Annamalai's Binomial Identity and Theorem

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Abstract: This paper presents Annamalai's binomial theorem, coefficient, identity, and binomial expansion developed by Chinnaraji Annamalai of the Indian Institute of Technology Kharagpur. Also, an extended geometric series is introduced with innovative summation of single terms and more successive terms of the series in this article.

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1. Introduction

In the earlier days, geometric series with positive exponents served as a vital role in the development of differential and integral calculus and as an introduction to Taylor series and Fourier series. Geometric series have significant applications in physics, engineering, biology, economics, finance, management, queueing theory, computer science and medicine [1]. An extended finite geometric series that is a geometric series with and without negative exponents [3, 7] denotes the summation of a finite number of terms as follows:

$$\sum_{i=-k}^{n} x^i = x^{-k} + x^{-k+1} + x^{-k+2} + \cdots + x^{-2} + x^{-1} + 1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - x^{-k}}{x-1},$$

that is, $$\sum_{i=-k}^{-1} x^i + \sum_{j=0}^{n} x^j = \frac{x^{-1+1} - x^{-k}}{x-1} + \frac{x^{n+1} - 1}{x-1} = \frac{x^{n+1} - x^{-k}}{x-1} \ (x \neq 1 \ \& k, n \in N),$$

where N is the set of natural numbers including zero.

In general, the summations of one terms or more successive terms of an extended geometric series with and without negative exponents [1-8] are mathematically expressed as

$$x^{-n} = \frac{x^{-n+1} - x^{-n}}{x-1}, \qquad \sum_{i=-q}^{-p} x^i = \frac{x^{-p+1} - x^{-q}}{x-1} \ (x \neq 1, -q \geq -p, and\ p, q \in N),$$

$$x^n = \frac{x^{n+1} - x^n}{x-1}, \qquad \sum_{j=k}^{n} x^j = \frac{x^{n+1} - x^k}{x-1} \ (x \neq 1, n \geq k, and\ n, k \in N).$$

This novel idea was generated by Chinnaraji Annamalai of the Indian Institute of Technology Kharagpur. The extended geometric series is used in science, management, and medicine [1]. Using the extended geometric series, Annamalai's binomial identity [2-6] and theorem and its proofs are constituted in this article.

2. Annamalai's Binomial Identity and Coefficients

The following identity [2-6] is named as Annamalai's binomial identity that is the binomial series developed by the multiple summations of the extended geometric series:

$$(i) \qquad \sum_{i_1=0}^{n} \sum_{i_2=i_1}^{n} \sum_{i_3=i_2}^{n} \cdots\cdots\cdots \sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i.$$

When substituting $r = 1$ in (i), it becomes double summation of a geometric series,

$$\sum_{i_1=0}^{n} \sum_{i_2=i_1}^{n} x^{i_2} = \sum_{i_2=0}^{n} x^{i_2} + \sum_{i_2=1}^{n} x^{i_2} + \sum_{i_2=2}^{n} x^{i_2} + \cdots + \sum_{i_2=n}^{n} x^{i_2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n.$$

that is, $$1 + 2x + 3x^2 + \cdots + (n+1)x^n = \sum_{i=0}^{n} (i+1)x^i = \sum_{i=0}^{n} V_i^1 x^i.$$

When substituting $r = 2$ in (i), it becomes triple summation of a geometric series,

$$\sum_{i_1=0}^{n} \sum_{i_2=i_1}^{n} \sum_{i_3=i_2}^{n} x^{i_3} = \sum_{i_2=0}^{n} \sum_{i_3=i_2}^{n} x^{i_3} + \sum_{i_2=1}^{n} \sum_{i_3=i_2}^{n} x^{i_3} + \sum_{i_2=2}^{n} \sum_{i_3=i_2}^{n} x^{i_3} + \cdots + \sum_{i_2=n}^{n} \sum_{i_3=i_2}^{n} x^{i_3} = \sum_{i=0}^{n} V_i^2 x^i.$$

Similarly, if this computation continues up to r times, the r^{th} equation becomes as follows:

$$\sum_{i=0}^{n} V_i^r x^i = \sum_{i_1=0}^{n} \sum_{i_2=i_1}^{n} \sum_{i_3=i_2}^{n} \cdots\cdots\cdots \sum_{i_r=i_{r-1}}^{n} x^{i_r},$$

which is called Annamalai's binomial identity.

The following binomial coefficient is named as Annamalai's binomial coefficient (optimized combination) V_r^n, which is the extension of tradition coefficient.

$$V_r^n = \frac{(r+1)(r+2)\cdots\cdots(r+n)}{n!} = \prod_{i=1}^{n} \frac{r+i}{n!} \quad (n \geq 1, r \geq 0 \;\&\; n, r \in N)$$

OR

$$V_n^r = \frac{(n+1)(n+2)\cdots\cdots(n+r)}{r!} = \prod_{j=1}^{r} \frac{n+j}{r!} \quad (n \geq 0, r \geq 1 \;\&\; n, r \in N).$$

$$V_r^n = \prod_{i=1}^{n} \frac{r+i}{n!} = V_n^r = \prod_{j=1}^{r} \frac{n+j}{r!} \ (n, r \geq 1 \ \& \ n, r \in N),$$

$$\text{where } V_n^0 = V_0^n \ \ and \ V_0^n = 1.$$

The traditional combination of combinatorics or traditional coefficient is given below:

$$nC_r = \frac{n!}{r!\,(n-r)!} = \frac{(r+1)(r+2)\cdots\cdots(r-(n-1)}{(n-r)!} = \prod_{i=1}^{n-1} \frac{r+i}{(n-r)!}, (n \geq r \ \& \ n, r \in N)$$

where $N = \{1, 2, 3, \cdots\cdots\}$, V_r^n is Annamalai′s binomial coefficient, and $n!$ is the factorial of n.

The comparison between optimized combination and traditional combination are given below:

$V_x^y = V_y^x$ denotes the otimized combination. Let $z = x + y$. Then, $zC_x = zC_y$.

For example,

$$V_3^5 = V_5^3 = (5+3)C_3 = (5+3)C_5 = 56.$$

Also, $\quad V_n^0 = V_0^n = nC_0 = nC_n = \dfrac{n!}{n!\,0!} = 1 \ \ and \ \ V_0^0 = 0C_0 = \dfrac{0!}{0!} = 1(\because 0! = 1).$

Note that both V_r^n and nC_r are not equal.

Some results (Annamalai's binomial identities) are provided below [5, 6]:

(ii) $\quad V_r^m = V_m^r \ (m, r \geq 1 \ \& \ n, r \in N)$

(iii) $\quad V_0^p + V_1^p + V_2^p + V_3^p + \cdots + V_r^p = V_r^{p+1} \Rightarrow \sum_{i=0}^{r} V_i^p = V_r^{p+1} \quad (p, r \in N)$

3. Annamalai's Binomial Theorem

Annamalai's binomial theorem states that multiple summations of extended geometric series with binomial coefficients are a binomial series,

$$\sum_{i=0}^{n} V_i^{r+1} x^i = \sum_{i=0}^{n} V_i^r x^i + \sum_{i=1}^{n} V_{i-1}^r x^i + \sum_{i=2}^{n} V_{i-2}^r x^i + \cdots + \sum_{i=n-1}^{n} V_{i-(n-1)}^r x^i + \sum_{i=n}^{n} V_{i-n}^r x^i$$

Proof: Let's show that the computation of addition of extended geometric series (right-hand side of the theorem) is equal to the summation of binomial series for upper limit $r+1$in Annamalai's binomial coefficient [4-6].

$$\sum_{i=0}^{n} V_i^r x^i + \sum_{i=1}^{n} V_{i-1}^r x^i + \sum_{i=2}^{n} V_{i-2}^r x^i + \cdots + \sum_{i=n-1}^{n} V_{i-(n-1)}^r x^i + \sum_{i=n}^{n} V_{i-n}^r x^i$$

$$= (V_0^r + V_1^r x + V_2^r x^2 + V_3^r x^3 + \cdots + V_n^r x^n) + (V_0^r x + V_1^r x^2 + V_2^r x^3 + V_3^r x^4 + \cdots + V_{n-1}^r x^n)$$

$$+(V_0^r x^2 + V_1^r x^3 + V_2^r x^4 + V_3^r x^5 + \cdots + V_{n-2}^r x^n) + \cdots + (V_0^r x^{n-1} + V_1^r x^n) + V_0^r x^n$$

$$= V_0^r + (V_0^r + V_1^r)x + (V_0^r + V_1^r + V_2^r)x^2 + \cdots + (V_0^r + V_1^r + V_2^r + V_3^r + \cdots + V_n^r)x^n$$

(Note that $V_0^p + V_1^p + V_2^p + \cdots + V_r^p = V_r^{p+1} for\ r = 1, 2, 3, \cdots$, and $V_0^p = V_0^{p+1} = 1[4, 5]$)

$$= V_0^{r+1} + V_1^{r+1} x + V_2^{r+1} x^2 + V_3^{r+1} x^3 + V_4^{r+1} x^4 + \cdots + V_{n-1}^{r+1} x^{n-1} + V_n^{r+1} x^n$$

$$= \sum_{i=0}^{n} V_i^{r+1} x^i$$

Hence, theorem is proved.

For Example,

$$\sum_{i=0}^{5} V_i^2 x^i = \sum_{i=0}^{5} V_i^1 x^i + \sum_{i=1}^{5} V_{i-1}^1 x^i + \sum_{i=2}^{5} V_{i-2}^1 x^i + \sum_{i=3}^{5} V_{i-3}^1 x^i + \sum_{i=4}^{5} V_{i-4}^1 x^i + \sum_{i=5}^{5} V_{i-5}^1 x^i$$

The computation of $\sum_{i=0}^{5} V_i^2 x^i$ is equal to $1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5$.

Let's compute the yield of other side,

$$\sum_{i=0}^{5} V_i^1 x^i + \sum_{i=1}^{5} V_{i-1}^1 x^i + \sum_{i=2}^{5} V_{i-2}^1 x^i + \sum_{i=3}^{5} V_{i-3}^1 x^i + \sum_{i=4}^{5} V_{i-4}^1 x^i + \sum_{i=5}^{5} V_{i-5}^1 x^i$$

$$= (1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^{5)} + (x + 2x^2 + 3x^3 + 4x^4 + 5x^5)$$

$$+(x^2 + 2x^3 + 3x^4 + 4x^5) + (x^3 + 2x^4 + 3x^5) + (x^4 + 2x^5) + x^5$$

$$= 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5$$

Thus, both sides are equal.
As per the above example, we can prove the theorem for *r* =1,2,3, . . .

4. Annamalai's Binomial Expansion

The following binomial expansions[9], named as Annamalai's binomial expansions, are derived from the Annamalai's (iii) binomial identity $\sum_{i=0}^{r} V_i^p = V_r^{p+1}$.

$$(1).\quad \sum_{i=0}^{n}\frac{(i+1)}{1!}=1+2+3+\cdots+n+(n+1)=\frac{(n+1)(n+2)}{2!}.$$

$$(2).\quad \sum_{i=0}^{n}\frac{(i+1)(i+2)}{2!}=1+3+6+\cdots+\frac{(n+1)(n+2)}{2!}=\frac{(n+1)(n+2)(n+3)}{3!}.$$

$$(3).\quad \sum_{i=0}^{n}\frac{(i+1)(i+2)(i+3)}{3!}=\frac{(n+1)(n+2)(n+3)(n+4)}{4!}.$$

$$(4).\quad \sum_{i=0}^{n}\frac{(i+1)(i+2)(i+3)(i+4)}{4!}=\frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{5!}.$$

Similarly, this process continues upto *r* times. The r^{th} binomial expansion is as follows:

$$(r).\quad \sum_{i=0}^{n}\frac{(i+1)(i+2)(i+3)\cdots(i+r)}{r!}=\frac{(n+1)(n+2)\cdots(n+r)(n+r+1)}{(r+1)!}$$

$$i.e.,\sum_{i=0}^{n}\prod_{j=1}^{r}\frac{i+j}{r!}=\prod_{i=1}^{r+1}\frac{n+i}{(r+1)!}.$$

This Annamalai's binomial expansion is applied into the following binomial series:

$$\sum_{i=0}^{n}V_i^r x^i=\sum_{i=0}^{n}\prod_{j=1}^{r}\frac{i+j}{r!}x^i$$

5. Conclusion

In this paper, Annamalai's binomial identities, coefficients, expansions, theorems, and their proofs have been introduced for researchers that will be useful for future research and development. Also, an extended geometric series has been developed with innovative summation of single terms and more successive terms of the series in this article.

References

[1] Annamalai C (2010) "Application of Exponential Decay and Geometric Series in Effective Medicine", Advances in Bioscience and Biotechnology, Vol. 1(1), pp 51-54. https://doi.org/10.4236/abb.2010.11008

[2] Annamalai C (2017) "Computational modelling for the formation of geometric series using Annamalai computing method", Jñānābha, Vol. 47(2), pp 327-330. https://doi.org/10.31219/osf.io/kwmcv

[3] Annamalai C (2018) "Annamalai's Computing Model for Algorithmic Geometric Series and its Mathematical Structures", Journal of Mathematics and Computer Science, Vol.3 (1), pp 1-6. https://www.doi.org/10.11648/j.mcs.20180301.11

[4] Annamalai C (2019) "A Model of Iterative Computations for Recursive Summability", Oxford Journal of Information and Mathematical Sciences, Vol. 33(1), pp 75-77. https://doi.org/10.22541/au.165167013.33544978/v1

[5] Annamalai C (2020) "Optimized Computing Technique for Combination in Combinatorics", hal-0286583. https://doi.org/10.31219/osf.io/9p4ek

[6] Annamalai C (2020) "Novel Computing Technique in Combinatorics", hal-02862222. https://doi.org/10.31219/osf.io/m9re5

[7] Annamalai C (2022) "Sum of Geometric Series with Negative Exponents", OpenAIRE. https://doi.org/10.5281/zenodo.6472535

[8] Annamalai C (2022) "Comparison between Optimized and Traditional Combinations of Combinatorics", OSF Preprints. https://doi.org/10.31219/osf.io/2qdyz

[9] Annamalai C (2022) "Novel Binomial Series and its Summations", Authorea Preprints. https://doi.org/10.22541/au.164933885.53684288/v2