A Fractional Cubic Spline for Solving Fractional Volterra-Integral Equations with Convergence Analysis

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Abstract

In this work, we present a new boundary conditions for fractional cubic spline (FCS) model for solving fractional Volterra-integral equations. We reduced the problem to a set of a linear systems by using fractional continuity conditions. Convergence analysis proved to solve fractional Volterra-integral equations by obtained linear systems, to determine fractional spline derivatives, we applied the Caputo fractional derivative. The process is detailed and computed with three computational examples, and the results show that it is both effective and simple to use. Moreover, the results are compared with the methods in [1], [2] and [4].

Keywords: Fractional spline function, derivative of fractional order, analysis of convergence, Fractional Volterra integral equations.

1 Introduction

Integral equations have generated a lot of interest due to their numerous uses, specifically in sciences related to physics and engineering, functional, mechanic, numerical optimization and dynamical systems, etc. Heat conduction problems [9], diffusion problems [7], engineering [24] and physics [16]. In mathematical modeling, integral equations are used in special ways. For such equations, it is almost hard to obtain an analytical solution. As a result, numerical approaches for obtaining approximate solutions to such equations have been developed. A large number of authors have written about the numerical solution of integral equations. some of these solutions are as follows: In [5] the author used Quintic spline polynomial for solving FIE, an application of the Homotopy analysis method for solving the nonlinear and linear integral equations of the second kind presented in [28] also Grigorieff in [8] presented a periodic pseudo differential operators with multiple knot splines, author in [23] solved integro-differential equations by using Exponential spline function, in [6] authors presented a modified variational iteration method for solving Fredholm integro-differential equations, author in [10] used Bernstein polynomial for solving Volterra integral equation.
We want to use fractional Cubic spline (FCS) to propose a numerical procedure for the solution of the linear integral equation of the second kind

$$y(t) = f(t) + \int_{g(t)}^{h(t)} k(t,v)y(v)dv \tag{1}$$

The kernel function of two variables x and t is $k(t,x)$, $h(t)$ and $g(t)$ are the limit of integration's and can be constants, variables, or mixed (for this paper we take $h(t) = t$) , $y(t)$ is the unknown function, and $f(t)$ is given.
Fractional calculus is one of the most trustworthy procedures for managing complex systems, and it is still an area where many models are still to be proposed, studied, and applied to real world applications in many areas of science and engineering where no locality significantly contributes. In recent, several definitions of fractional derivatives and integrals have been proposed. In [25] authors introduce two classes of lacunary fractional spline functions by using the Liouville–Caputo fractional Taylor expansion, Debnath in [20] presented applications of fractional calculus to dynamical systems, some important pioneers that started to apply fractional calculus to scientific and engineering problems during the nineteenth and twentieth century's presented in [18].
The following sections of this paper are organized in the following order: in Sect. 2 we derives the fractional spline function and matrix representation of the system, Sect. 3 and 4 provided methodology for integral equations and convergence analysis is investigated, respectively. In Sect. 5 by solving some fractional examples, the proposed method's efficiency is illustrated and comparison of the numerical solutions with some other existing methods in [1],[2] and [4]. Finally, we arrive at the conclusion of our paper.

2 FRACTIONAL SPLINE MODEL AND MATRIX REPRESENTATION

Definition 1 *[21] (β is the order of Fractional Integral) For all $\beta > 0$ and $f(t)$ be a local integrable function, the left FI can be defined as:*

$$ {}_tI_b^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_t^b (u-t)^{\beta} f(u)du, \quad -\infty < t < b \leq \infty, $$

Also the right FI defined:

$$ {}_aI_t^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-u)^{\beta} f(u)du, \quad -\infty \leq a < t < \infty. $$

Alternatively

Definition 2 *[21] (Caputo fractional derivative of order λ) is defined as:*

$$ {}_a^C D_t^{\lambda} f(t) = \frac{1}{\Gamma(n-\lambda)} \int_a^t (t-u)^{n-\lambda} \left(\frac{d}{du}\right)^n f(u)du \quad, n = \lceil \lambda \rceil and \lambda > 0. $$

For $a = 0$, we introduce the notation:

$$ {}^C D_t^{\lambda} f(t) = D^{\lambda} f(t). $$

In this work we used the fractional spline function to derive an approximate solution to the integral equation in this section. We introduce a set $\Omega = [a,b]$ with partitions $\Delta : a = v_0 < \ldots < v_n = b$, where $v_j = a + jh$ (h is step size).
we are modified the construction in [22] it consists of a new scheme the boundary conditions such as based on the fractional cubic spline function with new fractional continuity conditions, defined on $[v_j, v_{j+1}], j = 0, \ldots, n-1$ as:

$$ S_j(v) = a_j + b_j (v - v_j)^{\frac{1}{2}} + c_j(v - v_j) + d_j(v - v_j)^{\frac{3}{2}} + e_j(v - v_j)^2 + f_j(v - v_j)^{\frac{5}{2}} + g_j(v - v_j)^3. \quad (2) $$

where $a_j, b_j, c_j, d_j, e_j,\ f_j$ and g_j are real numbers.
To find all coefficients $a_j, b_j, c_j, d_j, e_j,\ f_j$ and g_j , we define boundary conditions: $\mathrm{S}_j\left(v_j\right) = y_j, S_j\left(v_{j+1}\right) = y_{j+1}, S_j'\left(v_j\right) = M_j, S_j'\left(v_{j+1}\right) = M_{j+1}, S_j''\left(v_i\right) = F_j,$

$$ S_j''\left(v_{j+1}\right) = F_{j+1} and S_j'''\left(v_j\right) = \mathrm{y}_j'''. \quad (3) $$

Then we get: $\mathrm{a}_j = y_j$, $\mathrm{b_j} = -\frac{y_j}{\sqrt{h}} + \frac{y_{j+1}}{\sqrt{h}} - h^{\frac{5}{2}}\frac{y_j'''}{90} - \sqrt{h}\frac{M_j + 2M_{j+1}}{3} - \mathrm{h}^{\frac{3}{2}}\frac{F_j - 16F_{i+1}}{90}, c_j = M_j$, $\mathrm{d_j} = \frac{h^{\frac{3}{2}} y_j'''}{9} + \frac{2M_{j+1} - 2M_j}{3\sqrt{h}} - \sqrt{h}\frac{2F_j + 4F_{j+1}}{9}, e_j = \frac{F_j}{2},$

$$ f_j = -\sqrt{h}\frac{4}{15}y_j''' - \frac{4F_j - 4F_{j+1}}{15\sqrt{h}} and g_j = \frac{y_j'''}{6} \quad (4) $$

By using fractional derivative, we derived the new continuity conditions $S_j^{\left(\frac{1}{2}\right)}\left(v_j\right) = S_{j-1}^{\left(\frac{1}{2}\right)}\left(v_j\right)$ and $S_j^{\left(\frac{3}{2}\right)}\left(v_j\right) = S_{j-1}^{\left(\frac{3}{2}\right)}\left(v_j\right)$ we get:

$$ \left(\frac{2\pi - 6}{\pi}\right) M_{j-1} + 2M_j - M_{j+1} = \frac{3}{h}\left[y_j - \frac{1}{2}y_{j-1} - \frac{1}{2}y_{j+1}\right] + 3h\left[\frac{60 - 19\pi}{45\pi}F_{j-1} + \frac{F_j}{90} - \frac{8}{90}F_{j+1}\right] $$

$$ + \frac{h^2}{12}\left[13y_{j-1}''' + \frac{1}{5}y_j'''\right] \quad (5) $$

$$ M_{j-1} - 2M_j + M_{j+1} = h\left[\left(\frac{4}{\pi} - \frac{4}{3}\right)F_{j-1} + \frac{2}{3}F_j + \frac{2}{3}F_{j+1}\right] + h^2\left[-\frac{y_j'''}{6} + y_{j-1}'''\left(\frac{16}{6\pi} - \frac{5}{6}\right)\right] \quad (6) $$

Subtracting (5) in (6) and multiplying by $\left(\frac{\pi}{3\pi - 6}\right)$ we get: $\mathrm{M}_j = \frac{\pi}{2h(\pi - 2)}\left[-y_j + 2y_{j+1} - y_{j+2}\right] + \frac{h\pi}{(15)(\pi - 2)}\left[\left(\frac{40 - 13\pi}{\pi}\right)F_j + \frac{7}{2}F_{j+1} + 2F_{j+2}\right]$

$$ + \frac{h^2\pi}{36(\pi - 2)}\left[\left(\frac{3\pi + 32}{\pi}\right)y_j''' - \frac{9}{5}y_{j+1}'''\right] \quad (7) $$

Eliminating of M_j's between (5) and (6) to get the following system: $\left(\frac{7\pi^2 - 60\pi + 120}{715 - 30\pi}\right)F_j + \left(\frac{49\pi - 120}{30\pi - 60}\right)F_{j+1} + \left(\frac{60 - 28\pi}{15\pi - 30}\right)F_{j+2} - \frac{\pi}{30\pi - 60}F_{j+3} + \frac{4\pi}{30\pi - 60}F_{j+4}$
$= \pi\overline{2h^2(\pi - 2)\left[y_j - 4y_{j+1} + 6y_{j+2} - 4y_{j+3} + y_{j+4}\right] -}$

$$ \mathrm{h}\left[\left(\frac{27\pi^2 - 188\pi + 192}{36\pi^2 - 72\pi}\right)y_j''' + \left(\frac{69\pi + 260}{180\pi - 360}\right)y_{j+1}''' - \left(\frac{33\pi + 160}{180\pi - 360}\right)y_{j+2}''' + \left(\frac{\pi}{20\pi - 40}\right)y_{j+3}'''\right] \quad (8) $$

We need two more equations, to get a unique solution for a system (8).
For this purpose by using the method of undetermined coefficients for fifth-order technique and the Taylor series can be calculated , as shown below: $\sum_{j=1}^{3}\gamma_j y_j'' = \frac{1}{h^2}\sum_{j=0}^{5}\eta_j y_j + O\left(h^5\right)$

$$ \sum_{j=2}^{4}\gamma_j y_j'' = \frac{1}{h^2}\sum_{j=1}^{5}\sigma_j y_j + O\left(h^5\right) \quad (9) $$

Where $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \left(\frac{49\pi - 120}{30\pi - 60}, \frac{60 - 28\pi}{15\pi - 30}, \frac{-\pi}{30\pi - 60}, \frac{4\pi}{30\pi - 60}\right)$,
$(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5) = \left(-\frac{49\pi - 120}{30\pi - 60}, \frac{203\pi - 480}{30\pi - 60}, -\frac{157\pi - 360}{15\pi - 30}, \frac{107\pi - 240}{15\pi - 30}, -\frac{53\pi - 120}{30\pi - 60}, \frac{-\pi}{30\pi - 60}\right)$,
$(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = \left(\frac{60 - 28\pi}{15\pi - 30}, \frac{37\pi - 80}{10\pi - 20}, -\frac{5\pi - 12}{3\pi - 6}, -\frac{3\pi}{10\pi - 20}, \frac{4\pi}{30\pi - 60}\right)$.

The system (8) is non-singular with (3) are matrix that have a unique solution to get $F_0, F_1, \ldots, F_N$. In the matrix notation, the relations (8) can be expressed as follows:

$$RF = L_1 Y + L_2 \bar{Y} \tag{10}$$

$$\text{R}=\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \beta_5 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_5 & \beta_1 & \beta_2 & \beta_3 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \beta_5 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_5 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_5 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix},$$

$$L_1 = \frac{\pi}{2h^2(\pi-2)}\begin{bmatrix} 1 & -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 6 & -4 & 1 \end{bmatrix},$$

$$and L_2 = -h\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_3 & \alpha_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix},$$

$Where \beta_1 = \frac{49\pi-120}{30\pi-60}, \beta_2 = \frac{60-28\pi}{15\pi-30}, \beta_3 = \frac{-\pi}{30\pi-60}, \quad \beta_4 = \frac{4\pi}{30\pi-60}, \beta_5 = \frac{7\pi^2-60\pi+120}{715-30\pi}, \alpha_1 = \frac{27\pi^2-188\pi+192}{36\pi^2-72\pi}$, $\alpha_2 = \frac{69\pi+260}{180\pi-360}, \alpha_3 = -\frac{33\pi+160}{180\pi-360}$ and $\alpha_4 = \frac{\pi}{20\pi-40}$.

Also the relations (7) can be expressed as follows:

$$M = L_4 Y + L_5 F + L_6 \bar{Y} \tag{11}$$

Where:

$$\text{L}_4 = \frac{\pi}{2\ \text{h}(\pi-2)}\begin{bmatrix} -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \end{bmatrix},$$

$$L_5=\frac{h\pi}{15(\pi-2)}\begin{bmatrix} \frac{40-13\pi}{\pi} & 7/2 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{40-13\pi}{\pi} & 7/2 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{40-13\pi}{\pi} & 7/2 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \frac{40-13\pi}{\pi} & 7/2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{40-13\pi}{\pi} & 7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{40-13\pi}{\pi} & 7/2 & 2 \end{bmatrix},$$

$$L_6=\frac{h^2\pi}{36\pi-72)}\begin{bmatrix} \frac{3\pi+32}{\pi} & -9/5 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{3\pi+32}{\pi} & -9/5 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{3\pi+32}{\pi} & -9/5 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 & \frac{3\pi+32}{\pi} & -9/5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3\pi+32}{\pi} & -9/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3\pi+32}{\pi} & -9/5 \end{bmatrix},$$

Y=$(y_0,y_1,y_2,\ldots,y_N)^T$, $\bar{F}=\left(\bar{F}_0,\bar{F}_1,\bar{F}_2,\ldots,\bar{F}_N\right)^T$ and $\bar{Y}=(y_0''',y_1''',y_2''',\ldots,y_N''')^T$
From expanding equation (8) by Taylor series expansion about (v_j), we get the local truncation error T_i of FCS method.

$$T_j=\rho_1 y_j''+\rho_2 h y_j'''+\rho_3 h^2 y_j^{(4)}+\rho_4 h^3 y_j^{(5)}+\rho_5 h^4 y_j^{(6)}+\rho_6 h^5 y_j^{(7)}+O\left(h^7\right) \tag{12}$$

Where $\rho_1=\beta_1+\beta_2+\beta_3+\beta_4+\beta_5, \rho_2=\beta_1+2\beta_2-2\beta_3-\beta_4+\alpha_1-\alpha_3+\alpha_4,\ \rho_3=\frac{\beta_1}{2}+2\beta_2+2\beta_3+\frac{\beta_4}{2}-\alpha_5-2\alpha_1-\alpha_2+\alpha_4, \rho_4=\frac{\beta_1}{6}+\frac{4\beta_2}{3}-\frac{4\beta_3}{3}-\frac{\beta_4}{6}+$
$2\alpha_1+\frac{\alpha_2}{2}+\frac{\alpha_4}{2}, \rho_5=\frac{\beta_1}{24}+\frac{2\beta_2}{3}+\frac{2\beta_3}{3}+\frac{\beta_4}{24}-\frac{4\alpha_1}{3}-\frac{\alpha_2}{6}+\frac{\alpha_4}{6} and \rho_6=\frac{2\alpha_1}{3}+\frac{\alpha_2}{24}+\frac{\alpha_4}{24}$.
By substituting ρ_k in (18), we get:

$$\|\mathrm{T}\|\leq \mu h^5 y_j^{(7)}, where \mu is constant and$$

k=0,1,...,6.(13)

3 METHODOLOGY

We derive problem for the equation (1), using a fractional spline polynomial technique.
$y(t)=f(t)+\int_a^t k(t,v)y(v)dv$
From (2) and (4) we obtained:

$\mathrm{S}_j(v)=y_j-\left[\frac{y_j-y_{j+1}}{\sqrt{h}}+h^{\frac{5}{2}}\frac{y_j'''}{90}+\sqrt{h}\frac{M_j+2M_{j+1}}{3}+h^{\frac{3}{2}}\frac{F_j-16F_{j+1}}{90}\right](v-v_j)^{\frac{1}{2}}+M_j(v-v_j)+$
$\left[h^{\frac{3}{2}}\frac{y_j'''}{9}+2\frac{M_{j+1}-M_j}{3\sqrt{h}}-2\sqrt{h}\frac{F_j+2F_{j+1}}{9}\right](v-v_j)^{\frac{3}{2}}+\frac{F_j}{2}(v-v_j)^2-\left[\sqrt{h}\frac{4y_j'''}{15}+4\frac{F_j-F_{j+1}}{15\sqrt{h}}\right](v-v_j)^{\frac{5}{2}}+$
$\frac{y_j'''}{6}(v-v_j)^3$. (14)

Hence, from (1), we obtain for all $j=0,\ldots,n$.
$\mathrm{y}_j=f_j+\int_a^t k(t_j,v)\,y(v)dv\approx f_j+\sum_{j=0}^{n-1}\int_{v_j}^{\mathrm{t}} k(t_j,v)\,S_j(v)dv+O\left(h^3\right)$
Above relation is equivalent to
$\rightarrow\quad y_j\approx f_j+\sum_{j=0}^{n-1}y_j\int_{v_j}^{\mathrm{t}}k(t_j,x)\,dv-\sum_{j=0}^{\mathrm{n}-1}\left[\frac{y_j}{\sqrt{h}}+\sqrt{h}\frac{M_j}{3}+\mathrm{h}^{\frac{3}{2}}\frac{\mathrm{F_j}}{90}+\mathrm{h}^{\frac{5}{2}}\frac{\mathrm{y_j'''}}{90}\right]\int_{v_j}^{\mathrm{t}}k(t_j,v)(v-v_j)^{\frac{1}{2}}dv+$
$\sum_{j=0}^{n-1}\left[\frac{y_{j+1}}{\sqrt{h}}-\sqrt{h}\frac{2M_{j+1}}{3}+h^{\frac{3}{2}}\frac{16F_{j+1}}{90}\right]\int_{v_j}^{\mathrm{t}}k(t_j,v)(v-v_j)^{\frac{1}{2}}dv+\sum_{j=0}^{n-1}M_j\int_{v_j}^{\mathrm{t}}k(t_i,v)(v-v_j)\,dv-$
$\sum_{j=0}^{n-1}\left[\frac{2M_j}{3\sqrt{h}}+\sqrt{h}\frac{2F_j}{9}-h^{\frac{3}{2}}\frac{y_j'''}{9}\right]\int_{v_j}^{\mathrm{t}}k(t_j,v)(v-v_j)^{\frac{3}{2}}dv+\sum_{j=0}^{n-1}\left[\frac{2M_{j+1}}{3\sqrt{h}}-\sqrt{h}\frac{4F_{j+1}}{9}\right]\int_{v_j}^{\mathrm{t}}k(t_j,v)(v-v_j)^{\frac{3}{2}}dv$
$+\sum_{j=0}^{n-1}\frac{F_j}{2}\int_{v_j}^{\mathrm{t}}k(t_j,v)(v-v_j)^2dv-\sum_{j=0}^{n-1}\left[\sqrt{h}\frac{4y_j'''}{15}+\frac{4F_j}{15\sqrt{h}}\right]\int_{v_j}^{\mathrm{t}}k(t_j,v)(v-v_j)^{\frac{5}{2}}dv+\sum_{j=0}^{n-1}\frac{4F_{j+1}}{15\sqrt{h}}\int_{v_j}^{\mathrm{t}}k(t_i,v)$
$(v-v_j)^{\frac{5}{2}}dv+\sum_{j=0}^{n-1}\frac{y_j'''}{6}\int_{v_j}^{\mathrm{t}}k(t_j,v)(v-v_j)^3dv$
Let $\mathrm{a}(j,i)=\int_{v_j}^t k(t_j,v)\,dv, b(j,i)=\int_{v_j}^t k(t_j,v)\sqrt{v-v_j}dv=c(j,i+1), d(j,i)=\int_{v_j}^t k(t_j,v)(v-v_j)\,dv$,
$g(j,i)=\int_{v_j}^t k(t_j,v)\sqrt{(v-v_j)^3}dv=r(j,i+1), p(j,i)=\int_{v_j}^t k(t_j,v)(v-v_j)^2dv, q(j,i)=\int_{v_j}^t k(t_j,v)$
$\sqrt{(v-v_j)^5}dv=z(j,i+1) and s(j,i)=\int_{v_j}^t k(t_j,v)(v-v_j)^3dv$.
The defined notations can be written as a matrix as shown below:
$\mathrm{A}=\mathrm{a}(j,i), \mathrm{B}=\mathrm{b}(j,i), \mathrm{C}=\mathrm{c}(j,i+1), \mathrm{D}=\mathrm{d}(j,i), \mathrm{G}=g(j,i), \mathrm{R}=r(j,i+1), \mathrm{P}=p(j,i), \mathrm{Q}=q(j,i)$,
$\mathrm{Z}=z(j,i+1), \mathrm{S}=s(j,i), Y\approx\widehat{\mathrm{Y}}=(\hat{y}_0,\widehat{y}_1,\widehat{y}_2,\ldots,\widehat{y}_N)^T, M\approx\widehat{\mathrm{M}}=\left(\widehat{M}_0,\widehat{M}_1,\widehat{M}_2,\ldots,\widehat{M}_N\right)^T$,
$F\approx\widehat{\mathrm{F}}=\left(\widehat{F}_0 and \widehat{F}_1,\widehat{F}_2,\ldots,\widehat{F}_N\right)^T$.

$$\rightarrow \widehat{\mathrm{Y}} = \mathrm{F} + \left(\mathrm{A} - \tfrac{\mathrm{B}}{\sqrt{\mathrm{h}}} + \tfrac{\mathrm{C}}{\sqrt{\mathrm{h}}}\right)\widehat{\mathrm{Y}} + \left(-\tfrac{\sqrt{\mathrm{h}}\mathrm{B}}{3} - \tfrac{\sqrt{\mathrm{h}}C}{3} + \mathrm{D} - \tfrac{2\mathrm{G}}{3\sqrt{\mathrm{h}}} + \tfrac{2\mathrm{R}}{3\sqrt{\mathrm{h}}}\right)$$
$$\widehat{\mathrm{M}} + \left(-\tfrac{\sqrt{\mathrm{h}^3}\ \mathrm{B}}{90} + \tfrac{6\sqrt{\mathrm{h}^3}\mathrm{C}}{90} - \tfrac{2\sqrt{\mathrm{h}}G}{9} - \tfrac{4\sqrt{\mathrm{h}}R}{9} + \tfrac{\mathrm{P}}{2} - \tfrac{4\mathrm{Q}}{15\sqrt{\mathrm{h}}} + \tfrac{4\mathrm{Z}}{15\sqrt{\mathrm{h}}}\right)\widehat{\mathrm{F}} + \left(-\tfrac{\sqrt{\mathrm{h}^5}\ \mathrm{B}}{90} + \tfrac{\sqrt{\mathrm{h}^3}\mathrm{G}}{9} - \tfrac{4\sqrt{\mathrm{h}Q}}{15} + \tfrac{\mathrm{S}}{6}\right)\breve{Y}. \quad (15)$$

$$\rightarrow \widehat{\mathrm{Y}} = \mathrm{F} + \mathrm{A}_1\widehat{\mathrm{Y}} + \mathrm{A}_2\widehat{\mathrm{M}} + \mathrm{A}_3\widehat{\mathrm{F}} + \mathrm{A}_4\breve{\mathrm{Y}}. \quad (16)$$

Where $A_1 = \mathrm{A} - \frac{\mathrm{B}}{\sqrt{\mathrm{h}}} + \frac{\mathrm{C}}{\sqrt{\mathrm{h}}}, A_2 = -\frac{\sqrt{\mathrm{h}}\mathrm{B}}{3} - \frac{\sqrt{\mathrm{h}}\mathrm{C}}{3} + \mathrm{D} - \frac{2\mathrm{G}}{3\sqrt{\mathrm{h}}} + \frac{2\mathrm{R}}{3\sqrt{\mathrm{h}}}, A_3 = -\frac{\sqrt{\mathrm{h}^3}\ \mathrm{B}}{90} + \frac{16\sqrt{\mathrm{h}^3}\mathrm{C}}{90} - \frac{2\sqrt{\mathrm{h}}\mathrm{G}}{9} - \frac{4\sqrt{\mathrm{h}}R}{9} + \frac{\mathrm{P}}{2} - \frac{4\mathrm{Q}}{15\sqrt{\mathrm{h}}} + \frac{4\mathrm{Z}}{15\sqrt{\mathrm{h}}}, A_4 = -\frac{\sqrt{\mathrm{h}^5}\ \mathrm{B}}{90} + \frac{\sqrt{\mathrm{h}^3}\mathrm{G}}{9} - \frac{4\sqrt{\mathrm{h}}\mathrm{Q}}{15} + \frac{\mathrm{S}}{6}$.
An approximation solution of equation (1), will be obtained by solving the above system, we can use FCS $\widehat{S}_j$ to find approximation solution of y_j function, where
$$\hat{\mathrm{S}}_j(v) = \hat{y}_j - \left[\tfrac{\hat{y}_j - \hat{y}_{j+1}}{\sqrt{h}} + h^{\frac{5}{2}}\tfrac{\breve{y}_j}{90} + \sqrt{h}\tfrac{\hat{M}_j + 2\hat{M}_{j+1}}{3} + h^{\frac{3}{2}}\tfrac{\hat{F}_j - 16\hat{F}_{j+1}}{90}\right](v - v_j)^{\frac{1}{2}}$$
$$+ M_j(v - v_j) + \left[h^{\frac{3}{2}}\tfrac{\breve{y}_j}{9} + 2\tfrac{\hat{M}_{j+1} - \hat{M}_j}{3\sqrt{h}} - 2\sqrt{h}\tfrac{\hat{F}_j + 2\hat{F}_{j+1}}{9}\right](v - v_j)^{\frac{3}{2}}$$
$$+ \tfrac{\hat{F}_j}{2}(v - v_j)^2 - \left[\sqrt{h}\tfrac{4\breve{y}_j}{15} + 4\tfrac{\hat{F}_j - \hat{F}_{j+1}}{15\sqrt{h}}\right](v - v_j)^{\frac{5}{2}} + \tfrac{\breve{y}_j}{6}(v - v_j)^3 + \mathrm{O}\left(\mathrm{h}^7\right)$$
$\forall j = 0(1)n - 1 andv \in (v_j, v_{j+1}), weget: \quad \mathrm{S}_j(v) - \widehat{\mathrm{S}}_j(v) \equiv \delta_1\ \mathrm{h}^7$
$$andhence \left\|S_j - \hat{S}_j\right\| \leq \delta_1\left(h^7\right), \quad (17)$$

From equations (10) and (11) we get:

$$\widehat{\mathrm{F}} = R^{-1}L_1\hat{\mathrm{Y}} + R^{-1}L_2\breve{\mathrm{Y}} \quad and \quad \widehat{\mathrm{M}} = L_4\hat{\mathrm{Y}} + L_5\hat{\mathrm{F}} + L_6\breve{\mathrm{Y}}$$

Let
$$\bar{Y} = L_3\hat{Y} \rightarrow \hat{\mathrm{F}} = R^{-1}L_1\hat{\mathrm{Y}} + R^{-1}L_2L_3\hat{Y} \quad (18)$$

$$\rightarrow \hat{\mathrm{M}} = L_4\hat{Y} + R^{-1}L_1L_5\hat{Y} + R^{-1}L_2L_3L_5\hat{Y} + L_3\hat{Y} \quad (19)$$

Putting (18) and (19) in (16) we get:
$\rightarrow \widehat{\mathrm{Y}} = \mathrm{F} + \mathrm{A}_1\widehat{\mathrm{Y}} + \mathrm{A}_2L_4\widehat{\mathrm{Y}} + \mathrm{A}_2R^{-1}L_1L_5\hat{\mathrm{Y}} + \mathrm{A}_2R^{-1}L_2L_3L_5\widehat{Y} + \mathrm{A}_2L_3\widehat{Y} + \mathrm{A}_3R^{-1}L_1\widehat{\mathrm{Y}} + \mathrm{A}_3R^{-1}L_2L_3$
$\widehat{Y} + \mathrm{A}_4L_3\widehat{Y}$,
$\rightarrow \left[I - \left(\mathrm{A}_1 + \mathrm{A}_2L_4 + \mathrm{A}_2R^{-1}L_1L_5 + \mathrm{A}_2R^{-1}L_2L_3L_5 + \mathrm{A}_2L_3 + \mathrm{A}_3R^{-1}L_1 + \ \mathrm{A}_3R^{-1}L_2L_3 + \mathrm{A}_4L_3\right)\right]\widehat{\mathrm{Y}}$
$= \mathrm{F}$,

$$\rightarrow \widehat{\mathrm{Y}} = \left[I - \left(\mathrm{A}_1 + \mathrm{A}_2L_4 + \mathrm{A}_2R^{-1}L_1L_5 + \mathrm{A}_2R^{-1}L_2L_3L_5 + \mathrm{A}_2L_3 + \mathrm{A}_3R^{-1}L_1 + \ \mathrm{A}_3R^{-1}L_2L_3 + \mathrm{A}_4L_3\right)\right]^{-1}\mathrm{F}, \quad (20)$$

We consider that the exact matrix form solution of equation (20) is

$$\left[I - \left(\mathrm{A}_1 + \mathrm{A}_2L_4 + \mathrm{A}_2R^{-1}L_1L_5 + \mathrm{A}_2R^{-1}L_2L_3L_5 + \mathrm{A}_2L_3 + \mathrm{A}_3R^{-1}L_1 + \mathrm{A}_3R^{-1}L_2L_3 + \ \mathrm{A}_4L_3\right)\right]\dot{\mathrm{Y}} = \mathrm{F} + \mathrm{T}. \quad (21)$$

Where $\dot{\mathrm{Y}} = (\dot{\mathrm{y}}_0, \dot{\mathrm{y}}_1, \ldots, \dot{\mathrm{y}}_\mathrm{n})^\mathrm{T}$ is column vector of the exact solution (n+1) dimensional. $\mathrm{T} = (\mathrm{t}_0, \mathrm{t}_1, \ldots, \mathrm{t}_\mathrm{n})^\mathrm{T}$ is the vector of the local truncation error.
By subtracting (20) and (21) we get:

$$\left[I - \mathrm{A}_1 - \mathrm{A}_2L_4 - \mathrm{A}_2R^{-1}L_1L_5 - \mathrm{A}_2R^{-1}L_2L_3L_5 - \mathrm{A}_2L_3 - \mathrm{A}_3R^{-1}L_1 - \ \mathrm{A}_3R^{-1}L_2L_3 - \mathrm{A}_4L_3\right]\mathrm{E} = \mathrm{T} \quad (22)$$

Where

$$\mathrm{E} = \left[I - \mathrm{A}_1 - \mathrm{A}_2L_4 - \mathrm{A}_2R^{-1}L_1L_5 - \mathrm{A}_2R^{-1}L_2L_3L_5 - \mathrm{A}_2L_3 - \mathrm{A}_3R^{-1}L_1 - \ \mathrm{A}_3R^{-1}L_2L_3 - \mathrm{A}_4L_3\right]^{-1}\ \mathrm{T} \quad (23)$$

Where $E = (e_0, e_1, \ldots, e_n)^T$ is the column vector of errors, $e_j = y_j - \hat{y}_j, y_j$ is the exact solution and $\hat{y}_j$ is the approximation solution, for $j = 0, 1, \ldots, n$.

4 CONVERGENCE ANALYSIS

In this section, we studied some important lemmas and theorems, as well as the fractional cubic spline's convergence analysis.

Lemma 1 *[22] If M is a square Matrix with $\|M\|_\infty < 1$, then the matrix $(I - M)^{-1}$ is exist. And $\left\|(I - M)^{-1}\right\|_\infty \leq \frac{1}{1 - \|M\|_\infty}$.*

Lemma 2 *The matrix*
$\left[I - \left(A_1\widehat{Y} + A_2L_4\widehat{Y} + A_2R^{-1}L_1L_5\widehat{Y} + A_2R^{-1}L_2L_3L_5\widehat{Y} + A_2L_3\widehat{Y} + \ \mathrm{A}_3R^{-1}L_1\widehat{Y} + \mathrm{A}_3R^{-1}L_2L_3\widehat{Y} + \right.\right.$
$\left.\left.\mathrm{A}_4L_3\widehat{Y}\right)\right]$ *in equation (20) is invertible, if*

$$\varepsilon\|k\|_\infty(b - a)\left(\theta + \omega_3\omega_6\frac{2\ \mathrm{h}(\pi - 2)}{\pi} + (\omega_6 + \omega_7)\,\omega_1\frac{2\ \mathrm{h}^2(\pi - 2)}{\pi} + \omega_2\omega_5\omega_6\omega_7\frac{36\pi - 72}{\mathrm{h}^3\pi} + \frac{\omega_2\omega_7 + \mathrm{h}\omega_8}{\mathrm{h}}\right) < 1.$$

for $i = 1, \ldots, n$, we have:

$\|A\|_\infty \leq \|k\|_\infty(b - a)\theta$
$\|B\|_\infty = \|C\|_\infty \leq \|k\|_\infty(b - a)\frac{2\sqrt{\theta^3}}{3}$
$\|D\|_\infty \leq \|k\|_\infty(b - a)\frac{\theta^2}{2}$,
$\|G\|_\infty = \|R\|_\infty \leq \|k\|_\infty(b - a)\frac{2\sqrt{\theta^5}}{5}$
$\|P\|_\infty \leq \|k\|_\infty(b - a)\frac{\theta^3}{3}$,

$\|Q\|_\infty=\|L\|_\infty \leq \|k\|_\infty (b-a)\frac{2\sqrt{\theta^7}}{7}$,
$\|S\|_\infty \leq \|k\|_\infty (b-a)\frac{\theta^4}{4}$,
$\|L_1\|_\infty \leq \frac{2h^2(\pi-2)}{\pi}\omega_1, \quad \|L_2\|_\infty \leq \frac{\omega_2}{\mathrm{h}}$,
$\|L_5\|_\infty \leq \frac{15(\pi-2)\omega_4}{\mathrm{h}\pi}, \quad \|L_6\|_\infty \leq \frac{(36\pi-72)\omega_5}{\mathrm{h}^2\pi}, \quad \|L_4\|_\infty \leq \frac{2\ \mathrm{h}(\pi-2)\omega_3}{\pi}$,
$\|A_1\|_\infty \leq \|k\|_\infty (x-a)\theta, \quad \|A_2\|_\infty \leq \|k\|_\infty (b-a)\omega_6$,
$\|A_3\|_\infty \leq \|k\|_\infty (x-a)\omega_7, \quad \|A_4\|_\infty \leq \|k\|_\infty (b-a)\omega_8$,
$\omega_6=\frac{\theta^2-2\sqrt{\mathrm{h}\theta^3}}{6}, \omega_7=\frac{2\sqrt{(\mathrm{h}\theta)^3}-12\sqrt{\mathrm{h}\theta^5}}{45}+\frac{\theta^3}{6}, \omega_8=\frac{6\sqrt{\mathrm{h}^3\theta^5}-\sqrt{\mathrm{h}^5\theta^3}}{135}-\frac{8\sqrt{\mathrm{h}\theta^7}}{105}+\frac{\theta^4}{4}.(24)$

From equation (21) of the matrix representation we get:
$\left(\|\mathrm{A}_1\|+\|\mathrm{A}_2\|\,\|L_4\|+\|\mathrm{A}_2\|\,\left\|R^{-1}\right\|\,\|L_1\|\,\|L_5\|+\|\mathrm{A}_2\|\,\left\|R^{-1}\right\|\,\|L_2\|\,\|L_3\|\,\|L_5\|+\|\mathrm{A}_2\|\,\|L_3\|+\right.$
$\left.\|\mathrm{A}_3\|\,\left\|R^{-1}\right\|\,\|L_1\|+\|\mathrm{A}_3\|\,\left\|R^{-1}\right\|\,L_2\,\|L_3\|+\|\mathrm{A}_4\|\,\|L_3\|\right)<1$.
Then we use lemma(1), the matrix
$\left[I-\mathrm{A}_1-\mathrm{A}_2L_4-\mathrm{A}_2R^{-1}L_1L_5-\ \mathrm{A}_2R^{-1}L_2L_3L_5-\mathrm{A}_2L_3-\mathrm{A}_3R^{-1}L_1-\mathrm{A}_3R^{-1}L_2L_3-\mathrm{A}_4L_3\right]$
is invertible, if
$\|\mathrm{A}_1+\mathrm{A}_2L_4+\mathrm{A}_2R^{-1}L_1L_5+\mathrm{A}_2R^{-1}L_2L_3L_5+\mathrm{A}_2L_3+\mathrm{A}_3R^{-1}L_1+\mathrm{A}_3R^{-1}L_2L_3+\ \mathrm{A}_4L_3\|_\infty<1$, we get:
$\varepsilon\|k\|_\infty(b-a)\left(\theta+\omega_3\omega_6\frac{2\ \mathrm{h}(\pi-2)}{\pi}+(\omega_6+\omega_7)\,\omega_1\frac{2h^2(\pi-2)}{\pi}+\ \omega_2\omega_5\omega_6\omega_7\frac{36\pi-72}{h^3\pi}+\frac{\omega_2\omega_7+h\omega_8}{h}\right)<1.$

Theorem 1 *Let $y(x)\in C^8(I), k(t,x)\in C^8(I\times I)$ such that*
$\varepsilon\|k\|_\infty(x-a)\left(\theta+\omega_3\omega_6\frac{2\ \mathrm{h}(\pi-2)}{\pi}+(\omega_6+\omega_7)\,\omega_1\frac{2h^2(\pi-2)}{\pi}+\ \omega_2\omega_5\omega_6\omega_7\frac{36\pi-72}{h^3\pi}+\frac{\omega_2\omega_7+h\omega_8}{h}\right)<1$
Then develop a unique approximating solution and the error obtained, equation (23), satisfies $\|E\|\equiv O\left(h^5\right), \Omega:=[a,x]$ *Where x,θ,h,ω_i for $i=1,\ldots,8$ are constants. satisfies* $\|E\|\equiv O\left(h^5\right), \Omega:=[a,x]$ *Where x,θ,h,ω_i for $i=1,\ldots,8$ are constants.*

From (23) and lemma (24), we get $\|\mathrm{E}\|\leq$
$$\frac{\|\mathrm{T}\|}{1-\left(\|\mathrm{A}_1\|+\mathrm{A}_2\|L_4\|+\|\mathrm{A}_2\|\|L_3\|+\|\mathrm{A}_4\|\|L_3\|+\left\|R^{-1}\right\|(\|\mathrm{A}_2\|\|\mathrm{L}_2\|\|L_5\|+\|\mathrm{A}_2\|\|L_2\|\|L_3\|\|L_5\|+\|\mathrm{A}_3\|\|L_1\|+\|\mathrm{A}_3\|\|L_2\|\|L_3\|)\right.}(25)$$
By substituting $\|\mathrm{T}\|\leq\mu\mathrm{h}^5\delta_2$ and (24) in (25) we obtained: $\|\mathrm{E}\|\leq\mathrm{O}\left(\mathrm{h}^5\right)$.
Where $\delta_2=\max_{x_i\leq v\leq x_{i+1}}y^{(7)}(v), \mu$ and v are constants.
Therefore, we have:
$$\|y-S\|_\infty\leq\delta_3\left(h^5\right). \tag{26}$$
Where $\delta_3=\frac{\mu\delta_1}{\delta_4}$,
$\delta_4=1-\varepsilon\|k\|_\infty(x-a)\left(\theta+\omega_3\omega_6\frac{2h(\pi-2)}{\pi}+(\omega_6+\omega_7)\,\omega_1\frac{2h^2(\pi-2)}{\pi}+\omega_2\omega_5\omega_6\omega_7\frac{36\pi-72}{h^3\pi}+\right.$
$\left.\frac{\omega_2\omega_7+h\omega_8}{h}\right)<1$
Since $\|y-\hat{S}\|_\infty\leq\|y-S\|_\infty+\|S-\hat{S}\|_\infty$, thus applying (26) and (17), we get: $\|y-\hat{S}\|_\infty\leq\delta_3\left(\ \mathrm{h}^5\right)+\delta_1\left(\ \mathrm{h}^7\right)\equiv\delta\left(h^5\right)$, where $\delta=\delta_1+\delta_3$.

5 NUMERICAL RESULTS

The proposed technique is applied to three fractional Volterra integral equations, and a comparison of FCS method and the exact solutions to show the correctness and effectiveness of the proposed approach with figures, all of the calculations are obtained by the Python software. The maximum absolute error is computed, and the results are compared to well-known values in [1],[2] and [4].

Example 1 *[2] Consider the fractional V-FIE equation* $\mathrm{y(t)}=f(t)+\frac{t}{\Gamma\left(\frac{3}{2}\right)}\int_0^t(t-s)^{\frac{1}{2}}y^2(s)ds+\frac{t^2}{\Gamma\left(\frac{3}{2}\right)}\int_0^1(1-s)^{\frac{1}{2}}y(s)ds, t\in[0,1]$, *where f(t)=*$t^3-\frac{64t^2\left(143+64t^{13/2}\right)}{45045\sqrt{\pi}}$, *and the exact solution is $y(t)=t^3$.*

h	$\|E\|$ of FCS	Best $\|E\|$ in [2]
0.25	$5.306150311984864\times10^{-4}$	4.015179×10^{-3}
0.0625	$2.2544240725426358\times10^{-4}$	2.603753×10^{-4}
0.03125	$3.659526172419494\times10^{-5}$	6.671122×10^{-5}
0.015625	$5.052937917680365\times10^{-6}$	1.831474×10^{-5}
0.0078125	$3.347020452462105\times10^{-7}$	6.260511×10^{-6}

comparison of the FCS's solution and method in [2], of example 1.

Example 2 *[1],[4] Consider the FVIE* $\mathrm{y(t)}=f(t)+\frac{0.01}{\Gamma\left(\frac{1}{2}\right)}\mathrm{t}^{\frac{5}{2}}\int_0^t(t-s)^{-\frac{1}{2}}\mathrm{y}(s)ds$, *where,* $\mathrm{f(t)}=\frac{\sqrt{\pi}}{\sqrt{(1+\mathrm{t})^3}}+0.02\frac{t^3}{1+\mathrm{t}}$ *and the exact solution is* $y(t)=\frac{\sqrt{\pi}}{\sqrt{(1+t)^3}}$.

h	$\|E\|$ of FCS	Best $\|E\|$ in [1]	Best $\|E\|$ in [4]
0.2	$3.290319631998884\times10^{-5}$	4.226282×10^{-4}	0.1
0.1	$2.559823498726743\ 10^{-6}$	4.446827×10^{-5}	0.001

comparison of the FCS's solution and method in [1] and [4], of example 2.

Example 3 *[4] Consider the FVIE* $\mathrm{y(t)}=f(t)+\frac{1}{27\Gamma\left(\frac{2}{3}\right)}\int_0^{\mathrm{t}}\frac{sy(s)}{\sqrt{(t-s)^3}}ds$, *where* $f(\mathrm{t})=\Gamma\left(\frac{2}{3}\right)\mathrm{t}-\frac{t^{\frac{8}{3}}}{40'}$, *and the exact solution is* $y(t)=\Gamma\left(\frac{2}{3}\right)\mathrm{t}$.

h	$\|E\|$ of FCS	Best $\|E\|$ in [4]
0.2	0.0022799512383703535	0.003
0.05	$4.336918367485454\times10^{-5}$	$>10^{-5}$

comparison of the FCS's solution and method in [4], of example 3.

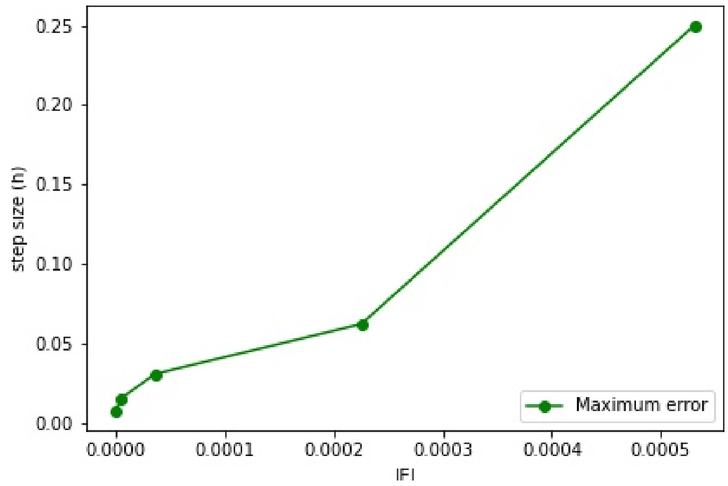

Figure 1: Maximum errors of example 1.

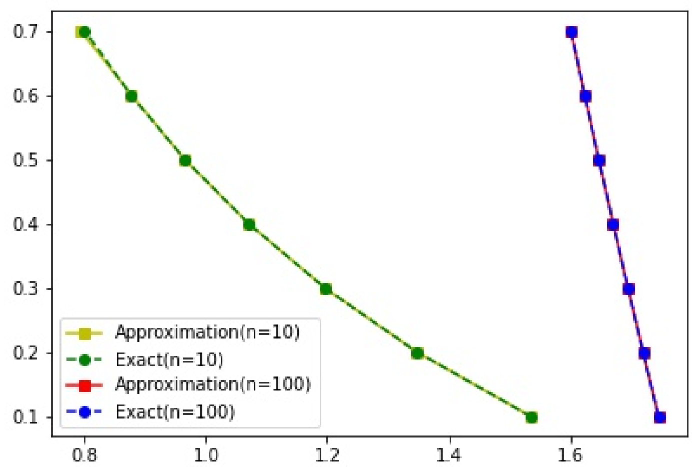

Figure 2: Comparison of FCS solutions with exact solution of example 2.

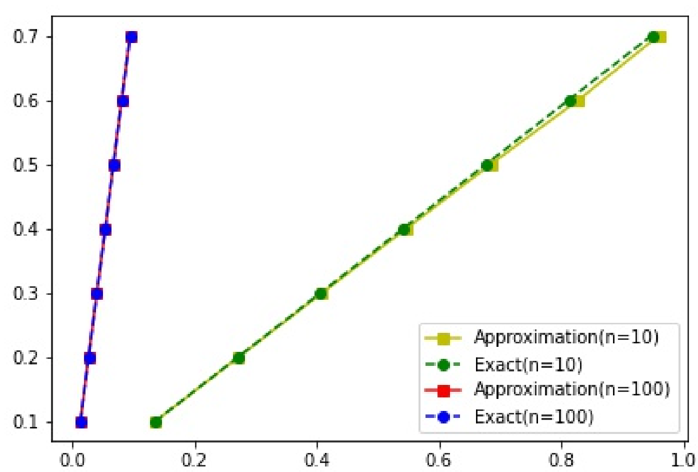

Figure 3: Comparison of FCS solutions with exact solution of example 3.

6 CONCLUSION

In this paper, according to the fractional cubic spline, a method for approximating FVIE is presented. The proposed approach for solving fractional Volterra-integral equations is simple and effective.We calculated the proposed method's fifth-order convergence, and the computational illustration was found to be comparable with theoretical expectations. The current technique was developed using three different examples by Python program, and the outcomes were compared to the exact solution. In 2D, for various points, and Our method shows more accuracy compared to the existing method in [1],[2] and [4].

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