COVID-19 Mathematical Study with Environmental Reservoirs and Three General Functions for Transmissions

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Abstract

In this paper, the ongoing new coronavirus (COVID-19) epidemic is being investigated using a mathematical model. The model depicts the dynamics of infection with several transmission pathways and general infection functions plus it highlights the significance of the environment as a reservoir for the disease's propagation and dissemination. We have studied the qualitative behavior of the proposed model representing a system of fractional order differential equations. Under a set of conditions on the general functions and the parameters, we have proven the global asymptotic stability of all equilibria by using Lyapunov method and LaSalle's invariance principle. We also carried some numerical results to confirm the analytical results we obtained.

Keywords: COVID-19 pandemic; Global stability; Fractional-order differential equations; Environmental reservoir; Basic reproduction number; Numerical simulations.

1 Introduction

Coronavirus disease 2019 (COVID-19) is an infectious disease caused by severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2). In the beginning, it was isolated December 2019 in Wuhan, China from some people who have pneumonia connected to the cluster of acute respiratory illness cases, and since then, it has spread over the world, culminating in the pandemic of 2020. When an infected individual cough, sneezes or exhales, the virus that causes COVID-19 is mostly transferred by droplets. These droplets are too heavy to float in the air and fall to the ground or other surfaces. COVID-19 affects different groups of people of different ages, but it is more prevalent in the less immune groups and those with chronic diseases. Also, most of the infected people have mild to moderate symptoms and recover without going to the hospital.

Because of thousands of confirmed infections and thousands of fatalities throughout the world, the COVID-19 pandemic is now regarded as the greatest global threat. In the weekly epidemiological update-8 December 2020 received by World Health Organization from national authorities, COVID-19 cases have remained stable at over 4 million new cases, but new fatalities have risen to around 73 000. Since the beginning of the pandemic, there have been about 65.8 million recorded illnesses and 1.5 million fatalities worldwide [1]. Many countries have followed China's lead and imposed curfews, closed borders, and halted all normal daily operations, such as school and workplace closures. The use of mathematical models to study social distancing techniques has proved their efficiency in limiting the spread of COVID-19 infection.

Infectious disease transmission dynamics mathematical models are increasingly widely used. Models like this are useful for quantifying potential infectious disease prevention and mitigation techniques. For infectious diseases, there are a variety of models available, ranging from the very simple SIR model to more complicated ideas. Many researchers in the scientific community have conducted multidisciplinary investigations using various mathematical models to understand the virus spread pattern (see [2], [3], [4], [5]). However, a comprehensive approach of mathematical instrumentalization models in the characterization of the COVID-19 growth curve and its containment strategies are remaining drastically understudied in current literature. We supplement existing studies on the topic by extending SIR models and relying on conclusions gained from extant studies using SIR model extensions [6], [7], [8], [9]. These models were only concerned with direct human-to-human transmission [10], [11]. In addition, the role of the environment in COVID-19 transmission has been largely ignored in contemporary clinical and theoretical investigations and is seldom studied in modeling and simulation. As a result, our understanding of COVID-19 transmission mechanism and epidemiological features is still restricted. COVID-19 may be transferred between humans through direct touch, and both symptomatic and asymptomatic persons can infect others [12], [13], [14], [15]. Furthermore, the environment to human hosts indirect transmission is a highly likely method for propagate of coronavirus. Coughing and sneezing of infected people released respiratory droplets containing the coronavirus, and the majority of these droplets land on neighbouring surfaces and items. By contacting infected surfaces or items and then touching their faces, other people might get the virus. Meanwhile, coronaviruses generated by sick persons might float in the air as aerosols and be inhaled by those who pass by. Such environment-to-human transmission channels, and the effectiveness of such a form of transmission, are primarily dependent on the coronavirus's capacity to live and remain in the environment. The viability and duration of SARS-CoV in the environment were verified in [12] and [16]. New coronavirus (SARS-CoV-2) can stay alive and infectious in aerosols for hours and on surfaces for days, according to experimental research published in March 2020, indicating a high likelihood and large danger of environmental transmission. C. Yang and J. Wang [17] studied the effect of the environmental reservoirs by incorporating it into a model represented by a system of ordinary differential equations, however using fractional derivative to model a real process has piqued the interest of a number of authors from different fields (see e.g. [18], [19]) as fractional derivative is an ideal tool for describing real-world phenomena with memory, such as most biological systems.

The manuscript is structured as follows: In Section 2, we introduce fractional order differential equations preliminaries. In section 3, a new fractional-order COVID-19 mathematical model is proposed and takes into account the influence of environmental reservoirs with three general functions of the transition, which are susceptible-exposed, susceptible-infected and susceptible-environmental transmissions, furthermore qualitative analysis of the model is introduced in subsections 3.1 and 3.2, also in Section 3.2, we calculate the basic reproduction number $\Re_0$ for the system. Section 4 deals local asymptotical stability for the disease-free and endemic equilibriums in terms of $\Re_0$. Global stability of both disease-free equilibrium and chronic equilibrium point is given in Section 5. The effect of parameters on the system dynamics illustrated in Section 6 of numerical simulations. Section 7 brings us to a close with conclusions and discussion.

2 Preliminaries

In this section, we introduce the basic definitions and lemma of fraction calculus which is an important tool in modeling processes of biological systems, and has the ability to provide an exact description not only of the current state of the disease but also of all its historical states.

Definition 1. Define a function $f : [0, \infty) \longrightarrow R$ then fractional integral of it of order $\alpha \in (0, 1]$ given as follows:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx,$$

where $\Gamma(.)$ is the gamma function [20], and the Caputo fractional derivative of order α is given by:

$$D^{\alpha} f(t) = I^{n-\alpha} D^n f(t),$$

where $n - 1 < \alpha \leq n$ and f(t) is a continuous function [21]. In particular, when $0 < \alpha \leq 1$, one has

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(x)}{(t-x)^{\alpha}} dx.$$

For more properties of the fraction order derivatives (see e.g. [22] and [23]).

Lemma 1. Consider a fraction order system

$$D^{\alpha}(x) = f(x), \qquad x(0) = x_0,$$

with $0 < \alpha \leq 1$ and $x \in \{R^n\}$, evaluate the equilibrium points of the system by let $D^{\alpha}(x) = 0$ then this points are locally asymptotically stable if all eigenvalues λ_i of the Jacobian matrix of the system evaluated at the equilibrium points satisfy the following conditions: [24]

$$|\arg(\lambda_i)| > \alpha \frac{\pi}{2}. \tag{1}$$

3 Proposed COVID-19 fractional-order model

We will institute new COVID-19 model as a system of fractional order differential equations that includes five elements S, E, I, R and V represent the concentrations of the susceptible, exposed, infected and recovered individuals, respectively and V is the concentration of the coronavirus in the environment as follows:

$$D^{\alpha} S(t) = \lambda - \mu S - L_1(S, V) - L_2(S, E) - L_3(S, I), \tag{2}$$

$$D^{\alpha} E(t) = L_1(S, V) + L_2(S, E) + L_3(S, I) - (\delta + \mu) E, \tag{3}$$

$$D^{\alpha} I(t) = \delta E - (\omega + \gamma + \mu) I, \tag{4}$$

$$D^{\alpha} R(t) = \gamma I - \mu R, \tag{5}$$

$$D^{\alpha} V(t) = \rho_1 E + \rho_2 I - \sigma V. \tag{6}$$

The parameter λ is the population influx, μ is the pace at which human hosts die naturally, δ^{-1} is the time between infection and emergence of symptoms (incubation period), ω represents the death rate as a result of disease, γ symbolizes the recovery from infection rate, ρ_1 and ρ_2 denote the contribution of exposed and infected individuals with coronavirus to the environmental reservoir rates, respectively, and σ denote the pace at which the virus is removed from the environment. The $SEIRV$ model scheme are shown in Figure 1. For simplicity, we suggest the following conditions:

(C1) $L_k(S,W) > 0$ and $L_k(S,0) = L_k(0,W) = 0$ for all $S > 0, W > 0$ where $k = 1, 2, 3$.

(C2) $\frac{\partial L_k(S,W)}{\partial S} > 0$, $\frac{\partial L_k(S,W)}{\partial W} > 0$, $\frac{\partial L_k(S,W)}{\partial W}|_{W=0} > 0$, and $\frac{d}{dS}\left(\frac{\partial L_k(S,W)}{\partial W}|_{W=0}\right)$ for all $S > 0, W > 0$ where $k = 1, 2, 3$.

(C3) $\frac{\partial}{\partial V}\left(\frac{L_1(S,V)}{V}\right) \le 0$, $\frac{\partial}{\partial E}\left(\frac{L_2(S,E)}{E}\right) \le 0$ and $\frac{\partial}{\partial I}\left(\frac{L_3(S,I)}{I}\right) \le 0$ for all $S, E, I, V > 0$.

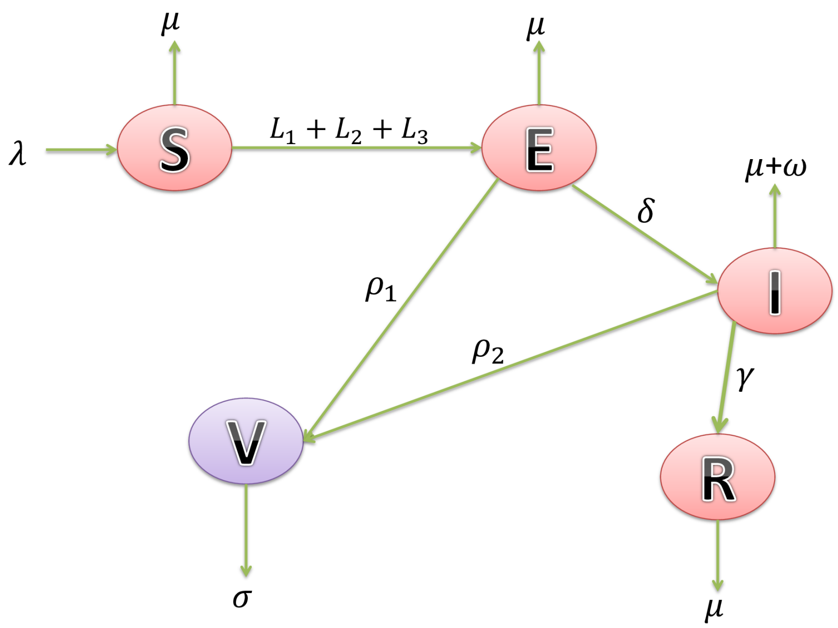

Figure 1: $SEIRV$ model scheme.

3.1 Nonnegativity and boundedness

Proposition 1. Suppose that for system (2)-(6) the conditions **C1- C3** are satisfied. Then the compact set

$$\Psi = \{(S, E, I, R, V) \in R^5_{\ge 0}, 0 \le S(t), E(t) \le \Gamma_1, 0 \le I(t), R(t) \le \Gamma_2, 0 \le V(t) \le \Gamma_3\}.$$

is positively invariant.

Proof. We have

$$\begin{aligned} D^\alpha S(t)|_{S=0} &= \lambda > 0, \\ D^\alpha E(t)|_{E=0} &= L_1(S,V) + L_3(S,I) \ge 0, \quad \text{for all} \quad S, I, V \ge 0 \\ D^\alpha I(t)|_{I=0} &= \delta E \ge 0, \quad \text{for} \quad E \ge 0 \\ D^\alpha R(t)|_{R=0} &= \gamma I \ge 0, \quad \text{for} \quad I \ge 0 \\ D^\alpha V(t)|_{V=0} &= \rho_1 E + \rho_2 I \ge 0, \quad \text{for} \quad E, I \ge 0. \end{aligned}$$

To prove the boundedness of the state variables we let

$$\Upsilon = S + E + \frac{\rho_2\delta}{\rho_2\delta + \rho_1\mu} I + \frac{\rho_2\delta}{\rho_2\delta + \rho_1\mu} R + \frac{\mu\delta}{\rho_2\delta + \rho_1\mu} V.$$

Let $\rho_2\delta + \rho_1\mu = \mathbb{B}$, then

$$
\begin{aligned}
D^\alpha \Upsilon &= D^\alpha S + D^\alpha E + \frac{\rho_2\delta}{\mathbb{B}} D^\alpha I + \frac{\rho_2\delta}{\mathbb{B}} D^\alpha R + \frac{\mu\delta}{\mathbb{B}} D^\alpha V \\
&= \lambda - \mu S - (\delta+\mu)E + \frac{\rho_2\delta}{\mathbb{B}}\Big[\delta E - (\omega+\gamma+\mu)I\Big] + \frac{\rho_2\delta}{\mathbb{B}}\Big[\gamma I - \mu R\Big] + \frac{\mu\delta}{\mathbb{B}}\Big[\rho_1 E + \rho_2 I - \sigma V\Big] \\
&= \lambda - \mu S - \delta E - \mu E + \frac{\rho_2\delta^2}{\mathbb{B}} E - \frac{\rho_2\delta[\omega+\mu]I}{\mathbb{B}} - \frac{\rho_2\mu\delta}{\mathbb{B}} R + \frac{\rho_1\mu\delta}{\mathbb{B}} E + \frac{\rho_2\mu\delta}{\mathbb{B}} I - \frac{\mu\delta\sigma}{\mathbb{B}} V \\
&= \lambda - \mu S - \mu E - \frac{\rho_2\delta\omega}{\mathbb{B}} I - \frac{\rho_2\mu\delta}{\mathbb{B}} R - \frac{\mu\delta\sigma}{\mathbb{B}} V \\
&= \lambda - \tau\left[S + E + \frac{\rho_2\delta}{\mathbb{B}} I + \frac{\rho_2\delta}{\mathbb{B}} R + \frac{\mu\delta}{\mathbb{B}} V\right] \\
&= \lambda - \tau\Upsilon,
\end{aligned}
$$

where, $\tau = \min\{\omega, \mu, \sigma\}$. Then

$$\Upsilon(t) \le e^{-\tau t}\left(\Upsilon(0) - \frac{\lambda}{\tau}\right) + \frac{\lambda}{\tau}.$$

This yields, $0 \le \Upsilon(t) \le \Gamma_1$ for all $t \ge 0$ if $\Upsilon(0) \le \Gamma_1$, where $\Gamma_1 = \frac{\lambda}{\tau}$. It follows that $0 \le S(t), E(t) \le \Gamma_1, 0 \le I(t), R(t) \le \Gamma_2$ and $0 \le V(t) \le \Gamma_3$ for all $t \ge 0$ if $S(0) + E(0) + \frac{\rho_2\delta}{\mathbb{B}} I(0) + \frac{\rho_2\delta}{\mathbb{B}} R(0) + \frac{\mu\delta}{\mathbb{B}} V(0) \le \Gamma_1$, where $\Gamma_2 = \frac{\lambda\mathbb{B}}{\rho_2\delta\tau}$, $\Gamma_3 = \frac{\lambda\mathbb{B}}{\mu\delta\tau}$ and $\mathbb{B} = \rho_2\delta + \rho_1\mu$. This prove the boundedness of S, E, I, R and V.

3.2 Steady states

This section researches the steady states of the model and extract the criteria for its existence. It is the positive solutions of the next equations

$$0 = \lambda - \mu S - L_1(S,V) - L_2(S,E) - L_3(S,I), \tag{7}$$

$$0 = L_1(S,V) + L_2(S,E) + L_3(S,I) - (\delta+\mu)E, \tag{8}$$

$$0 = \delta E - (\omega+\gamma+\mu)I, \tag{9}$$

$$0 = \gamma I - \mu R, \tag{10}$$

$$0 = \rho_1 E + \rho_2 I - \sigma V. \tag{11}$$

Model (2)-(6) has a disease-free steady state $Q_0 = (S_0, 0, 0, 0, 0)$ which is always exist and $S_0 = \frac{\lambda}{\mu}$. The other positive steady state is evaluated as follows:

From equation (7)-(11), we obtain

$$\lambda - \mu S = L_1(S,V) + L_2(S,E) + L_3(S,I) = (\delta+\mu)E = \frac{A(\delta+\mu)}{\delta} I = \frac{\mu A(\delta+\mu)}{\gamma\delta} R = \frac{\sigma A(\delta+\mu)}{\rho_2\delta + \rho_1 A} V,$$

where $A = \omega + \gamma + \mu$.

From the last equation, we get

$$V = h_1(S), \quad E = h_2(S), \quad I = h_3(S), \quad R = h_4(S), \tag{12}$$

where

$$h_1(S) = \frac{(\lambda-\mu S)(\rho_2\delta+\rho_1 A)}{\sigma A(\delta+\mu)}, \quad h_2(S) = \frac{(\lambda-\mu S)}{\delta+\mu}, \quad h_3(S) = \frac{\delta(\lambda-\mu S)}{A(\delta+\mu)}, \quad h_4(S) = \frac{\gamma\delta(\lambda-\mu S)}{\mu A(\delta+\mu)}. \tag{13}$$

It is clear that, $h_j(S) > 0$ for $S \in [0, S_0)$ and $h_j(S_0) = 0$, $j = 1,2,3,4$. Let

$$F_1(S) = L_1(S, h_1(S)) + L_2(S, h_2(S)) + L_3(S, h_3(S)) - \frac{\sigma A(\delta+\mu)}{\rho_2\delta + \rho_1 A} h_1(S).$$

From Condition **C1**, we have

$$F_1(0) = -\frac{\sigma A(\delta+\mu)}{\rho_2\delta+\rho_1 A} h_1(0) = -\lambda < 0, \qquad F_1(S_0) = 0.$$

Moreover,

$$F_1^{'}(S) = \frac{\partial L_1}{\partial S} + h_1^{'}(S)\frac{\partial L_1}{\partial V} + \frac{\partial L_2}{\partial S} + h_2^{'}(S)\frac{\partial L_2}{\partial E} + \frac{\partial L_3}{\partial S} + h_3^{'}(S)\frac{\partial L_3}{\partial I} - \frac{\sigma(\delta+\mu)A}{\rho_2\delta+\rho_1 A}h_1^{'}(S),$$

$$\begin{aligned} F_1^{'}(S_0) =& \frac{\partial L_1(S_0,0)}{\partial S} + h_1^{'}(S_0)\frac{\partial L_1(S_0,0)}{\partial V} + \frac{\partial L_2(S_0,0)}{\partial S} + h_2^{'}(S_0)\frac{\partial L_2(S_0,0)}{\partial E} + \frac{\partial L_3(S_0,0)}{\partial S} \\ &+ h_3^{'}(S_0)\frac{\partial L_3(S_0,0)}{\partial I} - \frac{\sigma(\delta+\mu)A}{\rho_2\delta+\rho_1 A}h_1^{'}(S_0). \end{aligned}$$

Condition **C2** implies that $\frac{\partial L_k(S_0,0)}{\partial S} = 0$, $k = 1,2,3$, then

$$F_1^{'}(S_0) = h_1^{'}(S_0)\frac{\partial L_1(S_0,0)}{\partial V} + h_2^{'}(S_0)\frac{\partial L_2(S_0,0)}{\partial E} + h_3^{'}(S_0)\frac{\partial L_3(S_0,0)}{\partial I} - \frac{\sigma A(\delta+\mu)}{\rho_2\delta+\rho_1 A}h_1^{'}(S_0).$$

From equation (13), we obtain

$$F_1^{'}(S_0) = -\mu\left[\frac{(\rho_2\delta+\rho_1 A)}{\sigma A(\delta+\mu)}\frac{\partial L_1(S_0,0)}{\partial V} + \frac{1}{\delta+\mu}\frac{\partial L_2(S_0,0)}{\partial E} + \frac{\delta}{A(\delta+\mu)}\frac{\partial L_3(S_0,0)}{\partial I} - 1\right].$$

If $\frac{(\rho_2\delta+\rho_1 A)}{\sigma A(\delta+\mu)}\frac{\partial L_1(S_0,0)}{\partial V} + \frac{1}{\delta+\mu}\frac{\partial L_2(S_0,0)}{\partial E} + \frac{\delta}{A(\delta+\mu)}\frac{\partial L_3(S_0,0)}{\partial I} > 1$, then $F_1^{'}(S_0) < 0$ and there exists $S^* \in (0, S_0)$ as well as $F_1(S^*) = 0$. From equations (12) and (13), we get

$$V^* = \frac{(\lambda-\mu S^*)(\rho_2\delta+\rho_1 A)}{\sigma A(\delta+\mu)}, \qquad E^* = \frac{(\lambda-\mu S^*)}{\delta+\mu}, \qquad I^* = \frac{\delta(\lambda-\mu S^*)}{A(\delta+\mu)}, \qquad R^* = \frac{\gamma\delta(\lambda-\mu S^*)}{\mu A(\delta+\mu)}. \tag{14}$$

It follows that system (2)-(6) has an endemic steady state $Q_1 = (S^*, E^*, I^*, R^*, V^*)$ if

$$\frac{(\rho_2\delta+\rho_1 A)}{\sigma A(\delta+\mu)}\frac{\partial L_1(S_0,0)}{\partial V} + \frac{1}{\delta+\mu}\frac{\partial L_2(S_0,0)}{\partial E} + \frac{\delta}{A(\delta+\mu)}\frac{\partial L_3(S_0,0)}{\partial I} > 1.$$

Now, we compute the basic reproduction number $\Re_0$ for our model by using next generation matrix method. Let $X = (E, I, V)^T$, then system (2)-(6) can be written as:

$$D^\alpha X = \aleph(X) - \hbar(X),$$

where

$$\aleph(X) = \begin{pmatrix} L_1(S,V) + L_2(S,E) + L_3(S,I) \\ 0 \\ 0 \end{pmatrix}, \qquad \hbar(X) = \begin{pmatrix} (\delta+\mu)E \\ -\delta E + AI \\ -\rho_1 E - \rho_2 I + \sigma V \end{pmatrix}.$$

Jacobian matrices of $\aleph$ and $\hbar$ at Q_0 are

$$\mathcal{F} = \begin{pmatrix} \frac{\partial L_2(S_0,0)}{\partial E} & \frac{\partial L_3(S_0,0)}{\partial I} & \frac{\partial L_1(S_0,0)}{\partial V} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} (\delta+\mu) & 0 & 0 \\ -\delta & A & 0 \\ -\rho_1 & -\rho_2 & \sigma \end{pmatrix}.$$

Then, the next generation matrix is

$$\mathcal{F}\mathcal{V}^{-1} = \begin{pmatrix} \frac{1}{\delta+\mu}\frac{\partial L_2(S_0,0)}{\partial E} + \frac{\delta}{A(\delta+\mu)}\frac{\partial L_3(S_0,0)}{\partial I} + \frac{(\rho_2\delta+\rho_1 A)}{\sigma A(\delta+\mu)}\frac{\partial L_1(S_0,0)}{\partial V} & \frac{1}{A}\frac{\partial L_3(S_0,0)}{\partial I} + \frac{\rho_2}{\sigma A}\frac{\partial L_1(S_0,0)}{\partial V} & \frac{1}{\sigma}\frac{\partial L_1(S_0,0)}{\partial V} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The basic reproduction number of system (2)-(6) is the spectral radius of $(\mathcal{F}\mathcal{V}^{-1})$ which is given as:

$$\Re_0 = \frac{(\rho_2\delta+\rho_1 A)}{\sigma(\delta+\mu)A}\frac{\partial L_1(S_0,0)}{\partial V} + \frac{1}{\delta+\mu}\frac{\partial L_2(S_0,0)}{\partial E} + \frac{\delta}{(\delta+\mu)A}\frac{\partial L_3(S_0,0)}{\partial I} = \Re_{01} + \Re_{02} + \Re_{03}.$$

Based on the above review, we have the next outcome:

Lemma 2. Suppose that for the system (2)-(6) the conditions **C1-C3** are satisfied. Then there exists basic reproduction number $\Re_0$ satisfies that

(i) if $\Re_0 \leq 1$; then there exists only one steady state Q_0,

(ii) if $\Re_0 > 1$; then there exist two steady states Q_0 and Q_1.

4 Local Stability

This section discusses local stability of the proposed model (2)-(6). The fourth equation of system (2)-(6) is independent from the rest of the system equations, so we can reduce system (2)-(6) to the following subsystem for ease of analysis.

$$D^\alpha S(t) = \lambda - \mu S - L_1(S,V) - L_2(S,E) - L_3(S,I), \tag{15}$$

$$D^\alpha E(t) = L_1(S,V) + L_2(S,E) + L_3(S,I) - (\delta+\mu)E, \tag{16}$$

$$D^\alpha I(t) = \delta E - AI, \tag{17}$$

$$D^\alpha V(t) = \rho_1 E + \rho_2 I - \sigma V. \tag{18}$$

The Jacobian matrix of system (15)-(18) at a point $Q = (S,E,I,V)$ is

$$J(Q) = \begin{pmatrix} -\mu - \frac{\partial L_1(S,V)}{\partial S} - \frac{\partial L_2(S,E)}{\partial S} - \frac{\partial L_3(S,I)}{\partial S} & -\frac{\partial L_2(S,E)}{\partial E} & -\frac{\partial L_3(S,I)}{\partial I} & -\frac{\partial L_1(S,V)}{\partial V} \\ \frac{\partial L_1(S,V)}{\partial S} + \frac{\partial L_2(S,E)}{\partial S} + \frac{\partial L_3(S,I)}{\partial S} & \frac{\partial L_2(S,E)}{\partial E} - (\delta+\mu) & \frac{\partial L_3(S,I)}{\partial I} & \frac{\partial L_1(S,V)}{\partial V} \\ 0 & \delta & -A & 0 \\ 0 & \rho_1 & \rho_2 & -\sigma \end{pmatrix}. \tag{19}$$

Theorem 1. If $\Re_0 < 1$, then the disease-free equilibrium point $Q_0 = (S_0,0,0,0)$ of system (15)-(18) is locally asymptotically stable if the condition $A > \sigma$ hold.

Proof:

The Jacobian matrix (19) at $Q_0 = (S_0,0,0,0)$ takes the following form

$$J(Q_0) = \begin{pmatrix} -\mu & -\frac{\partial L_2(S_0,0)}{\partial E} & -\frac{\partial L_3(S_0,0)}{\partial I} & -\frac{\partial L_1(S_0,0)}{\partial V} \\ 0 & \frac{\partial L_2(S_0,0)}{\partial E} - (\delta+\mu) & \frac{\partial L_3(S_0,0)}{\partial I} & \frac{\partial L_1(S_0,0)}{\partial V} \\ 0 & \delta & -A & 0 \\ 0 & \rho_1 & \rho_2 & -\sigma \end{pmatrix}.$$

The disease-free equilibrium point Q_0 is locally asymptotically stable if all eigenvalues $\xi_i, i = 1,2,3,4$ of $J(Q_0)$ satisfy the condition given in (1). These eigenvalues are the roots of the characteristic equation of $J(Q_0)$ which is given from $det(J(Q_0) - \xi I) = 0$, as follows:

$$|J(Q_0) - \xi I| = (\mu + \xi) \begin{vmatrix} -c_{11} - \xi & c_{12} & c_{13} \\ \delta & -A - \xi & 0 \\ \rho_1 & \rho_2 & -\sigma - \xi \end{vmatrix} = 0,$$

where,

$$\begin{aligned} c_{11} &= (\delta + \mu) - \frac{\partial L_2(S_0, 0)}{\partial E} > 0, \quad if \quad \Re_0 < 1, \\ c_{12} &= \frac{\partial L_3(S_0, 0)}{\partial I} > 0, \qquad c_{13} = \frac{\partial L_1(S_0, 0)}{\partial V} > 0. \end{aligned} \tag{20}$$

Hence,

$$det(J(Q_0) - \xi I) := (\xi + \mu)(\xi^3 + F\xi^2 + G\xi + H) = 0. \tag{21}$$

Clearly, one of the roots of $J(Q_0)$ is $-\mu$ which is negative. The remaining roots can be obtained from the following equation

$$\Phi(\xi) := \xi^3 + F\xi^2 + G\xi + H = 0, \tag{22}$$

where,

$$\begin{aligned} F &= \sigma + A + c_{11}, \\ G &= A(c_{11} + \sigma) + \sigma c_{11} - \delta c_{12} - \rho_1 c_{13} \\ &= \sigma A + \sigma c_{11} + \frac{\rho_2 \delta}{\sigma} c_{13} + \frac{\rho_1}{\sigma} c_{13}(A - \sigma) + A(\delta + \mu)(1 - \Re_0) \\ H &= \sigma A c_{11} - \delta \sigma c_{12} - c_{13}(\delta \rho_2 + \rho_1 A) \\ &= \sigma A(\delta + \mu)(1 - \Re_0). \end{aligned}$$

Now,

$$\begin{aligned} FG - H =& \big(\sigma + A + c_{11}\big)\left(\sigma A + \sigma c_{11} + \frac{\rho_2 \delta}{\sigma} c_{13} + \frac{\rho_1}{\sigma} c_{13}(A - \sigma) + A(\delta + \mu)(1 - \Re_0)\right) \\ & - \sigma A(\delta + \mu)(1 - \Re_0) \\ =& \big(\sigma + A + c_{11}\big)\left(\sigma A + \sigma c_{11} + \frac{\rho_2 \delta}{\sigma} c_{13} + \frac{\rho_1}{\sigma} c_{13}(A - \sigma)\right) \\ & + A(\delta + \mu)\big(A + c_{11}\big)(1 - \Re_0). \end{aligned}$$

The discriminant $D(\Phi)$ of $\Phi(\xi)$ given in (22) is:

$$D(\Phi) = \begin{pmatrix} 1 & F & G & H & 0 \\ 0 & 1 & F & G & H \\ 3 & 2F & G & 0 & 0 \\ 0 & 3 & 2F & G & 0 \\ 0 & 0 & 3 & 2F & G \end{pmatrix} = 18FGH + F^2G^2 - 4HF^3 - 4G^3 - 27H^2.$$

It is clear that $F > 0$. Also, we have $G > 0, \quad H > 0 \quad$ and $FG - H > 0, \quad$ if $\Re_0 < 1 \quad$ and $A > \sigma$. Following Ahmed et al. [25], for the fractional Routh-Hurwitz conditions, then all the eigenvalues associated to $J(Q_0)$ have negative real parts and therefore Q_0 is locally asymptotically stable if $D(\Phi) > 0 \;\; for \;\; 0 < \alpha \leq 1$. This ends the proof. ■

Now, we are analysing the stability of the endemic steady state Q_1 of the model (15)-(18). The Jacobian matrix (19), calculated at the endemic steady states Q_1, is shown as below.

$$J(Q_1) = \begin{pmatrix} -\mu - \frac{\partial L_1(S^*,V^*)}{\partial S} - \frac{\partial L_2(S^*,E^*)}{\partial S} - \frac{\partial L_3(S^*,I^*)}{\partial S} & -\frac{\partial L_2(S^*,E^*)}{\partial E} & -\frac{\partial L_3(S^*,I^*)}{\partial I} & -\frac{\partial L_1(S^*,V^*)}{\partial V} \\ \frac{\partial L_1(S^*,V^*)}{\partial S} + \frac{\partial L_2(S^*,E^*)}{\partial S} + \frac{\partial L_3(S^*,I^*)}{\partial S} & \frac{\partial L_2(S^*,E^*)}{\partial E} - (\delta+\mu) & \frac{\partial L_3(S^*,I^*)}{\partial I} & \frac{\partial L_1(S^*,V^*)}{\partial V} \\ 0 & \delta & -A & 0 \\ 0 & \rho_1 & \rho_2 & -\sigma \end{pmatrix}$$

$$\sim \begin{pmatrix} -\mu & -(\delta+\mu) & 0 & 0 \\ 0 & -d_{11} & d_{12} & d_{13} \\ 0 & \delta & -A & 0 \\ 0 & \rho_1 & \rho_2 & -\sigma \end{pmatrix}, \tag{23}$$

where,

$$\begin{aligned} d_{11} &= \Big(\frac{\delta+\mu}{\mu}\Big)\frac{\partial}{\partial S}\Big(L_1(S^*,V^*) + L_2(S^*,E^*) + L_3(S^*,I^*)\Big) - \frac{\partial L_2(S^*,E^*)}{\partial E} + (\delta+\mu), \\ d_{12} &= \frac{\partial L_3(S^*,I^*)}{\partial I}, \qquad d_{13} = \frac{\partial L_1(S^*,V^*)}{\partial V}. \end{aligned} \tag{24}$$

Then the characteristic equation of $J(Q_1)$ is

$$(\xi+\mu)(\xi^3 + L\xi^2 + M\xi + N) = 0. \tag{25}$$

One of the roots is obviously negative which is $-\mu$. The rest roots can be extracted from the equation:

$$\Psi(\xi) := \xi^3 + L\xi^2 + M\xi + N = 0, \tag{26}$$

where,

$$L = \sigma + A + d_{11}, \qquad M = \sigma A + (\sigma + A)d_{11} - \delta d_{12} - \rho_1 d_{13}, \tag{27}$$

$$N = \sigma A d_{11} - \sigma\delta d_{12} - (\delta\rho_2 + \rho_1 A)d_{13}. \tag{28}$$

The discriminant $D(\Psi)$ of $\Psi(\xi)$ reads:

$$D(\Psi) = \begin{pmatrix} 1 & L & M & N & 0 \\ 0 & 1 & L & M & N \\ 3 & 2L & M & 0 & 0 \\ 0 & 3 & 2L & M & 0 \\ 0 & 0 & 3 & 2L & M \end{pmatrix} = 18LMN + L^2M^2 - 4NL^3 - 4M^3 - 27N^2.$$

Following Ahmed et al. [25], we have the following result.

Theorem 2. The endemic steady state Q_1 is locally asymptotically stable if one of the following requirements is met:

(i) $D(\Psi) > 0, L > 0, N > 0, LM > N$;

(ii) $D(\Psi) < 0, L \geq 0, M \geq 0, N > 0, \quad for \quad \alpha < 2/3$;

(ii) $D(\Psi) < 0, L > 0, M > 0, LM = N for \quad \alpha \in [0, 1)$.

Also, Q_1 is unstable if $\quad D(\Psi) < 0, L < 0, M < 0, \alpha > 2/3$.

5 Global Stability

In this section, we develop Lyapunov functionals to demonstrate the global asymptotic stability of free-disease and endemic steady states, define

$$G_1(S) = \lim_{V \to 0^+} \frac{L_1(S,V)}{V}, \quad G_2(S) = \lim_{E \to 0^+} \frac{L_2(S,E)}{E}, \quad G_3(S) = \lim_{I \to 0^+} \frac{L_3(S,I)}{I}. \tag{29}$$

From conditions **C1** and **C2**, we obtain

$$G_1(S) = \frac{\partial L_1(S,0)}{\partial V} > 0, \qquad G_2(S) = \frac{\partial L_2(S,0)}{\partial E} > 0, \qquad G_3(S) = \frac{\partial L_3(S,0)}{\partial I} > 0, \tag{30}$$

For any $S > 0$.

Also

$$\acute{G_k}(S) > 0 \quad \text{for all} \quad k = 1, 2, 3. \tag{31}$$

Therefore, the basic reproduction number is rewritten as

$$\Re_0 = \frac{(\rho_2\delta + \rho_1 A)G_1(S_0)}{\sigma A(\delta + \mu)} + \frac{\delta G_3(S_0)}{A(\delta + \mu)} + \frac{\delta G_2(S_0)}{(\delta + \mu)}.$$

The following condition is required to survey the next theorem [26]:

Condition **(C4)**:

(i) The supremum of $\frac{G_2(S)}{G_1(S)}$ is accomplished at $S = S_0$ for all $S \in (0, S_0)$,

(ii) The supremum of $\frac{G_3(S)}{G_1(S)}$ is accomplished at $S = S_0$ for all $S \in (0, S_0)$,

Theorem 3. If $\Re_0 < 1$ and constraints **C1-C4** for system (2)-(6) are met, then Q_0 is globally asymptotically stable (G.A.S).

Proof. Investigating a Lyapunov function as follows:

$$P_0(t) = S - S_0 - \int_{S_0}^{S} \frac{G_1(S_0)}{G_1(\zeta)} d\zeta + E + \frac{[\rho_2 G_1(S_0) + \sigma G_3(S_0)]}{A\sigma} I + \frac{G_1(S_0)}{\sigma} V.$$

We note that, $P_0(S, E, I, V) > 0$ for all $S, E, I, V > 0$ and $P_0(S_0, 0, 0, 0) = 0$. We calculate $D^\alpha P_0$ along the system (2)-(6) solutions as:

$$\begin{aligned} D^\alpha P_0(t) = &\left(1 - \frac{G_1(S_0)}{G_1(S)}\right) [\lambda - \mu S(t) - L_1(S,V) - L_2(S,E) - L_3(S,I)] + L_1(S,V) + L_2(S,E) + L_3(S,I) \\ &- (\delta + \mu)E + \frac{[\rho_2 G_1(S_0) + \sigma G_3(S_0)]}{A\sigma} [\delta E - AI] + \frac{G_1(S_0)}{\sigma} [\rho_1 E + \rho_2 I - \sigma V]. \end{aligned} \tag{32}$$

From **(C3)** and Eq. (29)

$$\begin{aligned}\frac{L_1(S,V)}{V} &\leq \lim_{V\to 0^+}\frac{L_1(S,V)}{V} = G_1(S),\\ \frac{L_2(S,E)}{E} &\leq \lim_{E\to 0^+}\frac{L_2(S,E)}{E} = G_2(S),\\ \frac{L_3(S,I)}{I} &\leq \lim_{I\to 0^+}\frac{L_3(S,I)}{I} = G_3(S).\end{aligned}$$

Then Eq. (32) can rewritten as

$$\begin{aligned}D^\alpha P_0(t) \leq &\left(1-\frac{G_1(S_0)}{G_1(S)}\right)(\lambda-\mu S)+\frac{G_1(S_0)}{G_1(S)}\left[G_1(S)V+G_2(S)E+G_3(S)I\right]-(\delta+\mu)E+\frac{\rho_2\delta}{A\sigma}G_1(S_0)E\\ &+\frac{\delta}{A}G_3(S_0)E-\frac{\rho_2}{\sigma}G_1(S_0)I-G_3(S_0)I+\frac{\rho_1}{\sigma}G_1(S_0)E+\frac{\rho_2}{\sigma}G_1(S_0)I-G_1(S_0)V.\end{aligned}$$

From condition **(C4)** and Eq. (31)

$$\begin{aligned}\frac{G_1(S_0)}{G_1(S)}G_2(S) &\leq G_1(S_0)\frac{G_2(S_0)}{G_1(S_0)} = G_2(S_0),\\ \frac{G_1(S_0)}{G_1(S)}G_3(S) &\leq G_1(S_0)\frac{G_3(S_0)}{G_1(S_0)} = G_3(S_0), \quad \text{for any} \quad 0<S\leq S_0.\end{aligned}$$

Applying free-disease equilibrium condition $\lambda = \mu S_0$

Finally, we obtain

$$\begin{aligned}D^\alpha P_0(t) &\leq -\mu\left(1-\frac{G_1(S_0)}{G_1(S)}\right)(S-S_0)+\left[G_2(S_0)+\frac{\rho_2\delta}{A\sigma}G_1(S_0)+\frac{\delta}{A}G_3(S_0)+\frac{\rho_1}{\sigma}G_1(S_0)-(\delta+\mu)\right]E\\ &= -\mu\left(1-\frac{G_1(S_0)}{G_1(S)}\right)(S-S_0)+(\delta+\mu)(R_0-1)E.\end{aligned}$$

From condition **C1** and **C2**, we have $\left(1-\frac{G_1(S_0)}{G_1(S)}\right)\left(1-\frac{S}{S_0}\right)\leq 0$. Clearly if $\Re_0<1$, then $D^\alpha P_0(t)\leq 0$ for all $S,E,I,R,V>0$. Moreover $D^\alpha P_0(t)=0$ if and only if $S(t)=S_0$ and $E(t)=0$.

Let $\mathcal{F}_0 = \{(S,E,I,R,V) : D^\alpha P_0(t)=0\}$ and $\acute{\mathcal{F}}_0$ be largest invariant subset of $\mathcal{F}_0$. The solutions of the model (2)-(6) tend to $\acute{\mathcal{F}}_0$. For each element in $\acute{\mathcal{F}}_0$ we set $S(t)=S_0$ and $E(t)=0$. Thus Eq. (4) yields

$$D^\alpha I = 0 = \delta E(t) - AI(t),$$

hence $I(t)=0$. From Eq. (5) we have

$$D^\alpha R(t) = 0 = \gamma I(t) - \mu R(t)$$

then $R(t)=0$. Also from Eq. (6), we conclude that $V(t)=0$. It follows that $\acute{\mathcal{F}}_0$ contains a single point which is $(S_0,0,0,0,0)$. LaSalle's invariance principle (LIP) implies that Q_0 is G.A.S when $\Re_0<1$.

Remark 1. From conditions **C1- C3** we obtain

$$\Big(L_1(S,V)-L_1(S,V^*)\Big)\left(\frac{L_1(S,V)}{V}-\frac{L_1(S,V^*)}{V^*}\right)\leq 0, \quad \text{for all} \;\; S,V,V^*>0,$$

and this leads to

$$\left(1-\frac{L_1(S,V^*)}{L_1(S,V)}\right)\left(\frac{L_1(S,V)}{L_1(S,V^*)}-\frac{V}{V^*}\right)\leq 0, \quad textforall \;\; S,V,V^*>0.$$

Define the next functions [5]:

$$\mathcal{H}_E(S,E)=\frac{L_2(S,E)}{L_1(S,V^*)},\qquad \mathcal{H}_I(S,I)=\frac{L_3(S,I)}{L_1(S,V^*)}.$$

We put the following condition

Condition(C5)

$$\textbf{(i)}\ \Big(\mathcal{H}_E(S,E)-\mathcal{H}_E(S^*,E^*)\Big)\left(\frac{\mathcal{H}_E(S,E)}{E}-\frac{\mathcal{H}_E(S^*,E^*)}{E^*}\right)\leq 0,$$
$$\textbf{(ii)}\ \Big(\mathcal{H}_I(S,I)-\mathcal{H}_I(S^*,I^*)\Big)\left(\frac{\mathcal{H}_I(S,I)}{I}-\frac{\mathcal{H}_I(S^*,I^*)}{I^*}\right)\leq 0,$$

for all $E,E^*,I,I^*>0$ and $S\in(0,S_0]$. Hence, we get the following remark:

Remark 2.

$$\left(1-\frac{\mathcal{H}_E(S^*,E^*)}{\mathcal{H}_E(S,E)}\right)\left(\frac{\mathcal{H}_E(S,E)}{\mathcal{H}_E(S^*,E^*)}-\frac{E}{E^*}\right)\leq 0,\qquad E,E^*>0, S\in(0,S_0],$$
$$\left(1-\frac{\mathcal{H}_I(S^*,I^*)}{\mathcal{H}_I(S,I)}\right)\left(\frac{\mathcal{H}_I(S,I)}{\mathcal{H}_I(S^*,I^*)}-\frac{I}{I^*}\right)\leq 0,\qquad I,I^*>0, S\in(0,S_0].$$

Theorem 4. For the model (2)-(6) if the chronic equilibrium Q_1 exists, then its global asymptomatic stable if the conditions **(C1)-(C3)** and **(C5)** are hold.

Proof: Constructing a function $P_1(S,E,I,V)$ as:

$$\begin{aligned}
P_1(S,E,I,V)=&\ S-S^*-\int_{S^*}^{S}\frac{L_1(S^*,V^*)}{L_1(\zeta,V^*)}d\zeta+E-E^*-E^*\ln\left(\frac{E}{E^*}\right)\\
&+\left(\frac{L_3(S^*,I^*)}{\delta E^*}+\frac{\rho_2 L_1(S^*,V^*)}{(\rho_1 A+\rho_2\delta)E^*}\right)\left(I-I^*-I^*\ln\left(\frac{I}{I^*}\right)\right)\\
&+\frac{AL_1(S^*,V^*)}{(\rho_1 A+\rho_2\delta)E^*}\left[V-V^*-V^*\ln\left(\frac{V}{V^*}\right)\right].
\end{aligned}$$

Clearly $P_1(S,E,I,V)>0$ for all $S,E,I,V>0$ and $P_1(S^*,E^*,I^*,V^*)=0$. Moreover,

$$\begin{aligned}
D^\alpha P_1(t)=&\left(1-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\right)\Big(\lambda-\mu S-L_1(S,V)-L_2(S,E)-L_3(S,I)\Big)\\
&+\left(1-\frac{E^*}{E}\right)\Big(L_1(S,V)+L_2(S,E)+L_3(S,I)-(\delta+\mu)E\Big)\\
&+\left(\frac{L_3(S^*,I^*)}{\delta E^*}+\frac{\rho_2 L_1(S^*,V^*)}{(\rho_1 A+\rho_2\delta)E^*}\right)\left(1-\frac{I^*}{I}\right)\Big(\delta E-AI\Big)\\
&+\frac{AL_1(S^*,V^*)}{(\rho_1 A+\rho_2\delta)E^*}\left(1-\frac{V^*}{V}\right)\Big(\rho_1 E+\rho_2 I-\sigma V\Big)\\
=&\left(1-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\right)\Big(\lambda-\mu S\Big)+\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\Big[L_1(S,V)+L_2(S,E)+L_3(S,I)\Big]\\
&-(\delta+\mu)E-\frac{E^*}{E}\Big[L_1(S,V)+L_2(S,E)+L_3(S,I)\Big]+(\delta+\mu)E^*\\
&+\left(\frac{L_3(S^*,I^*)}{\delta E^*}+\frac{\rho_2 L_1(S^*,V^*)}{(\rho_1 A+\rho_2\delta)E^*}\right)\left(1-\frac{I^*}{I}\right)\Big(\delta E-AI\Big)\\
&+\frac{AL_1(S^*,V^*)}{(\rho_1 A+\rho_2\delta)E^*}\left(1-\frac{V^*}{V}\right)\Big(\rho_1 E+\rho_2 I-\sigma V\Big).
\end{aligned}$$

Applying the steady state conditions for Q_1:

$$\begin{aligned}
\lambda &= \mu S^* + L_1(S^*,V^*) + L_2(S^*,E^*) + L_3(S^*,I^*),\\
(\delta+\mu)E^* &= L_1(S^*,V^*) + L_2(S^*,E^*) + L_3(S^*,I^*),\\
\delta E^* &= AI^*,\\
\sigma V^* &= \rho_1 E^* + \rho_2 I^*.
\end{aligned}$$

we get

$$\begin{aligned}
D^\alpha P_1(t) &= \mu S^*\left(1-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\right)\left(1-\frac{S}{S^*}\right) + 2L_1(S^*,V^*) + 2L_2(S^*,E^*) + 2L_3(S^*,I^*)\\
&\quad - \frac{L_1(S^*,V^*)}{L_1(S,V^*)}\Big[L_1(S^*,V^*) + L_2(S^*,E^*) + L_3(S^*,I^*)\Big] + \frac{L_1(S^*,V^*)}{L_1(S,V^*)}\Big[L_1(S,V) + L_2(S,E) + L_3(S,I)\Big]\\
&\quad - \frac{E}{E^*}\Big[L_1(S^*,V^*) + L_2(S^*,E^*) + L_3(S^*,I^*)\Big] - \frac{E^*}{E}\Big[L_1(S,V) + L_2(S,E) + L_3(S,I)\Big]\\
&\quad + \delta E^*\left(\frac{L_3(S^*,I^*)}{\delta E^*} + \frac{\rho_2 L_1(S^*,V^*)}{(\rho_1 A + \rho_2\delta)E^*}\right)\left(1+\frac{E}{E^*}-\frac{I}{I^*}-\frac{I^*E}{IE^*}\right)\\
&\quad + \frac{AL_1(S^*,V^*)}{(\rho_1 A+\rho_2\delta)E^*}\left[\rho_1 E + \rho_2 I - \frac{V}{V^*}\left(\rho_1 E^* + \frac{\rho_2\delta E^*}{A}\right) - \rho_1 E\frac{V^*}{V} - \rho_2 I\frac{V^*}{V} + \rho_1 E^* + \frac{\rho_2\delta E^*}{A}\right]\\
&= \mu S^*\left(1-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\right)\left(1-\frac{S}{S^*}\right) + L_1(S^*,V^*)\left[3 - \frac{V}{V^*} - \frac{L_1(S^*,V^*)}{L_1(S,V^*)} + \frac{L_1(S,V)}{L_1(S,V^*)} - \frac{E}{E^*}\right.\\
&\quad \left. - \frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}\right] + L_2(S^*,E^*)\left[2 - \frac{L_1(S^*,V^*)}{L_1(S,V^*)} + \frac{L_1(S^*,V^*)L_2(S,E)}{L_1(S,V^*)L_2(S^*,E^*)} - \frac{E}{E^*} - \frac{E^*L_2(S,E)}{EL_2(S^*,E^*)}\right]\\
&\quad + L_3(S^*,I^*)\left[2 - \frac{L_1(S^*,V^*)}{L_1(S,V^*)} + \frac{L_1(S^*,V^*)L_3(S,I)}{L_1(S,V^*)L_3(S^*,I^*)} - \frac{E}{E^*} - \frac{E^*L_3(S,I)}{EL_3(S^*,I^*)}\right]\\
&\quad + L_3(S^*,I^*)\left[1 + \frac{E}{E^*} - \frac{I}{I^*} - \frac{EI^*}{E^*I}\right] + \frac{\rho_2\delta L_1(S^*,V^*)}{\rho_1 A + \rho_2\delta}\left[1 + \frac{E}{E^*} - \frac{I}{I^*} - \frac{EI^*}{E^*I}\right]\\
&\quad + \frac{\rho_1 AL_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\frac{E}{E^*} + \frac{\rho_2\delta L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\frac{I}{I^*} - \frac{\rho_1 AL_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\frac{EV^*}{E^*V} - \frac{\rho_2\delta L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\frac{IV^*}{I^*V}. \qquad (33)
\end{aligned}$$

Eq. (33) can be simplified as

$$\begin{aligned}
D^\alpha P_1(t) &= \mu S^*\left(1-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\right)\left(1-\frac{S}{S^*}\right) + \frac{\rho_1 AL_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left[3 - \frac{V}{V^*} - \frac{L_1(S^*,V^*)}{L_1(S,V^*)} + \frac{L_1(S,V)}{L_1(S,V^*)} - \frac{E}{E^*}\right.\\
&\quad \left. - \frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}\right] + \frac{\rho_2\delta L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left[4 - \frac{V}{V^*} - \frac{L_1(S^*,V^*)}{L_1(S,V^*)} + \frac{L_1(S,V)}{L_1(S,V^*)} - \frac{EI^*}{E^*I} - \frac{IV^*}{I^*V}\right.\\
&\quad \left. - \frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}\right] + L_2(S^*,E^*)\left[3 - \frac{L_1(S^*,V^*)}{L_1(S,V^*)} - \frac{E^*L_2(S,E)}{EL_2(S^*,E^*)} - \frac{EL_2(S^*,E^*)L_1(S,V^*)}{E^*L_2(S,E)L_1(S^*,V^*)}\right]\\
&\quad + L_3(S^*,I^*)\left[4 - \frac{L_1(S^*,V^*)}{L_1(S,V^*)} - \frac{EI^*}{E^*I} - \frac{E^*L_3(S,I)}{EL_3(S^*,I^*)} - \frac{IL_1(S,V^*)L_3(S^*,I^*)}{I^*L_1(S^*,V^*)L_3(S,I)}\right]\\
&\quad + L_2(S^*,E^*)\left[-1 + \frac{L_1(S^*,V^*)L_2(S,E)}{L_1(S,V^*)L_2(S^*,E^*)} - \frac{E}{E^*} + \frac{EL_2(S^*,E^*)L_1(S,V^*}{E^*L_2(S,E)L_1(S^*,V^*)}\right]\\
&\quad + L_3(S^*,I^*)\left[-1 + \frac{L_1(S^*,V^*)L_3(S,I)}{L_1(S,V^*)L_3(S^*,I^*)} - \frac{I}{I^*} + \frac{IL_1(S,V^*)L_3(S^*,I^*)}{I^*L_1(S^*,V^*)L_3(S,I)}\right].
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
D^\alpha P_1(t) &= \mu S^*\left(1-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\right)\left(1-\frac{S}{S^*}\right)+\frac{\rho_1 A L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left[4-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{EV^*}{E^*V}\right.\\
&\left.-\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}-\frac{VL_1(S,V^*)}{V^*L_1(S,V)}\right]+\frac{\rho_1 A L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left(1-\frac{L_1(S,V^*)}{L_1(S,V)}\right)\left(\frac{L_1(S,V)}{L_1(S,V^*)}-\frac{V}{V^*}\right)\\
&+\frac{\rho_2\delta L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left[5-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{EI^*}{E^*I}-\frac{IV^*}{I^*V}-\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}-\frac{VL_1(S,V^*)}{V^*L_1(S,V)}\right]\\
&+\frac{\rho_2\delta L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left(1-\frac{L_1(S,V^*)}{L_1(S,V)}\right)\left(\frac{L_1(S,V)}{L_1(S,V^*)}-\frac{V}{V^*}\right)\\
&+L_2(S^*,E^*)\left[3-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{E^*L_2(S,E)}{EL_2(S^*,E^*)}-\frac{EL_2(S^*,E^*)L_1(S,V^*)}{E^*L_2(S,E)L_1(S^*,V^*)}\right]\\
&+L_3(S^*,I^*)\left[4-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{EI^*}{E^*I}-\frac{E^*L_3(S,I)}{EL_3(S^*,I^*)}-\frac{IL_1(S,V^*)L_3(S^*,I^*)}{I^*L_1(S^*,V^*)L_3(S,I)}\right]\\
&+L_2(S^*,E^*)\left(1-\frac{L_1(S,V^*)L_2(S^*,E^*)}{L_1(S^*,V^*)L_2(S,E)}\right)\left(\frac{L_1(S^*,V^*)L_2(S,E)}{L_1(S,V^*)L_2(S^*,E^*)}-\frac{E}{E^*}\right)\\
&+L_3(S^*,I^*)\left(1-\frac{L_1(S,V^*)L_3(S^*,I^*)}{L_1(S^*,V^*)L_3(S,I)}\right)\left(\frac{L_1(S^*,V^*)L_3(S,I)}{L_1(S,V^*)L_3(S^*,I^*)}-\frac{I}{I^*}\right).
\end{aligned}$$

We can rewrite it as

$$\begin{aligned}
D^\alpha P_1(t) &= \mu S^*\left(1-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\right)\left(1-\frac{S}{S^*}\right)+\frac{\rho_1 A L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left[4-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{EV^*}{E^*V}\right.\\
&\left.-\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}-\frac{VL_1(S,V^*)}{V^*L_1(S,V)}\right]+L_1(S^*,V^*)\left(1-\frac{L_1(S,V^*)}{L_1(S,V)}\right)\left(\frac{L_1(S,V)}{L_1(S,V^*)}-\frac{V}{V^*}\right)\\
&+\frac{\rho_2\delta L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left[5-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{EI^*}{E^*I}-\frac{IV^*}{I^*V}-\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}-\frac{VL_1(S,V^*)}{V^*L_1(S,V)}\right]\\
&+L_2(S^*,E^*)\left[3-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{E^*L_2(S,E)}{EL_2(S^*,E^*)}-\frac{EL_2(S^*,E^*)L_1(S,V^*)}{E^*L_2(S,E)L_1(S^*,V^*)}\right]\\
&+L_3(S^*,I^*)\left[4-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{EI^*}{E^*I}-\frac{E^*L_3(S,I)}{EL_3(S^*,I^*)}-\frac{IL_1(S,V^*)L_3(S^*,I^*)}{I^*L_1(S^*,V^*)L_3(S,I)}\right]\\
&+L_2(S^*,E^*)\left(1-\frac{L_1(S,V^*)L_2(S^*,E^*)}{L_1(S^*,V^*)L_2(S,E)}\right)\left(\frac{L_1(S^*,V^*)L_2(S,E)}{L_1(S,V^*)L_2(S^*,E^*)}-\frac{E}{E^*}\right)\\
&+L_3(S^*,I^*)\left(1-\frac{L_1(S,V^*)L_3(S^*,I^*)}{L_1(S^*,V^*)L_3(S,I)}\right)\left(\frac{L_1(S^*,V^*)L_3(S,I)}{L_1(S,V^*)L_3(S^*,I^*)}-\frac{I}{I^*}\right).
\end{aligned}$$

Using the geometrical and arithmetical means relationship we obtain

$$\begin{aligned}
4 &\le \frac{L_1(S^*,V^*)}{L_1(S,V^*)}+\frac{EV^*}{E^*V}+\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}+\frac{VL_1(S,V^*)}{V^*L_1(S,V)},\\
5 &\le \frac{L_1(S^*,V^*)}{L_1(S,V^*)}+\frac{EI^*}{E^*I}+\frac{IV^*}{I^*V}+\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}+\frac{VL_1(S,V^*)}{V^*L_1(S,V)},\\
3 &\le \frac{L_1(S^*,V^*)}{L_1(S,V^*)}+\frac{E^*L_2(S,E)}{EL_2(S^*,E^*)}+\frac{EL_2(S^*,E^*)L_1(S,V^*)}{E^*L_2(S,E)L_1(S^*,V^*)},\\
4 &\le \frac{L_1(S^*,V^*)}{L_1(S,V^*)}+\frac{EI^*}{E^*I}+\frac{E^*L_3(S,I)}{EL_3(S^*,I^*)}+\frac{IL_1(S,V^*)L_3(S^*,I^*)}{I^*L_1(S^*,V^*)L_3(S,I)}.
\end{aligned}$$

Also, from condition **(C5)** we have

$$\begin{aligned}
0 &\ge \left(1-\frac{L_1(S,V^*)L_2(S^*,E^*)}{L_1(S^*,V^*)L_2(S,E)}\right)\left(\frac{L_1(S^*,V^*)L_2(S,E)}{L_1(S,V^*)L_2(S^*,E^*)}-\frac{E}{E^*}\right)\\
0 &\ge \left(1-\frac{L_1(S,V^*)L_3(S^*,I^*)}{L_1(S^*,V^*)L_3(S,I)}\right)\left(\frac{L_1(S^*,V^*)L_3(S,I)}{L_1(S,V^*)L_3(S^*,I^*)}-\frac{I}{I^*}\right).
\end{aligned}$$

We conclude that $D^\alpha P_1(t)\le 0$ and $D^\alpha P_1(t)=0$ at the point (S^*,E^*,I^*,R^*,V^*). Let $\acute{\mathcal{W}}_1$ be the largest invariant subset of the set $\{(S,E,I,R,V): D^\alpha P_1(t)=0\}$. Thus, the solutions of model tend to $\acute{\mathcal{W}}_1$. It is clear that $\acute{\mathcal{F}}_1$ contains unique point which is Q_1. The global asymptotic stable of Q_1 follows from LIP.

6 Numerical Simulations

In this section, we introduce the following COVID-19 model example as a special case of system (2)-(6):

$$D^\alpha S(t) = \lambda - \mu S - \frac{S}{1+\epsilon S}\left(\frac{\beta_1 V}{1+\kappa_1 V} + \frac{\beta_2 E}{1+\kappa_2 E} + \frac{\beta_3 I}{1+\kappa_3 I}\right), \tag{34}$$

$$D^\alpha E(t) = \frac{S}{1+\epsilon S}\left(\frac{\beta_1 V}{1+\kappa_1 V} + \frac{\beta_2 E}{1+\kappa_2 E} + \frac{\beta_3 I}{1+\kappa_3 I}\right) - (\delta+\mu)E, \tag{35}$$

$$D^\alpha I(t) = \delta E - (\omega+\gamma+\mu)I, \tag{36}$$

$$D^\alpha R(t) = \gamma I - \mu R, \tag{37}$$

$$D^\alpha V(t) = \rho_1 E + \rho_2 I - \sigma V. \tag{38}$$

The three functions for the transmission rates of infection are given by:

$$L_1(S,V) = \frac{\beta_1 SV}{(1+\epsilon S)(1+\kappa_1 V)}, \quad L_2(S,E) = \frac{\beta_2 SE}{(1+\epsilon S)(1+\kappa_2 E)}, \quad L_3(S,I) = \frac{\beta_3 SI}{(1+\epsilon S)(1+\kappa_3 I)}. \tag{39}$$

The parameters β_j indicate maximum transmission rates and κ_j allow transmission speeds to be adjusted and are all positive constants where $j = 1,2,3$. Parameters $\lambda, \mu, \epsilon, \delta, \omega, \gamma, \rho_1, \rho_2$ and σ are positive constants. Checking the conditions **(C1) - (C5)**

(C1) Obviously,

$$L_1(S,V) > 0, \quad L_2(S,E) > 0, \quad L_3(S,I) > 0 \quad for\ all \quad S,E,I,V > 0,$$
$$L_1(S,0) = L_2(S,0) = L_3(S,0) = 0 \ for\ S > 0,$$
$$L_1(0,V) = L_2(0,E) = L_3(0,I) = 0 \quad for\ all \quad E,I,V > 0.$$

(C2)

$$\begin{aligned}
&\frac{\partial L_1(S,V)}{\partial S} = \frac{\beta_1 V}{(1+\epsilon S)^2(1+\kappa_1 V)} > 0, \quad \frac{\partial L_2(S,E)}{\partial S} = \frac{\beta_2 E}{(1+\epsilon S)^2(1+\kappa_2 E)} > 0\\
&\frac{\partial L_3(S,I)}{\partial S} = \frac{\beta_3 I}{(1+\epsilon S)^2(1+\kappa_3 I)} > 0, \quad \frac{\partial L_1(S,V)}{\partial V} = \frac{\beta_1 S}{(1+\epsilon S)(1+\kappa_1 V)^2} > 0,\\
&\frac{\partial L_2(S,E)}{\partial E} = \frac{\beta_2 S}{(1+\epsilon S)(1+\kappa_2 E)^2} > 0, \quad \frac{\partial L_3(S,I)}{\partial I} = \frac{\beta_3 S}{(1+\epsilon S)(1+\kappa_3 I)^2} > 0,\\
&\frac{\partial L_1(S,0)}{\partial V} = \frac{\beta_1 S}{(1+\epsilon S)} > 0, \quad \frac{\partial L_2(S,0)}{\partial E} = \frac{\beta_2 S}{(1+\epsilon S)} > 0, \quad \frac{\partial L_3(S,0)}{\partial I} = \frac{\beta_3 S}{(1+\epsilon S)} > 0,
\end{aligned}$$

for all $S,E,I,V > 0$, furthermore,

$$\begin{aligned}
&\frac{d}{dS}\left(\frac{\partial L_1(S,0)}{\partial V}\right) = \frac{\beta_1}{(1+\epsilon S)^2} > 0, \quad \frac{d}{dS}\left(\frac{\partial L_2(S,0)}{\partial E}\right) = \frac{\beta_2}{(1+\epsilon S)^2} > 0,\\
&\frac{d}{dS}\left(\frac{\partial L_3(S,0)}{\partial I}\right) = \frac{\beta_3}{(1+\epsilon S)^2} > 0, \quad for\ all \quad S,E,V,I > 0.
\end{aligned}$$

(C3)

$$\begin{aligned}
&\frac{\partial}{\partial V}\left(\frac{L_1(S,V)}{V}\right) = \frac{\partial}{\partial V}\left(\frac{\beta_1 S}{(1+\epsilon S)(1+\kappa_1 V)}\right) = \frac{-\kappa_1\beta_1 S}{(1+\epsilon S)(1+\kappa_1 V)^2} \le 0,\\
&\frac{\partial}{\partial E}\left(\frac{L_2(S,E)}{E}\right) = \frac{\partial}{\partial E}\left(\frac{\beta_2 S}{(1+\epsilon S)(1+\kappa_2 E)}\right) = \frac{-\kappa_2\beta_2 S}{(1+\epsilon S)(1+\kappa_2 E)^2} \le 0,\\
&\frac{\partial}{\partial I}\left(\frac{L_3(S,I)}{I}\right) = \frac{\partial}{\partial I}\left(\frac{\beta_3 S}{(1+\epsilon S)(1+\kappa_3 I)}\right) = \frac{-\kappa_3\beta_3 S}{(1+\epsilon S)(1+\kappa_3 I)^2} \le 0, \quad \text{for all} \quad S,E,I,V > 0.
\end{aligned}$$

(C4) We have

$$G_1(S) = \frac{\partial L_1(S,0)}{\partial V} = \frac{\beta_1 S}{(1+\epsilon S)},$$
$$G_2(S) = \frac{\partial L_2(S,0)}{\partial E} = \frac{\beta_2 S}{(1+\epsilon S)},$$
$$G_3(S) = \frac{\partial L_3(S,0)}{\partial I} = \frac{\beta_3 S}{(1+\epsilon S)}.$$

Thus, $\frac{G_2(S)}{G_1(S)} = \frac{\beta_2}{\beta_1}$ and $\frac{G_3(S)}{G_1(S)} = \frac{\beta_3}{\beta_1}$.

(C5)

$$\mathcal{H}_E(S,E) = \frac{L_2(S,E)}{L_1(S,V^*)} = \frac{\beta_2(1+\kappa_1 V^*)E}{\beta_1(1+\kappa_2 E)V^*}, \quad \mathcal{H}_E(S^*,E^*) = \frac{L_2(S^*,E^*)}{L_1(S^*,V^*)} = \frac{\beta_2(1+\kappa_1 V^*)E^*}{\beta_1(1+\kappa_2 E^*)V^*},$$
$$\mathcal{H}_I(S,I) = \frac{L_3(S,I)}{L_1(S,V^*)} = \frac{\beta_3(1+\kappa_1 V^*)I}{\beta_1(1+\kappa_3 I)V^*}, \quad \mathcal{H}_I(S^*,I^*) = \frac{L_3(S^*,I^*)}{L_1(S^*,V^*)} = \frac{\beta_3(1+\kappa_1 V^*)I^*}{\beta_1(1+\kappa_3 I^*)V^*},$$

$$\Big(\mathcal{H}_E(S,E) - \mathcal{H}_E(S^*,E^*)\Big)\left(\frac{\mathcal{H}_E(S,E)}{E} - \frac{\mathcal{H}_E(S^*,E^*)}{E^*}\right) = -\frac{\kappa_2\beta_2^2(1+\kappa_1 V^*)^2(E-E^*)^2}{\beta_1^2(V^*)^2(1+\kappa_2 E^*)^2(1+\kappa_2 E)^2} \le 0,$$
$$\Big(\mathcal{H}_I(S,I) - \mathcal{H}_I(S^*,I^*)\Big)\left(\frac{\mathcal{H}_I(S,I)}{I} - \frac{\mathcal{H}_I(S^*,I^*)}{I^*}\right) = -\frac{\kappa_3\beta_3^2(1+\kappa_1 V^*)^2(I-I^*)^2}{\beta_1^2(V^*)^2(1+\kappa_3 I^*)^2(1+\kappa_3 I)^2} \le 0,$$

for all $E, I > 0,\ S \in (0, S_0]$.

As a result, the validity of conditions **C1-C5** ensure that the results of global stability shown in theorems 3 and 4 are true in this example. Therefore the basic reproduction number of model (34)-(38) is:

$$\Re_0 = \frac{S_0}{\sigma A(\delta+\mu)(1+\epsilon S_0)}\Big(\beta_1(\rho_2\delta + \rho_1 A) + \sigma(A\beta_2 + \delta\beta_3)\Big) = \Re_{01} + \Re_{02} + \Re_{03}.$$

Specifically,

$$\Re_{01} = \frac{\beta_1(\rho_2\delta+\rho_1 A)S_0}{\sigma A(\delta+\mu)(1+\epsilon S_0)}, \quad \Re_{02} = \frac{\beta_2 S_0}{(\delta+\mu)(1+\epsilon S_0)}, \quad \Re_{03} = \frac{\beta_3\delta S_0}{A(\delta+\mu)(1+\epsilon S_0)}.$$

Case (I): In this case, we run computational simulations for real-world data beginning from June 5, 2021 to September 11, 2021. We assume that the global influx and death rates in 2021 are 18.077 and 7.612 per 1000 people, respectively as the same as 2020 [27], [28]. On June 5^{th}, 2021, the total population of the world was N = 7794798739. So, $\lambda = \frac{18.077\times N}{1000\times 365} = 3.8605\times 10^5$ and $\mu = \frac{7.612}{1000\times 365} = 2.0855\times 10^{-5}$. According to [1], the initial condition is set as $I(0) = 13032161$, $R(0) = 157029051$ and we assume $E(0) = 3.3\times 10^7$ then from $S(0)+E(0)+I(0)+R(0) = N(0)$, we have $S(0) = 7.59174\times 10^9$. Figure 2 depicts the fitted curve and the reported global cumulative number of COVID-19 from 5 Jun to 11 Sep 2021. A comparison is also provided in Figure 3 between the integer-order one when $\alpha = 1$, fractional order model with $\alpha = 0.95$, and the actual active infected cases with COVID-19 in the world at the same period.

Parameter	Value	Source	Parameter	Value	Source
β_1	$1 \times 10^{-}11$	[17],[27]	ρ_1	0.2	Assumed
β_2	$3 \times 10^{-}11$	[17],[27]	ρ_2	2	Assumed
β_3	$1 \times 10^{-}11$	[17],[27]	κ_1	1×10^{-9}	fitting by data
ω	0.0003	fitting by data	κ_2	2.2×10^{-10}	fitting by data
δ	0.0000435	fitting by data	κ_3	5×10^{-11}	fitting by data
γ	0.009	fitting by data	ϵ	0.00064	fitting by data
σ	1	[17]			

Table 1: Worldwide approximate parameters for the COVID-19 model (34)-(38).

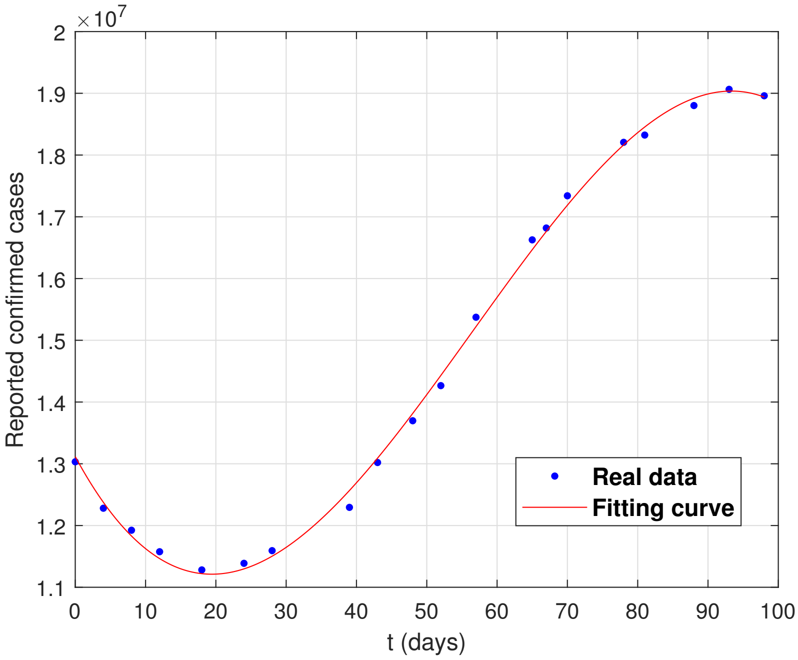

Figure 2: The fitted curve and the reported COVID-19 cases in the world from 5 June to 11 September 2021.

The achieved results show that the response of the fractional order model matches real data and show the benefit of using the derivative of the fractional-order instead of integer order in conjunction with the results of Table 1.

Case (II): To study the effect of environmental reservoirs on the COVID-19 transmissions, we will assume new set of initial conditions as $I(0) = 24545$, $R(0) = 907$ and $S(0) = 7.6100 \times 10^9$ [27] and a set of parameters in Table 2. We can calculate the basic reproduction number $\Re_0 = 5.3837$ using the parameter values from Table 2. We find, in particular, that

$\Re_{01} = 1.5558, \qquad \Re_{02} = 2.5304 \qquad \Re_{03} = 1.2974 \qquad .$

The largest of these three components $\Re_{02}$ comes from exposed-to-susceptible transmission, since exposed persons display no symptoms and can transmit the disease to others easily in close proximity, even without their awareness. In the meantime, if we suppose that the infected to susceptible transmission rate is equal

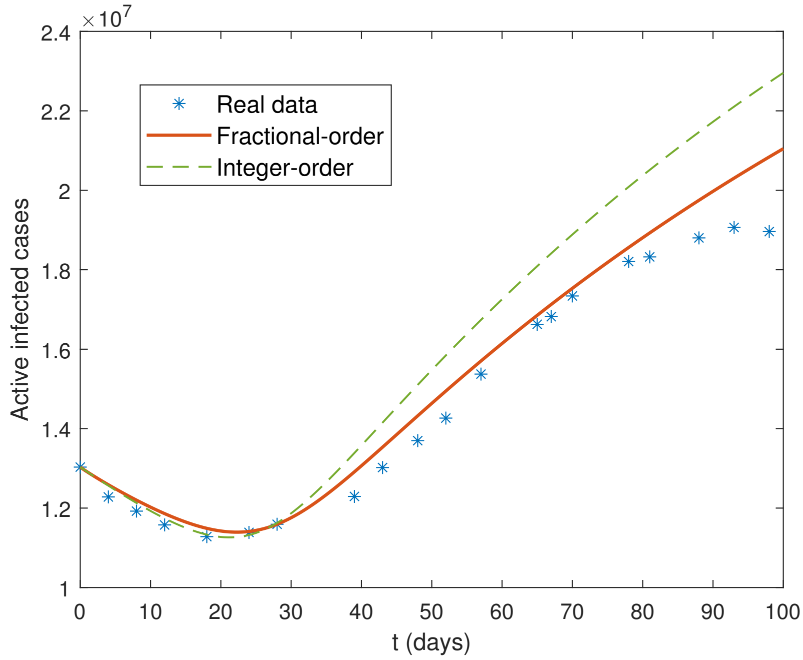

Figure 3: Comparison between the results of the fractional-order derivative $\alpha = 0.95$ and the integer-order derivative $\alpha = 1$ and with real data.

to transmission rate from environmental factors to susceptible persons, we find that the rate of impact of transmission from the environment is greater than its effect from infected persons on basic reproduction number, likely as a result of the symptomatic infected persons being isolated. Furthermore, this indicating that the environmental reservoir plays a significant role in the overall risk for infection.

7 Conclusion and discussion

In this paper, a mathematical model has been introduced to examine the ongoing novel coronavirus pandemic. We have proposed a fractional-order $SEIRV$ epidemic model that uses general infection rates and incorporates

Parameter	Value	Parameter	Value
λ	3.7690×10^5	ρ_1	2.3
μ	2.0855×10^{-5}	ρ_2	0.2
β_1	1.012×10^-11	κ_1	1.01×10^{-4}
β_2	2.11×10^-11	κ_2	1.01×10^{-4}
β_3	0.82×10^-11	κ_3	1.01×10^{-4}
δ	1/7	ϵ	0.00064
ω	0.034	σ	2
γ	1/15		

Table 2: Approximate parameters for the COVID-19 model (34)-(38).

the environmental reservoir into the dynamics of disease transmission that alter with the epidemiological state and environmental factors. We started by applying the Caputo derivative to create a general $SEIRV$ model that is appropriate for initial-value problems. We have shown the system's feasibility area and calculated its steady states. The basic reproduction number $\Re_0$ is obtained using the next generation technique, and it is made up of three components that represent the three different mechanisms of infection, namely, exposed people, infected people, and environmental reservoirs, to susceptible people. Based on characteristic equations and suitable Lyapunov functions, we examined local and global stability analyzes of equilibrium points in detail. In the disease-free case, the underlying model is locally and globally asymptotically stable when $\Re_0 < 1$, when $\Re_0 > 1$, the positive endemic steady state is both locally and globally asymptotically stable. Based on real-world data, numerical simulations of an example with general infection functions have been presented. The results of our simulations show that our model can be applied to the COVID-19 outbreak in the world as it fits the supplied data really well. We can estimate the fundamental reproduction number using data fitting. Moreover, the environmental reservoir is important in shaping the outbreak risk. In the numerical portion, we also looked at the benefits of applying the fractional-order instead of the integer-order by comparing the results of our model with real data and integer-order and fractional-order, we obtained that the fractional-order has the better result that depends on the historical states of the disease and increases the stability region of the solution.

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