Exotic hybrid rogue wave and breather solutions for a complex mKdV equation in shallow water wave and optics

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Abstract

We report exotic hybrid rogue wave and breather solutions for a complex mKdV equation describing soliton dynamics in shallow water wave and optics. Starting from the Lax pair, the higher-order Darboux transformations for the equation are constructed. Further, we obtain a series of theorems to compute the hybrid rogue wave and breather solutions. We also demonstrate the interaction features between rogue waves and breathers by controlling parameters. These results are of importance to understand nonlinear wave dynamics in water wave systems.

Key words: complex mKdV equation; Darboux transformation; rogue wave; breather; hybrid solution

1. Introduction

KdV equation has been an extremely well-known model to describe nonlinear wave dynamics in shallow water wave since it was initially developed in the mid of 1960s [1]. In order to simulate more specific situations in various application scenarios, a variety of KdV-type models have continuously developed, such as modified KdV (mKdV) [2, 3], coupled mKdV [4], higher-dimensional KdV [5], KdV-mKdV [6], modified KdV-Calogero-Bogoyavlenskii-Schiff [7], KdV-Burgers [8], complex KdV [9], and so on.

Soliton theory play a critical role in nonlinear science and engineering because soliton can be used to manifest the essential characteristics of the nonlinear systems [10]. Rogue wave and breather are two special solitons. Rogue wave generally depicts local features in narrow space and short time [11–13]. Breather can propagate periodically [14–16]. In a complicated nonlinear system, there usually co-exist multiple solitons, such as stripe-like solitons, rogue waves and/or breathers. As a result, the study on the dynamical properties between these solitons is becoming more and more important [17–22].

Complex mKdV equation is a completely integrable equation, which reads [9]

$$u_t + u_{xxx} + 6|u|^2 u_x = 0. \tag{1}$$

Eq. (1) has been derived from many physical applications, e.g. nonlinear lattices, plasmas, fluids, ultrashort pulses in nonlinear optics [9]. He et. al. got the higher-order rogue waves by parameterized Darboux transformation (DT) for Eq. (1), and the intrinsic structures were analyzed and classified [23]. Novel higher-order soliton molecules and breather-positon of Eq. (1) were obtained by DT [24].

Recently, our team extended DT to construct the hybrid rogue wave and breather solutions for some nonlinear models, for instances, a generalized nonlinear Schrödinger system with two higher-order dispersion operators [25], a classical Schrödinger equations [26]. These hybrid solutions are instructive to reveal rich interaction features.

In this article, our object is to construct the hybrid rogue wave and breather solutions for Eq. (1).

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2. Lax pair for Eq. (1)

Eq. (1) has the following Lax pair [9, 23, 24]

$$\varphi_x = U\varphi, \varphi_t = V\varphi, \tag{2}$$

with

$$U = i\lambda\sigma_1 + Q, V = \begin{pmatrix} V_1 & V_2 \\ V_3 & -V_1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q = \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix}, \tag{3}$$

where

$$V_1 = 4i\lambda^3 - 2i\lambda|u|^2 + (uu_x^* - u_x u^*), \tag{4}$$

$$V_2 = 4\lambda^2 u - 2i\lambda u_x - 2|u|^2 u - u_{xx}, \tag{5}$$

$$V_3 = -4\lambda^2 u^* - 2i\lambda u_x^* + 2|u|^2 u^* + u_{xx}^*. \tag{6}$$

In this work, we will investigate the hybrid solutions of rogue wave and breather under the initial solution as

$$u = u^{[0]} = ae^{i\theta}, \tag{7}$$

where $\theta = k\left(x + \left(k^2 - 6a^2\right)t\right)$, a and k are arbitrary real values.

3. The first- and second-order rogue wave solutions for Eq. (1)

According to DT, we can expand the solutions of Eqs. (2) at $f = 0$ under the conditions $\lambda_1 = \frac{1}{2}k + ia\left(1 + f^2\right), \lambda_2 = \frac{1}{2}k - ia\left(1 + f^2\right)$ as

$$\Phi_j = \Phi_j^{(0)} + \Phi_j^{(1)} f^2 + \Phi_j^{(2)} f^4 + \cdots, j = 1, 2, \tag{8}$$

where

$$\Phi_1^{(0)} \triangleq \begin{pmatrix} \phi_{11}^{[0]} \\ \phi_{12}^{[0]} \end{pmatrix} = \begin{pmatrix} (-1 + 2a\xi)\, e^{\frac{1}{2}i\theta} \\ (1 + 2a\xi)\, e^{-\frac{1}{2}i\theta} \end{pmatrix}, \tag{9}$$

$$\Phi_2^{(0)} \triangleq \begin{pmatrix} \phi_{21}^{[0]} \\ \phi_{22}^{[0]} \end{pmatrix} = \begin{pmatrix} -\left(1 + 2a\xi^*\right) e^{\frac{1}{2}i\theta} \\ \left(-1 + 2a\xi^*\right) e^{-\frac{1}{2}i\theta} \end{pmatrix}, \tag{10}$$

$$\Phi_1^{(1)} \triangleq \begin{pmatrix} \phi_{11}^{(1)} \\ \phi_{12}^{(1)} \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{4} - E^2 + \frac{2}{3}E^3 + \frac{1}{2}E + H\right) e^{\frac{1}{2}i\theta} \\ \left(-\frac{1}{4} + E^2 + \frac{2}{3}E^3 + \frac{1}{2}E + H\right) e^{-\frac{1}{2}i\theta} \end{pmatrix}, \tag{11}$$

$$\Phi_2^{(1)} \triangleq \begin{pmatrix} \phi_{21}^{(1)} \\ \phi_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{4} - E^{*2} - \frac{2}{3}E^{*3} - \frac{1}{2}E^* - H^*\right) e^{\frac{1}{2}i\theta} \\ \left(\frac{1}{4} - E^{*2} + \frac{2}{3}E^{*3} + \frac{1}{2}E^* + H^*\right) e^{-\frac{1}{2}i\theta} \end{pmatrix}, \tag{12}$$

with $\xi = x + \left(3k^2 - 6a^2 + i6ka\right)t, E = a\xi, H = 4a^2\left(3ik - 4a\right)t$, the asterisk $*$ on the upper-right corner of the identifier represents its conjugate complex number.

We can verify that $\Phi_1^{(0)}$ in (9) and $\Phi_2^{(0)}$ in (10) are the solutions of (2) and (7) under $\lambda = \lambda_1 = \frac{1}{2}k + ia$ and $\lambda = \lambda_2 = \lambda_1^* = \frac{1}{2}k - ia$, respectively. As a result, we have the following proposition on the first rogue wave solution.

Proposition 1. If we denote

$$\Delta_{1r}^{[1]} = \phi_{11}^{[0]}\phi_{22}^{[0]} - \phi_{12}^{[0]}\phi_{21}^{[0]}, \tag{13}$$

and

$$M_{1r}^{[1]} \triangleq \begin{pmatrix} m_{11r}^{[1]} & m_{12r}^{[1]} \\ m_{21r}^{[1]} & m_{22r}^{[1]} \end{pmatrix} = \frac{1}{\Delta_{1r}^{[1]}} \begin{pmatrix} \lambda_1 \phi_{11}^{[0]} \phi_{22}^{[0]} - \lambda_2 \phi_{12}^{[0]} \phi_{21}^{[0]} & (\lambda_2 - \lambda_1) \phi_{11}^{[0]} \phi_{21}^{[0]} \\ (\lambda_1 - \lambda_2) \phi_{12}^{[0]} \phi_{22}^{[0]} & \lambda_2 \phi_{11}^{[0]} \phi_{22}^{[0]} - \lambda_1 \phi_{12}^{[0]} \phi_{21}^{[0]} \end{pmatrix}, \tag{14}$$

then the first-order rogue wave solution of Eq. (1) with the first-order generalized DT is

$$u_{1r} = u^{[0]} + 2im_{12r}^{[1]}. \tag{15}$$

Subsequently, we compute the second-order rogue wave solution of Eq. (1). Now, it is necessary to obtain $M_r^{[2]}$ in the second-order DT defined by

$$T[2] = \lambda I - M_r^{[2]}, M_r^{[2]} \triangleq \begin{pmatrix} m_{11r}^{[2]} & m_{12r}^{[2]} \\ m_{21r}^{[2]} & m_{22r}^{[2]} \end{pmatrix}. \tag{16}$$

We let

$$M_{2r}^{[2]} = L_{2r} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} L_{2r}^{-1}, L_{2r} = \begin{pmatrix} \phi_{11}^{[1]} & \phi_{21}^{[1]} \\ \phi_{12}^{[1]} & \phi_{22}^{[1]} \end{pmatrix}, \tag{17}$$

and L_{2r}^{-1} represents the inverse matrix of L_{2r}^{-1} (This identifier is still valid in the following text).

Then

$$M_{2r}^{[2]} = \begin{pmatrix} m_{11r}^{[2]} & m_{12r}^{[2]} \\ m_{21r}^{[2]} & m_{22r}^{[2]} \end{pmatrix} = \frac{1}{\Delta_{2r}^{[2]}} \begin{pmatrix} \lambda_1 \phi_{11}^{[1]} \phi_{22}^{[1]} - \lambda_2 \phi_{12}^{[1]} \phi_{22}^{[1]} & (\lambda_2 - \lambda_1) \phi_{11}^{[1]} \phi_{21}^{[1]} \\ (\lambda_1 - \lambda_2) \phi_{12}^{[1]} \phi_{22}^{[1]} & \lambda_2 \phi_{11}^{[1]} \phi_{22}^{[1]} - \lambda_1 \phi_{12}^{[1]} \phi_{21}^{[1]} \end{pmatrix}, \tag{18}$$

with

$$\Delta_{2r}^{[2]} = \phi_{11}^{[1]} \phi_{22}^{[1]} - \phi_{12}^{[1]} \phi_{21}^{[1]},$$

$$\begin{pmatrix} \phi_{11}^{[1]} \\ \phi_{12}^{[1]} \end{pmatrix} = ia \begin{pmatrix} \phi_{11}^{[0]} \\ \phi_{12}^{[0]} \end{pmatrix} + \left(\lambda_1 I - M_r^{[1]}\right) \begin{pmatrix} \phi_{11}^{(1)} \\ \phi_{12}^{(1)} \end{pmatrix},$$

$$\begin{pmatrix} \phi_{21}^{(1)} \\ \phi_{22}^{(1)} \end{pmatrix} = -ia \begin{pmatrix} \phi_{21}^{[0]} \\ \phi_{22}^{[0]} \end{pmatrix} + \left(\lambda_2 I - M_r^{[1]}\right) \begin{pmatrix} \phi_{21}^{(1)} \\ \phi_{22}^{(1)} \end{pmatrix}.$$

Proposition 2. The second-order rogue wave solution of Eq. (1) from the generalized DT is

$$u_{2r}^{[2]} = u_{1r}^{[1]} + 2im_{12r}^{[2]}, \tag{19}$$

where $u_{1r}^{[1]}$ is the first-order rogue wave given by (15), and $m_{12r}^{[2]}$ is defined by (14).

4. The first- and second-order breather solutions of Eq. (1)

When we take

$$\lambda_3 = \alpha_3 + i\beta_3, \lambda_4 = \lambda_3^* = \alpha_3 - i\beta_3,$$

$$\lambda_5 = \alpha_5 + i\beta_5, \lambda_6 = \lambda_5^* = \alpha_5 - i\beta_5,$$

where λ_j $(j = 1, 2, 3, 4, 5, 6)$ are different each other, we are able to get the solutions of the Lax pair equation (2), under the initial solution (7) with $\lambda_j, j = 3, 4, 5, 6$, are respectively as follows

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix} = \begin{pmatrix} \left(ae^{A_j} + \left(\sqrt{-\left(\lambda_j - \frac{1}{2}k\right)^2 - a^2} - i\left(\lambda_j - \frac{1}{2}k\right)\right) e^{-A_j}\right) e^{\frac{1}{2}i\theta} \\ \left(ae^{-A_j} + \left(\sqrt{-\left(\lambda_j - \frac{1}{2}k\right)^2 - a^2} - i\left(\lambda_j - \frac{1}{2}k\right)\right) e^{A_j}\right) e^{-\frac{1}{2}i\theta} \end{pmatrix}, j = 3, 5, \tag{20}$$

$$\left(\begin{array}{c}\phi_{1j}\\ \phi_{2j}\end{array}\right)=\left(\begin{array}{c}-\left(ae^{-A_j}+\left(\sqrt{-\left(\lambda_j-\frac{1}{2}k\right)^2-a^2}+i\left(\lambda_j-\frac{1}{2}k\right)\right)e^{A_j}\right)e^{\frac{1}{2}i\theta}\\ \left(ae^{A_j}+\left(\sqrt{-\left(\lambda_j-\frac{1}{2}k\right)^2-a^2}+i\left(\lambda_j-\frac{1}{2}k\right)\right)e^{-A_j}\right)e^{-\frac{1}{2}i\theta}\end{array}\right),j=4,6, \tag{21}$$

where $A_j=\sqrt{-\left(\lambda_j-\frac{1}{2}k\right)^2-a^2}\left(x+\left(4\lambda_j^2+2k\lambda_j+k^2-2a^2\right)t\right),j=3,4,5,6.$

We can expand the first DT to get the following proposition.

Proposition 3. The first-order breather solution of Eq. (1) is

$$u_{1b}^{[1]}=u^{[0]}+2im_{12b}^{[1]}, \tag{22}$$

where $u^{[0]}$ has been given by (7),

$$m_{12b}^{[1]}=\frac{(\lambda_4-\lambda_3)\,\phi_{13}\phi_{14}}{\Delta_{1b}^{[1]}},\Delta_{1b}^{[1]}=\phi_{13}\phi_{24}-\phi_{14}\phi_{23}. \tag{23}$$

If we denote

$$M_{1b}^{[1]}=\left(\begin{array}{cc}m_{11b}^{[1]} & m_{12b}^{[1]}\\ m_{21b}^{[1]} & m_{22b}^{[1]}\end{array}\right), \tag{24}$$

where

$$m_{11b}^{[1]}=\frac{\lambda_3\phi_{13}\phi_{24}-\lambda_4\phi_{23}\phi_{14}}{\Delta_{1b}^{[1]}}, \tag{25}$$

$$m_{12b}^{[1]}=\frac{(\lambda_4-\lambda_3)\,\phi_{13}\phi_{14}}{\Delta_{1b}^{[1]}}, \tag{26}$$

$$m_{21b}^{[1]}=\frac{(\lambda_3-\lambda_4)\,\phi_{23}\phi_{24}}{\Delta_{1b}^{[1]}}, \tag{27}$$

$$m_{22b}^{[1]}=\frac{\lambda_4\phi_{13}\phi_{24}-\lambda_3\phi_{23}\phi_{14}}{\Delta_{1b}^{[1]}}. \tag{28}$$

Proposition 4. Stemming from the classical second-order DT, the second-order breather solution of Eq. (1) can be expressed as

$$u_{2b}^{[2]}=u_{1b}^{[1]}+2im_{12b}^{[2]}, \tag{29}$$

where $u_{1b}^{[1]}$ is given by (22), and

$$m_{12b}^{[2]}=\frac{(\lambda_6-\lambda_5)\,\phi_{15b}^{[1]}\phi_{16b}^{[1]}}{\Delta_{2b}^{[2]}}, \tag{30}$$

$$\Delta_{2b}^{[2]}=\phi_{15b}^{[1]}\phi_{26b}^{[1]}-\phi_{25b}^{[1]}\phi_{16b}^{[1]}, \tag{31}$$

with

$$\left(\begin{array}{c}\phi_{15b}^{[1]}\\ \phi_{25b}^{[1]}\end{array}\right)=\left(\lambda_5 I-M_{1b}^{[1]}\right)\left(\begin{array}{c}\phi_{15}\\ \phi_{25}\end{array}\right), \tag{32}$$

$$\left(\begin{array}{c}\phi_{16b}^{[1]}\\ \phi_{26b}^{[1]}\end{array}\right)=\left(\lambda_6 I-M_{1b}^{[1]}\right)\left(\begin{array}{c}\phi_{16}\\ \phi_{26}\end{array}\right). \tag{33}$$

In order to facilitate constructing the hybrid solution, we denote

$$M_{2b}^{[2]}=\left(\begin{array}{cc}m_{11b}^{[2]} & m_{12b}^{[2]}\\ m_{21b}^{[2]} & m_{22b}^{[2]}\end{array}\right)=\frac{1}{\Delta_{2b}^{[2]}}\left(\begin{array}{cc}\lambda_5\phi_{15b}^{[1]}\phi_{26b}^{[1]}-\lambda_6\phi_{25b}^{[1]}\phi_{16b}^{[1]} & (\lambda_6-\lambda_5)\,\phi_{15b}^{[1]}\phi_{16b}^{[1]}\\ (\lambda_5-\lambda_6)\,\phi_{25b}^{[1]}\phi_{26b}^{[1]} & \lambda_6\phi_{15b}^{[1]}\phi_{26b}^{[1]}-\lambda_5\phi_{25b}^{[1]}\phi_{16b}^{[1]}\end{array}\right). \tag{34}$$

5. The hybrid rogue wave and breather solution of Eq. (1)

5.1. *The hybrid first-order rogue wave and first-order breather solution*

Based on the first-order rogue wave solution and the first-order breather solution, we modify the second-order DT, and let

$$\begin{pmatrix} \phi_{13h}^{[1]} \\ \phi_{23h}^{[1]} \end{pmatrix} = \left(\lambda_3 I - M_{1r}^{[1]}\right) \begin{pmatrix} \phi_{13} \\ \phi_{23} \end{pmatrix}, \tag{35}$$

$$\begin{pmatrix} \phi_{14h}^{[1]} \\ \phi_{24h}^{[1]} \end{pmatrix} = \left(\lambda_4 I - M_{1r}^{[1]}\right) \begin{pmatrix} \phi_{14} \\ \phi_{24} \end{pmatrix}, \tag{36}$$

then, we have the following theorem.

Theorem 1. Eq. (1) has the hybrid first-order rogue wave and first-order breather solution as

$$u_{1r1b}^{[2]} = u_{1r}^{[1]} + 2im_{12h}^{[2]}, \tag{37}$$

where $u_{1r}^{[1]}$ has been given by (15), and

$$m_{12h}^{[2]} = \frac{(\lambda_4 - \lambda_3)\, \phi_{13h}^{[1]} \phi_{14h}^{[1]}}{\Delta_{1r1b}^{[2]}},$$

$$\Delta_{1r1b}^{[2]} = \phi_{13h}^{[1]} \phi_{24h}^{[1]} - \phi_{23h}^{[1]} \phi_{14h}^{[1]}.$$

Proof. On the base of the first-order rogue wave solution (15) and $M_r^{[1]}$, we can set $T_h^{[2]} = \lambda I - M_h^{[2]}$ as applying the second-order DT. If we denote

$$M_h^{[2]} \triangleq \begin{pmatrix} m_{11h}^{[2]} & m_{12h}^{[2]} \\ m_{21h}^{[2]} & m_{22h}^{[2]} \end{pmatrix}, \tag{38}$$

and

$$M_h^{[2]} = L_{1h} \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{pmatrix} L_{1h}^{-1}, \tag{39}$$

where

$$L_{1h} = \begin{pmatrix} \phi_{13h}^{[1]} & \phi_{14h}^{[1]} \\ \phi_{23h}^{[1]} & \phi_{24h}^{[1]} \end{pmatrix}. \tag{40}$$

Through computation, it leads

$$m_{12h}^{[2]} = \frac{(\lambda_4 - \lambda_3)\, \phi_{13h}^{[1]} \phi_{14h}^{[1]}}{\Delta_{1r1b}^{[2]}}. \tag{41}$$

Therefore, the second-order iteration DT solution can be derived as

$$u_{1r1b}^{[2]} = u_{1r}^{[1]} + 2im_{12h}^{[2]}. \tag{42}$$

At this point, **Theorem 1** is proved.

Noticing the fact that there are four free parameters: k, a, α_1 and β_2 in the hybrid first-order rogue wave and first-order breather solution (37), we can observe the interaction structures and dynamics between the rogue wave and breather by adjusting and controlling these parameters. Figs. 1-4 illustrate the impacts of these parameters, respectively. It is easy to see the propagating directions, positions, amplitudes and crossed patterns are determined the parameters.

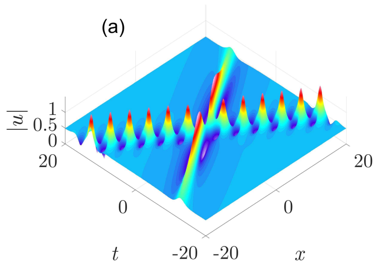

Fig. 1: The hybrid first-order rogue wave and first-order breather solution (37) with the settings: $a = 0.5, \alpha_1 = 0.35, \beta_1 = 0.45$, and different k, (a) $k = 0.1$; (b) $k = 0.2$, (b) $k = 0.4$.

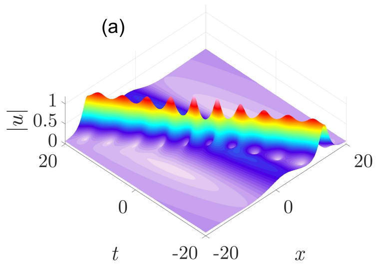

Fig. 2: The hybrid first-order rogue wave and first-order breather solution (37) with the settings: $k = 0.3, \alpha_1 = 0.35, \beta_1 = 0.45$, and different a, (a) $a = 0.1$; (b) $a = 0.2$, (b) $a = 0.4$.

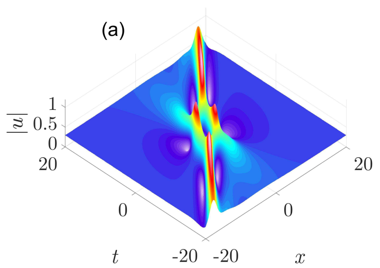

Fig. 3: The hybrid first-order rogue wave and first-order breather solution (37) with the settings: $a = 0.3, k = 0.3, \beta_1 = 0.45$, and different α_1, (a) $\alpha_1 = 0.1$; (b) $\alpha_1 = 0.2$, (b) $\alpha_1 = 0.4$.

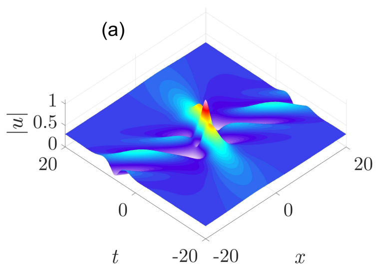

Fig. 4: The hybrid first-order rogue wave and first-order breather solution (37) with the settings: $a = 0.3, k = 0.3, \alpha_1 = 0.45$, and different β_1, (a) $\beta_1 = -0.1$; (b) $\beta_1 = -0.2$, (b) $\beta_1 = -0.4$.

5.2. The second-order rogue wave and first-order breather solution

There are two ways to construct the second-order rogue wave and first-order breather solution of Eq. (1) by using the third-order DT, namely, (I) based on the second-order rogue wave solution (19) and the first-order breather solution (22), and (II) based on the first-order rogue wave solution (15) and the second-order breather solution (29). Their results are the same. In this article, we only investigate the former case.

We take

$$T_1^{[3]} = \lambda I - M_{2r1b}^{[3]}, \tag{43}$$

and

$$M_{2r1b}^{[3]} = \begin{pmatrix} m_{11rb}^{[3]} & m_{12rb}^{[3]} \\ m_{21rb}^{[3]} & m_{22rb}^{[3]} \end{pmatrix}. \tag{44}$$

Now, we let

$$M_{2r1b}^{[3]} = L_{2r1b} \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{pmatrix} L_{2r1b}^{-1}, \tag{45}$$

where

$$L_{2r1h} = \begin{pmatrix} \phi_{13}^{[2]} & \phi_{14}^{[2]} \\ \phi_{23}^{[2]} & \phi_{24}^{[2]} \end{pmatrix},$$

with

$$\begin{pmatrix} \phi_{13}^{[2]} \\ \phi_{23}^{[2]} \end{pmatrix} = \left(\lambda_3 I - M_{2r}^{[2]}\right) \begin{pmatrix} \phi_{13h}^{[1]} \\ \phi_{23h}^{[1]} \end{pmatrix},$$

$$\begin{pmatrix} \phi_{14}^{[2]} \\ \phi_{24}^{[2]} \end{pmatrix} = \left(\lambda_4 I - M_{2r}^{[2]}\right) \begin{pmatrix} \phi_{14h}^{[1]} \\ \phi_{24h}^{[1]} \end{pmatrix},$$

here $M_{2r}^{[2]}$ is given by (18), $\phi_{13h}^{[1]}, \phi_{23h}^{[1]}, \phi_{14h}^{[1]}$ and $\phi_{24h}^{[1]}$ are given by (35)-(36).

Through computation, it generates

$$m_{12rb}^{[3]} = \frac{(\lambda_4 - \lambda_3)\, \phi_{13}^{[2]} \phi_{14}^{[2]}}{\Delta_{2r1b}^{[3]}},$$

with

$$\Delta_{2r1b}^{[3]} = \phi_{13}^{[2]} \phi_{24}^{[2]} - \phi_{23}^{[2]} \phi_{14}^{[2]}.$$

Thereby, we have the following theorem.

Theorem 2. The second-order rogue wave and first-order breather solution of Eq. (1) can be expressed as

$$u_{2r1b}^{[3]} = u_{2r}^{[2]} + 2im_{12rb}^{[3]}, \tag{46}$$

where $u_{2r}^{[2]}$ is given by (19).

The graphs of the solution (46) is shown in Fig. 5.

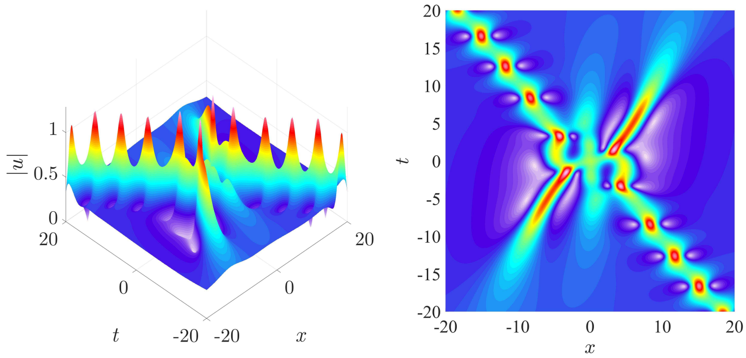

Fig. 5: The hybrid second-order rogue wave and first-breather solution (46) with the settings: $a = 0.32, k = 0.3, \alpha_1 = 0.35, \beta_1 = 0.45, \alpha_2 = 0.55$ and $\beta_2 = 0.4$. The right plot is the projecting view of the left on the $x - t$ plane.

5.3. The first-order rogue wave and second-order breather solution

By the similar way used in the last subsection, we construct the third-order DT based on the second-order breather solution (29).

By letting

$$T_2^{[3]} = \lambda I - M_{1r2b}^{[3]}, \tag{47}$$

and

$$M_{1r2b}^{[3]} = \begin{pmatrix} m_{11br}^{[3]} & m_{12br}^{[3]} \\ m_{21br}^{[3]} & m_{22br}^{[3]} \end{pmatrix}, M_{1r2b}^{[3]} = L_{1r2b} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} L_{1r2b}^{-1}, L_{1r2b} = \begin{pmatrix} \phi_{11h}^{[2]} & \phi_{12h}^{[2]} \\ \phi_{21h}^{[2]} & \phi_{22h}^{[2]} \end{pmatrix},$$

where

$$\begin{pmatrix} \phi_{11h}^{[2]} \\ \phi_{12h}^{[2]} \end{pmatrix} = \left(\lambda_1 I - M_{2b}^{[2]}\right) \begin{pmatrix} \phi_{11h}^{[1]} \\ \phi_{12h}^{[1]} \end{pmatrix},$$

$$\begin{pmatrix} \phi_{21h}^{[2]} \\ \phi_{22h}^{[2]} \end{pmatrix} = \left(\lambda_2 I - M_{2b}^{[2]}\right) \begin{pmatrix} \phi_{21h}^{[1]} \\ \phi_{22h}^{[1]} \end{pmatrix},$$

with

$$\begin{pmatrix} \phi_{11h}^{[1]} \\ \phi_{12h}^{[1]} \end{pmatrix} = \left(\lambda_1 I - M_{1b}^{[1]}\right) \begin{pmatrix} \phi_{11}^{[0]} \\ \phi_{12}^{[0]} \end{pmatrix},$$

$$\begin{pmatrix} \phi_{21h}^{[1]} \\ \phi_{22h}^{[1]} \end{pmatrix} = \left(\lambda_2 I - M_{1b}^{[1]}\right) \begin{pmatrix} \phi_{21}^{[0]} \\ \phi_{22}^{[0]} \end{pmatrix}.$$

Here, $M_{1b}^{[1]}, M_{2b}^{[2]}$ are respectively given by (24) and (34); $\phi_{11}^{[0]}, \phi_{12}^{[0]}, \phi_{21}^{[0]}$ and $\phi_{22}^{[0]}$ are referred by (9) and (10).

Theorem 3. The first-order rogue wave and second-order breather solution of Eq. (1) can be expressed as

$$u_{1r2b}^{[3]} = u_{2b}^{[2]} + 2im_{12br}^{[3]}, \tag{48}$$

where $u_{2b}^{[2]}$ is given by (29), and

$$m_{12rb}^{[3]} = \frac{(\lambda_2 - \lambda_1)\, \phi_{11h}^{[2]} \phi_{12h}^{[2]}}{\phi_{11h}^{[2]} \phi_{22h}^{[2]} - \phi_{12h}^{[2]} \phi_{21h}^{[2]}}.$$

We draw the plots of the solution (46) in Fig. 6.

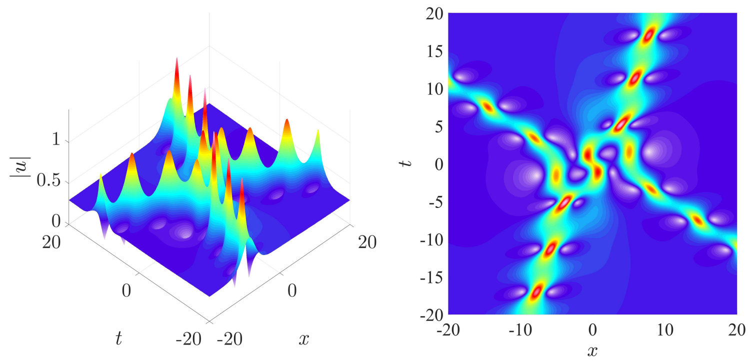

Fig. 6: The hybrid first-order rogue wave and second-order breather solution (48) with the settings: $a = 0.3, k = 0.4, \alpha_1 = 0.4, \beta_1 = 0.45, \alpha_5 = 0.55$ and $\beta_5 = 0.5$. The right plot is the projecting view of the left on the $x - t$ plane.

6. Conclusions

The rogue wave and breather are significant solitons in nonlinear evolution systems. The complex mKdV equation is a critical model describing wave movements in many nonlinear systems. In this work, we start from the Lax pair of this equation, then construct its higher-order Darboux transformation. Further, we obtained and prove a series of theorems to compute the hybrid rogue wave and breather solutions. The interaction features between rogue waves and breathers are exhibited by adjusting and controlling parameters in the solutions. Our study will contribute more understandings to the systems governed by the complex mKdV equation.

Compliance with ethical standards

The authors ensure the compliance with ethical standards for this work.

Conflict of interest:

The authors declare that there are no conflicts of interests with publication of this work.

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