The Einstein's Mass-Energy Equivalence and the Relativistic Mass and Momentum derived from the Newton's Second Law of Motion

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Abstract: In the Einstein's theory of special relativity, the relativistic mechanics is concerned with the motion of bodies whose velocities approach the speed of light. It is understood the velocity of a moving particle with mass is less than the speed of velocity and velocity of a massless particle like photon is equal to the speed of light. This paper presents the Einstein's mass-energy equivalence and the equations of relativistic mass, momentum and energy from the Newton's second law of motion.

Keywords: relativistic motion, momentum equation, kinetic energy, rest mass

1. Introduction

The equation of Einstein's mass-energy equivalence [1-4] is $E = mc^2$, where E, m, and c denote electromagnetic/light energy, mass, and speed of light respectively. The equations of relativistic mass, energy, and momentum [5-9] and the mass-energy equivalence are derived from the Newton's second law of motion by differentiation and integration [10].

2. Relativistic Mass, Momentum and Energy

The Einstein mass-energy equivalence denotes relativistic energy. In the theory of special relativity, the relativistic momentum is concerned with the motion of a particle whose velocity approaches the speed of light.

The Newton's second law of motion states that the force (F) acting on a particle is equal to the rate of change of its momentum (p).

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt}, \text{where momentum } p = mv.$$

According to Einstein's relativity theory, mass (m) is a variable over time. By differentiating the force equation, we get

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} + v\frac{dv}{dt}, \text{where } v \text{ is velocity.}$$

The differential equation for work and kinetic energy is derived as follows:

$$dK = dW = Fds$$

$$dK = Fds = \left(m\frac{dv}{dt} + v\frac{dv}{dt}\right)ds$$

$$dK = Fds = m\frac{ds}{dt}dv + v\frac{ds}{dt}dv, \text{where } \frac{ds}{dt} = v$$

Note that the term c^2dm allows the hypothesis of variable mass as it actually occurs at high speed. Also, c^2dm is equal to the kinetic energy [7].

$$dK = mvdv + v^2dm, \text{where } dK \text{ is kinetic energy.}$$

$$c^2dm = mvdv + v^2dm$$

$$\frac{dm}{m} = \frac{v}{c^2 - v^2}dv$$

$$\int_{m_0}^{m} \frac{dm}{m} = \int_0^v \frac{v}{c^2 - v^2}dv$$

$$[\ln(m)]_{m_0}^{m} = -\frac{1}{2}[\ln(c^2 - v^2)]_0^v$$

$$\ln m - \ln m_0 = -\frac{1}{2}\ln(c^2 - v^2) + \frac{1}{2}\ln c^2$$

$$\ln\frac{m}{m_0} = \frac{1}{2}\ln\frac{c^2}{c^2 - v^2}$$

$$\frac{m}{m_0} = \sqrt{\frac{c^2}{c^2 - v^2}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\text{Relativistic mass } (m) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\text{Relativistic momentum } (p) = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\text{Relativistic Force } (F) = \frac{m_0 a}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\text{Relativistic Energy } (E) = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Where,

- the rest mass of the body is m_0
- the velocity of the body in motion is v
- the acceleration of the body in motion is a
- the speed of the light is c

The relation between relativistic energy and momentum show below:

$$E^2 = \frac{m_0^2 c^4}{1 - \frac{v^2}{c^2}} \Rightarrow E^2 = \frac{m_0^2 c^2 (v^2 - v^2 + c^2)}{1 - \frac{v^2}{c^2}} \Rightarrow E^2 = \frac{m_0^2 c^2 v^2 - m_0^2 c^2 v^2 + m_0^2 c^4}{1 - \frac{v^2}{c^2}}$$

$$Then, E^2 = \left(\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} \right)^2 c^2 + \frac{m_0^2 c^2 (c^2 - v^2)}{\frac{c^2 - v^2}{c^2}}$$

From the above expression, we obtain the energy-momentum relation

$$E^2 = p^2 c^2 + m_0^2 c^4.$$

If particle is at rest, then $p = 0$. Thus, the rest energy is that $E = m_0 c^2$.

Now, it is understood that the relativistic mass, momentum, and energy are derived from the Newton's second law of motion.

3. Mass-Energy Equivalence

$$\text{Relativistic mass } (m) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$m^2 = \frac{{m_0}^2}{1 - \frac{v^2}{c^2}} \Rightarrow m^2 (c^2 - v^2) = {m_0}^2 c^2, \text{where rest mass energy} = {m_0}^2 c^2$$

$$m^2 c^2 - m^2 v^2 = {m_0}^2 c^2$$

By differentiating the equation with respect to time, we get

$$2mc^2 \frac{dm}{dt} - 2mv \frac{d(mv)}{dt} = 0 \Rightarrow c^2 \frac{dm}{dt} = v \frac{d(mv)}{dt}$$

$$\frac{dE}{dt} = Fv = v \frac{d(mv)}{dt} = c^2 \frac{dm}{dt}$$

$$dE = c^2 dm$$

$$\int_0^K dK = \int_{m_0}^m c^2 dm$$

$$K = c^2 (m - m_0), \text{where } K \text{ is kinetic energy.}$$

Total Energy (E) = Kinetic Energy (K) + Rest Mass-Energy ($m_0 c^2$)

$$E = c^2 (m - m_0) + m_0 c^2$$

$$E = c^2 m - c^2 m_0 + m_0 c^2$$

$$E = c^2 m$$

$$\text{Therefore}, E = mc^2.$$

Hence, the equation of Einstein's mass-energy equivalence is derived from Newton's Second Law of Motion.

4. Conclusion

The mass-energy equivalence along with relativistic mass, momentum, and energy play an important role in the Einstein's theory of special relativity. Also, the relativistic mechanics is concerned with the motion of bodies whose velocities approach the speed of light. In this article, the equation of Einstein's mass-energy equivalence and the relativistic mass and momentum equation have been derived from Newton's Second Law of Motion.

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