On the hypercomplex numbers of all finite dimensions: Beyond quaternions and octonians

Pushpendra Singh*, Anubha Gupta, and Shiv Dutt Joshi

Abstract

In search of a real three-dimensional, normed, associative, division algebra, Hamilton discovered quaternions that form a non-commutative division algebra of quadruples. Later works showed that there are only four real division algebras with 1, 2, 4, or 8 dimensions. This work overcomes this limitation and introduces generalized hypercomplex numbers of all dimensions that are extensions of the traditional complex numbers. The space of these numbers forms non-distributive normed division algebra that is extendable to all finite dimensions. To obtain these extensions, we defined a unified multiplication, designated as scaling and rotative multiplication, fully compatible with the existing multiplication. Therefore, these numbers and the corresponding algebras reduce to distributive normed algebras for dimensions 1 and 2. Thus, this work presents a generalization of $\mathbb{C}$ in higher dimensions along with interesting insights into the geometry of the vectors in the corresponding spaces.

Index Terms

Generalized hypercomplex numbers; Real numbers; Complex numbers; Quaternions; Octonians; Scaling and rotative multiplication (SRM); Non-distributive Field.

I. Introduction

Real numbers ($\mathbb{R}$) form a field, wherein addition, subtraction, multiplication, and division are well defined. Complex numbers ($\mathbb{C}$) or imaginary numbers emerged in the quest of finding the solution of the polynomial equation $x^2 + 1 = 0$. While $\mathbb{R}$ is also a vector space of dimension 'one' over itself (i.e., over the field of real numbers), $\mathbb{C}$ is a vector space of dimension 'two' defined over the field $\mathbb{R}$. In particular, $\mathbb{C}$ is an interesting space where one can deal with elements as complex or imaginary numbers (of the form of $a + ib$) or work with them as in abstract algebra, and at the same time can also visualize the elements in the 2-dimensional (2D) space as in traditional geometry with the notion of the length of the vectors, the distance between vectors, and the angle between vectors. This beautiful connection of complex numbers and 2D geometry inspired William Rowan Hamilton to look for a solution of a 3D algebra with a similarly associated 3D geometry. In modern mathematical language, Hamilton was trying for a 3D normed division algebra. In October 1843, Hamilton discovered quaternions ($\mathbb{H}$), and in a very famous act of scientific vandalism, he instantly carved the fundamental equations of quaternions into the stone of the Brougham Bridge as: $i^2 = j^2 = k^2 = ijk = -1$.

Now, it is well-established that a 3D normed division algebra does not exist. Frobenius [1] in 1878 obtained the classification of associative normed division algebras and proved that there are only three such algebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. Hurwitz in 1898 [2] proved that there are only four normed division algebras, namely $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ (octonions are also known as Cayley numbers) with a natural embedding as $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$, where multiplication by a unit vector is distance-preserving. Likewise, Zorn in 1930 [3] had shown that if associativity condition is relaxed with alternativity, then there are only four normed division algebras: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. The theorems of Adams (1958, 1960) [4], [5], Kervaire (1958) [6], and Bott–Milnor (1958) [7] reveal that the finite-dimensional normed division algebra can have only 1, 2, 4, and 8 dimensions.

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The work of Hamilton was seminal because quaternions find applications in various areas such as astronautics, robotics, computer graphics, and animation. They are found to be useful in modern physics, particularly in the general theory of relativity, because they can express the Lorentz transform [8]. The quaternion calculus is useful in crystallography, the kinematics of rigid body motion, classical electromagnetism, and quantum mechanics [8]. This is to note that Hamilton was searching for a real, normed, three-dimensional, associative, division algebra that does not exist. In order to equate the Euclidean length of the product of a pair of triples to the product of their lengths, he dropped the property of commutative multiplication and also added a fourth dimension defined by k. Hence, he moved to a 4D hypercomplex number system while trying to define a space of a 3D hypercomplex number system. Similarly, the space of octonions, i.e., the 8D hypercomplex number system, drops not only the property of commutativity but also the additional property of associativity in multiplication. In these hypercomplex number systems (4D and 8D), a polynomial of degree *n* can have infinitely many quaternion or octonion roots, unlike the result of the fundamental theorem of algebra that guarantees that a polynomial of degree *n* with complex coefficients has precisely *n* complex roots (counting multiplicity) for 2D complex number system. Furthermore, only four real division algebras with 1, 2, 4, or 8 dimensions can exist, where this existing framework cannot be extended to other finite dimensions.

Intrigued by the above limitation and inspired by the works of Hamilton where he thought of an entirely different out-of-the-box solution of those times, this work is an attempt to look for a different solution that can work for all finite dimensions. Similar to the works on quaternions and octonions, we have also dropped a property. In addition, we have defined a new multiplication operator. We have proposed a solution with non-distributive normed division algebra along with the definition of a new multiplication operation. The theory turns out to be interesting that is generalizable to all finite higher dimensions. In sum, this work makes the below significant contributions:

1) This work introduces generalized hypercomplex numbers of all dimensions $(\mathbb{S}^M)$ that are extensions of the traditional complex numbers with a natural nesting as $\mathbb{S} \subset \mathbb{S}^2 \subset \mathbb{S}^3 \cdots \subset \mathbb{S}^M \subset \mathbb{S}^{M+1} \cdots$, where $\mathbb{S} = \mathbb{R}$, $\mathbb{S}^2 = \mathbb{C}$ and $M \in \mathbb{Z}^+$.
2) The space of the defined hypercomplex numbers forms *non-distributive normed division algebra* that holds applicability and generalizability to all finite higher dimensions.
3) In order to be consistent with the traditional theory of the $\mathbb{R}$ and $\mathbb{C}$ spaces along with the geometry of the vectors in the corresponding spaces, we introduced a new multiplication operation called scaling and rotative (SR) multiplication that is a natural inhabitant of the Spherical Coordinate System (SCS). Unlike the traditional multiplication over the Cartesian space that appears to be derived from addition, the introduced SR multiplication is completely different from addition. As a consequence, it does not follow the distributive property leading to non-distributive normed division algebra.
4) Unlike the quaternions and octonions, these generalized hypercomplex number systems do not have infinite roots for polynomials of degree n, but have a minimum of n real roots and a maximum of $(M-1)n$ complex roots, where *M* is the dimension of the number system. This is an interesting result because it *updates* the fundamental theorem of algebra for hypercomplex number systems with more than *n* but a finite number of roots, such that this theorem reduces to the original theorem for dimensions 1 and 2.
5) These hypercomplex numbers and the corresponding algebras reduce to distributive normed algebras for dimensions 1 and 2. Likewise, the *updated* fundamental theorem of algebra reduces to the original theorem for dimensions 1 and 2. This shows backward compatibility. In other words, the introduced concept appears to be a true generalization of $\mathbb{C}$ in higher dimensions.

II. Preliminaries

A field is a set F with two binary operations on F called addition $(+)$ and multiplication $(\cdot)$ where binary operation on F is a mapping $F \times F \to F$ such that it satisfies the following field axioms for all $z_1, z_2, z_3 \in F$

1) Associativity of addition and multiplication: $z_1+(z_2+z_3) = (z_1+z_2)+z_3,$ and $z_1\cdot(z_1\cdot z_2) = (z_1\cdot z_2)\cdot z_3$
2) Commutativity of addition and multiplication: $z_1 + z_2 = z_2 + z_1$ and $z_1 \cdot z_2 = z_2 \cdot z_1$
3) Additive and multiplicative identities: for every $z \in F$, there exist two different elements 0 and 1 in F such that $z + 0 = z$ and $z \cdot 1 = z$
4) Additive and multiplicative inverses: for every $z \in F$, $\exists - z \in F$, called the additive inverse of z, such that $z + (-z) = 0$; and for every $z \in F$, $\exists\, z^{-1}$ or $1/z$ in F, called the multiplicative inverse of z, such that $z \cdot z^{-1} = 1$
5) Distributivity of multiplication over addition:
$z_1 \cdot (z_2 + z_3) = (z_1 \cdot z_2) + (z_1 \cdot z_3)$.

If multiplication is not commutative in a field, it is known as the skew field. Moreover, if multiplication is not distributive over addition, we designate it as a non-distributive field (NDF).

III. Proposed Generalized Hypercomplex Number System

In this section, first we define three dimensional (3D) numbers (denoted as set $\mathbb{S}^3$) as a true extension of existing two dimensional numbers ($\mathbb{C}$) that we denote as $\mathbb{S}^2$, i.e., $\mathbb{C} = \mathbb{S}^2$.

A. *Proposed* 3D *Hypercomplex Number System*

We consider 3D numbers from the set $\mathbb{S}^3$ as

$$z = a + ib + jc, \tag{1}$$

such that $i^2 = -1$ and $a, b, c \in \mathbb{R}$. Here, we have assumed $j^2 = -1$, the proof for which is provided later before *Remark 1*. First, we write (1) in the spherical coordinate system (SCS) as

$$a = r\cos(\phi)\cos(\theta), \quad b = r\cos(\phi)\sin(\theta), \quad c = r\sin(\phi), \tag{2}$$

$$r = \sqrt{a^2+b^2+c^2}, \quad \theta = \tan^{-1}\left(\frac{b}{a}\right), \quad \phi = \tan^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right), \tag{3}$$

where the azimuth angle $\theta \in (-\pi, \pi]$ and the elevation angle $\phi \in [-\pi/2, \pi/2]$ as shown in Fig. 1. This is to note that the conventional notation of spherical coordinate system is not considered in this work[1]. The elevation angle ϕ is measured from the x-y plane, i.e., if elevation $\phi = 0$, the point is in the x-y plane, and if elevation $\phi = \pi/2$, then the point is on the positive z-axis. If $\phi \in (0, \pi/2]$, the point is above the x-y plane, and if $\phi \in [-\pi/2, 0)$, the point is below the x-y plane.

To obtain the generalized multiplication of these numbers, we write (1) using SCS in triplet notations as

$$z_1 = \begin{bmatrix} r_1 \\ \theta_1 \\ \phi_1 \end{bmatrix}, \quad z_2 = \begin{bmatrix} r_2 \\ \theta_2 \\ \phi_2 \end{bmatrix}, \quad \text{and} \quad z_3 = \begin{bmatrix} r_3 \\ \theta_3 \\ \phi_3 \end{bmatrix}. \tag{4}$$

Two special cases of (2) are (i) when $z = jc$, both a and b are zeros. In this case, θ is indeterminate. However, since $\phi = \frac{\pi}{2}$ in (3), any value of θ in (2) will lead to $a = b = 0$, and we consider $\theta = 0$ to make it unique. (ii) when $z = a + jc$, then $\theta = 0$ if $a > 0$, and $\theta = \pi$ if $a < 0$.

Further, we define a new multiplication operation, named hereby the scaling and rotative (SR) multiplication (SRM), as

$$z_1 z_2 = \begin{bmatrix} r_1 r_2 \\ \theta_1+\theta_2 \\ \phi_1+\phi_2 \end{bmatrix}, \quad z_1 z_3 = \begin{bmatrix} r_1 r_3 \\ \theta_1+\theta_3 \\ \phi_1+\phi_3 \end{bmatrix}, \quad z_2 z_3 = \begin{bmatrix} r_2 r_3 \\ \theta_2+\theta_3 \\ \phi_2+\phi_3 \end{bmatrix}, \tag{5}$$

[1]One can also use the conventional notation of spherical coordinate system with $z = c + ia + jb$ where $a = r\cos(\theta)\sin(\phi)$, $b = r\sin(\theta)\sin(\phi)$, $c = r\cos(\phi)$, $r = \sqrt{a^2+b^2+c^2}$, $\theta = \tan^{-1}(b/a)$, $\phi = \tan^{-1}(\sqrt{a^2+b^2}/c)$, and $\phi \in [0, \pi]$ is the angle from the c coordinate axis (or the real axis). This will also lead to the above 3D hypercomplex number system, but any arbitrary choice of system will not yield the desired results.

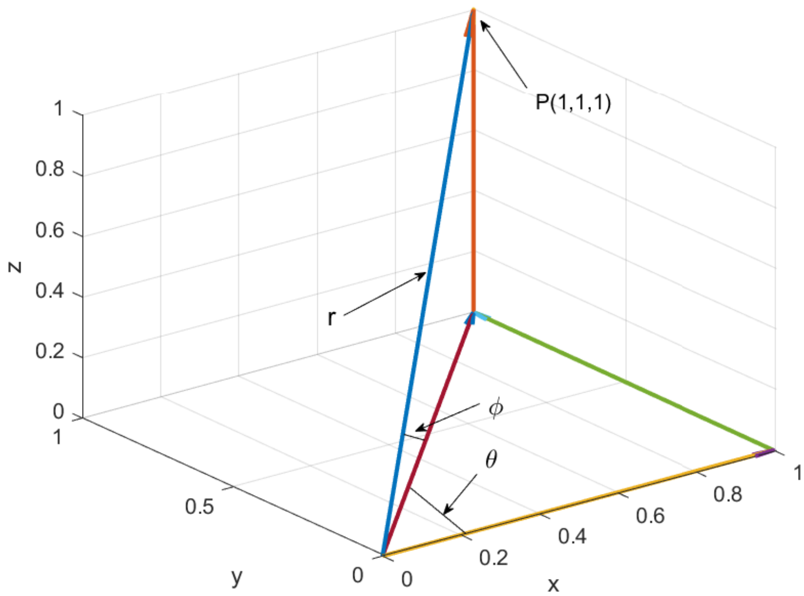

Fig. 1. A point $P = 1 + i + j$ in the considered spherical co-ordinate system where radius ($r = \sqrt{3}$), azimuth angle ($\theta = \pi/4$ rad) and elevation angle ($\phi = \arctan\big(1/\sqrt{2}\big) = 0.615479709$ rad) are shown.

and likewise, SR division (SRD) as

$$z_1/z_2 = \begin{bmatrix} r_1/r_2 \\ \theta_1-\theta_2 \\ \phi_1-\phi_2 \end{bmatrix},\ z_1/z_3 = \begin{bmatrix} r_1/r_3 \\ \theta_1-\theta_3 \\ \phi_1-\phi_3 \end{bmatrix},\ z_2/z_3 = \begin{bmatrix} r_2/r_3 \\ \theta_2-\theta_3 \\ \phi_2-\phi_3 \end{bmatrix}, \tag{6}$$

where we have assumed that $r_2, r_3 \neq 0$ in (6). The multiplication operation defined in (2) consists of scaling and rotation operations such that $\|z_1 z_2\| = \|z_1\|\|z_2\|$. The defined SRM operation reduces to the traditional multiplication when we move from 3D to 2D by considering $c = 0$ in (1).

The complex conjugate of (1) is defined as $\bar{z} = a - ib - jc$. This can be written in the triplet notation as

$$\bar{z} = \begin{bmatrix} r \\ -\theta \\ -\phi \end{bmatrix}, \tag{7}$$

such that $\|z\bar{z}\| = \|z\|^2 = r^2 \implies \|z\| = r$. Similarly, the multiplicative inverse of (1) is defined as $z^{-1} = \frac{\bar{z}}{z\bar{z}}$. This can be written in the triplet notation as

$$z^{-1} = \begin{bmatrix} 1/r \\ -\theta \\ -\phi \end{bmatrix}, \tag{8}$$

where this result can also be obtained from (6).

Addition of two complex numbers (e.g., $z_2 + z_3$) can be written as

$$\begin{aligned} z_2 + z_3 &= (a_2 + a_3) + i(b_2 + b_3) + j(c_2 + c_3) = \begin{bmatrix} r_2 \\ \theta_2 \\ \phi_2 \end{bmatrix} + \begin{bmatrix} r_3 \\ \theta_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix}, \\ \text{where} \quad r &= \sqrt{r_2^2 + r_3^2 + 2r_2 r_3 \left[\cos(\phi_2)\cos(\phi_3)\cos(\theta_2 - \theta_3) + \sin(\phi_2)\sin(\phi_3)\right]}, \\ \theta &= \tan^{-1}\left(\frac{r_2\cos(\phi_2)\sin(\theta_2) + r_3\cos(\phi_3)\sin(\theta_3)}{r_2\cos(\phi_2)\cos(\theta_2) + r_3\cos(\phi_3)\cos(\theta_3)}\right), \\ \text{and} \quad \phi &= \tan^{-1}\left(\frac{r_2\sin(\phi_2) + r_3\sin(\phi_3)}{r_2^2\cos^2(\phi_2) + r_3^2\cos^2(\phi_3) + 2r_2 r_3\cos(\phi_2)\cos(\phi_3)\cos(\theta_2 - \theta_3)}\right). \end{aligned} \tag{9}$$

The additive inverse of an element z_2 is given by

$$-z_2 = -a_2 - ib_2 - jc_2 = \begin{bmatrix} r_2 \\ \theta_2+\pi \\ -\phi_2 \end{bmatrix} = \begin{bmatrix} r_2 \\ \theta_2 \\ \phi_2 \end{bmatrix} \begin{bmatrix} 1 \\ \pi \\ \pi \end{bmatrix} = z_2(-1). \tag{10}$$

Thus, we conclude that $(\phi \pm \pi)$ is same as $-\phi$ due to related geometry as shown in Fig. 2. This operation is captured with the help of a new modulo operation as shown in Fig. 3 defined on angle ϕ as $(\phi \pm \pi) \bmod_c \pi = -\phi$. Because the above defined SR multiplication is not derived from addition, it does not follow the distributive property, and thus, in general, $z_1(z_2+z_3) \neq z_1z_2 + z_1z_3$. This is a desired property of the defined SRM because, geometrically, the operation on the left side is different from that on the right side.

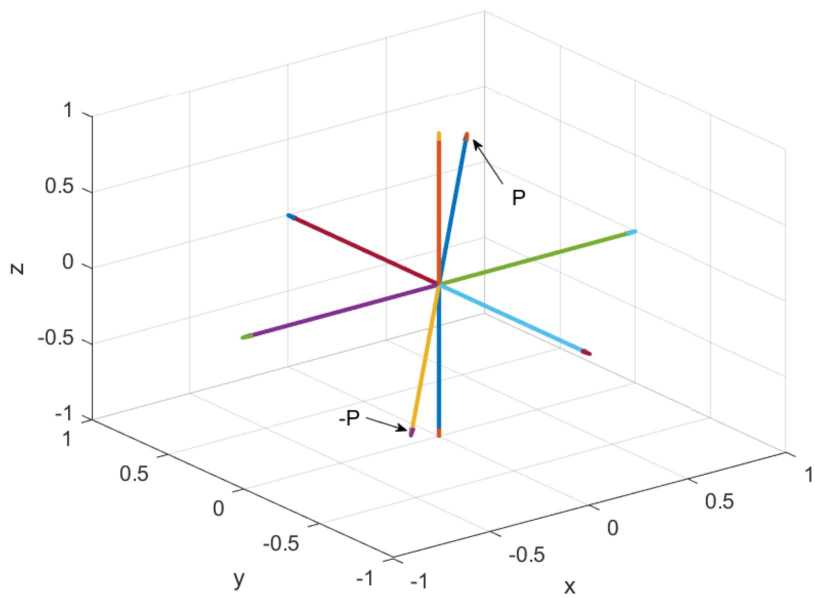

Fig. 2. A point $P = \frac{1}{2\sqrt{2}}(\sqrt{3} + i\sqrt{3} + j\sqrt{2})$ in the considered spherical co-ordinate system where radius $r = 1$, azimuth angle $\theta = \pi/4$ and elevation angle $\phi = \pi/6$ are shown, and point $-P$ with $r = 1$, $\theta = 5\pi/4$ and elevation angle $\phi = -\pi/6$ are also shown.

Result 1. *The distributive property of the defined SR multiplication over addition (i.e., $z_1(z_2+z_3) = z_1z_2 + z_1z_3$) can be guaranteed if $z_1 \in \mathbb{R}$.*

Here, it is interesting to observe that, $i^2 = -1 = \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \pi \\ 0 \end{bmatrix}$ and $j^2 = -1 = \begin{bmatrix} 1 \\ \pi/2 \\ \pi/2 \end{bmatrix} \begin{bmatrix} 1 \\ \pi/2 \\ \pi/2 \end{bmatrix} = \begin{bmatrix} 1 \\ \pi \\ \pi \end{bmatrix} = \begin{bmatrix} 1 \\ \pi \\ 0 \end{bmatrix}$ due to modulo π operation on the elevation angle ϕ and following the geometry of multiplication, and thus, $-1 = \begin{bmatrix} 1 \\ \pi \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \pi \\ \pi \end{bmatrix}$.

Further, if $z_1 = \cos(\phi_1) + j\sin(\phi_1)$ and $z_2 = \cos(\phi_2) + j\sin(\phi_2)$, then $z_1z_2 = \begin{bmatrix} 1 \\ 0 \\ \phi_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \phi_1+\phi_2 \end{bmatrix}$, and, in general, $z_1^m z_2^n = \begin{bmatrix} 1 \\ 0 \\ m\phi_1+n\phi_2 \end{bmatrix} = \cos(m\phi_1 + n\phi_2) + j\sin(m\phi_1 + n\phi_2)$. In fact, the new imaginary number j can be written as $j = \begin{bmatrix} 1 \\ \theta \\ \pi/2 \end{bmatrix}$ for any $\theta \in [0, 2\pi)$ which implies $j^2 = \begin{bmatrix} 1 \\ 2\theta \\ 0 \end{bmatrix}$, and thus, it can have infinite number of representations. For examples (i) $j^2 = 1$ when $\theta = 0$, (ii) $j^2 = -1$ when $\theta = \pi/2$, (iii) $j^2 = i$ when $\theta = \pi/4$, (iv) $j^2 = -i$ when $\theta = 3\pi/4$, and (v) in general $j^2 = \cos(2\theta) + i\sin(2\theta)$ for $\theta \in [0, 2\pi)$. However, to obtain uniqueness in the presentation, one may consider $\theta = \pi/2$, i.e., $j = \begin{bmatrix} 1 \\ \pi/2 \\ \pi/2 \end{bmatrix}$, which provides $j^2 = -1$, and $-j = \begin{bmatrix} 1 \\ -\pi/2 \\ -\pi/2 \end{bmatrix}$ that leads to $j(-j) = 1$.

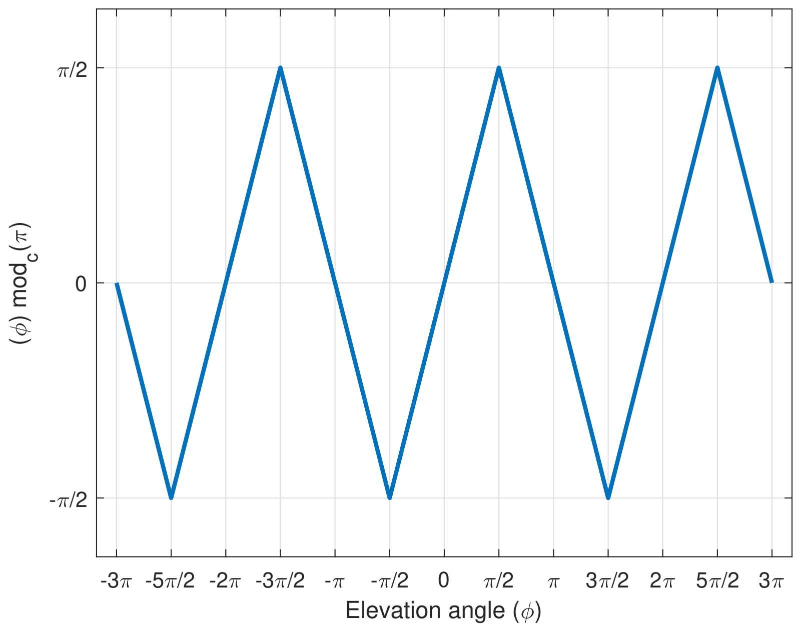

Fig. 3. Elevation phase angle (ϕ) and corresponding defined modulo operation.

Remark 1. *By not considering the units, the traditional multiplication can be considered as a repetitive addition up to rational numbers. In all practical applications, we always consider real numbers up to a finite precision only. Thus, in practice, real numbers are used as rational numbers. Therefore, the traditional multiplication works as if derived from addition. This leads to the distributivity of multiplication over addition. Thus, one can observe that of the two binary operations (i.e., $+$ and $\cdot$), one seems redundant.*

It is pertinent to note that the defined SR multiplication is backward compatible with the traditional (existing) multiplication for the complex number system. In fact, it is a generalization of the traditional multiplication to higher dimensional hypercomplex number systems. To demonstrate this, we present the following results.

Theorem 1. *A non-distributive normed division (ND2) algebra, as defined in* (1)–(10)*, is a number system where one can add, subtract, multiply and divide, and satisfy the norm $\|z_1 z_2\| = \|z_1\|\|z_2\|$. Further, this algebra is of dimension $M = 3$, and becomes distributive when $M \in [1, 2]$.*

Proof. To prove the theorem, we have to prove that the 3D numbers given in (1)–(10) satisfy the axioms of *non-distributive field* for all $z_1, z_2, z_3 \in \mathbb{S}^3$:

1) Associativity of addition and multiplication: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 = (a_1 + a_2 + a_3) + i(b_1 + b_2 + b_3) + j(c_1 + c_2 + c_3)$ and $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3 = \begin{bmatrix} r_1 r_2 r_3 \\ \theta_1+\theta_2+\theta_3 \\ \phi_1+\phi_2+\phi_3 \end{bmatrix}$
2) Commutativity of addition and multiplication: $z_1 + z_2 = z_2 + z_1 = (a_1 + a_2) + i(b_1 + b_2) + j(c_1 + c_2)$ and $z_1 \cdot z_2 = z_2 \cdot z_1 = \begin{bmatrix} r_1 r_2 \\ \theta_1+\theta_2 \\ \phi_1+\phi_2 \end{bmatrix}$
3) Additive and multiplicative identities: For every $z \in \mathbb{S}^3$, there exist two different elements 0 and 1 in $\mathbb{S}^3$ such that $z + 0 = z$ and $z \cdot 1 = z$.
4) Additive and multiplicative inverses: For every $z \in F$, $\exists - z \in F$ called the additive inverse of z such that $z + (-z) = 0$; and for every $z \in F$, $\exists\, z^{-1}$ or $1/z$ in F called the multiplicative inverse of z such that $z \cdot z^{-1} = 1$
5) Distributivity of multiplication over addition: In general, this is not true because
$z_1 \cdot (z_2 + z_3) \neq (z_1 \cdot z_2) + (z_1 \cdot z_3)$

$\begin{bmatrix} r_1 \\ \theta_1 \\ \phi_1 \end{bmatrix} \cdot \left(\begin{bmatrix} r_2 \\ \theta_2 \\ \phi_2 \end{bmatrix} + \begin{bmatrix} r_3 \\ \theta_3 \\ \phi_3 \end{bmatrix} \right) \neq \begin{bmatrix} r_1 r_2 \\ \theta_1+\theta_2 \\ \phi_1+\phi_2 \end{bmatrix} + \begin{bmatrix} r_1 r_3 \\ \theta_1+\theta_3 \\ \phi_1+\phi_3 \end{bmatrix}$.

Thus, 3D numbers given in (1)–(10) satisfy the axioms of non-distributive field, which completes the proof. □

Result 2. *Now, we present two important results of using* $j^2 = \pm 1 \Leftrightarrow j^3 = \pm j$ *and* $j^2 = \pm 1 \implies j^4 = 1$ *as follows:*

$$e^{j\phi} = 1 + \frac{j\phi}{1!} + \frac{(j\phi)^2}{2!} + \frac{(j\phi)^3}{3!} + \frac{(j\phi)^4}{4!} + \frac{(j\phi)^5}{5!} + \cdots + \frac{(j\phi)^n}{n!} + \cdots \tag{11}$$

On using $j^2 = -1$, $j^3 = -j$ *and* $j^4 = 1$, *one can easily obtain Euler identity as*

$$e^{j\phi} = \left[1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \cdots\right] + j\left[\frac{\phi}{1!} - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \cdots\right], \tag{12}$$

$$= \cos(\phi) + j\sin(\phi). \tag{13}$$

Interestingly, on using $j^2 = 1$, $j^3 = j$ *and* $j^4 = 1$, *we obtain hyperbolic Euler type identity as*

$$e^{j\phi} = \left[1 + \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \frac{\phi^6}{6!} + \cdots\right] + j\left[\frac{\phi}{1!} + \frac{\phi^3}{3!} + \frac{\phi^5}{5!} + \frac{\phi^7}{7!} + \cdots\right], \tag{14}$$

$$= \cosh(\phi) + j\sinh(\phi). \tag{15}$$

Unless otherwise stated, $j^2 = -1$ is considered in the theory presented in this work.

Example. *Now, we consider an interesting example that will be useful in obtaining next result as follows: Let* $z_1 = r_1[\cos(\theta) + i\sin(\theta)] = r_1 e^{i\theta} \implies z_1 = \begin{bmatrix} r_1 \\ \theta \\ 0 \end{bmatrix}$ *and* $z_2 = r_2[\cos(\phi) + j\sin(\phi)] = r_2 e^{j\phi} \implies z_1 = \begin{bmatrix} r_2 \\ 0 \\ \phi \end{bmatrix}$, *where* $r_1, r_2 > 0$. *Using new multiplication defined in* (5) *we obtain* $z_1 z_2 = \begin{bmatrix} r_1 \\ \theta \\ 0 \end{bmatrix}\begin{bmatrix} r_1 \\ 0 \\ \phi \end{bmatrix} = \begin{bmatrix} r_1 r_2 \\ \theta \\ \phi \end{bmatrix}$. *Using* (1) *to* (4), *this can be written as* $z_1 z_2 = r_1 r_2 e^{i\theta} e^{j\phi} = r_1 r_2[\cos(\phi)\cos(\theta) + i\cos(\phi)\sin(\theta) + j\sin(\phi)]$, *and thus,*

$$re^{i\theta} e^{j\phi} = r[\cos(\phi)\cos(\theta) + i\cos(\phi)\sin(\theta) + j\sin(\phi)]. \tag{16}$$

Result 3. *We observe that if* $z = a + ib + jc$, *then* $e^z = e^a e^{ib} e^{jc} = e^a(\cos(b) + i\sin(b))(\cos(c) + j\sin(c)) = \begin{bmatrix} e^a \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ b \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} e^a \\ b \\ c \end{bmatrix}$. *Thus,* $\log(e^z) = z \implies \log\left(\begin{bmatrix} e^a \\ b \\ c \end{bmatrix}\right) = a + ib + jc$. *Therefore, if we consider any* $3D$ *hypercomplex number* $z = a + ib + jc \implies z = \begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix}$, *then* $\log(z) = \log(r) + i\theta + j\phi \implies z = re^{i\theta} e^{j\phi}$, *and it will reduce to the traditional* $2D$ *complex number system if* $c = 0$ *and hence,* $\phi = 0$.

Thus, the proposed $3D$ hypercomplex number system is a true generalization of the existing $2D$ complex number system. To obtain the multiplication of two numbers, we can use *Result 3* as follows: $\log(z_1) = \log(r_1) + i\theta_1 + j\phi_1$ and $\log(z_2) = \log(r_2) + i\theta_2 + j\phi_2$ and thus, $\log(z_1) + \log(z_2) = \log(z_1 z_2) = \log(r_1 r_2) + i(\theta_1 + \theta_2) + j(\phi_1 + \phi_2) \implies z_1 z_2 = \begin{bmatrix} r_1 r_2 \\ \theta_1+\theta_2 \\ \phi_1+\phi_2 \end{bmatrix}$. Therefore, we conclude that the addition of hypercomplex numbers is naturally defined in the Cartesian coordinates and *multiplication is naturally defined in the spherical coordinates through the natural logarithmic addition*.

B. Examples of the Proposed 3D *Hypercomplex Numbers*

Example 1: For a quadratic equation (QE), $ax^2 + bx + c = 0$, $a, b, c \in \mathbb{R}$, $a \neq 0$, there are either two real roots $x = (-b \pm \sqrt{b^2 - 4ac})/2a$ when $(b^2 - 4ac) \geq 0$, or four $3D$ complex roots, corresponding to the case $((b^2 - 4ac) < 0)$, which are $x = \alpha \pm i\beta$ and $x = \alpha \pm j\beta$, where $\alpha, \beta \in \mathbb{R}$ and $(x - (\alpha + i\beta))(x - (\alpha - i\beta)) = 0$. Thus, one can obtain $x^2 + x(-2\alpha) + \alpha^2 + \beta^2 = 0$, which provides four $3D$ complex roots of the QE where $\alpha = -b/2a$ and $\alpha^2 + \beta^2 = (c/a) \implies \beta^2 = (4ac - b^2)/4a^2 \implies \beta = \pm\sqrt{(4ac - b^2)}/2a$.

For example, let us consider the QE $x^2 + 1 = 0$. If $x \in \mathbb{R}$, then there are no real roots. If $x \in \mathbb{S}^2$ (traditionally, $x \in \mathbb{C}$), then there are two roots $x = \pm i$ where $i^2 = -1$. If $x \in \mathbb{S}^3$, then there are four roots

$x = \pm i, \pm j$ as $\alpha = 0$ and $\beta^2 = 1 \implies \beta = \pm 1$. This can be generalized as follows: if $x \in \mathbb{S}^M$, then there are $2(M-1)$ roots, two in each of the j_m imaginary axis, i.e., $x = \pm j_m^2 = -1$ for $m = 1, 2, \cdots, M-1$. We can conclude the above observation as follows:

Result 4. *A quadratic equation can have either two real roots that are minimum or $2(M-1)$ complex roots for $x \in \mathbb{S}^M$.*

Next, we present the fundamental theorem of algebra for reference, and provide its extension for MD complex numbers.

Theorem 2. *(Fundamental Theorem of Algebra). Let $p(z)$ be a polynomial with real coefficients of degree $n \geq 1$. Then, $p(z)$ has n roots.*

Since the complex roots of the polynomial $p(z)$ are always in pairs, let us assume that r out of n roots are real. Then, $(n-r)$ roots are complex such that $(n-r) \geq 0$ is an even number. Using this, we can easily extend the fundamental theorem of algebra as follows:

Theorem 3. *(Updated Fundamental Theorem of Algebra). Let $p(z)$ be a polynomial with real coefficients of degree $n \geq 2$, where $z \in \mathbb{S}^M$ with $M \geq 2$. Then, $p(z)$ has r real roots and $(M-1)(n-r)$ complex roots such that $(n-r) \geq 0$ is an even number. Thus, there are a minimum of n real roots, and a maximum of $(M-1)n$ complex roots.*

Example 2: In this example, we demonstrate that the SR multiplication does not distribute over addition in this generalized hypercomplex number system. Let us consider $z_1 = 1$, $z_2 = i$, $z_3 = j$, which can be written using (3), (4), and (9) as

$$z_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix}, \quad z_3 = \begin{bmatrix} 1 \\ \pi/2 \\ \pi/2 \end{bmatrix} \implies z_1 + z_2 = \begin{bmatrix} \sqrt{2} \\ \pi/4 \\ 0 \end{bmatrix}, \quad z_1 + z_3 = \begin{bmatrix} \sqrt{2} \\ 0 \\ \pi/4 \end{bmatrix}, \quad z_2 + z_3 = \begin{bmatrix} \sqrt{2} \\ \pi/2 \\ \pi/4 \end{bmatrix}.$$

Therefore, using (5) and (9), one can compute

$$z_1 z_2 = \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix}, \quad z_1 z_3 = \begin{bmatrix} 1 \\ \pi/2 \\ \pi/2 \end{bmatrix}, \quad z_2 z_3 = \begin{bmatrix} 1 \\ \pi \\ \pi/2 \end{bmatrix}, \quad z_3(z_1 + z_2) = \begin{bmatrix} \sqrt{2} \\ 3\pi/4 \\ \pi/2 \end{bmatrix}, \quad z_1 z_3 + z_2 z_3 = \begin{bmatrix} 2 \\ \pi/2 \\ \pi/2 \end{bmatrix},$$

and thus, $z_3(z_1 + z_2) \neq z_1 z_3 + z_2 z_3$.

C. Geometrical Insights into the Generalized Hypercomplex Number System

We note that algebraically, the additional imaginary axis j considered in $\mathbb{S}^{\mathbb{M}}$ behaves similar to i. For example, $i^2 = -1$ and $j^2 = -1$. Similarly, one can also show that $(1+j)^2 = 2j$ and $(1-j)^2 = -2j$. Similar identities are satisfied by i. Moreover, this j axis geometrically plays interestingly on the hypercomplex numbers. If there is a point $P = a + ib$ in the complex x-y plane and if it is multiplied by i, then that point will rotate counterclockwise by $\pi/2$, i.e., new point $Q = (a+ib)i = re^{i(\theta+\pi/2)} = \begin{bmatrix} r \\ \theta+\pi/2 \\ 0 \end{bmatrix}$. Similarly, $P = a + ib \implies P = \begin{bmatrix} r \\ \theta \\ 0 \end{bmatrix}$ and $Q = (a+ib)j \implies Q = \begin{bmatrix} r \\ \theta \\ \pi/2 \end{bmatrix}$. Thus in the proposed 3D hypercomplex number system, one can rotate a point in both θ and ϕ directions with desired angles.

D. *Generalized (*MD*) Hypercomplex Number System*

The 3D hypercomplex number system can be generalized to the MD hypercomplex number $\mathbb{S}^M$ system by using the generalized MD spherical coordinate system as

$$\begin{aligned}
&d_0 = r\cos(\theta_3)\cos(\theta_2)\cos(\theta_1),\\
&d_1 = r\cos(\theta_3)\cos(\theta_2)\sin(\theta_1), \quad \theta_1 \in (-\pi,\pi],\\
&d_2 = r\cos(\theta_3)\sin(\theta_2), \quad \theta_2 \in [-\pi/2,\pi/2],\\
&d_3 = r\sin(\theta_3), \quad \theta_3 \in [-\pi/2,\pi/2],\\
&\theta_1 = \tan^{-1}\left(\frac{d_1}{d_0}\right), \quad \theta_2 = \tan^{-1}\left(\frac{d_2}{\sqrt{d_0^2+d_1^2}}\right), \quad \theta_3 = \tan^{-1}\left(\frac{d_3}{\sqrt{d_0^2+d_1^2+d_2^2}}\right),\\
&r = \sqrt{d_0^2+d_1^2+d_2^2+d_3^2},
\end{aligned} \tag{17}$$

and, in general,

$$\begin{aligned}
d_0 &= r\cos(\theta_{M-1})\cos(\theta_{M-2})\cdots\cos(\theta_2)\cos(\theta_1),\\
d_1 &= r\cos(\theta_{M-1})\cos(\theta_{M-2})\cdots\cos(\theta_2)\sin(\theta_1), \quad \theta_1 \in (-\pi,\pi],\\
d_2 &= r\cos(\theta_{M-1})\cos(\theta_{M-2})\cdots\sin(\theta_2), \quad \theta_2 \in [-\pi/2,\pi/2],\\
&\vdots\\
d_{M-3} &= r\cos(\theta_{M-1})\cos(\theta_{M-2})\sin(\theta_{M-3}), \quad \theta_{M-3} \in [-\pi/2,\pi/2],\\
d_{M-2} &= r\cos(\theta_{M-1})\sin(\theta_{M-2}), \quad \theta_{M-2} \in [-\pi/2,\pi/2],\\
d_{M-1} &= r\sin(\theta_{M-1}), \quad \theta_{M-1} \in [-\pi/2,\pi/2],
\end{aligned} \tag{18}$$

$$\begin{aligned}
\theta_1 &= \tan^{-1}\left(\frac{d_1}{d_0}\right), \quad \theta_2 = \tan^{-1}\left(\frac{d_2}{\sqrt{d_0^2+d_1^2}}\right), \cdots,\\
\theta_{M-1} &= \tan^{-1}\left(\frac{d_{M-1}}{\sqrt{d_0^2+d_1^2+\cdots+d_{M-2}^2}}\right),\\
r &= \sqrt{d_0^2+d_1^2+\cdots+d_{M-2}^2+d_{M-1}^2},
\end{aligned} \tag{19}$$

and thus, we write MD hypercomplex number as

$$z = d_0 + j_1 d_1 + \cdots + j_{M-2} d_{M-2} + j_{M-1} d_{M-1}, \tag{20}$$

where $j_m^2 = -1$ for $m = 1, 2, \cdots, M-1$, and M-tuple representations are

$$1 = \begin{bmatrix}1\\0\\0\\0\\\vdots\\0\end{bmatrix}, \quad j_1 = \begin{bmatrix}1\\\pi/2\\0\\0\\\vdots\\0\end{bmatrix}, \quad j_2 = \begin{bmatrix}1\\\pi/2\\\pi/2\\0\\\vdots\\0\end{bmatrix}, \quad j_3 = \begin{bmatrix}1\\\pi/2\\\pi/2\\\pi/2\\\vdots\\0\end{bmatrix}, \quad \cdots, \quad j_{M-1} = \begin{bmatrix}1\\\pi/2\\\pi/2\\\pi/2\\\vdots\\\pi/2\end{bmatrix}, \quad -1 = \begin{bmatrix}1\\\pi\\\pi\\\pi\\\vdots\\\pi\end{bmatrix}. \tag{21}$$

To obtain the generalized multiplication of these numbers we write (19) and (20) using SCS in M-tuple notations as

$$z_1 = \begin{bmatrix}r_1\\\theta_{1,1}\\\theta_{2,1}\\\vdots\\\theta_{M-1,1}\end{bmatrix}, \quad z_2 = \begin{bmatrix}r_2\\\theta_{1,2}\\\theta_{2,2}\\\vdots\\\theta_{M-1,2}\end{bmatrix}, \quad z_3 = \begin{bmatrix}r_3\\\theta_{1,3}\\\theta_{2,3}\\\vdots\\\theta_{M-1,3}\end{bmatrix} \tag{22}$$

and hereby define the scaling and rotative (SR) multiplication (SRM) as

$$z_1 z_2 = \begin{bmatrix}r_1 r_2\\\theta_{1,1}+\theta_{1,2}\\\theta_{2,1}+\theta_{2,2}\\\vdots\\\theta_{M-1,1}+\theta_{M-1,2}\end{bmatrix}. \tag{23}$$

IV. Conclusion

In this work, we have introduced generalized hypercomplex numbers and the associated algebra that exist for all finite higher dimensions. Interestingly, this framework reduces to the traditional theory of $\mathbb{R}$ and $\mathbb{C}$ spaces along with the geometry of the vectors in the corresponding spaces. In order to ensure this generalizability, an out-of-the-box solution is proposed with 1) non-distributive normed division algebra, 2) a new multiplication operation defined in the spherical coordinate system, which is also backward compatible to the multiplication operation of numbers in $\mathbb{C}$, and 3) an update on the fundamental theorem of algebra.

References

[1] Frobenius FG. Über lineare Substitutionen und bilineare. Formen J Reine Angew Math. 84:1–63, 1878.

[2] Hurwitz, A. Ueber die Composition der quadratischen Formen von beliebig vielen Variabeln, Nachr. Königl. Gesell. Wiss. Göttingen. Math.-Phys. Klasse, 309–316, 1898.

[3] Zorn M. Theorie der alternativen Ringe. Abh Math Sem Univ Hamburg. 8:123–147, 1930.

[4] Adams, J.F. On the Nonexistence of Elements of Hopf Invariant One, Bull. Amer. Math. Soc. 64, 279–282, 1958.

[5] Adams, J.F. On the Non-Existence of Elements of Hopf Invariant One, Ann. of Math. 72, 20–104, 1960.

[6] Kervaire M. Non-parallelizability of the n sphere for $n > 7$, Proc. Nat. Acad. Sci. USA 44 (1958), 280–283.

[7] Bott R., Milnor J. On the parallelizability of the spheres. Bull Amer Math Soc. 64:87–89, 1958.

[8] Girard, Patrick R. The quaternion group and modern physics, European Journal of Physics. 5:25–31, 1984.

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What was the motivation to explore hyper-complex numbers of all dimensions? In October 2021, Singh was introducing complex numbers to his daughter Prisha Singh (a student of High School). He was explaining that like $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, real numbers are also a subset of complex numbers ($\mathbb{R} \subset \mathbb{C}$). Then she asked a simple question, are complex numbers a subset of some other numbers? Where does it stop? If yes, then how and why? He told her that he would try to answer this question as soon as possible. Therefore, in order to answer this simple question, he started looking at an extension of complex numbers in the mathematical literature and found that it is not as simple as it looks. After about six months of steady and determined thinking on a general extension of complex numbers, he obtained a simple answer to the posed question. For this, he collaborated with A. Gupta (IIIT Delhi) and SD Joshi (IIT Delhi) and had many extensive discussions on the proposed solution ($\mathbb{S} \subset \mathbb{S}^2 \subset \mathbb{S}^3 \subset \cdots \subset \mathbb{S}^M$, where $\mathbb{S} = \mathbb{R}$, $\mathbb{S}^2 = \mathbb{C}$). In between, Prisha kept keen watch and asked intermittently about the progress to answer the posed question? This question is of fundamental importance, and it may seed new dimensions in the theory of number systems. We hope that finally it has been answered after many months of hard efforts.

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