Approximation factor of the piecewise linear functions in Mamdani fuzzy systems and its realization process

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Abstract: A piecewise linear function is not only an extension of a segmented linear function of one variable in the case of multivariate variables, but also an important bridge to study the approximation of a continuous function by Mamdani and Takagi-Sugeno fuzzy systems. In this paper, the concepts of a piecewise linear function and subdivision number are introduced in hyperplane, the analytic expression of the piecewise linear function is given by matrix determinant, and a new definition of the approximation factor is first proposed by *m*-mesh subdivision. Secondly, by the method of generating small polyhedron from three-dimensional cube, the change rule of vertex coordinates of *n*-dimensional subdivision polyhedron is studied, the vertex coordinates of small polyhedron are obtained by rotating component coordinates of their respective coordinate axes. Furthermore, the calculation methods of algebraic cofactor and matrix norm for the corresponding determinant are given. Finally, according to the method of solving algebraic cofactors and matrix norms, it is proved that the approximation factor has nothing to do with the subdivision number, but the approximation precision has something to do with the subdivision number. In addition, the realization process of a specific binary piecewise linear function approaching a continuous function according to infinite norm in two dimensions space is given by an example.

Key words: piecewise linear function; mesh subdivision; approximation factor; Mamdani fuzzy system; matrix norm

1 Introduction

The core of fuzzy system is to bypass the precise mathematical model to carry out logical reasoning and calculation for fuzzy information. The main method is to process data information and language information based on a set of If -Then rules. Generally, Mamdani and T-S fuzzy systems are two kinds of common models, in which Mamdani fuzzy system is the simplest kind of model, the main characteristic of which is that the output of each rule is a fuzzy set, while the output of T-S fuzzy system is a multivariate linear function of the input variables. Although it does not depend on the accurate mathematical model, it has better logical reasoning and numerical calculation and nonlinear function approximation ability. In the late 1990s, fuzzy systems as approximators have been widely used in the fields of system identification, pattern recognition, nonlinear system design and fuzzy control. See Ref. [1-3]. Especially in 1998, Ying [4] used the linear programming method to study the general approximation of T-S fuzzy system, and then Zeng [5] gave the sufficient conditions for the approximation of the fuzzy system. These results provide some new ideas and methods for further study of the approximation performance of generalized fuzzy systems.

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The essence of a piecewise linear function is an extension of a segmented linear function of one variable in the case of multivariate variables. It can not only approximate an unknown continuous function with arbitrary precision, but also play an important role in the approximation theory of fuzzy system. In 2000, the concept of multivariate piecewise linear function based on the subdivision input space was first proposed by Prof. Liu in [6], and it is an important bridge to study the approximation of Takagi- Sugeno fuzzy system to continuous function and integrable function in [7]. Subsequently, in 2001, he studied the approximation performance of generalized Mamdani fuzzy system to a class of p-integrable functions in [8], for further information. See Ref. [9-11]. In 2006, Zhang and Li [12] proved that Mamdani fuzzy system is a universal approximator of integrable function by means of a square piecewise linear function, and provided the necessary conditions for Mamdani fuzzy system to be an approximator in [13]. In recent years, the convergence of fuzzy transformation and the solution of fuzzy dual complex linear systems have been studied. See [14,15]. However, these results only take piecewise linear function as a bridge to complete the proof, and there is no specific method to obtain it, which naturally limits the wide application of piecewise linear functions.

In 2014, Wang proved the universal approximation of Mamdani fuzzy system by introducing the piecewise linear function in [16]. In the same year, Peng [17] further gave the construction method and analytical formula of the piecewise linear function on the basis of Ref. [16], and proposed the solution formula of corresponding equation system through the matrix determinant. In 2015, Tao et al. [18] introduced K-quasi-subtraction operation to give the concept of Kp-integral norm, and then discussed the approximation of piecewise linear functions to a class of integrable functions in 2015. she utilized the piecewise linear functions as a tool to explore the approximation performance of generalized Mamdani fuzzy system to Kp-integrable functions in [19]. See [20,22]. In 2017, Wang et al.[21] proposed that the piecewise linear functions can approximate a continuous function to arbitrary precision in the sense of maximum norm based on the mesh subdivision of generalized cube. Unfortunately, he only guessed that the approximation factor part is a constant independent of the subdivision number, but its proof is not given. The main motivation of this paper is to give a complete proof of them by calculating the determinant algebraic cofactor and matrix norm.

The remainder of the paper is organized as follows. In Section 2, according to Ref. [14], the concepts of the piecewise linear function and subdivision number are introduced. Meanwhile, we provide the analytical expressions of coefficient matrixes of corresponding linear equation system. In Section 3, the concept of the approximation factor based on the piecewise linear functions is first proposed by applying m-mesh subdivision of a generalized cube, and the calculation method of algebraic cofactors and matrix norm for the corresponding determinant are given. In Section 4, we demonstrate that the approximation factor of a piecewise linear function is independent of the selection of subdivision number by solving algebraic cofactors and matrix norm. In Section 5, the realization process of the binary piecewise linear function approaching a continuous function is given by an example analysis, and it is verified that the approximation factor is independent of the subdivision number, but the approximation accuracy is related to the subdivision number.

2 Piecewise linear functions

A piecewise linear function is a generalization of a segmented linear function of one variable in the case of multivariate variables, so it has many excellent properties such as zero outside the compact set of $\mathbb{R}^n$, uniformly continuous on the compact set, existence of unilateral partial derivative and bounded. Next, we will give some related concepts of n-variables piecewise linear functions, where the word "n-variables" can be omitted.

In this paper, we use the symbols $\mathbb{R}^n$ and $\mathbb{N}$ to represent the n-dimensional Euclidean space and the set of natural numbers, respectively. For any $a>0$, let

$$\Delta(a)=\{(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n \mid 0\le x_i\le a\,;\ i=1,2,\cdots,n\},$$

then $\Delta(a)$ is called the generalized cube with side length a in $\mathbb{R}^n$, and it is abbreviated to $\Delta(a)=[0,a]\times[0,a]\times\cdots\times[0,a]\,\underline{\underline{\Delta}}\,[0,a]^n$.

Definition 2.1 [16,21] Let an n-variables continuous function $S:\mathbb{R}^n\to\mathbb{R}$. If the following conditions ① to ② are satisfied:

① There is a real number $a>0$ such that S is always zero outside the generalized cube $\Delta(a)$;

② If there is a group of n-dimensional polyhedrons $\{\Delta_1,\Delta_2,\cdots,\Delta_{N_S}\}\subset\Delta(a)$ with $\bigcup_{j=1}^{N_S}\Delta_j=\Delta(a)$, such that S takes n-variables linear function on each small polyhedron $\Delta_j\,(j=1,2,\cdots,N_s)$, that is to say, S can be expressed as

$$S(x)=\sum_{i=1}^{n}\beta_{ij}\cdot x_i+\lambda_j,\ \text{for all}\ \ x=(x_1,x_2,\cdots,x_n)\in\Delta_j,\ j=1,2,\ \cdots,N_s.$$

Then S is called a piecewise linear function on $\mathbb{R}^n$, where β_{ij} and λ_j are constants, $i=1,2,\cdots,n$.

In fact, the piecewise linear functions play an important role in studying the approximation of Mamdani fuzzy system and T-S fuzzy system. This is because the fuzzy system can approximate some piecewise linear function, and the piecewise linear function can approximate a continuous or integrable function, thus achieving the fuzzy system approximating to some unknown continuous function.

Definition 2.2 Let $b=(b_1,b_2,\cdots,b_n)\in\mathbb{R}^n$, $\Omega(\beta)=\{x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n \mid <x,b>=\beta\}$, then $\Omega(\beta)$ is call a $\beta-$hyperplane on $\mathbb{R}^n$, where real number $\beta\in\mathbb{R}$, symbol $<\cdot>$ is inner product. Clearly, the hyperplane $\Omega(\beta)$ can also be expressed in the form of linear combination of multivariate variables, that is,

$$\Omega(\beta)=\{(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n \mid b_1x_1+b_2x_2+\cdots+b_nx_n=\beta,\ b_i\in\mathbb{R},\ i=1,2,\cdots,n\}.$$

Definition 2.3 Let input space be $\Delta(a)=[0,a]^n$ ($a>0$), and $\Delta(a)$ is divided into m_i small closed intervls along each axis x_i ($i=1,2,\cdots,n$) in turn, $[0,a]$ is divided into $[0,a/m_i],[a/m_i,2a/m_i],$ $\cdots,[a(m_i-2)/m_i,a(m_i-1)/m_i],\ [a(m_i-1)/m_i,a]$. If all subdivision points can be uniformly listed as $t_j^i=ja/m_i$, $j=1,2,\cdots,m_i$, then m_i is called an isometry subdivision number of the axis x_i in the input space $\Delta(a)$, also referred to as m_i is the subdivision number on the x_i axis.

Note 1 For simplicity, this paper always assumes that the same subdivision number is taken on each axis x_i, i.e., $m_i=m$, $i=1,2,\cdots,n$. In this case, the isometry subdivision of input space $\Delta(a)$ is also called $m-$mesh subdivision. Under this convention, it is not difficult to obtain the generalized cube $\Delta(a)$ which can be decomposed into m^n generalized small polyhedrons Δ_j with right angle side length as $\frac{a}{m}$, and $\Delta(a)=\bigcup_{j=1}^{m^n}\Delta_j$.

Construction of a piecewise linear function S **:** According to Refs. [20,21], the input space $\Delta(a)$ is divided into *m*-mesh . Let $\Delta_{i_1 i_2 \cdots i_n}$ be a small $n-$ dimensional polyhedron after subdivision, and $x_k^* = (x_1^k, x_2^k, \cdots, x_n^k)$ is denoted as the k -th vertex coordinate of n -dimensional small polyhedron $\Delta_{i_1 i_2 \cdots i_n}$ after subdivision, and suppose that these $n+1$ vertices are not on the same hyperplane, so as to ensure that the following determinant $|D_n| \neq 0$. If $n+1$ vertices coordinates of $\Delta_{i_1 i_2 \cdots i_n}$ are briefly note as $x_1^*, x_2^*, \cdots, x_n^*, x_{n+1}^*$ in the specified order, then each $f(x_k^*)$ can take the corresponding value under the action of f, and the vertices coordinates of each small polyhedron $\Delta_{i_1 i_2 \cdots i_n}$ in $\mathbb{R}^{n+1}$ can be expressed as follows:

$$(x_1^1, x_2^1, \cdots, x_n^1, f(x_1^*)), (x_1^2, x_2^2, \cdots, x_n^2, f(x_2^*)), \cdots, (x_1^{n+1}, x_2^{n+1}, \cdots, x_n^{n+1}, f(x_{n+1}^*)). \tag{1}$$

Let the hyperplane S be determined by $n+1$ vertices of form (1) of $\Delta_{i_1 i_2 \cdots i_n}$ in $\mathbb{R}^{n+1}$, and the analytic expression of the linear function S is expressed as

$$S_{i_1 i_2 \cdots i_n}(x) = b_{i_1}^1 x_1 + b_{i_2}^2 x_2 + \cdots + b_{i_n}^n x_n + \lambda_{i_1 i_2 \cdots i_n}, \tag{2}$$

where $x = (x_1, x_2, \cdots, x_n) \in \Delta_{i_1 i_2 \cdots i_n}$, $i_j = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$. In addition, because the hyperplane $S_{i_1 i_2 \cdots i_n}(x)$ is formed by cutting surface $f(x)$ in the sense of $m-$mesh, and $n+1$ vertices x_k^* on $\mathbb{R}^{n+1}$ are the common intersection of hyperplane $S_{i_1 i_2 \cdots i_n}(x)$ and surface $f(x)$, it is necessary for each vertex x_k^* coordinate to have $S_{i_1 i_2 \cdots i_n}(x_k^*) = f(x_k^*), k = 1, 2, \cdots, n, n+1$.

Then, by substituting the coordinates of $n+1$ vertices of $\Delta_{i_1 i_2 \cdots i_n}$ into formula (2), a set of hyperplane linear equations on $\mathbb{R}^{n+1}$ can be obtained as follows:

$$\begin{cases} b_{i_1}^1 x_1^1 + b_{i_2}^2 x_2^1 + \cdots + b_{i_n}^n x_n^1 + \lambda_{i_1 i_2 \cdots i_n} = f(x_1^*) \\ b_{i_1}^1 x_1^2 + b_{i_2}^2 x_2^2 + \cdots + b_{i_n}^n x_n^2 + \lambda_{i_1 i_2 \cdots i_n} = f(x_2^*) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ b_{i_1}^1 x_1^n + b_{i_2}^2 x_2^n + \cdots + b_{i_n}^n x_n^n + \lambda_{i_1 i_2 \cdots i_n} = f(x_n^*) \\ b_{i_1}^1 x_1^{n+1} + b_{i_2}^2 x_2^{n+1} + \cdots + b_{i_n}^n x_n^{n+1} + \lambda_{i_1 i_2 \cdots i_n} = f(x_{n+1}^*) \end{cases}. \tag{3}$$

Because all vertex coordinates of each small polyhedron $\Delta_{i_1 i_2 \cdots i_n}$ in $\mathbb{R}^{n+1}$ shown in formula (1) are known in the sense of $m-$mesh subdivision. Therefore, all coefficients $\{b_{i_1}^1, b_{i_2}^2, \cdots, b_{i_n}^n, \lambda_{i_1 i_2 \cdots i_n}\}$ in equation group (3) can be regarded as unknown quantities, and then the values of these coefficients can be obtained by solving equation group (3). According to Ref. [21], the analytic expression of the piecewise linear function S on $\Delta(a)$ is

$$S(x_1, x_2, \cdots, x_n) = \begin{cases} \dfrac{|D_{i_1}^1|}{|D_n|} x_1 + \dfrac{|D_{i_2}^2|}{|D_n|} x_2 + \cdots + \dfrac{|D_{i_n}^n|}{|D_n|} x_n + \dfrac{|D_{i_{n+1}}^{n+1}|}{|D_n|}, \\ (x_1, x_2, \cdots, x_n) \in \Delta_{i_1 i_2 \cdots i_n}, i_1, i_2 \cdots, i_n \in \{1, 2, \cdots, m\} \\ 0, \qquad \text{Otherwise} \end{cases}. \tag{4}$$

Here, the coefficients $|D_j^{i_j}| \big/ |D_n|$ ($j = 1, 2, \cdots, n, n+1$) of the piecewise linear function S on each hyperplane $\Delta_{i_1 i_2 \cdots i_n}$ are given in the following matrix determinant form, i.e.,

$$|D_{i_1}^1| = \begin{vmatrix} f(x_1^*) & x_2^1 & \cdots & x_n^1 & 1 \\ f(x_2^*) & x_2^2 & \cdots & x_n^2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ f(x_n^*) & x_2^n & \cdots & x_n^n & 1 \\ f(x_{n+1}^*) & x_2^{n+1} & \cdots & x_n^{n+1} & 1 \end{vmatrix}, \quad |D_{i_2}^2| = \begin{vmatrix} x_1^1 & f(x_1^*) & \cdots & x_n^1 & 1 \\ x_1^2 & f(x_2^*) & \cdots & x_n^2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_1^n & f(x_n^*) & \cdots & x_n^n & 1 \\ x_1^{n+1} & f(x_{n+1}^*) & \cdots & x_n^{n+1} & 1 \end{vmatrix},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$|D_{i_n}^n| = \begin{vmatrix} x_1^1 & x_2^1 & \cdots & f(x_1^*) & 1 \\ x_1^2 & x_2^2 & \cdots & f(x_2^*) & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_1^n & x_2^n & \cdots & f(x_n^*) & 1 \\ x_1^{n+1} & x_2^{n+1} & \cdots & f(x_{n+1}^*) & 1 \end{vmatrix}, \quad |D_n| = \begin{vmatrix} x_1^1 & x_2^1 & \cdots & x_n^1 & 1 \\ x_1^2 & x_2^2 & \cdots & x_n^2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_1^n & x_2^n & \cdots & x_n^n & 1 \\ x_1^{n+1} & x_2^{n+1} & \cdots & x_n^{n+1} & 1 \end{vmatrix}. \tag{5}$$

Moreover, by applying $m-$mesh subdivision, it is not difficult to know that the vertex coordinates of each small polyhedron $\Delta_{i_1 i_2 \cdots i_n}$ can be written as $x_k^* = (\frac{ai_1}{m}, \frac{ai_2}{m}, \cdots, \frac{ai_n}{m})$, and the index of each coordinate axis is $i_1, i_2, \cdots, i_n \in \{1, 2, \cdots, m\}$. See [21,22].

3 Approximation of Mamdani fuzzy system

With the specific analytical expression of a piecewise linear function S , it is natural to think of whether piecewise linear function can approximate to a continuous function with any precision? Regarding to this question, an affirmative answer and a detailed proof have been given in Ref. [21], and it is studied that Mamdani fuzzy system can not only approximate a piecewise linear function, but also take the piecewise linear function as a bridge to prove that Mamdani fuzzy system can approach a continuous function on a compact set with any precision. Refer to Ref. [21].

In fact, Mamdani fuzzy system is one of the simplest fuzzy system models. Its main feature is that each rule's subsequent output is a fuzzy set, while the subsequent output of T-S fuzzy system is a multivariate linear function about input variables. Next, we first review some related knowledge of Mamdani fuzzy system, and assume that the rule base is composed of the following fuzzy rules

$$R_{i_1 i_2 \cdots i_n}: \text{ If } x_1 \text{ is } A_{i_1}^1,\ x_2 \text{ is } A_{i_2}^2, \cdots, x_n \text{ is } A_{i_n}^n, \text{then } u \text{ is } C_{i_1 i_2 \cdots i_n}.$$

The input-output relationship of Mamdani fuzzy system with single point fuzzification, product reasoning machine and central average fuzzification is as follows:

$$F(x_1, x_2, \cdots, x_n) = \frac{\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \cdots \sum_{i_n=1}^{N_n} \left(\prod_{k=1}^{n} A_{i_k}^k(x_k) \right) \cdot \overline{y}_{i_1 i_2 \cdots i_n}}{\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \cdots \sum_{i_n=1}^{N_n} \left(\prod_{k=1}^{n} A_{i_k}^k(x_j) \right)}, \tag{6}$$

where $x=(x_1,x_2,\cdots,x_n)\in U_1\times U_2\times\cdots U_n\subset\mathbb{R}^n$ is an input variable, $\{A^1_{i_1},A^2_{i_2},\cdots,A^n_{i_n}\}$ is a family of antecedent fuzzy sets of Mamdani fuzzy system corresponding to the i_j -th coordinate axis, i.e., $A^j_{i_j}\in F(U_j)$, $j=1,2,\cdots,n$, $i_j=1,2,\cdots,N_j$, and $C_{i_1i_2\cdots i_n}$ is a consequent fuzzy set corresponding to rule $R_{i_1i_2\cdots i_n}$ in the output domain $V\subset\mathbb{R}$, the $\overline{y}^{i_1,i_2\cdots i_n}$ is the center of the consequent fuzzy set $C_{i_1i_2\cdots i_n}$, the real number u is the output variable on V. See [21].

In addition, it is not hard to see that the total number of all possible fuzzy rules in Mamdani fuzzy system is $M=N_1\times N_2\times\cdots\times N_n$, and $\{A^1_1,A^1_2,\cdots,A^1_{N_1}\}$, $\{A^2_1,A^2_2,\cdots,A^2_{N_2}\}$, $\cdots$, $\{A^n_1,A^n_2,\cdots,A^n_{N_n}\}$ are the antecedent fuzzy sets on the i_j -th coordinate axis, respectively. For simplicity, $N_1=N_2=\cdots=N_n=m$ is chosen in this paper. See [21,22].

Lemma 1 [21] Let S be a piecewise linear function of form (4) on the generalized cube $\Delta(a)$, where $a>0$, then, for arbitrary $\varepsilon>0$, there is a $m_0\in\mathbb{N}$ such that Mamdani fuzzy system F_m of form (6) determined by subdivision number m satisfies $\|F_m-S\|_\infty<\varepsilon$ when $m>m_0$.

In accordance with the properties of determinant, it is not difficult to rewrite the above matrix determinant (5) as follows:

$$|D^1_{i_1}|=\begin{vmatrix} f(x_1^*)-f(x_2^*) & x_2^1-x_2^2 & \cdots & x_n^1-x_n^2 \\ f(x_2^*)-f(x_3^*) & x_2^2-x_2^3 & \cdots & x_n^2-x_n^3 \\ \vdots & \vdots & & \vdots \\ f(x_n^*)-f(x_{n+1}^*) & x_2^n-x_2^{n+1} & \cdots & x_n^n-x_n^{n+1} \end{vmatrix},\quad |D^2_{i_2}|=\begin{vmatrix} x_1^1-x_1^2 & f(x_1^*)-f(x_2^*) & \cdots & x_n^1-x_n^2 \\ x_1^2-x_1^3 & f(x_2^*)-f(x_3^*) & \cdots & x_n^2-x_n^3 \\ \vdots & \vdots & & \vdots \\ x_1^n-x_1^{n+1} & f(x_n^*)-f(x_{n+1}^*) & \cdots & x_n^n-x_n^{n+1} \end{vmatrix},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$|D^n_{i_n}|=\begin{vmatrix} x_1^1-x_1^2 & x_2^1-x_2^2 & \cdots & f(x_1^*)-f(x_2^*) \\ x_1^2-x_1^3 & x_2^2-x_2^3 & \cdots & f(x_2^*)-f(x_3^*) \\ \vdots & \vdots & & \vdots \\ x_1^n-x_1^{n+1} & x_2^n-x_2^{n+1} & \cdots & f(x_n^*)-f(x_{n+1}^*) \end{vmatrix},\quad |D_n|=\begin{vmatrix} x_1^1-x_1^2 & x_2^1-x_2^2 & \cdots & x_n^1-x_n^2 \\ x_1^2-x_1^3 & x_2^2-x_2^3 & \cdots & x_n^2-x_n^3 \\ \vdots & \vdots & & \vdots \\ x_1^n-x_1^{n+1} & x_2^n-x_2^{n+1} & \cdots & x_n^n-x_n^{n+1} \end{vmatrix}. \quad (7)$$

Note 2 According to the $m-$mesh subdivision, the difference of all adjacent coordinate components in these determinants on the same coordinate axis $x_i^j-x_i^{j+1}$ can only be $\pm\frac{a}{m}$ or zero. If these determinants (except f) are expanded in column 1, column 2, …, column n, and then n algebraic cofactors of $n-1$ order m can be obtained by combining and sorting out the factors $(f(x_i^*)-f(x_{i+1}^*))$ item by item, where the $n-1$ order algebraic cofactors corresponding to factor $(f(x_1^*)-f(x_2^*))$ are simply expressed as $|B_1^1|,|B_2^1|,\cdots,|B_n^1|$; factor $(f(x_2^*)-f(x_3^*))$ corresponds to $|B_1^2|,|B_2^2|,\cdots,|B_n^2|$. By analogy, the factor $(f(x_i^*)-f(x_{i+1}^*))$ corresponds to the $n-1$ order algebraic cofactors are $|B_1^i|,|B_2^i|,\cdots,|B_k^i|,\cdots,|B_n^i|$, where $|B_k^i|$ represents the algebraic cofactor of the i-th determinant, the column k and the i-th element. For example, $|B_1^1|$ is a cofactor of $|D^1_{i_1}|$, $|B_2^1|$ and $|B_2^2|$ are cofactors of $|D^2_{i_2}|$, $|B_n^2|$ is a cofactor of $|D^n_{i_n}|$, and

$$|B_1^1|=(-1)^2\begin{vmatrix} x_2^2-x_2^3 & \cdots & x_n^2-x_n^3 \\ \vdots & & \vdots \\ x_2^n-x_2^{n+1} & \cdots & x_n^n-x_n^{n+1}\end{vmatrix},\quad |B_2^1|=(-1)^3\begin{vmatrix} x_1^2-x_1^3 & \cdots & x_n^2-x_n^3 \\ \vdots & & \vdots \\ x_1^n-x_1^{n+1} & \cdots & x_1^n-x_1^{n+1}\end{vmatrix},$$

$$|B_2^2|=(-1)^4\begin{vmatrix} x_1^1-x_1^2 & \cdots & x_n^1-x_n^2 \\ \vdots & & \vdots \\ x_1^n-x_1^{n+1} & \cdots & x_n^n-x_n^{n+1}\end{vmatrix},\quad |B_n^2|=(-1)^{n+2}\begin{vmatrix} x_1^1-x_1^2 & \cdots & x_{n-1}^1-x_{n-1}^2 \\ \vdots & & \vdots \\ x_1^n-x_1^{n+1} & \cdots & x_{n-1}^n-x_{n-1}^{n+1}\end{vmatrix}.$$

It should be noted that in the sense of isometric subdivision, the vertex coordinates of each small polyhedron $\Delta_{i_1 i_2 \cdots i_n}$ as $x_k^* = (\frac{ai_1}{m}, \frac{ai_2}{m}, \cdots, \frac{ai_n}{m})$. Hence, the difference $x_i^j - x_i^{j+1}$ of all adjacent coordinate components on the same coordinate axis in these algebraic cofactor is only $\pm\frac{a}{m}$ or zero.

Definiton 3.1 Let matrix A be n square array, let $\|A\|=|(|A|)|$, then $\|A\|$ is called matrix norm of A, that is, matrix norm $\|A\|$ is absolute value of determinant of A. Obviously, the matrix norm of any square array A always satisfies $\|A\|\geq 0$.

Lemma 2 [21] Let f be continuous function on compact set $\Delta(a)\subset\mathbb{R}^n$, $(x; f(x))$ is a given data pair, but the analytic expression of f is unknown. Then, for any $\varepsilon>0$, there is a subdivision number $m\in\mathbb{N}$ and the piecewise linear function S of form (4), which satisfies the requirements in the sense of infinite norm $\|S-f\|_\infty < \left(\frac{a}{m\|D_n\|}\sum_{i=1}^{n}\sum_{k=1}^{n}\|B_k^i\|+1\right)\varepsilon$, where the infinite norm is defined as $\|S-f\|_\infty = \sup_{x\in\Delta(a)}|S(x)-f(x)|$.

It is not difficult to see that Lemma 1 and Lemma 2 can be utilized to obtain the following Lemma 3. That is to say, Mamdani fuzzy system F_m of form (6) can indeed approximate f to arbitrary accuracy with respect to infinite norm.

Lemma 3 [21] If f be a continuous function on compact set $\Delta(a)\subset\mathbb{R}^n$, then, for arbitrary $\varepsilon>0$, there is a subdivision number $m_0\in\mathbb{N}$ and the Mamdani fuzzy system of form (6), such that $\|F_m-f\|_\infty<\varepsilon$ when $m>m_0$, that is, F_m can approximate f to any precision by infinite norm.

However, it must be said that it is a pity, because the Ref. [21] does not give strict proof in theory that the sum factor $\left(\frac{a}{m\|D_n\|}\sum_{i=1}^{n}\sum_{k=1}^{n}\|B_k^i\|+1\right)$ in Lemma 2 is a constant independent of the subdivision number m, but simply expounds it in language. Thus, in this paper we will prove that the sum factor is a constant independent of subdivision number m on $\mathbb{R}^n$.

Definition 3.2 Assume that matrix $|B_k^i|$ ($i,k=1,2,\cdots,n$) are the corresponding $n-1$ order algebraic cofactors as described above, and $\|B_k^i\|$ and $\|D_n\|$ are matrix norms, then the expression $\left(\frac{a}{m\|D_n\|}\sum_{i=1}^{n}\sum_{k=1}^{n}\|B_k^i\|+1\right)$ is called the approximation factor of $\|f-S\|_\infty$, also referred to as the approximation factor of a piecewise linear function S.

In fact, the expression of the approximation factor is more complex with the increasing dimension of the input variable n. Moreover, the isometric subdivision of the generalized cube $\Delta(a)$ no longer exists when $n \geq 4$, which makes it difficult to find the $n-1$ order algebraic cofactors.

4 Approximation factor and subdivision number

According to Lemma 2, only if the approximation factor is a constant independent of the subdivision number, the piecewise linear function S has the approximation, so it is very important whether the approximation factor is a constant independent of the subdivision number. This conclusion is only a conjecture in Ref. [21], but has not been proved in detail. Therefore, in this paper we will prove that the approximation factor is indeed a constant independent of the subdivision number m on $\mathbb{R}^n$.

In order to verify whether the approximation factor is constant, it is necessary to determin the vertex coordinates of each small polyhedron $\Delta_{i_1 i_2 \cdots i_n}$ in $\mathbb{R}^n$ and its order, and then find the $n-1$ order algebraic cofactor and its matrix norm $\| B_k^i \|$ corresponding to factor $(f(x_i^*) - f(x_{i+1}^*))$ according to Note 2. Next, take $n=3$ as an example to continue to explore the vertex coordinates of small tetrahedron $\Delta_{i_1 i_2 \cdots i_n}$ on $\mathbb{R}^n$, and its sorting problem. See Fig. 1.

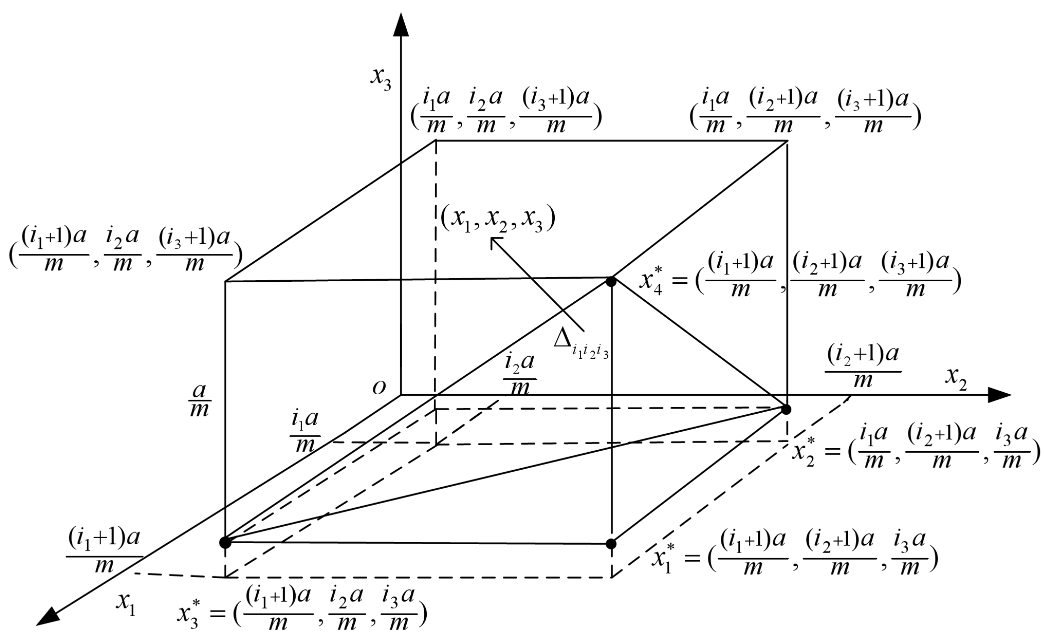

Fig.1 Subdivision image of a small cube with a side length of $\frac{a}{m}$ when $n = 3$

In fact, for any input variable $(x_1, x_2, x_3) \in [0, a]^3$, there are the indexes $i_1, i_2, i_3 \in \mathbb{N}$, such that $(x_1, x_2, x_3) \in \Delta_{i_1 i_2 i_3}$. Suppose all coordinates of four vertices of the small tetrahedron $\Delta_{i_1 i_2 i_3}$ are $x_1^* = (\frac{(i_1+1)a}{m}, \frac{(i_2+1)a}{m}, \frac{i_3 a}{m})$, $x_2^* = (\frac{i_1 a}{m}, \frac{(i_2+1)a}{m}, \frac{i_3 a}{m})$, $x_3^* = (\frac{(i_1+1)a}{m}, \frac{i_2 a}{m}, \frac{i_3 a}{m})$ and $x_4^* = (\frac{(i_1+1)a}{m}, \frac{(i_2+1)a}{m}, \frac{(i_3+1)a}{m})$ respectively., as shown in Fig.1.

Note 3 It is not difficult to find from Fig. 1 that if x_1^* is taken as the base point, the order of vertices x_1^* and x_2^*, x_3^*, x_4^* conforms to the right-hand rule, that is, x_1^* is taken as the base point, and the bending direction of the four fingers of the right hand is from $\overrightarrow{x_1^* x_2^*}$ to $\overrightarrow{x_2^* x_3^*}$, so the order of x_2^*, x_3^*, x_4^* is determined in turn. According to this sort, we find that the corresponding coordinates have the following rules: the straight-line distance of vertices x_1^* and x_2^*, x_3^*, x_4^* along their respective coordinate axes is $\frac{a}{m}$, where the difference between the x_2^*-first component, x_3^*-second component and x_4^*-third component of vertices and the x_1^*-corresponding coordinate component are all $\frac{a}{m}$, while the other coordinate components have no change (difference is zero), that is, the difference between the components of adjacent vertices on the same coordinate axis is $\pm\frac{a}{m}$ or zero.

Similarly, the method can be extended to n-dimensional polyhedron. Specific method: firstly, select a vertex on n-dimensional polyhedron $\Delta_{i_1 i_2 \cdots i_n}$ as the base point $x^*_{i_1 i_2 \cdots i_n}$, let the base point be $x^*_{i_1 i_2 \cdots i_n} = (\frac{i_1 a}{m}, \frac{i_2 a}{m}, \cdots, \frac{i_{n-1} a}{m}, \frac{i_n a}{m})$, and then rotate each coordinate component $\frac{i_j a}{m}$ to $\frac{(i_j - 1)a}{m}$ or $\frac{(i_j + 1)a}{m}$ ($j = 1, 2, \cdots, n$) along its coordinate axis i_j in turn. If the other components are invariant, then the total $n+1$ vertex coordinates of $\Delta_{i_1 i_2 \cdots i_n}$ can be obtained.

Thorem4.1 Let f be a continuous function on compact set $\Delta(a) \subset \mathbb{R}^n$, $\Delta_{i_1 i_2 \cdots i_n}$ be the small polyhedron of the above n-dimensional subdivision, and the coordinates of all vertices are denoted as $x^*_{i_1 i_2 \cdots i_n} = (\frac{i_1 a}{m}, \frac{i_2 a}{m}, \cdots, \frac{i_n a}{m})$, $i_1, i_2, \cdots, i_n \in \{1, 2, \cdots, m\}$. Matrix determinant $|B_k^i|$ is the algebraic cofactor of $n-1$ order defined in Note 2, $i, k = 1, 2, \cdots, n$, $\| B_k^i \|$ and $\| D_n \|$ are matrix norms, then the approximation factor $\left(\frac{a}{m \| D_n \|} \sum_{i=1}^{n} \sum_{k=1}^{n} \| B_k^i \| + 1 \right)$ is independent of the subdivision number m.

Proof According to Ref. [19], for any input variable $(x_1, x_2, \cdots, x_n) \in \Delta(a)$, there is m--mesh subdivision and index $i_1, i_2, \cdots, i_n \in \{1, 2, \cdots, m\}$ on cube $\Delta(a)$, such that $(x_1, x_2, \cdots, x_n) \in \Delta_{i_1 i_2 \cdots i_n}$, where the right angle side length of each small polyhedron $\Delta_{i_1 i_2 \cdots i_n}$ is $\frac{a}{m}$.

Without losing generality, we can choose vertex $(\frac{(1+i_1)a}{m}, \frac{(1+i_2)a}{m}, \frac{(1+i_3)a}{m}, \cdots, \frac{(1+i_{n-1})a}{m}, \frac{i_n a}{m})$ as the base point of $\Delta_{i_1 i_2 \cdots i_n}$, the other n vertex coordinates are

$$(\frac{i_1 a}{m}, \frac{(1+i_2)a}{m}, \frac{(1+i_3)a}{m}, \cdots, \frac{(1+i_{n-1})a}{m}, \frac{i_n a}{m}), (\frac{(1+i_1)a}{m}, \frac{i_2 a}{m}, \frac{(1+i_3)a}{m}, \cdots, \frac{(1+i_{i-1})a}{m}, \frac{i_n a}{m}), (\frac{(1+i_1)a}{m}, \frac{(1+i_2)a}{m}, \frac{i_3 a}{m}, \cdots, \frac{(1+i_{n-1})a}{m}, \frac{i_n a}{m}),$$
$$\cdots, (\frac{(1+i_1)a}{m}, \frac{(1+i_2)a}{m}, \frac{(1+i_3)a}{m}, \cdots, \frac{(1+i_{n-1})a}{m}, \frac{(1+i_n)a}{m}).$$

By Note 3, these vertex coordinates are directly substituted into the formula (7) to obtain

$$|D_n| = \begin{vmatrix} \frac{a}{m} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{a}{m} & \frac{a}{m} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{a}{m} & \frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{a}{m} & 0 \\ 0 & 0 & 0 & \cdots & -\frac{a}{m} & -\frac{a}{m} \end{vmatrix} = -(\frac{a}{m})^n \Rightarrow \|D_n\| = (\frac{a}{m})^n$$

According to Note 2, the difference $x_i^j - x_i^{j+1}$ of adjacent coordinate components on the same coordinate axis in matrix determinant $|D_{i_1}^1|, |D_{i_2}^2|, \cdots, |D_{i_n}^n|$ can only be $\pm\frac{a}{m}$ or zero. If the above vertex coordinates are directly substituted into formula (7), it can be obtained immediately.

$$|D_{i_1}^1| = \begin{vmatrix} f(x_1^*)-f(x_2^*) & 0 & 0 & \cdots & 0 & 0 \\ f(x_2^*)-f(x_3^*) & \frac{a}{m} & 0 & \cdots & 0 & 0 \\ f(x_3^*)-f(x_4^*) & -\frac{a}{m} & \frac{a}{m} & \cdots & 0 & 0 \\ f(x_4^*)-f(x_5^*) & 0 & -\frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ f(x_n^*)-f(x_{n+1}^*) & 0 & 0 & \cdots & -\frac{a}{m} & -\frac{a}{m} \end{vmatrix}, \quad |D_{i_2}^2| = \begin{vmatrix} \frac{a}{m} & f(x_1^*)-f(x_2^*) & 0 & \cdots & 0 & 0 \\ -\frac{a}{m} & f(x_2^*)-f(x_3^*) & 0 & \cdots & 0 & 0 \\ 0 & f(x_3^*)-f(x_4^*) & \frac{a}{m} & \cdots & 0 & 0 \\ 0 & f(x_4^*)-f(x_5^*) & -\frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & f(x_n^*)-f(x_{n+1}^*) & 0 & \cdots & -\frac{a}{m} & -\frac{a}{m} \end{vmatrix},$$

$$|D_{i_3}^3| = \begin{vmatrix} \frac{a}{m} & 0 & f(x_1^*)-f(x_2^*) & \cdots & 0 & 0 \\ -\frac{a}{m} & \frac{a}{m} & f(x_2^*)-f(x_3^*) & \cdots & 0 & 0 \\ 0 & -\frac{a}{m} & f(x_3^*)-f(x_4^*) & \cdots & 0 & 0 \\ 0 & 0 & f(x_4^*)-f(x_5^*) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & f(x_n^*)-f(x_{n+1}^*) & \cdots & -\frac{a}{m} & -\frac{a}{m} \end{vmatrix}, \cdots\cdots, \quad |D_{i_n}^n| = \begin{vmatrix} \frac{a}{m} & 0 & 0 & \cdots & 0 & f(x_1^*)-f(x_2^*) \\ -\frac{a}{m} & \frac{a}{m} & 0 & \cdots & 0 & f(x_2^*)-f(x_3^*) \\ 0 & -\frac{a}{m} & \frac{a}{m} & \cdots & 0 & f(x_3^*)-f(x_4^*) \\ 0 & 0 & -\frac{a}{m} & \cdots & 0 & f(x_4^*)-f(x_5^*) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{a}{m} & f(x_n^*)-f(x_{n+1}^*) \end{vmatrix}.$$

By Note 2, the $n-1$ order algebraic cofactors are obtained by expanding $|D_{i_1}^1|$ with column 1, i.e.,

$$|B_1^1| = \begin{vmatrix} \frac{a}{m} & 0 & \cdots & 0 & 0 \\ -\frac{a}{m} & \frac{a}{m} & \cdots & 0 & 0 \\ 0 & -\frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{a}{m} & -\frac{a}{m} \end{vmatrix} = -(\tfrac{a}{m})^{n-1}, \quad |B_1^2| = \begin{vmatrix} 0 & 0 & \cdots & 0 & 0 \\ -\frac{a}{m} & \frac{a}{m} & \cdots & 0 & 0 \\ 0 & -\frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{a}{m} & -\frac{a}{m} \end{vmatrix} = 0,$$

$$|B_1^3| = \begin{vmatrix} 0 & 0 & \cdots & 0 & 0 \\ \frac{a}{m} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{a}{m} & -\frac{a}{m} \end{vmatrix} = 0, \cdots\cdots, \quad |B_1^n| = \begin{vmatrix} 0 & 0 & \cdots & 0 & 0 \\ -\frac{a}{m} & 0 & \cdots & 0 & 0 \\ \frac{a}{m} & -\frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{a}{m} & 0 \end{vmatrix} = 0 \cdot$$

In the same way, the $n-1$ order algebraic cofactors are obtained by expanding $|D_{i_2}^2|$ with column 2, that is,

$$|B_2^1| = \begin{vmatrix} -\frac{a}{m} & 0 & \cdots & 0 & 0 \\ 0 & \frac{a}{m} & \cdots & 0 & 0 \\ 0 & -\frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{a}{m} & -\frac{a}{m} \end{vmatrix} = (\tfrac{a}{m})^{n-1}, \quad |B_2^2| = \begin{vmatrix} \frac{a}{m} & 0 & \cdots & 0 & 0 \\ 0 & \frac{a}{m} & \cdots & 0 & 0 \\ 0 & -\frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{a}{m} & -\frac{a}{m} \end{vmatrix} = -(\tfrac{a}{m})^{n-1},$$

$$|B_2^3|=\begin{vmatrix} \frac{a}{m} & 0 & \cdots & 0 & 0 \\ -\frac{a}{m} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{a}{m} & -\frac{a}{m} \end{vmatrix}=0, \cdots\cdots, \ |B_2^n|=\begin{vmatrix} \frac{a}{m} & 0 & \cdots & 0 & 0 \\ -\frac{a}{m} & 0 & \cdots & 0 & 0 \\ 0 & \frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{a}{m} & 0 \end{vmatrix}=0 .$$

Analogously, the determinant $|D_{i_n}^n|$ is expanded by column n to obtain $n-1$ order algebraic cofactors as follows:

$$|B_n^1|=\begin{vmatrix} -\frac{a}{m} & \frac{a}{m} & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\frac{a}{m} & \frac{a}{m} & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\frac{a}{m} & \frac{a}{m} & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{a}{m} & \frac{a}{m} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{a}{m} \end{vmatrix} =(-1)^{n-1}(\tfrac{a}{m})^{n-1},$$

$$|B_n^2|=\begin{vmatrix} \frac{a}{m} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\frac{a}{m} & \frac{a}{m} & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\frac{a}{m} & \frac{a}{m} & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{a}{m} & \frac{a}{m} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{a}{m} \end{vmatrix} =(-1)^{n-2}(\tfrac{a}{m})^{n-1},$$

$$|B_n^3|=\begin{vmatrix} \frac{a}{m} & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{a}{m} & \frac{a}{m} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\frac{a}{m} & \frac{a}{m} & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{a}{m} & \frac{a}{m} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{a}{m} \end{vmatrix} =(-1)^{n-3}(\tfrac{a}{m})^{n-1}, \cdots\cdots,$$

$$|B_n^n|=\begin{vmatrix} \frac{a}{m} & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{a}{m} & \frac{a}{m} & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\frac{a}{m} & \frac{a}{m} & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\frac{a}{m} & \frac{a}{m} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{a}{m} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{a}{m} & \frac{a}{m} \end{vmatrix} =(\tfrac{a}{m})^{n-1}.$$

In summary, after the determinant $|D_{i_1}^1|$ is expanded according to the column 1, except for the $n-1$ order algebraic cofactor $|B_1^1|=-(\frac{a}{m})^{n-1}$, all other $n-1$ order algebraic cofactors are zero, that is, $|B_1^i|=0$, $i=2,3,\cdots,n$; the $|D_{i_2}^2|$ is expanded according to the column 2, except for the $n-1$ order algebraic cofactors $|B_2^1|=(\frac{a}{m})^{n-1}$ and $|B_2^2|=-(\frac{a}{m})^{n-1}$, all other $n-1$ order algebraic cofactors are zero, i.e., $|B_2^i|=0$, $i=3,4,\cdots,n$. Similarly, the $|D_{i_3}^3|$ is expanded as column 3, except for $|B_3^1|=-(\frac{a}{m})^{n-1}$, $|B_3^3|=-(\frac{a}{m})^{n-1}$ and $|B_3^2|=(\frac{a}{m})^{n-1}$, all others have $|B_3^4|=|B_3^5|=\cdots=|B_3^n|=0$. Generally, the determinant $|D_{i_n}^n|$ is expanded by column n, we always have

$$|B_n^1|=(-1)^{n-1}(\tfrac{a}{m})^{n-1}, \ |B_n^2|=(-1)^{n-2}(\tfrac{a}{m})^{n-1}, \ |B_n^3|=(-1)^{n-3}(\tfrac{a}{m})^{n-1}, \cdots, \ |B_n^n|=(\tfrac{a}{m})^{n-1}.$$

Finally, by Definition 3.2 we can immediately get that

$$\frac{a}{m\|D_n\|}\sum_{i=1}^{n}\sum_{k=1}^{n}\|B_k^i\|=\frac{a}{m\|D_n\|}\Big[\big(\|B_1^1\|+\|B_1^2\|+\cdots+\|B_1^n\|\big)+\big(\|B_2^1\|+\|B_2^2\|+\cdots+\|B_2^n\|\big)+ \big(\|B_3^1\|+\|B_3^2\|+\cdots+\|B_3^n\|\big)+\cdots+\big(\|B_n^1\|+\|B_n^2\|+\cdots+\|B_n^n\|\big)\Big]$$

$$= \frac{a}{m\,\| D_n \|}\left[(\tfrac{a}{m})^{n-1} + 2(\tfrac{a}{m})^{n-1} + 3(\tfrac{a}{m})^{n-1} + \cdots + n(\tfrac{a}{m})^{n-1} \right]$$
$$= \frac{a}{m\,(a/m)^n}\,\frac{n(n+1)}{2}(\tfrac{a}{m})^{n-1}$$
$$= \frac{n(n+1)}{2}.$$

Hence, the approximation factor is $\frac{a}{m\,\| D_n \|}\sum_{i=1}^{n}\sum_{k=1}^{n}\| B_k^i \| + 1 = \frac{n(n+1)+2}{2}$, and it is indeed a constant independent of the subdivision fraction m.

Actually, although the subdivision number m of input space $\Delta(a)$ is independent of the approximation factor, the subdivision number m is closely related to the corresponding piecewise linear function, Mamdani fuzzy system and approximation accuracy. For example, piecewise linear functions are mainly constructed by the subdivision of input space. Generally speaking, the larger the m value, the finer the subdivision, the better the approximation accuracy, but the greater the complexity. Conversely, if the m value is too small, although the complexity is reduced, it may not achieve the required approximation accuracy . See Fig. 2 below.

In addition, from the construction of Mamdani fuzzy system (6), the antecedent fuzzy sets $\{A_1^1, A_2^1, \cdots, A_{N_1}^1\}$, $\{A_1^2, A_2^2, \cdots, A_{N_2}^2\}$ and $\{A_1^3, A_2^3, \cdots, A_{N_3}^3\}$ on the three coordinate axes also depend on the subdivision number m, but in this paper we assume that there is always $N_1 = N_2 = N_3 = m$.

5 An example analysis

In this part, the important role of approximation factor of a piecewise linear function S is illustrated through a practical example. For simplicity, we may extend the condition of the given binary pair $((x,y); f(x,y))$ to the analytic expression of the known function $f(x,y)$, and assume that the binary function $f(x,y)$ is a continuous differentiable function.

An Example Let $n = 2$, $a = 1$, $f(x,y) = e^{-(x^2+y^2)/40}$, $(x,y) \in [0,1]\times[0,1]$, and the precision of a given piecewise linear function S approximate to f is $\sigma = 0.1$. Please give the realization process of this approximation by the approximation factor.

In fact, by Definition 2.3 and $m-$ mesh subdivision of Note 1, for arbitrary $(x,y) \in [0,1]\times[0,1]$, there are $i_1, i_2 \in \mathbb{N}$ such that $(x,y) \in \Delta_{i_1 i_2}$, and the length of the right angle side of triangle $\Delta_{i_1 i_2}$ is $\frac{a}{m} = \frac{1}{m}$, where m is a subdivision number waiting to be determined, and each component coordinate on $\Delta_{i_1 i_2}$ meets $| x_1 - x_2 | < \frac{1}{m}$ and $| y_1 - y_2 | < \frac{1}{m}$. See Fig. 1 or Fig. 2 below.

Obviously, the function $f(x,y) = e^{-(x^2+y^2)/40}$ is uniform continuity on closed set $\Delta_{i_1 i_2}$, then, for arbitrary $\varepsilon > 0$, there is $\delta > 0$, such that $\left| f(x_1, y_1) - f(x_2, y_2) \right| < \varepsilon$ when for any $(x_1, y_1), (x_2, y_2) \in \Delta_{i_1 i_2}$ with $\| (x_1, y_1) - (x_2, y_2) \| < \delta$. Now, if it is satisfies

$$\| (x_1, y_1) - (x_2, y_2) \| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \le \sqrt{2(\tfrac{1}{m})^2} = \frac{\sqrt{2}}{m} < \delta , \tag{8}$$

then, we only need to select the natural number with the subdivision number $m \geq \sqrt{2}/\delta$.

Next, we will determine the value of the minimum subdivision number m according to the path $\sigma \to \varepsilon \to \delta \to m$ in turn with approximation accuracy $\sigma = 0.1$.

Firstly, for the above $\varepsilon > 0$, $n = 2$ and given accuracy $\sigma = 0.1$, by Theorem 4.1 and Lemma 2, if

$$\left\| S - f \right\|_{\infty} < \left(\frac{a}{m \, || \, D_2 \, ||} \sum_{i=1}^{2} \sum_{k=1}^{2} || \, B_k^{\,i} \, || + 1 \right) \varepsilon = \frac{n(n+1)+2}{2} \varepsilon = 4\varepsilon \leq \sigma = 0.1 .$$

The solution is $\varepsilon \leq 0.1/4 = 1/40$. So we may take $\varepsilon = 1/40$.

Secondly, as $f(x,y)$ is a continuous differentiable function on the closed set $\Delta_{i_1 i_2}$, and the partial derivatives of f satisfy $\left|\frac{\partial f}{\partial x}\right| = \frac{1}{20} x e^{-(x^2+y^2)/40} \leq \frac{1}{20}$ and $\left|\frac{\partial f}{\partial y}\right| = \frac{1}{20} y e^{-(x^2+y^2)/40} \leq \frac{1}{20}$.

By the binary Taylor formula, we can expand $f(x,y)$ at point (x_1, y_1) in the first order, that is, for all $(x,y) \in \Delta_{i_1 i_2}$, there is $\theta \in (0,1)$, such that

$$f(x,y) = f(x_1,y_1) + \frac{\partial f\left(x_1 + \theta(x-x_1), y_1 + \theta(y-y_1)\right)}{\partial x}(x-x_1) + \frac{\partial f\left(x_1 + \theta(x-x_1), y_1 + \theta(y-y_1)\right)}{\partial y}(y-y_1).$$

Let $(x,y) = (x_2, y_2)$, then we have

$$\begin{aligned} \left|f(x_2,y_2) - f(x_1,y_1)\right| &= \left| \frac{\partial f\left(x_1 + \theta(x_2-x_1), y_1 + \theta(y_2-y_1)\right)}{\partial x}(x_2-x_1) + \frac{\partial f\left(x_1 + \theta(x_2-x_1), y_1 + \theta(y_2-y_1)\right)}{\partial y}(y_2-y_1) \right| \\ &\leq \left| \frac{\partial f\left(x_1 + \theta(x_2-x_1), y_1 + \theta(y_2-y_1)\right)}{\partial x} \right| \left|x_2-x_1\right| + \left| \frac{\partial f\left(x_1 + \theta(x_2-x_1), y_1 + \theta(y_2-y_1)\right)}{\partial y} \right| \left|y_2-y_1\right| \end{aligned}$$

Especially, when $\left|x_1 - y_1\right| < \delta$ and $\left|x_2 - y_2\right| < \delta$, it is not hard to get that

$$\left|f(x_2,y_2) - f(x_1,y_1)\right| \leq \frac{1}{20}\left(\left|x_2 - x_1\right| + \left|y_2 - y_1\right|\right) < \frac{1}{20}(\delta + \delta) = \frac{\delta}{10}.$$

According to the uniform continuity of $f(x,y) = e^{-(x^2+y^2)/40}$ on $\Delta_{i_1 i_2}$, for $\varepsilon = 1/40 > 0$, if

$$\left|f(x_2,y_2) - f(x_1,y_1)\right| < \frac{\delta}{10} \leq \frac{1}{40} = \varepsilon .$$

The solution is $\delta \leq 1/4$, we take $\delta = 1/4$. Finally, we can gain the subdivision number based on formula (8) $m \geq \frac{\sqrt{2}}{\delta} = 4\sqrt{2} \approx 5.656$. Thus, the minimum subdivision number is $m = 6$.

Next, we will construct a specific bivariate piecewise linear function on $\mathbb{R}^2$ with the subdivision number $m = 6$ as follows:

Firstly, the compact set $[0,1]\times[0,1] = \Delta(1)$ is divided into 6-meshes, and 36 small squares with side length of $1/6$ are obtained. In order to satisfy the rule of three points to determine a plane, then divide each small square into two parts with the diagonal, so as to get 72 small isosceles right angle triangles. We denote they as $\Delta_{i_1 i_2}$, where $i_1 = 1,2,\cdots,6; i_2 = 1,2,\cdots,12$. See Fig. 2.

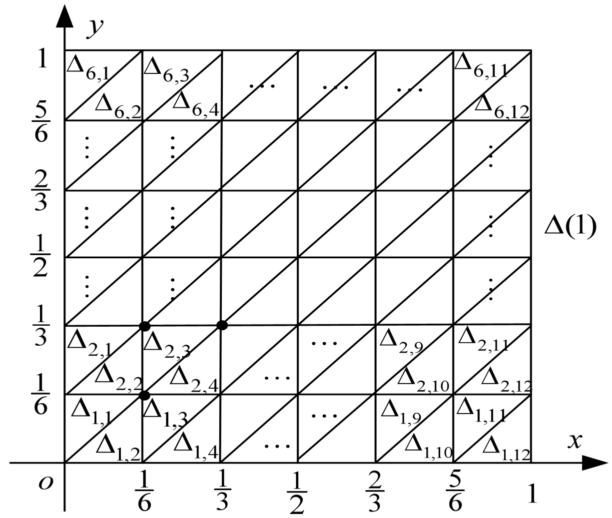

Fig. 2 6-mesh subdivision graph of $\Delta(1)$ when $n=2$

It is not difficult to see that the vertex coordinates of all these small triangles can be determined. For example,

$$\Delta_{1,2}:(0,0),(\frac{1}{6},0),(\frac{1}{6},\frac{1}{6})\ ;\quad \Delta_{2,3}:(\frac{1}{6},\frac{1}{6}),(\frac{1}{6},\frac{1}{3}),(\frac{1}{3},\frac{1}{3})\ ;\quad \Delta_{6,12}:(\frac{5}{6},\frac{5}{6}),(1,\frac{5}{6}),(1,1)\,.$$

Now, the vertex coordinates of the surface of each triangular prism that each small triangle $\Delta_{j,i}$ on $\mathbb{R}^3$ under the action of f are also determined in turn. For example, the vertex coordinates of the triangular prism of $\Delta_{2,3}$ on $\mathbb{R}^3$ are in turn

$$(\frac{1}{6},\frac{1}{6},e^{-\frac{1}{720}}),(\frac{1}{6},\frac{1}{3},e^{-\frac{5}{1440}}),(\frac{1}{3},\frac{1}{3},e^{-\frac{1}{180}})\,.$$

According to the formulas (5) and (7), it is not difficult to obtain the plane equation determined by these three points. The specific steps are as follows:

$$\left|D_2\right|=\begin{vmatrix}\frac{1}{6}&\frac{1}{6}&1\\ \frac{1}{6}&\frac{1}{3}&1\\ \frac{1}{3}&\frac{1}{3}&1\end{vmatrix}=\begin{vmatrix}\frac{1}{6}&\frac{1}{6}&1\\ 0&\frac{1}{6}&0\\ \frac{1}{6}&\frac{1}{6}&0\end{vmatrix}=-\frac{1}{36}\ ,\quad \left|D_{21}^3\right|=\begin{vmatrix}e^{-\frac{1}{720}}&\frac{1}{6}&1\\ e^{-\frac{5}{1440}}&\frac{1}{3}&1\\ e^{-\frac{1}{180}}&\frac{1}{3}&1\end{vmatrix}=\frac{1}{6}e^{-\frac{5}{1440}}-\frac{1}{6}e^{-\frac{1}{180}}\ ;$$

$$\left|D_{22}^3\right|=\begin{vmatrix}\frac{1}{6}&e^{-\frac{1}{720}}&1\\ \frac{1}{6}&e^{-\frac{5}{1440}}&1\\ \frac{1}{3}&e^{-\frac{1}{180}}&1\end{vmatrix}=\frac{1}{6}e^{-\frac{1}{720}}-\frac{1}{6}e^{-\frac{5}{1440}},\qquad \left|D_{23}^3\right|=\begin{vmatrix}\frac{1}{6}&\frac{1}{6}&e^{-\frac{1}{720}}\\ \frac{1}{6}&\frac{1}{3}&e^{-\frac{5}{1440}}\\ \frac{1}{3}&\frac{1}{3}&e^{-\frac{1}{180}}\end{vmatrix}=\frac{1}{36}e^{-\frac{1}{180}}-\frac{1}{18}e^{-\frac{1}{720}}.$$

We will immediately obtain the plane equation determined by $\Delta_{2,3}$ in $\mathbb{R}^3$ with plugging these determinant values into the formula (4), it's not hard to get that

$$S_{2,3}(x,y)=6\left(e^{-\frac{1}{180}}-e^{-\frac{5}{1440}}\right)x+6\left(e^{-\frac{5}{1440}}-e^{-\frac{1}{720}}\right)y+2e^{-\frac{1}{720}}-e^{-\frac{1}{180}},\quad (x,y)\in\Delta_{2,3}$$

Similarly, the analytic expression of each piecewise linear function of $\Delta_{1,1},\Delta_{1,2},\cdots,\Delta_{2,1},\cdots,\Delta_{6,1},\cdots,\Delta_{6,12}$ in $\mathbb{R}^3$ can be determined in turn. Therefore, the analytic expression of the piecewise linear function $S(x,y)$ on $[0,1]\times[0,1]$ as follows:

$$S(x,y)=\begin{cases}6\left(e^{-\frac{1}{720}}-e^{-\frac{1}{1440}}\right)x-6\left(1-e^{-\frac{1}{1440}}\right)y+1, & (x,y)\in\Delta_{1,1},\\ 6\left(e^{-\frac{1}{1440}}-1\right)x+6\left(e^{-\frac{1}{720}}-e^{-\frac{1}{1440}}\right)y+1, & (x,y)\in\Delta_{1,2},\\ \cdots\cdots & \\ 6\left(e^{-\frac{1}{180}}-e^{-\frac{5}{1440}}\right)x+6\left(e^{-\frac{5}{1440}}-e^{-\frac{1}{720}}\right)y+2e^{-\frac{1}{720}}-e^{-\frac{1}{180}}, & (x,y)\in\Delta_{2,3},\\ \cdots\cdots & \\ 6\left(e^{-\frac{61}{1440}}-e^{-\frac{5}{144}}\right)x+6\left(e^{-\frac{1}{20}}-e^{-\frac{61}{1440}}\right)y+6e^{-\frac{5}{144}}-5e^{-\frac{1}{20}}, & (x,y)\in\Delta_{6,12}.\end{cases}$$

With the analytic expression of the piecewise linear function $S(x,y)$, it is not difficult to draw the spatial surface graph and mixed surface graph of $f(x,y)$ and $S(x,y)$ on $[0,1]\times[0,1]$ by using MATLAB software programming, as shown in Figs. 3-4.

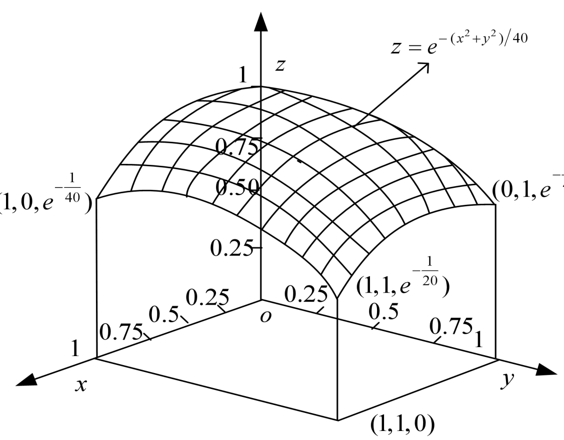

Fig. 3 Surface graph of a given function f on $\Delta(1)$

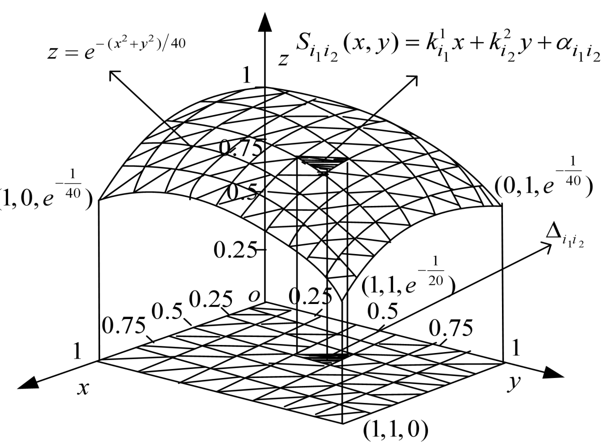

Fig. 4 Mixed surface graphs of f and S on $\Delta(1)$

However, only from the mixed Fig. 4, we have not enough reason to say that S can approximate f to arbitrary accuracy. Hence, we will test the approximation ability of the piecewise linear function S by sample points. We may randomly take 30 samples on $[0,1]\times[0,1]$, and calculate their values and errors at these sample points according to the analytical expressions of $f(x,y)$ and $S(x,y)$, respectively, as follows:

Table 1 The corresponding values and errors of $f(x,y)$ and $S(x,y)$ at 30 sample points

Number i	Sample point (x_1,x_2)	$f(x_1,x_2)=X_i$	$S(x_1,x_2)=Y_i$	$D_i=X_i-Y_i$
1	(1 / 6, 1/ 6)	0.998612075	0.998612075	0
2	(1/ 3, 1 / 3)	0.994459848	0.994459848	0
3	(1/ 2, 1/ 3)	0.991012850	0.991012850	0
4	(2/ 3, 1/ 6)	0.988263857	0.988263857	0
5	(2/ 3, 1 / 2)	0.982788725	0.982788725	0
6	(5 / 6, 1/ 3)	0.980062544	0.980062544	0
7	(5/ 6, 5 / 6)	0.965873677	0.965873677	0
8	(1/ 6, 1/ 3)	0.996533799	0.996533799	0
9	(1/ 2, 1 / 2)	0.987577800	0.987577800	0
10	(5/6, 2 / 3)	0.971929292	0.971929292	0
11	(1/6, 1/12)	0.999132321	0.998958935	0.000173386
12	(1/4, 1/ 3)	0.995669128	0.995496825	0.000172303
13	(1/12, 1/8)	0.999435923	0.999132486	0.000303437
14	(1/24,1/12)	0.999783010	0.999479468	0.000303542
15	(1/12,1/24)	0.999783010	0.999479468	0.000303542
16	(1/ 8, 1/12)	0.999435923	0.999132487	0.000303436
17	(1/12,1/12)	0.999652838	0.999306037	0.000346801
18	(1/ 6, 1/ 4)	0.997745601	0.996532355	0.001213246
19	(11/12, 5/6)	0.962358674	0.962198631	0.000160043
20	(1, 11 / 12)	0.955035330	0.954876504	0.000476796
21	(1/ 4, 1/ 8)	0.998048781	0.997746368	0.000302413
22	(1/ 4, 1 /24)	0.998395386	0.998092867	0.001302519
23	(11/12,11/12)	0.958856463	0.958551551	0.000304912
24	(1 / 4, 1 / 4)	0.996879878	0.996535960	0.000343918
25	(1/ 12, 5/24)	0.998742111	0.997572213	0.001169898
26	(1/12,23/24)	0.977131852	0.976841083	0.000290769
27	(11/12,23/24)	0.956985524	0.956714027	0.000271497
28	(11/12,21/24)	0.960647666	0.960375091	0.000272575
29	(5/ 6,11 /12)	0.962358674	0.962198631	0.000160043
30	(1/ 4, 1/ 12)	0.998265395	0.997919798	0.000345597

It is not difficult to see from Table 1 that the values of the function $f(x,y)$ and $S(x,y)$ are the same at the vertex coordinates, and their error values D_i are all zero, which is consistent with the condition $S_{i_1 i_2 \cdots i_n}(x_k^*) = f(x_k^*)$ under the piecewise linear function is constructed in Section 2. However, it is not enough to judge that S can approximate f to arbitrary precision according to infinite norm only by randomly selecting the values of these 30 sample points in Table 1.

Next, we will apply the $t-$test method in statistics to verify that the piecewise linear function S can indeed approximate to a continuous function f by the subdivision number $m=6$.

Assuming that the error data $D(i) = Y_i - X_i$ ($i = 1, 2, \cdots, 30$) in Table 1 are from a sample from the normal population distribution $N(\mu_D, \sigma_D^2)$, where both mean value μ_D and variance σ_D^2 are unknown. According to the $t-$hypothesis test method in statistical inference, we can easily calculate the values of the mean value $\bar{D}$ and variance s_D as follows:

$$\bar{D} = \frac{1}{30}\sum_{i=1}^{30} D(i) = \frac{0.008520673}{30} = 0.000284022,$$

$$s_D = \sqrt{\frac{1}{30-1}\sum_{i=1}^{30}(D_i - \bar{D})^2} = \sqrt{\frac{0.0000035988}{29}} \approx 0.000352274.$$

For the error data $\{Y_i - X_i\}$ in Table 1, we can test the hypothesis $\{H_0, H_1\}$ under the significance level $\alpha = 0.05$, where the hypothesis (acceptance domain H_0 and rejection domain) satisfies $H_0 : \mu_D = 0$, $H_1 : \mu_D \neq 0$.

Adopting the approach of the $t-$ test, we select the test statistic $t = (\bar{D} - 0)/(s_D/\sqrt{n})$, let $n = 30, \mu = 0$ and $\alpha = 0.05$. By looking up to the $t-$distribution Table for $t-$hypothesis test, we can obtain $t_\alpha(n-1) = t_{0.05}(29) = 1.6991$. Therefore, the rejection domain of the hypothesis test is

$$t = \frac{\bar{D} - 0}{s_D/\sqrt{n}} \geq t_\alpha(n-1) = 1.6991.$$

On the other hand, according to the above mean value $\bar{D}$ and variance s_D we can easily calculate the observational value of t is

$$t = \frac{\bar{D} - 0}{s_D/\sqrt{30}} = \frac{0.000284022}{0.000352274/\sqrt{30}} \approx 4.41603003 > 1.6991.$$

Clearly, the observation value t falls within the rejection region H_1. Hence, we must reject the hypothesis H_0 under significance level $\alpha = 0.05$. Therefore, the piecewise linear function S can approximate to the continuous function f with arbitrary accuracy.

6 Conclusion

In this paper, the vertex coordinates of each small polyhedron are analyzed by mesh generation of three-dimensional cubes, and then it is analyzed that the approximation factor is indeed a constant independent of the subdivision number in the case of *n*-dimension, so as to give a complete answer in [21]. It is not difficult to see from Lemma 2 and Theorem 4.1 that when the piecewise linear functions approach an unknown continuous function, the approximation factor is only related to the space dimension *n*, but not to the subdivision number m. Moreover, the larger the value is, the larger the approximation factor is, while the approximation accuracy is reduced. Therefore, it is not enough to increase the approximation accuracy of piecewise linear functions only by increasing the subdivision number m when the dimension of input space is determined. In addition, how to select the vertex coordinates of polyhedron is also a key issue when calculating algebraic cofactors of determinant in Theorem 4.1. In three-dimensional space, it may change the approximation factor and affect the approximation accuracy if we select the vertex coordinates according to the left-hand rule. Therefore, it will be the next focus of the study on how to select the optimal vertex coordinates to minimize the approximation factor and improve the approximation accuracy.

Reference

[1] Takagi T, Sugeno M. Fuzzy identification of system and its application to modeling and control. IEEE Transactions on Systems, Man, Cybern. 1985;15(1):116-132.

[2] Buckley JJ. Sugeno type controllers are universal controllers. Fuzzy Sets and Systems, 1993;53(3):299-303.

[3] Kosko B. Fuzzy systems are universal approximators. IEEE Transactions on Computers, 1994;43(11):1329-1333.

[4] Ying H. General SISO Takagi-Sugeno fuzzy systems with linear rule consequents are universal approximators. IEEE Transactions on Fuzzy Systems, 1998;6(4):582-587.

[5] Zeng K, Zhang NR, Xu WL. A comparative study on sufficient conditions for Takagi-Sugeno fuzzy systems as universal approximators. IEEE Transactions on Fuzzy Systems, 2000;8(6):773-780.

[6] Liu PY, Li HX. Approximation of generalized fuzzy system to integrable function. Science in China (Series E), 2000;43(6):618-628.

[7] Liu PY, Li HX. Analyses for Lp(u) norm approximation capability of the generalized Mamdani fuzzy systems. Information Sciences, 2001;138(2):195-210.

[8] Liu PY. Mamdani fuzzy system is universal approximator to a class of random process. IEEE Transactions on Fuzzy Systems, 2002;10(6):756-766.

[9] Wang YL, Li L, Li Y, et al. Nonlinear observer design using piecewise-linear functions based approximations. Control and Decision, 2008;48(1):153-156.

[10] Zhang H, Wang SN. Piecewise linear approximations with a hypercube partition. Journal of Tsinghua University (Science and Technology), 2008;48(1):153-156.

[11] Zeng K, Zahng NY, Xu WL. Sufficient condition for linear T-S fuzzy systems as universal approximators. Acta Automatica Sinica, 2001;27(5):606-612.

[12] Zhang YZ, Li HX. Generalized hierarchical Mamdani fuzzy systems and their universal approximation. Control Theory and Applications, 2006;23(3):449-454.

[13] Sun FC, Yang J, Liu HP. Preconditions for SISO Mamdani fuzzy systems to perform as function approximators. Transactions on Intelligent Systems, 2009;4(4):288-294.

[14] Verma1 A K, Kayenat1 S, Jha G J. A note on the convergence of fuzzy transformed fifinite difference methods. Journal of Applied Mathematics and Computing, 2020;63:143-170.

[15] Chehlabi M. Solving fuzzy dual complex linear systems. Journal of Applied Mathematics and Computing, 2019;60: 87-112.

[16] Wang GJ, Li XP, Sui XL. Universal approximation and its realization of a generalized Mamdani fuzzy system based on K-integral norms. Acta Automatica Sinica, 2014;40(1):143-148.

[17] Peng WL. Structure and approximation of a multi-dimensional piecewise linear function based on the input space of subdivision fuzzy systems. Journal System Science & Mathematical Science, 2014;34(3):340-351.

[18] Tao YJ,Wang HZ, Wang GJ. Approximation ability and its realization of the generalized Mamdani fzzzy system in the sense of Kp-integral norm. Acta Electronica Sinica, 2015;43(11):2284-2291.

[19] Tao YJ, Wang HZ, Wang GJ. Approximation of piecewise linear function in the sense of Kp-integral norm induced by K-quasi-arithmetic operations. Journal System Science & Mathematical Science, 2016;36(2):267-277.

[20] Wang HZ, Tao YJ,Wang GJ. Approximation analysis of non-homogeneous linear T-S fuzzy system based on grid piecewise linear function structure. Journal System Science & Mathematical Science, 2015;35(11):1276-1290.

[21] Wang GJ, Li XP. Mesh construction of PLF and its approximation process in Mamdani fuzzy system. Journal of Intelligent & Fuzzy Systems, 2017;32(6):4213-4225.

[22] Wang GJ. Polygonal fuzzy neural network and fuzzy system approximation. Beijing, Science Press, 2017.