, we have $\beta_0 \leq 1$, which implies $\alpha_0 = \beta_0 = 1$.

• Case $u \notin \mathcal{N}_b^\lambda$.

Assume there exist two other pairs of positive numbers (α_1, β_1) and (α_2, β_2) such that

$$
\sigma_1 = \alpha_1 u^+ + \beta_1 u^- \in \mathcal{N}_b^\lambda \quad \text{and} \quad \sigma_2 = \alpha_2 u^+ + \beta_2 u^- \in \mathcal{N}_b^\lambda.
$$

Then

$$
\sigma_2 = \left(\frac{\alpha_2}{\alpha_1}\right)\alpha_1 u^+ + \left(\frac{\beta_2}{\beta_1}\right)\beta_1 u^- = \left(\frac{\alpha_2}{\alpha_1}\right)\sigma_1^+ + \left(\frac{\beta_2}{\beta_1}\right)\sigma_1^- \in \mathcal{N}_b^\lambda.
$$

Since $\sigma_1 \in \mathcal{N}_b^\lambda$, it is clear that

$$
\frac{\alpha_2}{\alpha_1} = \frac{\beta_2}{\beta_1} = 1,
$$

which means $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$.

Claim 3. The pair (α_u, β_u) is the unique maximum point of the function φ on $\mathbb{R}_+ \times \mathbb{R}_+$.

We know from the above, (α_u, β_u) is the unique critical point of φ on $\mathbb{R}_+ \times \mathbb{R}_+$. By definition and (2.5), we have

$$
\begin{aligned}
\varphi(\alpha, \beta) &= I_b^\lambda(\alpha u^+ + \beta u^-) \\
&= \frac{\alpha^2}{2}\|u^+\|^2 + \frac{\beta^2}{2}\|u^-\|^2 + \alpha\beta B(u) + \frac{\alpha^4 b}{4}\left(A^+(u)\right)^2 + \frac{\beta^4 b}{4}\left(A^-(u)\right)^2 + \frac{\alpha^2\beta^2 b}{2}A^+(u)A^-(u) \\
&\quad - \lambda \int_{\mathbb{R}^N} F(\alpha u^+)dx - \lambda \int_{\mathbb{R}^N} F(\beta u^-)dx - \frac{\alpha^{2^{**}}}{2^{**}} \int_{\mathbb{R}^N} |u^+|^{2^{**}} dx - \frac{\beta^{2^{**}}}{2^{**}} \int_{\mathbb{R}^N} |u^-|^{2^{**}} dx \\
&< \frac{\alpha^2}{2}\|u^+\|^2 + \frac{\beta^2}{2}\|u^-\|^2 + \alpha\beta B(u) + \frac{\alpha^4 b}{4}\left(A^+(u)\right)^2 + \frac{\beta^4 b}{4}\left(A^-(u)\right)^2 + \frac{\alpha^2\beta^2 b}{2}A^+(u)A^-(u) \\
&\quad - \frac{\alpha^{2^{**}}}{2^{**}} \int_{\mathbb{R}^N} |u^+|^{2^{**}} dx - \frac{\beta^{2^{**}}}{2^{**}} \int_{\mathbb{R}^N} |u^-|^{2^{**}} dx
\end{aligned}
$$

as $|(\alpha,\beta)| \to \infty$. It implies $\lim_{|(\alpha,\beta)|\to\infty} \varphi(\alpha,\beta) = -\infty$ due to the fact that $2^{**} > 4$. Hence, it is adequate to claim that the maximum point cannot be achieved on the boundary of $\mathbb{R}_+ \times \mathbb{R}_+$.

We carry out our proof by contradiction. Assuming $(0,\bar\beta)$ is the global maximum point of φ with $\bar\beta \geq 0$, then we have

$$\begin{aligned}\varphi(\alpha,\bar\beta) = &\frac{\alpha^2}{2}\|u^+\|^2 + \frac{\bar\beta^2}{2}\|u^-\|^2 + \alpha\bar\beta B(u) + \frac{\alpha^4 b}{4}\left(A^+(u)\right)^2 + \frac{\bar\beta^4 b}{4}\left(A^-(u)\right)^2 + \frac{\alpha^2\bar\beta^2 b}{2}A^+(u)A^-(u)\\ &- \lambda\int_{\mathbb{R}^N} F(\alpha u^+)dx - \lambda\int_{\mathbb{R}^N} F(\bar\beta u^-)dx - \frac{\alpha^{2^{**}}}{2^{**}}\int_{\mathbb{R}^N}|u^+|^{2^{**}}dx - \frac{\bar\beta^{2^{**}}}{2^{**}}\int_{\mathbb{R}^N}|u^-|^{2^{**}}dx.\end{aligned}$$

Hence, it is clear that

$$\begin{aligned}\varphi'_\alpha(\alpha,\bar\beta) = &\alpha\|u^+\|^2 + \bar\beta B(u) + \alpha^3 b\left(A^+(u)\right)^2 + \alpha\bar\beta^2 bA^+(u)A^-(u)\\ &- \lambda\int_{\mathbb{R}^N} f(\alpha u^+)u^+dx - \alpha^{2^{**}-1}\int_{\mathbb{R}^N}|u^+|^{2^{**}}dx > 0,\end{aligned}$$

for α small enough. It means φ is an increasing function in respect of α if α is small enough, which is a contradiction. With a similar method, we deduce that φ cannot achieve its global maximum at $(\alpha,0)$ with $\alpha \geq 0$. Thus, we finish the proof. □

Lemma 2.2. *Assume that (V) and (f_1)-(f_4) hold. For any $u \in E$ with $u^\pm \neq 0$ such that $\langle (I_b^\lambda)'(u), u^\pm\rangle \leq 0$, the unique maximum point pair of φ on $\mathbb{R}_+ \times \mathbb{R}_+$ satisfies $0 < \alpha_u, \beta_u \leq 1$.*

Proof. Without loss of generality, suppose $\alpha_u \geq \beta_u > 0$. Due to the fact that $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_b^\lambda$, we have

$$\begin{aligned}&\alpha_u^2\|u^+\|^2 + \alpha_u\beta_u B(u) + \alpha_u^2\beta_u^2 bA^+(u)A^-(u) + \alpha_u^4 b\left(A^+(u)\right)^2\\ &= \lambda\int_{\mathbb{R}^N} f(\alpha_u u^+)\alpha_u u^+dx + \int_{\mathbb{R}^N}|\alpha_u u^+|^{2^{**}}dx.\end{aligned} \tag{2.13}$$

Furthermore, since $\langle (I_b^\lambda)'(u), u^+\rangle \leq 0$, we have

$$\|u^+\|^2 + B(u) + b\left(A^+(u)\right)^2 + bA^+(u)A^-(u) \leq \lambda\int_{\mathbb{R}^N} f(u^+)u^+dx + \int_{\mathbb{R}^N}|u^+|^{2^{**}}dx. \tag{2.14}$$

Based on (2.13) and (2.14), it follows that

$$[(\alpha_u)^{-2} - 1]\left(\|u^+\|^2 + B(u)\right) \geq \lambda\int_{\mathbb{R}^N}\left(\frac{f(\alpha_u u^+)}{(\alpha_u u^+)^3} - \frac{f(u^+)}{(u^+)^3}\right)(u^+)^4dx + [(\alpha_u)^{2^{**}-4} - 1]\int_{\mathbb{R}^N}|u^+|^{2^{**}}dx. \tag{2.15}$$

Obviously, the left side of (2.15) is negative for $\alpha_u > 1$ while the right side is positive, which is a contradiction. Accordingly, $0 < \alpha_u, \beta_u \leq 1$. □

Lemma 2.3. *Let $c_b^\lambda = \inf_{u\in\mathcal{N}_b^\lambda} I_b^\lambda(u)$, and we then have $\lim_{\lambda\to\infty} c_b^\lambda = 0$.*

Proof. For every $u \in \mathcal{N}_b^\lambda$, we have $\langle (I_b^\lambda)'(u), u\rangle = 0$, that is,

$$\|u^+\|^2 + \|u^-\|^2 + 2B(u) + b\left(A^+(u) + A^-(u)\right)^2 = \lambda\int_{\mathbb{R}^N} f(u)udx + \int_{\mathbb{R}^N}|u|^{2^{**}}dx. \tag{2.16}$$

Then, by (2.5) and (2.16), we have

$$\begin{aligned}\|u\|^2 &\leq \lambda\int_{\mathbb{R}^N} f(u^\pm)u^\pm dx + \int_{\mathbb{R}^N}|u^\pm|^{2^{**}}dx\\ &\leq \lambda\varepsilon\int_{\mathbb{R}^N}|u^\pm|^2dx + \lambda C_\varepsilon\int_{\mathbb{R}^N}|u^\pm|^pdx + \int_{\mathbb{R}^N}|u^\pm|^{2^{**}}dx.\end{aligned} \tag{2.17}$$

Choose ε small enough to meet $\lambda\varepsilon\int_{\mathbb{R}^N}|u^\pm|^2dx \le \frac{1}{2}\|u^\pm\|^2$, and then we can claim that there exists $\rho>0$ such that

$$\|u^\pm\|^2 \ge \rho \quad \text{for all } u\in\mathcal{N}_b^\lambda \tag{2.18}$$

due to $4<2^{**}$. And by (f_4), for $t\neq 0$, we have

$$\mathcal{F}(t) := tf(t) - 4F(t) \ge 0$$

and $\mathcal{F}(t)$ is increasing when $t>0$ and decreasing when $t<0$.

Therefore

$$\begin{aligned}
I_b^\lambda(u) &= I_b^\lambda(u) - \frac{1}{4}\langle (I_b^\lambda)'(u), u\rangle \\
&= \frac{1}{4}\|u\|^2 + (\frac{1}{4}-\frac{1}{2^{**}})\int_{\mathbb{R}^N}|u|^{2^{**}}dx + \frac{\lambda}{4}\int_{\mathbb{R}^N}[f(u)u-4F(u)]\,dx \\
&\ge \frac{1}{4}\|u\|^2 \ge \frac{\rho}{4} > 0.
\end{aligned} \tag{2.19}$$

So we have $I_b^\lambda(u)>0$ for all $u\in\mathcal{N}_b^\lambda$, which means $c_b^\lambda = \inf_{u\in\mathcal{N}_b^\lambda} I_b^\lambda(u)$ is well defined.

Fix $u\in E$ with $u^\pm\neq 0$. According to Lemma 2.1, for each $\lambda>0$, there exist $\alpha_\lambda,\beta_\lambda>0$ such that $\alpha_\lambda u^+ + \beta_\lambda u^- \in \mathcal{N}_b^\lambda$. Therefore, we have

$$\begin{aligned}
0\le c_b^\lambda = \inf_{u\in\mathcal{N}_b^\lambda} I_b^\lambda(u) &\le I_b^\lambda(\alpha_\lambda u^+ + \beta_\lambda u^-) \\
&\le \frac{1}{2}\|\alpha_\lambda u^+ + \beta_\lambda u^-\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla(\alpha_\lambda u^+ + \beta_\lambda u^-)|^2dx\right)^2 \\
&= \frac{\alpha_\lambda^2}{2}\|u^+\|^2 + \frac{\beta_\lambda^2}{2}\|u^-\|^2 + \alpha_\lambda\beta_\lambda B(u) + \frac{\alpha_\lambda^4 b}{4}\left(A^+(u)\right)^2 + \frac{\beta_\lambda^4 b}{4}\left(A^-(u)\right)^2 + \frac{\alpha_\lambda^2\beta_\lambda^2 b}{2}A^+(u)A^-(u).
\end{aligned}$$

To our end, we just prove that $\alpha_\lambda\to 0$ and $\beta_\lambda\to 0$, as $\lambda\to\infty$.

Let

$$\mathcal{T} = \{(\alpha_\lambda,\beta_\lambda)\in\mathbb{R}_+\times\mathbb{R}_+ : W(\alpha_\lambda,\beta_\lambda) = (0,0), \lambda>0\},$$

where W is defined as (2.2). Then we have

$$\begin{aligned}
\alpha_\lambda^{2^{**}}\int_{\mathbb{R}^N}|u^+|^{2^{**}}dx + \beta_\lambda^{2^{**}}\int_{\mathbb{R}^N}|u^-|^{2^{**}}dx &\le \alpha_\lambda^{2^{**}}\int_{\mathbb{R}^N}|u^+|^{2^{**}}dx + \beta_\lambda^{2^{**}}\int_{\mathbb{R}^N}|u^-|^{2^{**}}dx \\
&\quad + \lambda\int_{\mathbb{R}^N} f(\alpha_\lambda u^+)\alpha_\lambda u^+dx + \lambda\int_{\mathbb{R}^N} f(\beta_\lambda u^-)\beta_\lambda u^-dx \\
&= \|\alpha_\lambda u^+ + \beta_\lambda u^-\|^2 + b\left(\alpha_\lambda^2A^+(u) + \beta_\lambda^2 A^-(u)\right)^2.
\end{aligned}$$

Therefore, $\mathcal{T}$ is bounded since $4<2^{**}$. Let $\{\lambda_n\}\subset(0,\infty)$ be such that $\lambda_n\to\infty$ as $n\to\infty$. Accordingly, there exist α_0 and β_0 such that $(\alpha_{\lambda_n},\beta_{\lambda_n})\to(\alpha_0,\beta_0)$ as $n\to\infty$.

Now we claim $\alpha_0=\beta_0=0$. Assume, by contradiction, that $\alpha_0>0$ or $\beta_0>0$. Since $\alpha_{\lambda_n}u^+ + \beta_{\lambda_n}u^- \in\mathcal{N}_b^{\lambda_n}$, then for any $n\in\mathbb{N}$, we have

$$\begin{aligned}
&\|\alpha_{\lambda_n}u^+ + \beta_{\lambda_n}u^-\|^2 + b\left(\alpha_{\lambda_n}^2A^+(u) + \beta_{\lambda_n}^2A^-(u)\right)^2 \\
&= \lambda_n\int_{\mathbb{R}^N} f(\alpha_{\lambda_n}u^+ + \beta_{\lambda_n}u^-)(\alpha_{\lambda_n}u^+ + \beta_{\lambda_n}u^-)dx + \int_{\mathbb{R}^N}|\alpha_{\lambda_n}u^+ + \beta_{\lambda_n}u^-|^{2^{**}}dx.
\end{aligned} \tag{2.20}$$

Then, taking into consideration $\alpha_{\lambda_n}u^+\to\alpha_0u^+, \beta_{\lambda_n}u^-\to\beta_0u^-$ in E and Lebesgue dominated convergence theorem, we have

$$\int_{\mathbb{R}^N} f(\alpha_{\lambda_n}u^+ + \beta_{\lambda_n}u^-)(\alpha_{\lambda_n}u^+ + \beta_{\lambda_n}u^-)dx \to \int_{\mathbb{R}^N} f(\alpha_0u^+ + \beta_0u^-)(\alpha_0u^+ + \beta_0u^-)dx > 0$$

as $n\to\infty$. It is in contradiction to (2.20), given $\lambda_n\to\infty$ as $n\to\infty$ and $\{\alpha_{\lambda_n}u^+ + \beta_{\lambda_n}u^-\}$ bounded in E. Therefore, $\alpha_0=\beta_0=0$, which implies $\lim_{\lambda\to\infty}c_b^\lambda = 0$. □

Lemma 2.4. *There exists $\lambda^* > 0$ such that the infimum c_b^λ is achieved for all $\lambda \geq \lambda^*$.*

Proof. According to the definition of c_b^λ, there exists a sequence $\{u_n\} \subset \mathcal{N}_b^\lambda$ such that $\lim_{n\to\infty} I_b^\lambda(u_n) = c_b^\lambda$. Clearly, $\{u_n\}$ is bounded in E. By Lemma 2.1 and the properties of L^p space, up to a subsequence, we have

$$\begin{aligned} &u_n^\pm \rightharpoonup u^\pm \quad \text{in} \quad E, \\ &u_n^\pm \to u^\pm \quad \text{in} \quad L^p(\mathbb{R}^N) \quad \text{for } p \in [2, 2^{**}), \\ &u_n^\pm \to u^\pm \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

In view of Lemma 2.1, we also have

$$I_b^\lambda(\alpha u_n^+ + \beta u_n^-) \leq I_b^\lambda(u_n)$$

for all $\alpha, \beta \geq 0$. So, by Brézis-Lieb lemma, Fatou's lemma and the weak lower semicontinuity of norm, we may conclude

$$\begin{aligned} \liminf_{n\to\infty} I_b^\lambda(\alpha u_n^+ + \beta u_n^-) \geq &\frac{\alpha^2}{2}\lim_{n\to\infty}(\|u_n^+ - u^+\|^2 + \|u^+\|^2) + \frac{\beta^2}{2}\lim_{n\to\infty}(\|u_n^- - u^-\|^2 + \|u^-\|^2) \\ &+ \frac{\alpha^4 b}{4}\left[\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n^+ - \nabla u^+|^2 dx + \int_{\mathbb{R}^N} |\nabla u^+|^2 dx\right]^2 - \lambda \int_{\mathbb{R}^N} F(\alpha u^+) dx \\ &+ \frac{\beta^4 b}{4}\left[\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n^- - \nabla u^-|^2 dx + \int_{\mathbb{R}^N} |\nabla u^-|^2 dx\right]^2 - \lambda \int_{\mathbb{R}^N} F(\beta u^-) dx \\ &- \frac{\alpha^{2^{**}}}{2^{**}}\left[\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n^+ - u^+|^{2^{**}} dx + \lim_{n\to\infty}\int_{\mathbb{R}^N} |u^+|^{2^{**}} dx\right] \\ &- \frac{\beta^{2^{**}}}{2^{**}}\left[\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n^- - u^-|^{2^{**}} dx + \lim_{n\to\infty}\int_{\mathbb{R}^N} |u^-|^{2^{**}} dx\right] \\ &+ \frac{\alpha^2\beta^2 b}{2}\liminf_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n^+|^2 dx \int_{\mathbb{R}^N} |\nabla u_n^-|^2 dx \\ \geq &I_b^\lambda(\alpha u^+ + \beta u^-) + \frac{\alpha^2}{2}A_1 + \frac{\alpha^4 b}{4}A_3^2 + \frac{\alpha^4 b}{2}A_3 A^+(u) - \frac{\alpha^{2^{**}}}{2^{**}}B_1 \\ &+ \frac{\beta^2}{2}A_2 + \frac{\beta^4 b}{4}A_4^2 + \frac{\beta^4 b}{2}A_4 A^-(u) - \frac{\beta^{2^{**}}}{2^{**}}B_2, \end{aligned}$$

where

$$\begin{aligned} A_1 &= \lim_{n\to\infty}\|u_n^+ - u^+\|^2, & A_2 &= \lim_{n\to\infty}\|u_n^- - u^-\|^2, \\ A_3 &= \lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n^+ - \nabla u^+|^2 dx & A_4 &= \lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n^- - \nabla u^-|^2 dx, \\ B_1 &= \lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n^+ - u^+|^{2^{**}} dx, & B_2 &= \lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n^- - u^-|^{2^{**}} dx. \end{aligned}$$

That is,

$$\begin{aligned} c_b^\lambda \geq &I_b^\lambda(\alpha u^+ + \beta u^-) + \frac{\alpha^2}{2}A_1 + \frac{\alpha^4 b}{4}A_3^2 + \frac{\alpha^4 b}{2}A_3 A^+(u) - \frac{\alpha^{2^{**}}}{2^{**}}B_1 \\ &+ \frac{\beta^2}{2}A_2 + \frac{\beta^4 b}{4}A_4^2 + \frac{\beta^4 b}{2}A_4 A^-(u) - \frac{\beta^{2^{**}}}{2^{**}}B_2 \end{aligned} \tag{2.21}$$

for all $\alpha, \beta \geq 0$.

First, we prove that $u^\pm \neq 0$.

We carry out our proof by contradiction. Assume $u^+ = 0$. Let $\beta = 0$ in (2.21) and we have

$$c_b^\lambda \geq \frac{\alpha^2}{2}A_1 + \frac{\alpha^4 b}{4}A_3^2 - \frac{\alpha^{2^{**}}}{2^{**}}B_1 := \phi(\alpha) \tag{2.22}$$

for all $\alpha \geq 0$.

Case 1: $B_1 = 0$.

If $A_1 = 0$, it means, $u_n^+ \to u^+$ in E. By (2.17), we get $\|u^\pm\| > 0$, which is contradictory to our assumption. If $A_1 > 0$, by (2.22), we have $c_b^\lambda \geq \frac{\alpha^2}{2} A_1$ for all $\alpha \geq 0$, which contradicts Lemma 2.3.

Case 2: $B_1 > 0$.

Given the definition of S and Lemma 2.3, there exists $\lambda^* > 0$ such that

$$c_b^\lambda < \frac{2}{N} S^{-2/N} \tag{2.23}$$

for all $\lambda \geq \lambda^*$. According to Sobolev embedding and the fact that $B_1 > 0$, we get $A_1 > 0$. By (2.22), we have

$$\frac{2}{N} S^{-2/N} \leq \frac{2}{N} \left[\frac{A_1^{\frac{2^{**}}{2}}}{B_1} \right]^{\frac{2}{2^{**}-2}} \leq \max_{\alpha \geq 0} \left\{ \frac{\alpha^2}{2} A_1 - \frac{\alpha^{2^{**}}}{2^{**}} B_1 \right\} \leq \max_{\alpha \geq 0} \left\{ \frac{\alpha^2}{2} A_1 + \frac{\alpha^4 b}{4} A_3^2 - \frac{\alpha^{2^{**}}}{2^{**}} B_1 \right\} \leq c_b^\lambda,$$

which is a contradiction. Hence, we conclude $u^+ \neq 0$. Similarly, we have $u^- \neq 0$.

Secondly, we prove $B_1 = B_2 = 0$**.**

Given that the proof of $B_2 = 0$ is analogous, we just prove $B_1 = 0$. By contradiction, assume $B_1 > 0$.

Case 1: $B_2 > 0$.

Since $B_1, B_2 > 0$, we get $A_1, A_2 > 0$. Clearly, $\phi(\alpha) > 0$ for α small enough, where $\phi(\alpha)$ is given by (2.22), and $\phi(\alpha) < 0$ for α sufficiently large. Therefore, by continuity of $\phi(\alpha)$, there exists $\bar{\alpha} > 0$ such that

$$\frac{\bar{\alpha}^2}{2} A_1 + \frac{\bar{\alpha}^4 b}{4} A_3^2 - \frac{\bar{\alpha}^{2^{**}}}{2^{**}} B_1 = \max_{\alpha \geq 0} \left\{ \frac{\alpha^2}{2} A_1 + \frac{\alpha^4 b}{4} A_3^2 - \frac{\alpha^{2^{**}}}{2^{**}} B_1 \right\}.$$

Similarly, there exists $\bar{\beta} > 0$ such that

$$\frac{\bar{\beta}^2}{2} A_2 + \frac{\bar{\beta}^4 b}{4} A_4^2 - \frac{\bar{\beta}^{2^{**}}}{2^{**}} B_2 = \max_{\beta \geq 0} \left\{ \frac{\beta^2}{2} A_2 + \frac{\beta^4 b}{4} A_4^2 - \frac{\beta^{2^{**}}}{2^{**}} B_2 \right\}.$$

In view of the compactness of $[0, \bar{\alpha}] \times [0, \bar{\beta}]$ and the continuity of ϕ, there exists $(\alpha_u, \beta_u) \in [0, \bar{\alpha}] \times [0, \bar{\beta}]$ such that

$$\varphi(\alpha_u, \beta_u) = \max_{(\alpha, \beta) \in [0, \bar{\alpha}] \times [0, \bar{\beta}]} \varphi(\alpha, \beta),$$

where φ is defined as in Lemma 2.1.

Now we prove $(\alpha_u, \beta_u) \in (0, \bar{\alpha}) \times (0, \bar{\beta})$.

In fact, it is noticeable that if β is small enough, we have

$$\varphi(\alpha, 0) = I_b^\lambda(\alpha u^+) < I_b^\lambda(\alpha u^+) + I_b^\lambda(\beta u^-) \leq I_b^\lambda(\alpha u^+ + \beta u^-) = \varphi(\alpha, \beta)$$

for all $\alpha \in [0, \bar{\alpha}]$. Thus, there exists $\beta_0 \in [0, \bar{\beta}]$ such that $\varphi(\alpha, 0) \leq \varphi(\alpha, \beta_0)$, for all $\alpha \in [0, \bar{\alpha}]$. That is, $(\alpha_u, \beta_u) \notin [0, \bar{\alpha}] \times \{0\}$. With similar method, we get $(\alpha_u, \beta_u) \notin \{0\} \times [0, \bar{\beta}]$.

It is obvious that

$$\frac{\alpha^2}{2} A_1 + \frac{\alpha^4 b}{4} A_3^2 + \frac{\alpha^4 b}{2} A_3 A^+(u) - \frac{\alpha^{2^{**}}}{2^{**}} B_1 > 0, \quad \alpha \in (0, \bar{\alpha}], \tag{2.24}$$

$$\frac{\beta^2}{2} A_2 + \frac{\beta^4 b}{4} A_4^2 + \frac{\beta^4 b}{2} A_4 A^-(u) - \frac{\beta^{2^{**}}}{2^{**}} B_2 > 0, \quad \beta \in (0, \bar{\beta}]. \tag{2.25}$$

Then, we obtain

$$\frac{2}{N} S^{-2/N} \leq \frac{\bar{\alpha}^2}{2} A_1 + \frac{\bar{\alpha}^4 b}{4} A_3^2 - \frac{\bar{\alpha}^{2^{**}}}{2^{**}} B_1 + \frac{\bar{\alpha}^4 b}{2} A_3 A^+(u) + \frac{\beta^2}{2} A_2 + \frac{\beta^4 b}{4} A_4^2 + \frac{\beta^4 b}{2} A_4 A^-(u) - \frac{\beta^{2^{**}}}{2^{**}} B_2$$

and

$$\frac{2}{N}S^{-2/N} \leq \frac{\bar{\beta}^2}{2}A_2 + \frac{\bar{\beta}^4 b}{4}A_4^2 - \frac{\bar{\beta}^{2^{**}}}{2^{**}}B_2 + \frac{\bar{\beta}^4 b}{2}A_4 A^-(u) + \frac{\alpha^2}{2}A_1 + \frac{\alpha^4 b}{4}A_3^2 + \frac{\alpha^4 b}{2}A_3 A^+(u) - \frac{\alpha^{2^{**}}}{2^{**}}B_1$$

for all $\alpha \in [0, \bar{\alpha}]$ and all $\beta \in [0, \bar{\beta}]$.

Together with (2.21), we get $\varphi(\bar{\alpha}, \beta) \leq 0$, $\varphi(\alpha, \bar{\beta}) \leq 0$ for all $\alpha \in [0, \bar{\alpha}]$ and all $\beta \in [0, \bar{\beta}]$. Therefore, $(\alpha_u, \beta_u) \notin \{\bar{\alpha}\} \times [0, \bar{\beta}]$ and $(\alpha_u, \beta_u) \notin [0, \bar{\alpha}] \times \{\bar{\beta}\}$, which means $(\alpha_u, \beta_u) \in (0, \bar{\alpha}) \times (0, \bar{\beta})$. It follows that (α_u, β_u) is a critical point of φ.

So, $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_b^\lambda$. According to (2.21), we have

$$\begin{aligned} c_b^\lambda \geq{}& I_b^\lambda(\alpha_u u^+ + \beta_u u^-) + \frac{\alpha_u^2}{2}A_1 + \frac{\alpha_u^4 b}{4}A_3^2 + \frac{\alpha_u^4 b}{2}A_3 A^+(u) - \frac{\alpha_u^{2^{**}}}{2^{**}}B_1 \\ &+ \frac{\beta_u^2}{2}A_2 + \frac{\beta_u^4 b}{4}A_4^2 + \frac{\beta_u^4 b}{2}A_4 A^-(u) - \frac{\beta_u^{2^{**}}}{2^{**}}B_2 > I_b^\lambda(\alpha_u u^+ + \beta_u u^-) \geq c_b^\lambda, \end{aligned}$$

which is a contradiction. Thus, $B_1 = 0$.

Case 2: $B_2 = 0$.

In this case, we can maximize in $[0, \bar{\alpha}] \times [0, \infty)$. It is possible to show that there exists $\beta_0 \in [0, \infty)$ satisfying

$$I_b^\lambda(\alpha_u u^+ + \beta_u u^-) \leq 0 \quad \text{for all } (\alpha, \beta) \in [0, \bar{\alpha}] \times [\beta_0, \infty).$$

Therefore, there is $(\alpha_u, \beta_u) \in [0, \bar{\alpha}] \times [0, \infty)$ such that

$$\varphi(\alpha_u, \beta_u) = \max_{(\alpha,\beta) \in [0,\bar{\alpha}] \times [0,\infty)} \varphi(\alpha, \beta).$$

Now, we claim $(\alpha_u, \beta_u) \in (0, \bar{\alpha}) \times (0, \infty)$.

Indeed, $\varphi(\alpha, 0) < \varphi(\alpha, \beta)$ for $\alpha \in [0, \bar{\alpha}]$ and β small enough, while $\varphi(0, \beta) < \varphi(\alpha, \beta)$ for $\beta \in [0, \infty)$ and α sufficiently small, which implies $(\alpha_u, \beta_u) \notin [0, \bar{\alpha}] \times \{0\}$ and $(\alpha_u, \beta_u) \notin \{0\} \times [0, \infty)$.

However, it is noticeable that

$$\frac{2}{N}S^{-2/N} \leq \frac{\bar{\alpha}^2}{2}A_1 + \frac{\bar{\alpha}^4 b}{4}A_3^2 - \frac{\bar{\alpha}^{2^{**}}}{2^{**}}B_1 + \frac{\bar{\alpha}^4 b}{2}A_3 A^+(u) + \frac{\beta^2}{2}A_2 + \frac{\beta^4 b}{4}A_4^2 + \frac{\beta^4 b}{2}A_4 A^-(u),$$

for every $\beta \in [0, \infty)$.

Therefore, we have $\varphi(\bar{\alpha}, \beta) \leq 0$ for all $\beta \in [0, \infty)$, which means $(\alpha_u, \beta_u) \notin \{\bar{\alpha}\} \times [0, \infty)$. Based on the above, we get $(\alpha_u, \beta_u) \in (0, \bar{\alpha}) \times (0, \infty)$, that is, (α_u, β_u) is an inner maximizer of φ in $[0, \bar{\alpha}] \times [0, \infty)$. Therefore, $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_b^\lambda$. In that case, by (2.24), we have

$$\begin{aligned} c_b^\lambda &\geq I_b^\lambda(\alpha_u u^+ + \beta_u u^-) + \frac{\bar{\alpha}^2}{2}A_1 + \frac{\bar{\alpha}^4 b}{4}A_3^2 - \frac{\bar{\alpha}^{2^{**}}}{2^{**}}B_1 + \frac{\bar{\alpha}^4 b}{2}A_3 A^+(u) + \frac{\beta^2}{2}A_2 + \frac{\beta^4 b}{4}A_4^2 + \frac{\beta^4 b}{2}A_4 A^-(u) \\ &> I_b^\lambda(\alpha_u u^+ + \beta_u u^-) \geq c_b^\lambda, \end{aligned}$$

which is a contradiction. Hence, we have $B_1 = B_2 = 0$.

Finally, we prove that c_b^λ is achieved.

Given $u^\pm \neq 0$, according to Lemma 2.1, there exists $\alpha_u, \beta_u > 0$ such that $\hat{u} := \alpha_u u^+ + \beta_u u^- \in \mathcal{N}_b^\lambda$. Moreover, $\langle (I_b^\lambda)'(u), u^\pm \rangle \leq 0$. By Lemma 2.2, we have $0 < \alpha_u, \beta_u \leq 1$.

Combining $u_n \in \mathcal{N}_b^\lambda$ and Lemma 2.1, we obtain

$$I_b^\lambda(\alpha_u u_n^+ + \beta_u u_n^-) \leq I_b^\lambda(u_n^+ + u_n^-) = I_b^\lambda(u_n).$$

Taking into consideration $B_1 = B_2 = 0$ and the semicontinuity of the norm, we get

$$\begin{aligned}
c_b^\lambda \le I_b^\lambda(\hat{u}) &= I_b^\lambda(\hat{u}) - \frac{1}{4}\langle (I_b^\lambda)'(\hat{u}), \hat{u}\rangle \\
&= \frac{1}{4}\|\hat{u}\|^2 + (\frac{1}{4} - \frac{1}{2^{**}})\int_{\mathbb{R}^N} |\hat{u}|^{2^{**}} dx + \frac{\lambda}{4}\int_{\mathbb{R}^N} [f(\hat{u})\hat{u} - 4F(\hat{u})]\, dx \\
&\le \frac{1}{4}\|u\|^2 + (\frac{1}{4} - \frac{1}{2^{**}})\int_{\mathbb{R}^N} |u|^{2^{**}} dx + \frac{\lambda}{4}\int_{\mathbb{R}^N} [f(u)u - 4F(u)]\, dx \\
&\le \liminf_{n\to\infty}\left[I_b^\lambda(u_n) - \frac{1}{4}\langle (I_b^\lambda)'(u_n), u_n\rangle \right] \le c_b^\lambda.
\end{aligned}$$

Hence, we conclude $\alpha_u = \beta_u = 1$, and c_b^λ is achieved by $u_b := u^+ + u^- \in \mathcal{N}_b^\lambda$. □

3 Proof of Theorems

In this section, we prove our main results.

Proof of Theorem 1.1. In fact, thanks to Lemma 2.4, we just prove that the minimizer u_b for c_b^λ is indeed a nodal solution to problem (1.1).

Due to the fact that $u_b \in \mathcal{N}_b^\lambda$, we have $\langle (I_b^\lambda)'(u_b), u_b^+\rangle = \langle (I_b^\lambda)'(u_b), u_b^-\rangle = 0$. In view of Lemma 2.1, for $(\alpha,\beta) \in (\mathbb{R}_+ \times \mathbb{R}_+)\backslash(1,1)$, we have

$$I_b^\lambda(\alpha u_b^+ + \beta u_b^-) < I_b^\lambda(u_b^+ + u_b^-) = c_b^\lambda. \tag{3.1}$$

In the following, we proceed by an argument of contradiction. Suppose $(I_b^\lambda)'(u_b) \neq 0$, then there exist $\delta > 0$ and $\theta > 0$ such that

$$\|(I_b^\lambda)'(v)\| \ge \theta \qquad \text{for all } \|v - u_b\| \le 3\delta.$$

Choose $\tau \in (0, \min\{1/2, \frac{\delta}{\sqrt{2}\|u_b\|}\})$, and define

$$\begin{aligned}
D &:= (1-\tau, 1+\tau) \times (1-\tau, 1+\tau), \\
g(\alpha,\beta) &:= \alpha u_b^+ + \beta u_b^- \quad \text{for all } (\alpha,\beta) \in D.
\end{aligned}$$

By (3.1), we have

$$\bar{c}_\lambda := \max_{\partial D}(I_b^\lambda \circ g) < c_b^\lambda. \tag{3.2}$$

Let $\varepsilon := \min\{(c_b^\lambda - \bar{c}_\lambda)/2, \theta\delta/8\}$ and $S_\delta := B(u_b, \delta)$. According to Lemma 2.3 in [34], there exists a deformation $\eta \in C([0,1] \times D, D)$ such that

(a) $\eta(1, v) = v$ if $v \notin (I_b^\lambda)^{-1}([c_b^\lambda - 2\varepsilon, c_b^\lambda + 2\varepsilon] \cap S_{2\delta})$,

(b) $\eta(1, (I_b^\lambda)^{c_b^\lambda + \varepsilon} \cap S_\delta) \subset (I_b^\lambda)^{c_b^\lambda - \varepsilon}$,

(c) $I_b^\lambda(\eta(1, v)) \le I_b^\lambda(v)$ for all $v \in E$.

It is obvious that

$$\max_{(\alpha,\beta)\in\bar{D}} I_b^\lambda(\eta(1, g(\alpha,\beta))) < c_b^\lambda. \tag{3.3}$$

We then claim that $\eta(1, g(D)) \cap \mathcal{N}_b^\lambda \neq \emptyset$, which is contradictory to the definition of c_b^λ.

Define $h(\alpha,\beta) := \eta(1, g(\alpha,\beta))$,

$$\begin{aligned}
\Phi_0(\alpha,\beta) : &= (\langle (I_b^\lambda)'(g(\alpha,\beta)), u_b^+\rangle, \langle (I_b^\lambda)'(g(\alpha,\beta)), u_b^-\rangle) \\
&= (\langle (I_b^\lambda)'(\alpha u_b^+ + \beta u_b^-), u_b^+\rangle, \langle (I_b^\lambda)'(\alpha u_b^+ + \beta u_b^-), u_b^-\rangle)
\end{aligned}$$

and

$$\Phi_1(\alpha,\beta) := \left(\frac{1}{\alpha}\langle (I_b^\lambda)'(h(\alpha,\beta)),\,(h(\alpha,\beta))^+\rangle, \frac{1}{\beta}\langle (I_b^\lambda)'(h(\alpha,\beta)),(h(\alpha,\beta))^-\rangle\right).$$

In a similar approach to [19], using the degree theory, we can get $\deg(\Phi_0, D, 0) = 1$. We then, by (3.2), obtain

$$g(\alpha,\beta) = h(\alpha,\beta) \quad \text{on } \partial D,$$

as a result of which, we have $\deg(\Phi_1, D, 0) = \deg(\Phi_0, D, 0) = 1$. Hence, $\Phi_1(\alpha_0,\beta_0) = 0$ for some $(\alpha_0,\beta_0) \in D$ so that

$$\eta(1, g(\alpha_0,\beta_0)) = h(\alpha_0,\beta_0) \in \mathcal{N}_b^\lambda,$$

which contradicts (3.3). Hence, $(I_b^\lambda)'(u_b) = 0$, which implies u_b is a critical point of I_b^λ. Thus, we deduce u_b is a nodal solution to problem (1.1). □

By Theorem 1.1, we obtain a least energy nodal solution u_b to problem (1.1), contributing to the establishment of Theorem 1.2, where, we prove that the energy of u_b is strictly larger than twice the ground state energy.

Proof of Theorem 1.2. As in the proof of Lemma 2.3, there exists $\lambda_1^* > 0$ such that for all $\lambda \geq \lambda_1^*$, and for each $b > 0$, there exists $v_b \in \mathcal{M}_b^\lambda$ such that $I_b^\lambda(v_b) = c^* > 0$. By standard arguments (see Corollary 2.13 in Ref. [11]), the critical points of the functional I_b^λ on $\mathcal{M}_b^\lambda$ are critical points of I_b^λ in E, so we obtain $(I_b^\lambda)'(v_b) = 0$. That is, v_b is a ground state solution to (1.1).

As stated in Theorem 1.1, u_b is known as a least energy nodal solution to problem (1.1), which changes sign only once when $\lambda \geq \lambda^*$.

Let $\lambda^{**} = \max\{\lambda^*, \lambda_1^*\}$ and assume $u_b = u_b^+ + u_b^-$. Adopting the same approach to Lemma 2.1, we claim there exist $\alpha_{u_b^+} > 0$ and $\beta_{u_b^-} > 0$ such that $\alpha_{u_b^+}u_b^+ \in \mathcal{M}_b^\lambda$ and $\beta_{u_b^-}u_b^- \in \mathcal{M}_b^\lambda$. Then, through Lemma 2.2, we obtain $\alpha_{u_b^+}, \beta_{u_b^-} \in (0,1)$.

Hence, thanks to Lemma 2.1, we have

$$2c^* \leq I_b^\lambda(\alpha_{u_b^+}u_b^+) + I_b^\lambda(\beta_{u_b^-}u_b^-) \leq I_b^\lambda(\alpha_{u_b^+}u_b^+ + \beta_{u_b^-}u_b^-) < I_b^\lambda(u_b^+ + u_b^-) = c_b^\lambda.$$

Therefore, it follows that $c^* > 0$ cannot be achieved by a nodal function. □

Finally, we put an end to this section with the proof of Theorem 1.3. In the following, we regard $b > 0$ as a parameter in problem (1.1).

Proof of Theorem 1.3. We shall analyse in stages the convergence property of u_b as $b \to 0$, where u_b is the least energy nodal solution obtained in Theorem 1.1.

Step 1. For any sequence $\{b_n\}$, we prove $\{u_{b_n}\}$ is bounded in E, if $b_n \searrow 0$.

Let a nonzero function $\chi \in C_0^\infty(\mathbb{R}^N)$ with $\chi^\pm \neq 0$ fixed. Analogous to the process of Lemma 2.1, for any $b \in [0,1]$, there exists a pair of positive numbers (λ_1, λ_2) independent of b, such that

$$\langle (I_b^\lambda)'(\lambda_1\chi^+ + \lambda_2\chi^-), \lambda_1\chi^+\rangle < 0 \quad \text{and} \quad \langle (I_b^\lambda)'(\lambda_1\chi^+ + \lambda_2\chi^-), \lambda_2\chi^-\rangle < 0.$$

Then according to Lemma 2.2, for any $b \in [0,1]$, there exists a unique pair $(\alpha_\chi(b), \beta_\chi(b)) \in (0,1] \times (0,1]$ such that $\overline{\chi} := \alpha_\chi(b)\lambda_1\chi^+ + \beta_\chi(b)\lambda_2\chi^- \in \mathcal{N}_b^\lambda$. Therefore, by (2.5), it follows that, for any $b \in [0,1]$,

$$\begin{aligned}
I_b^\lambda(u_b) \le I_b^\lambda(\overline{\chi}) &= I_b^\lambda(\overline{\chi}) - \frac{1}{4}\langle (I_b^\lambda)'(\overline{\chi}), \overline{\chi}\rangle \\
&= \frac{1}{4}\|\overline{\chi}\|^2 + (\frac{1}{4} - \frac{1}{2^{**}})\int_{\mathbb{R}^N} |\overline{\chi}|^{2^{**}} dx + \frac{\lambda}{4}\int_{\mathbb{R}^N} [f(\overline{\chi})\overline{\chi} - 4F(\overline{\chi})]\, dx \\
&\le \frac{1}{4}\|\overline{\chi}\|^2 + (\frac{1}{4} - \frac{1}{2^{**}})\int_{\mathbb{R}^N} |\overline{\chi}|^{2^{**}} dx + \frac{\lambda}{4}\int_{\mathbb{R}^N} \left(C_1|\overline{\chi}|^2 + C_2|\overline{\chi}|^p\right) dx \\
&\le \frac{1}{4}\|\lambda_1\chi^+\|^2 + (\frac{1}{4} - \frac{1}{2^{**}})\int_{\mathbb{R}^N} |\lambda_1\chi^+|^{2^{**}} dx + \frac{\lambda}{4}\int_{\mathbb{R}^N} \left(C_1|\lambda_1\chi^+|^2 + C_2|\lambda_1\chi^+|^p\right) dx \\
&\quad + \frac{1}{4}\|\lambda_2\chi^-\|^2 + (\frac{1}{4} - \frac{1}{2^{**}})\int_{\mathbb{R}^N} |\lambda_2\chi^-|^{2^{**}} dx + \frac{\lambda}{4}\int_{\mathbb{R}^N} \left(C_1|\lambda_2\chi^-|^2 + C_2|\lambda_2\chi^-|^p\right) dx \\
&:= C^*,
\end{aligned}$$

where $C^* > 0$ is a constant independent of b. Thus, as $n \to \infty$, it follows that

$$C^* + 1 \ge I_{b_n}^\lambda(u_{b_n}) = I_{b_n}^\lambda(u_{b_n}) - \frac{1}{4}\langle (I_{b_n}^\lambda)'(u_{b_n}), u_{b_n}\rangle \ge \frac{1}{4}\|u_{b_n}\|^2,$$

that is, $\{u_{b_n}\}$ is bounded in E.

Step 2. In this step, we prove problem (1.10) possesses one nodal solution u_0.

Since $\{u_{b_n}\}$ is bounded in E, thanks to Step 1, up to a subsequence, there exists $u_0 \in E$ such that

$$\begin{aligned}
&u_{b_n} \rightharpoonup u_0 \quad \text{in} \quad E, \\
&u_{b_n} \to u_0 \quad \text{in} \quad L^p(\mathbb{R}^N) \quad \text{for } p \in [2, 2^{**}), \\
&u_{b_n} \to u_0 \quad \text{a.e. in } \mathbb{R}^N.
\end{aligned} \tag{3.4}$$

Given that $\{u_{b_n}\}$ is a weak solution to (1.1) with $b = b_n$, we have

$$\int_{\mathbb{R}^N} (\Delta u\Delta\phi + \nabla u\nabla\phi + V(x)u\phi)\, dx + b_n\int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u\nabla\phi dx = \lambda\int_{\mathbb{R}^N} f(u)\phi dx + \int_{\mathbb{R}^N} |u|^{2^{**}-2}u\phi dx \tag{3.5}$$

for all $\phi \in C_0^\infty(\mathbb{R}^N)$.

Combing (3.4), (3.5) and Step 1, we find that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\Delta u_0\Delta\phi + \nabla u_0\nabla\phi + V(x)u_0\phi)\, dx + b_n\int_{\mathbb{R}^N} |\nabla u_0|^2 dx \int_{\mathbb{R}^N} \nabla u_0\nabla\phi dx \\
&= \lambda\int_{\mathbb{R}^N} f(u_0)\phi dx + \int_{\mathbb{R}^N} |u_0|^{2^{**}-2}u_0\phi dx
\end{aligned} \tag{3.6}$$

for all $\phi \in C_0^\infty(\mathbb{R}^N)$, which in turn implies that u_0 is a weak solution to (1.10). Analogous to the process of Lemma 2.3, we obtain that $u_0^\pm \ne 0$. Thus, we complete the proof of this step.

Step 3. In this step, we prove problem (1.10) possesses a least energy nodal solution v_0, and that there exists a unique pair $(\alpha_{b_n}, \beta_{b_n}) \in \mathbb{R}^+ \times \mathbb{R}^+$ satisfying $\alpha_{b_n}v_0^+ + \beta_{b_n}v_0^- \in \mathcal{N}_{b_n}^\lambda$. Moreover, we prove $(\alpha_{b_n}, \beta_{b_n}) \to (1,1)$ as $n \to \infty$.

Similar to the proof of Theorem 1.1, we can reach a conclusion that problem (1.10) possesses a least energy nodal solution v_0, where $I_0^\lambda(v_0) = c_0$ and $(I_0^\lambda)'(v_0) = 0$. Then, in view of Lemma 2.1, we can obtain with ease the existence and uniqueness of the pair $(\alpha_{b_n}, \beta_{b_n})$ such that $\alpha_{b_n}v_0^+ + \beta_{b_n}v_0^- \in \mathcal{N}_{b_n}^\lambda$. Besides, we know $\alpha_{b_n} > 0$ and $\beta_{b_n} > 0$. To complete the proof, we just establish that $(\alpha_{b_n}, \beta_{b_n}) \to (1,1)$ as $n \to \infty$. Actually, given that

$\alpha_{b_n} v_0^+ + \beta_{b_n} v_0^- \in \mathcal{N}_{b_n}^\lambda$, we have

$$\begin{aligned}
&\alpha_{b_n}^2 \|v_0^+\|^2 + \alpha_{b_n}\beta_{b_n} \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- dx \\
&+ \alpha_{b_n}^2 b_n \int_{\mathbb{R}^N} |\nabla v_0^+|^2 dx \left(\alpha_{b_n}^2 \int_{\mathbb{R}^N} |\nabla v_0^+|^2 dx + \beta_{b_n}^2 \int_{\mathbb{R}^N} |\nabla v_0^-|^2 dx \right) \\
&= \lambda \int_{\mathbb{R}^N} f(\alpha_{b_n} v_0^+) \alpha_{b_n} v_0^+ dx + \int_{\mathbb{R}^N} |\alpha_{b_n} v_0^+|^{2^{**}} dx
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&\beta_{b_n}^2 \|v_0^-\|^2 + \alpha_{b_n}\beta_{b_n} \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- dx \\
&+ \beta_{b_n}^2 b_n \int_{\mathbb{R}^N} |\nabla v_0^-|^2 dx \left(\beta_{b_n}^2 \int_{\mathbb{R}^N} |\nabla v_0^-|^2 dx + \alpha_{b_n}^2 \int_{\mathbb{R}^N} |\nabla v_0^+|^2 dx \right) \\
&= \lambda \int_{\mathbb{R}^N} f(\beta_{b_n} v_0^-) \beta_{b_n} v_0^- dx + \int_{\mathbb{R}^N} |\beta_{b_n} v_0^-|^{2^{**}} dx.
\end{aligned} \tag{3.8}$$

Since $b_n \searrow 0$, we conclude that the sequences $\{\alpha_{b_n}\}$ and $\{\beta_{b_n}\}$ are bounded. Assume, up to a subsequence, $\alpha_{b_n} \to \alpha_0$ and $\beta_{b_n} \to \beta_0$. Then by (3.7) and (3.8), we have

$$\alpha_0^2 \|v_0^+\|^2 + \alpha_0\beta_0 \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- dx = \lambda \int_{\mathbb{R}^N} f(\alpha_0 v_0^+) \alpha_0 v_0^+ dx + \int_{\mathbb{R}^N} |\alpha_0 v_0^+|^{2^{**}} dx \tag{3.9}$$

and

$$\beta_0^2 \|v_0^+\|^2 + \alpha_0\beta_0 \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- dx = \lambda \int_{\mathbb{R}^N} f(\beta_0 v_0^-) \beta_0 v_0^- dx + \int_{\mathbb{R}^N} |\beta_0 v_0^-|^{2^{**}} dx. \tag{3.10}$$

Noticing that v_0 is a nodal solution to problem (1.10), we then get

$$\|v_0^+\|^2 + \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- dx = \lambda \int_{\mathbb{R}^N} f(v_0^+) v_0^+ dx + \int_{\mathbb{R}^N} |v_0^+|^{2^{**}} dx \tag{3.11}$$

and

$$\|v_0^+\|^2 \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- dx = \lambda \int_{\mathbb{R}^N} f(v_0^-) v_0^- dx + \int_{\mathbb{R}^N} |v_0^-|^{2^{**}} dx. \tag{3.12}$$

Therefore, from (3.9)-(3.12), we can obtain with ease that $(\alpha_0, \beta_0) = (1, 1)$, and thus Step 3 follows.

We can now give the proof of Theorem 1.3. We claim that u_0 obtained in Step 2 is a least energy solution to problem (1.10). In fact, according to Step 3 and Lemma 2.1, we find that

$$I_0^\lambda(v_0) \le I_0^\lambda(u_0) = \lim_{n\to\infty} I_{b_n}^\lambda(u_{b_n}) \le \lim_{n\to\infty} I_{b_n}^\lambda(\alpha_{b_n} v_0^+ + \beta_{b_n} v_0^-) = \lim_{n\to\infty} I_0^\lambda(v_0^+ + v_0^-) = I_0^\lambda(v_0),$$

which yields Theorem 1.3. □

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