WELL-POSEDNESS AND EXPONENTIAL STABILITY OF THE KAWAHARA EQUATION WITH A TIME-DELAYED LOCALIZED DAMPING

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Abstract. The aim of this article is to investigate the well-posedness and stability problems of the so-called Kawahara equation under the presence of an interior delayed damping. The system is shown to be well-posed. Furthermore, we prove that the trivial solution is exponentially stable in spite of the delay effect. Specifically, local and semi-global stability results are established according to the properties of the spatial distribution of the delay term.

Keywords: Nonlinear Kawahara equation; localized damping; time-delay; exponential stability.

1. Introduction and Preliminaries

The main concern of this article is to deal with the well-posedness and stability of an initial-boundary-value problem related to a fifth order dispersive partial differential equation (PDE) with localized time-delayed damping

$$
\begin{cases}
\partial_t u(x,t) + \partial_x^3 u(x,t) - \eta \partial_x^5 u(x,t) + u(x,t)\partial_x u(x,t) + \alpha \partial_x u(x,t) & \\
\qquad\qquad\qquad + a(x)u(x,t) + b(x)u(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty), \\
u(0,t) = u(\ell,t) = \partial_x u(0,t) = \partial_x u(\ell,t) = \partial_x^2 u(\ell,t) = 0, & t > 0, \\
u(x,0) = u_0(x), & x \in \Omega, \\
u(x,t) = z_0(x,t), & x \in \Omega,\ t \in \mathcal{T},
\end{cases}
\tag{1.1}
$$

where u represents the amplitude of the dispersive wave, $\ell > 0$, $\Omega = (0,\ell)$, $\alpha \geq 0$ and $\eta > 0$ are physical parameter of the dispersive equation. In turn, $\tau > 0$ is the time-delay and $\mathcal{T} = (-\tau, 0)$, while u_0 and z_0 are initial conditions. Finally, $a(\cdot)$ and $b(\cdot)$ are spatial distributions of the actuator and satisfy the following properties:

> **A1:** $a(\cdot)$ and $b(\cdot)$ are nonnegative functions of $L^\infty(\Omega)$ and b is positive on an open subset ω of Ω, that is, $b(x) \geq b_0$ a.e. in ω, for some positive constant b_0. In turn, a is positive on an open subset $\widehat{\omega}$.

The PDE in (1.1) is known in literature as Kawahara equation and commonly used to model several physical phenomena [18, 19]. As a matter of fact, the equation is a special case of the well-known Benney-Lin equation derived by Benney [3] and later by Lin [22]. With regard to the physical interpretation of the model, Kawahara equation may model a one-dimensional propagation of small-amplitude long waves in various problems of plasma physics and fluid dynamics [2, 18, 19, 32].

Concerning the mathematical endeavor, the Kawahara equation without delay has been the subject of numerous studies depending whether the spatial variable x belongs to the whole real line [9, 16] or half-line [10, 21] or a periodic domain [15, 17]. In addition of that, there exists a rich literature about the Kawahara equation in a non-periodic bounded domain [6, 10, 11, 12, 20, 21, 34], where the existence

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and uniqueness of solutions to the equation (without delay) along with its asymptotic behavior are proven. We also note that the equation has been studied in some papers from the control theory point of view [8, 13, 37, 38], whereas its generalized form is analysed in [39].

The novelty of this article is to take into account the time-delay phenomenon in the Kawahara system (1.1). To be more precise, the damping control proposed in [34] is supposed to have a delay. This idea is used for the Korteweg-de Vries equation [33]. Of course, the presence of a time-delay in the equation is motivated by the fact that in control systems, the sensors and actuators act under a delay. Thereby, it is crucial to study the impact of such a delay on the performance of the control proposed in [34] and attempt to eliminate or nullify any eventual negative effect of its presence.

The main contribution of the current work is to show that the Kawahara equation remains stable despite the presence of a localized interior damping. Such a desirable outcome is shown under two different circumstances of the spatial distribution function b. This result improves that of [34], where no delay occurs in the equation and also that of [33], where the equation is of third order. More importantly, unlike [33], we manage to obtain our stability results (see Theorem 2 and Theorem 3) without any smallness condition on the length ℓ.

Now, we briefly describe the plan of the paper. In section 2, we evoke an additional assumption **A2** and prove that the problem (1.1) is well-posed by combining semigroups theory and fixed point method. Then, two stability results are established. The first one is based on the energy method, while the second invokes an observability inequality and compactness arguments [29, 27]. Section 3 is devoted to the analysis of (1.1) with a sole assumption **A1**. Following a perturbation argument [26], well-posedness and stability findings are shown but at the expense of a condition on the parameters η and ℓ and the spatial distribution b of the delayed term.

2. The problem (1.1) with $\omega \subset \widehat{\omega}$

In addition of the assumption **A1**, we shall assume in this section that

A2: There exists a positive constant k such that $a(x) \geq k + b(x)$ a.e. in ω.

The energy of the system (1.1) takes the following form:

$$E(t) = \int_\Omega u^2(x,t)\,dx + \tau \int_\omega \int_0^1 \xi(x)(\partial_x u)^2(x, t-\tau\rho)\,d\rho dx, \tag{2.1}$$

where ξ is a nonnegative function of $L^\infty(\Omega)$, positive on ω and satisfies

$$b(x) + k \leq \xi(x) \leq 2a(x) - b(x) - k, \quad \text{a.e. in } \omega, \tag{2.2}$$

which is plausible thanks to **A2**.

Throughout this paper, $L^2(D)$ denotes the Hilbert space of square integrable scalar functions on an open set D of $\mathbb{R}^n$, whose norm will be denoted by $\|\cdot\|$. Then, we consider the Hilbert spaces

$$H := L^2(\Omega) \times L^2(\omega \times (0,1)), \quad \mathcal{H} := L^2(\Omega) \times L^2(\omega \times \mathcal{T}),$$

respectively equipped with the following inner product norms:

$$\begin{cases} \|(u,z)\|_H^2 = \|u\|^2 + \tau \displaystyle\int_\omega \int_0^1 \xi(x) z^2(x,\rho)\ d\rho dx, \\ \|(u,z)\|_{\mathcal{H}}^2 = \|u\|^2 + \displaystyle\int_\Omega \int_{\mathcal{T}} \xi(x) z^2(x,s)\,ds dx. \end{cases} \tag{2.3}$$

Obviously, the equivalence of the norms defined above and the standard ones is guaranteed by means of the assumptions **A1-A2** and (2.2).

In turn, we define, for $T > 0$, the Banach space

$$\mathcal{M} = C\Big([0,T];\, L^2(\Omega)\Big) \cap L^2\Big(0,T;\, H_0^2(\Omega)\Big),$$

endowed with the norm:

$$\|\cdot\|_{\mathcal{M}}^2 = \|\cdot\|^2_{C([0,T];\, L^2(\Omega))} + \|\cdot\|^2_{L^2(0,T;\, H_0^2(\Omega))}.$$

Now, we briefly present an overview of this paper. In Section 2.1, we consider the system (1.1) under the assumptions **A1-A2**. The well-posedness of the problem is established in H. The proof relies on the Fixed Point Theorem. Then, two exponential stability results of the trivial solution are obtained: the first one is local and its proof is based on the Lyapunov method, whereas the second one is semi-global and the proof invokes an observability result.

The Wirtinger's inequalities [14] (see also [Lemma 2.2 and Lemma 2.3, p. 838-839][35]):

$$\|\partial_x^i \varphi\| \leq (\ell/\pi)\|\partial_x^{i+1}\varphi\|,\ \forall \varphi \in H_0^{i+1}(\Omega), \quad i = 0, 1. \tag{2.4}$$

2.1. **Well-posedness of the problem (1.1).** This subsection is devoted to the local well-posedness result of the dispersive system (1.1).

2.1.1. *The linearized problem.* First, we consider the linear system associated to (1.1):

$$\begin{cases} \partial_t u(x,t) + \partial_x^3 u(x,t) - \eta\partial_x^5 u(x,t) + \alpha\partial_x u(x,t) + a(x)u(x,t) & \\ \qquad\qquad\qquad +b(x)u(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(0,t) = u(\ell,t) = \partial_x u(0,t) = \partial_x u(\ell,t) = \partial_x^2 u(\ell,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = z_0(x,t), & x \in \Omega,\ t \in \mathcal{T}. \end{cases} \tag{2.1}$$

Throughout this paper, we shall adopt the following notation: Given a function f (resp. g) on Ω (resp. ω), we denote by $\hat{f}$ (resp. $\tilde{g}$) the restriction of f on ω (resp. the extension of g by zero outside ω).

Whereupon, we define $z(x,\rho,t) := \widehat{u}(x, t-\tau\rho)$, for $(x,\rho,t) \in \omega \times (0,1) \times (0,\infty)$ [36] (see also [24]), which satisfies

$$\tau\partial_t z(x,\rho,t) + \partial_\rho z(x,\rho,t) = 0, \quad (x,\rho,t) \in \omega \times (0,1) \times (0,\infty). \tag{2.2}$$

Then, the system (2.1) can be formulated in H as follows:

$$\begin{cases} \Phi_t(t) = \mathcal{A}\Phi(t), \\ \Phi(0) = \Phi_0, \end{cases} \tag{2.3}$$

in which $\Phi = (u, z)$, $\Phi_0 = \big(u_0, \widehat{z_0}(\cdot, \tau\cdot)\big)$ and $\mathcal{A}$ is the operator defined by:

(2.4)

$$\mathcal{D}(\mathcal{A}) = \left\{\ (u,z) \in H; u \in H^5(\Omega) \cap H_0^2(\Omega), z \in L^2\Big(\omega, H^1(\Omega)\Big); \partial_x^2 u(\ell) = 0, z(x,0) = \widehat{u}(x) \text{ in } \omega\ \right\},$$

$$\mathcal{A}(u,z) = \left(\eta\partial_x^5 u - \partial_x^3 u - \alpha\partial_x u - a(\cdot)u - b(\cdot)\tilde{z}(\cdot,1), -\frac{1}{\tau}\partial_\rho z\right), \quad \text{for any } (u,z) \in \mathcal{D}(\mathcal{A}).$$

In the sequel, C denotes a positive constant that may depend on T, $\alpha, \eta, b_0, k, \|a\|_\infty$ and $\|b\|_\infty$ but independent of the initial data Φ_0.

The well-posedness and regularity estimates related to (2.1) are stated below:

Proposition 1. *Assume that the conditions **A1-A2** hold. Then, we have:*
(i) The linear operator $\mathcal{A}$ defined by (2.4) is densely defined in H and generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$. Thereby, given $\Phi_0 \in \mathcal{D}(\mathcal{A})$, the system (2.3) has a unique classical solution $\Phi(\cdot) \in C((0,\infty);\mathcal{D}(\mathcal{A})) \cap C^1((0,\infty);H)$. In turn, the system (2.3) admits a mild solution $\Phi(\cdot) \in C((0,\infty);H)$, whenever $\Phi_0 \in H$.
(ii) Given $\Phi_0 \in H$, we have the following estimates:

$$\begin{cases} \|z(\cdot,1,\cdot)\|^2_{L^2(\omega\times(0,T))} \le C\|\Phi_0\|^2_H, \\ \|\partial_x^2 u(0,\cdot)\|^2_{L^2(0,T)} + \|\sqrt{a(\cdot)}u\|^2_{L^2(0,T;L^2(\Omega))} \le C\|\Phi_0\|^2_H, \\ \|\widehat{z_0}(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))} \le C\left(\|z(\cdot,\cdot,T)\|^2_{L^2(\omega\times(0,1))} + \|z(\cdot,1,\cdot)\|^2_{L^2(\omega\times(0,T))}\right), \\ \|\partial_x^2 u\|^2_{L^2(0,T;L^2(\Omega))} \le C\|\Phi_0\|^2_H, \\ \|u_0\|^2 \le C\left(\|\partial_x^2 u(0,\cdot)\|^2_{L^2(0,T)} + \|u(\cdot)\|^2_{L^2(0,T;L^2(\Omega))} + \|\widehat{z_0}(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))}\right). \end{cases} \tag{2.5}$$

(iii) The map

$$\Upsilon : \Phi_0 \in H \to \Phi(\cdot) = e^{\cdot\mathcal{A}}\Phi_0 \in \mathcal{M} \times C\left([0,T];\, L^2(\omega\times(0,1))\right)$$

is continuous.

Proof. (i) Let $\Phi = (u,z) \in \mathcal{D}(\mathcal{A})$. In light of (2.4) and simple integration by parts, we have:

$$\begin{aligned} \langle \mathcal{A}\Phi,\Phi\rangle_H &= -\int_\Omega a(x)u^2(x)\,dx - \int_\omega b(x)u(x)z(x,1)\,dx - \frac{1}{2}\int_\omega \xi(x)u^2(x)\,dx \\ &\quad +\frac{1}{2}\int_\omega \xi(x)z^2(x,1)\,dx - \frac{\eta}{2}(\partial_x^2 u)^2(0). \end{aligned} \tag{2.6}$$

Thereafter, thanks to **A1-A2** and Young's inequality, we infer from (2.6) that

$$\langle \mathcal{A}\Phi,\Phi\rangle_H \le \int_\omega \left[\frac{b(x)+\xi(x)}{2} - a(x)\right]u^2(x)\,dx + \frac{1}{2}\int_\omega \big(b(x)-\xi(x)\big)z^2(x,1)\,dx - \frac{\eta}{2}(\partial_x^2 u)^2(0). \tag{2.7}$$

Thenceforth, $\mathcal{A}$ is dissipative in light of (2.2).

In turn, the reader can easily verify that the adjoint operator $\mathcal{A}^*$ is given by

$$\begin{aligned} \mathcal{D}(\mathcal{A}^*) &= \left\{ \begin{array}{l} (u,z) \in H;\ u \in H^5(\Omega)\cap H_0^2(\Omega), z \in L^2\Big(\omega, H^1(\Omega)\Big);\ \partial_x^2 u(0) = 0, \\ \xi(x)z(x,1) + b(x)\widehat{u}(x) = 0 \text{ in } \omega \end{array} \right\}, \\ \mathcal{A}^*(u,z) &= \left(-\eta\partial_x^5 u + \partial_x^3 u + \alpha\partial_x u - a(\cdot)u + \xi(\cdot)\tilde{z}(\cdot,0), \frac{1}{\tau}\partial_\rho z\right), \quad \text{for any } (u,z)\in\mathcal{D}(\mathcal{A}^*). \end{aligned} \tag{2.8}$$

Then, using (2.8) and arguing as above, one can obtain:

$$\langle \mathcal{A}^*\Phi,\Phi\rangle_H \le \int_\omega \left(\frac{\xi(x)}{2} + \frac{b^2(x)}{2\xi(x)} - a(x)\right)u^2(x)\,dx - \frac{\eta}{2}(\partial_x^2 u)^2(\ell).$$

The latter implies that $\mathcal{A}^*$ is also dissipative thanks to (2.2). Finally, $\mathcal{A}$ being densely defined, the proof of the assertion (i) follows from semigroups theory [5, 28].

(ii) Let $\Phi_0 = \big(u_0, \widehat{z_0}(\cdot,\tau\cdot)\big) \in H$. In view of the contraction of the semigroup $e^{t\mathcal{A}}$, we have:

$$\|(u(t),z(\cdot,t,\cdot))\|^2_H = \|u(t)\|^2 + \tau\|z(t)\|^2_{L^2(\omega\times(0,1))} \le \|u_0\|^2 + \|\widehat{z_0}(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))},\ \forall t\in[0,T]. \tag{2.9}$$

Next, we multiply the first differential equation of (2.3) by u, integrate over $\Omega\times[0,T]$ and use integrations by parts along with the boundary and initial conditions of (2.3) to obtain:

$$\frac{\eta}{2}\|\partial_x^2 u(0,\cdot)\|^2_{L^2(0,T)}+\|\sqrt{a(\cdot)}u\|^2_{L^2(0,T;L^2(\Omega))}\leq\frac{1}{2}\|u_0\|^2-\int_0^T\int_\omega b(x)u(x,t)u(x,t-\tau)\;dxdt. \tag{2.10}$$

Thanks to Young's inequality, we have

$$\int_0^T\int_\omega b(x)u(x,t)u(x,t-\tau)\;dxdt\leq\frac{\|b\|_\infty}{2}\left(\|u\|^2_{L^2(\omega\times(0,1))}+\|z(\cdot,1,\cdot)\|^2_{L^2(\omega\times(0,T))}\right), \tag{2.11}$$

which, together with (2.9), leads to rewrite (2.10) as follows:

$$\begin{aligned}\frac{\eta}{2}\|\partial_x^2 u(0,\cdot)\|^2_{L^2(0,T)}+\|\sqrt{a(\cdot)}u\|^2_{L^2(0,T;L^2(\Omega))}\;&\leq\; C\Big(\|u_0\|^2+\|\widehat{z_0}(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))}\\&\qquad+\|z(\cdot,1,\cdot)\|^2_{L^2(\omega\times(0,T))}\Big).\end{aligned} \tag{2.12}$$

Now, we multiply the second differential equation of (2.3) (see (2.2)) by $\rho\lambda(s)z$ and $\lambda(s)z$ respectively and then integrate by parts to obtain

$$\begin{aligned}\int_0^T\int_\omega z^2(x,1,t)\;dxdt\;&=\;\tau\int_0^1\int_\omega\rho\left(\widehat{z_0}^2(x,-\rho\tau)-z^2(x,\rho,T)\right)\;dxd\rho\\&\quad+\int_0^T\int_0^1\int_\omega z^2(x,\rho,t)\;dxd\rho dt,\end{aligned} \tag{2.13}$$

and

$$\tau\int_0^1\int_\omega\left(z^2(x,\rho,T)-\widehat{z_0}^2(x,-\rho\tau)\right)\;dxd\rho+\int_0^T\int_\omega\left(z^2(x,1,t)-u^2(x,t)\right)\;dxdt=0. \tag{2.14}$$

By virtue of **A1**, (2.2), (2.9), it follows from (2.13) that:

$$\int_0^T\int_\omega z^2(x,1,t)\;dxdt\leq\frac{T}{b_0}\left(\|u_0\|^2+\|\widehat{z_0}^2(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))}\right)+\frac{\tau}{b_0}\|\widehat{z_0}^2(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))},$$

and hence

$$\|z(\cdot,1,\cdot)\|^2_{L^2(\omega\times(0,T))}\leq C\|\Phi_0\|^2_H, \tag{2.15}$$

which is the estimate $(2.5)_1$. Inserting (2.15) in (2.12) yields $(2.5)_2$. In turn, $(2.5)_3$ is a direct consequence of (2.14) since the latter gives

$$\tau\int_0^1\int_\omega\widehat{z_0}^2(x,-\rho\tau)\;dxd\rho\leq\tau\int_0^1\int_\omega z^2(x,\rho,T)\;dxd\rho+\int_0^T\int_\omega z^2(x,1,t)\;dxdt. \tag{2.16}$$

Subsequently, we multiply the first differential equation of (2.3) by xu, integrate over $\Omega\times[0,T]$. Simple integrations by parts give:

$$\begin{aligned}5\eta\|\partial_x^2 u\|^2_{L^2(0,T);L^2(\Omega)}+3\|\partial_x u\|^2_{L^2(0,T;L^2(\Omega))}&=\int_\Omega xu_0^2(x)\;dx-\int_\Omega xu^2(x,T)\;dx+\alpha\|u\|^2_{L^2(0,T;L^2(\Omega))}\\&\quad-2\int_0^T\int_\Omega xa(x)u^2(x,t)\;dxdt-2\int_0^T\int_\omega xb(x)u(x,t)u(x,t-\tau)\;dxdt.\end{aligned} \tag{2.17}$$

Thereby, it suffices to use (2.9) and incorporate (2.11) in (2.17) to get $(2.5)_4$.

At last, we multiply the first differential equation in (2.3) by $(T-t)u$, integrate over $\Omega \times [0,T]$ and integrating by parts, a simple calculation yields:

$$\begin{aligned} T\|u_0\|^2 &= \int_0^T \|u(t)\|^2 \, dt + \eta \int_0^T (T-t) \left(\partial_x^2 u\right)^2 (0,t) \, dt \\ &+2 \int_0^T \int_0^1 (T-t) a(x) u^2(x,t) \, dxdt + 2 \int_0^T \int_0^1 (T-t) b(x) u(x,t) u(x,t-\tau) \, dxdt. \end{aligned} \tag{2.18}$$

By virtue of (2.9), $(2.5)_2$ and (2.11), the latter leads to the last estimate of (2.5). It is also noteworthy that the proof of the estimates remain valid for any initial data $\Phi_0 \in H$ by means of a standard argument of density.

(iii) Clearly, the proof of the continuity of Υ follows from $(2.5)_2$ and (2.9). $\square$

2.1.2. *Non-homogeneous linear system.* As for numerous dispersive equations, consider the linear system (2.1) with a source term $f(x,t)$:

$$\begin{cases} \Phi_t(t) = \mathcal{A}\Phi(t) + (f(\cdot,t), 0), \\ \Phi(0) = \Phi_0, \end{cases} \tag{2.19}$$

in which $\Phi = (u,z)$ and $\Phi_0 = (u_0, \widehat{z_0}(\cdot, -\tau \cdot))$. The proof of the next result resembles that of [30] for the Korteweg–de Vries equation (see also [29, 34]):

Proposition 2. *Suppose that the assumption **A1**-**A2** are fulfilled. The, we have:*
(i) Given $\Phi_0 \in H$ and $f \in L^1\Big(0,T; H_0^1(\Omega)\Big)$, the system (2.19) is well-posed, that is, there exists a unique mild solution $\Phi = (u,z) \in \mathcal{M} \times C\Big([0,T]; L^2\big(\omega \times (0,1)\big)\Big)$ of (2.19) such that:

$$\|(u,z)\|_{C([0,T];H)} \leq K \left(\|\Phi_0\|_H + \|f\|_{L^1(0,T;H_0^1(\Omega))} \right), \tag{2.20}$$

$$\|\partial_x^2 u\|_{L^2(0,T);L^2(\Omega)} + \|\partial_x u\|_{L^2(0,T;L^2(\Omega))} \leq K_T \left(\|\Phi_0\|_H + \|f\|_{L^1(0,T;H_0^1(\Omega))} \right), \tag{2.21}$$

where $K > 0$ (resp. $K_T > 0$) is independent of T, Φ_0 and f (resp. is independent of Φ_0 and f but depends on T).
(ii) If $u \in \mathcal{M}$, then we have: $u\partial_x u \in L^1\big(0,T; H_0^1(\Omega)\big)$ and the map

$$\Lambda : u \in \mathcal{M} \to u\partial_x u \in L^1\big(0,T; H_0^1(\Omega)\big)$$

is continuous.

Proof. (i) The well-posedness of (2.19) follows from the facts that $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ and $f \in L^1\Big(0,T; H_0^1(\Omega)\Big)$ [28] (see p. 106).

With regard to (2.20), consider a strong solution $\Phi \in \mathcal{D}(\mathcal{A})$ of (2.19) stemmed from $\Phi_0 \in \mathcal{D}(\mathcal{A})$. Next, recalling the energy $E(t)$ defined by (2.1), one can retrace the proof of (2.7) to get:

$$\begin{aligned} E'(t) &\leq \int_\omega (b(x) + \xi(x) - 2a(x)) \, u^2(x,t) \, dx + \int_\omega \big(b(x) - \xi(x)\big) z^2(x,1,t) \, dx - \eta(\partial_x^2 u)^2(0) \\ &+2 \int_\Omega f(x,t) u(x,t) \, dx. \end{aligned} \tag{2.22}$$

Using Cauchy-Schwarz, integrating over $[0,t]$ and applying Young's inequality, we obtain:

$$\max_{t\in[0,T]} \|\Phi(t)\|_H^2 \quad \leq \quad \|\Phi_0\|_H^2 + 2\delta \max_{t\in[0,T]} \|\Phi(t)\|_H^2 + \frac{1}{2\delta}\int_0^t \|f(\cdot,s)\|^2 \, ds, \tag{2.23}$$

for any $\delta > 0$. Finally, choosing δ small, the desired estimate (2.20) follows.

Analogously to $(2.5)_4$, one can get (2.21). Indeed, multiplying the first differential equation of (2.19) by xu, we obtain similarly to (2.17)

$$\begin{aligned}
5\eta\|\partial_x^2 u\|^2_{L^2(0,T);L^2(\Omega)} + 3\|\partial_x u\|^2_{L^2(0,T;L^2(\Omega))} &= \int_\Omega x u_0^2(x)\, dx - \int_\Omega x u^2(x,T)\, dx + \alpha\|u\|^2_{L^2(0,T;L^2(\Omega))} \\
-2\int_0^T\int_\Omega x a(x) u^2(x,t)\, dxdt &- 2\int_0^T\int_\omega x b(x) u(x,t) u(x,t-\tau)\, dxdt + 2\int_0^T\int_\Omega x u(x,t) f(x,t)\, dxdt \\
&\leq \ell\|u_0\|^2 + \alpha T\|u\|^2_{C([0,T];L^2(\Omega))} + \ell\|b\|_\infty T\|(u,z)\|^2_{C([0,T];H)} + \ell^2\|(u,z)\|^2_{C([0,T];H)} \\
&\quad +\|f\|^2_{L^1(0,T;L^2(\Omega))},
\end{aligned} \tag{2.24}$$

where Young's inequality and (2.11) are used. Clearly, (2.21) follows from (2.20) and (2.24). Note that (2.20)-(2.21) can be extended to the case when Φ_0 belongs to H via a density argument.

(ii) Given u in $\mathcal{M}$, we have $\partial_x u \in L^2(0,T;H_0^1(\Omega))$ and $u(\cdot,t)\partial_x u(\cdot,t) \in H_0^1(\Omega)$. In fact, $\partial_x(u\partial_x u)(x,t) = (\partial_x u(x,t))^2 + u(x,t)\partial_x^2 u(x,t)$, for any $t \in (0,T)$ [5]. This, together with [1, Theorem 4.39], implies that there exists a positive constant K

$$\int_0^T \|(u\partial_x u)(\cdot,t)\|_{H_0^1(\Omega)}\, dt \quad \leq \quad K\int_0^T \|u(\cdot,t)\|_{H_0^2(\Omega)}\, dt, \tag{2.25}$$

and hence $u\partial_x u \in L^2(0,T;H_0^1(\Omega))$.

Subsequently, let $u, v \in \mathcal{M}$. Invoking the triangle and Cauchy-Schwarz inequalities along with the Sobolev embedding $H_0^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we have:

$$\begin{aligned}
\|\Lambda(u) - \Lambda(v)\|_{L^1(0,T;H_0^1(\Omega))} &\leq \int_0^T \left[\|\partial_x u(t)(u(t)-v(t))\|_{H_0^1(\Omega)} + \|v(t)(\partial_x(u(t)-v(t)))\|_{H_0^1(\Omega)}\right] dt \\
&\leq C\left(\|u\|_{L^2(0,T;H_0^2(\Omega))}\|u-v\|_{L^2(0,T;H_0^2(\Omega))} + \|u-v\|_{L^2(0,T;H_0^2(\Omega))}\|v\|_{L^2(0,T;H_0^2(\Omega))}\right) \\
&\leq C\left(\|u\|_{L^2(0,T;H_0^2(\Omega))} + \|v\|_{L^2(0,T;H_0^2(\Omega))}\right)\|u-v\|_{\mathcal{M}},
\end{aligned} \tag{2.26}$$

for some positive constant C. Consequently, the mapping Λ is continuous with respect to the corresponding topologies.

□

2.1.3. *Well-posedness of the problem (1.1).* Our first main result is stated below:

Theorem 1. *Under the assumptions* ***A1-A2*** *hold and given $\Phi_0 \in H$, the problem (1.1) has a unique global mild solution $u \in C\left(0,\infty;L^2(\Omega)\right) \cap L^2_{loc}\left(0,\infty;H_0^2(\Omega)\right)$.*

Proof. Let $\Phi_0 = (u_0, \widehat{z_0}(\cdot,-\tau\cdot)) \in H$. Next, we define the map $\Theta : \mathcal{M} \to \mathcal{M}$ as follows: $\Theta(u) = v$, where v is the mild solution of (2.19) with $f = u\partial_x u$. It follows from Proposition 2 that the map Θ is well-defined and

$$\|\Theta(u) - \Theta(\tilde{u})\|_{\mathcal{M}} \leq C_1\left(\|u\partial_x u - \tilde{u}\partial_x\tilde{u}\|_{L^1(0,T;H_0^1(\Omega))}\right).$$

Analogously to (2.26), the latter gives:

$$\|\Theta(u) - \Theta(\tilde{u})\|_{\mathcal{M}} \leq C_2 T^{1/4} \left(\|u\|_{\mathcal{M}} + \|\tilde{u}\|_{\mathcal{M}}\right) \|u - \tilde{u}\|_{\mathcal{M}},\ \forall u, \tilde{u} \in \mathcal{M}, \tag{2.27}$$

for some $C_2 > 0$. Using once again Proposition 2 along with (2.27), we obtain:

$$\|\Theta(u)\|_{\mathcal{M}} \leq C\|\Phi_0\|_H + C_2 T^{1/4} \|u\|^2_{\mathcal{M}}. \tag{2.28}$$

Restricting the mapping Θ on the closed ball

$$\mathcal{M} = \left\{u \in \mathcal{M};\ \|u\|_{\mathcal{M}} \leq R := 2\max\{C, C_2\}\right\},$$

and taking T and R so that $2C_2 T^{1/4} R < 1$ and $4T^{1/4} \left(\max\{C, C_2\}\right)^2 \|\Phi_0\|_H < 1$, the mapping Θ is well-defined and contractive on $\mathcal{M}$. Applying the Banach Fixed Point Theorem, we conclude that there exists a unique fixed point u of Θ, which is the unique solution of (1.1). Lastly, taking $f = u\partial_x u$ in (2.22), we deduce that the system energy $E(t)$ of the problem (1.1) is decreasing, that is, there exists a positive constant K

$$E'(t) \quad \leq \quad -K\left(\eta(\partial_x^2 u)^2(0) + \int_\Omega a(x)u^2(x,t)\ dx + \int_\omega b(x)u^2(x,t-\tau)\ dx\right) \leq 0. \tag{2.29}$$

and hence the solution u is global. □

2.2. **Exponential stability for the problem (1.1).** This subsection is devoted to the proof of two stability results of the system (1.1). The first one is:

Theorem 2. *(Local stability) Assume that the conditions **A1-A2** hold. Then, there exists a positive constant $r > 0$ such that for every initial data $\Phi_0 = (u_0, \widehat{z_0}(\cdot, -\tau\cdot)) \in H$ satisfying $\|\Phi_0\|_H \leq r$, the trivial solution of the problem (1.1) is uniformly exponentially stable in H and hence the energy $E(t)$ uniformly exponentially decays.*

Proof. First, consider u as the regular solution of (1.1) stemmed from Φ_0 in $\mathcal{D}(\mathcal{A})$, with $\|\Phi_0\|_H \leq r$. In order to simplify the notation, the variables x and t will be omitted whenever it is unnecessary. Subsequently, we define the modified energy

$$\mathcal{E}(t) = E(t) + \lambda_1 V_1(t) + \lambda_2 V_2(t),\ V_1(t) := \int_\Omega e^{\lambda x} u^2\, dx,\ V_2(t) := \tau \int_\omega \int_0^1 \xi(x) e^{-\delta\rho\tau} (\partial_x u)^2(x, t - \tau\rho)\, d\rho dx, \tag{2.30}$$

where $E(t)$ is defined by (2.1), λ_1, λ_2 and λ are positive constants to de determined and δ is an arbitrary positive constant. Then, one can readily verify that

$$E(t) \leq \mathcal{E}(t) \leq \left(1 + \max\left\{\lambda_1 e^{\lambda\ell}, \frac{\lambda_2}{b_0}\right\}\right) E(t). \tag{2.31}$$

Next, differentiating (2.30), integrating by parts and using the same arguments as for (2.7), a lengthy computation permits to claim that for any $\gamma > 0$:

$$\begin{aligned}
\mathcal{E}'(t)+\gamma\mathcal{E}(t) \quad \leq \quad & -\eta(1+\lambda_1)(\partial_x^2 u)^2(0,t) + \underbrace{\int_\omega \Big(b(x)+\xi(x)-2a(x)+\lambda_1 b(x)e^{\lambda x}+\lambda_2\xi(x)\Big)\, u^2\ dx}_{B_1} \\
& + \underbrace{\int_\omega \Big(b(x)-\xi(x)-\lambda_2 e^{-\delta\tau}\xi(x)+\lambda_1 e^{\lambda x}b(x)\Big)\,(\partial_x u)^2(x,t-\tau)\,dx}_{B_2} \\
& + \underbrace{\gamma\int_\Omega \Big(1+\lambda_1 e^{\lambda x}\Big)\, u^2\ dx - 5\eta\lambda\lambda_1\int_\Omega e^{\lambda x}(\partial_x^2 u)^2\ dx}_{B_3} + \underbrace{\frac{2}{3}\lambda\lambda_1\int_\Omega e^{\lambda x}u^3\ dx}_{B_4} \\
& + \lambda_1\underbrace{\int_\omega e^{\lambda x}\left(-2a(x)+\alpha\lambda+\lambda^3-\eta\lambda^5\right)u^2\ dx}_{B_5} + \lambda\lambda_1\underbrace{\int_\Omega e^{\lambda x}(5\eta\lambda^2-3)(\partial_x u)^2\ dx}_{B_6} \\
& + \tau\underbrace{\int_\omega\int_0^1 \xi(x)\left(\gamma(1+\lambda_2 e^{-\delta\tau\rho})-\lambda_2\delta e^{-\delta\tau}\right)(\partial_x u)^2(x,t-\rho\tau)\,d\rho dx}_{B_7} \\
& + \int_{\widehat{\omega}\backslash\omega}\Big(\lambda_1 e^{\lambda x}\left[-2a(x)+\alpha\lambda+\lambda^3-\eta\lambda^5\right]-2a(x)\Big)\, u^2\ dx. \qquad (2.32)
\end{aligned}$$

The ultimate outcome is to choose, in (2.32), $\gamma, \lambda_1, \lambda_2$ and λ so that $\mathcal{E}'(t)+\gamma\mathcal{E}(t) \leq$, which implies the exponential stability of $\mathcal{E}(t)$ and hence that of $E(t)$ by virtue of (2.31). To do so, it suffices to handle the bad terms $B_1 - B_7$.

Thanks to (2.2), we know that $b(x)+\xi(x)-2a(x) < 0$. This together with the fact that $e^{\lambda x} \leq e^{\lambda\ell}$, for any $x \in \omega$, we can choose λ_1 and λ_2 as follows

$$\lambda_1 < \frac{k}{e^{\lambda\ell}\|b\|_\infty}, \quad \lambda_2 \leq \inf_{x\in\omega}\left\{\frac{2a(x)-b(x)-\lambda_1 e^{\lambda\ell}b(x)}{\xi(x)}\right\}, \qquad (2.33)$$

so that both of B_1 and B_2 are negative. With regard to B_3, we infer from (2.4)

$$B_3 \leq \left(\gamma\left[1+\lambda_1 e^{\lambda\ell}\right]\frac{\ell^4}{\pi^4} - 5\eta\lambda\lambda_1\right)\|\partial_x^2 u\|^2. \qquad (2.34)$$

The term B_4 can be handled by using standard arguments. Indeed, since $H_0^1(I)$ embeds into $L^\infty(I)$, that is, $\|u\|^2_{L^\infty(I)} \leq \ell\|\partial_x u\|^2$ and using Cauchy-Schwarz inequality as well as (2.4), we have:

$$B_4 \quad \leq \quad \frac{2}{3}\lambda\lambda_1\|u\|^2_{L^\infty(I)}\int_\Omega e^{\lambda x}|u|\ dx \leq \frac{\sqrt{2\lambda}\lambda_1\ell^3 e^{\lambda\ell}}{3\pi^2}\|\partial_x^2 u\|^2\|u\| \leq r\frac{\sqrt{2\lambda}\lambda_1\ell^3 e^{\lambda\ell}}{3\pi^2}\|\partial_x^2 u\|^2, \qquad (2.35)$$

where the dissipation property of the $E(t)$ allowed us to write $\|u\| \leq \|(u,z)\|_H \leq \|(u_0,z_0)\|_H$. Combining (2.34) and (2.35) yields:

$$B_3 + B_4 \leq \left(\gamma\left[1+\lambda_1 e^{\lambda\ell}\right]\frac{\ell^4}{\pi^4} - 5\eta\lambda\lambda_1 + r\frac{\sqrt{2\lambda}\lambda_1\ell^3 e^{\lambda\ell}}{3\pi^2}\right)\|\partial_x^2 u\|^2. \qquad (2.36)$$

Concerning B_5, we first have from **A1**-**A2** that $a(x) \geq b_0 + k$ and hence

$$-2a(x) + \alpha\lambda + \lambda^3 - \eta\lambda^5 < \Delta(\lambda) =: -2k + \alpha\lambda + \lambda^3 - \eta\lambda^5.$$

Obviously, Δ is continuous on $(0, \infty)$ and $\lim_{\lambda\to 0^+} \Delta(\lambda) = -2k < 0$. Therefore, one can claim that Δ remains negative on $(0, \lambda^*)$ for some $\lambda^* > 0$ (for instance one can choose $\lambda^* = \sqrt{\frac{3+\sqrt{9+20\alpha\eta}}{5\eta}}$ so that Δ is even decreasing). Thereafter, we choose $\lambda < \min\{\lambda^*, \sqrt{3/(5\eta)}\}$ so as B_5 and B_6 are negative. This also implies that the last term of the right-hand side in (2.32) is also negative. Finally, we choose γ small enough so that B_7 and $B_3 + B_4$ are negative. More precisely, we first choose r sufficiently small so that

$$r < \frac{15\pi^2\eta\sqrt{\lambda}}{\sqrt{2}\ell^3} e^{-\lambda\ell}$$

and then choose

$$\gamma < \min\left\{ \frac{\pi^4\lambda_1}{\ell^4(1+\lambda_1 e^{\lambda\ell})} \left[5\eta\lambda - r\frac{\sqrt{2\lambda}\ell^3 e^{\lambda\ell}}{3\pi^2}\right], (1+\lambda_2 e^{-\delta\tau\rho}) - \lambda_2\delta e^{-\delta\tau}, \frac{\lambda_2\delta e^{-\delta\tau}}{1+\lambda_2} \right\}. \tag{2.37}$$

Whereupon, we reach the desired result

$$\mathcal{E}'(t) + \gamma\mathcal{E}(t) < 0. \tag{2.38}$$

Note that a density argument allows to extend the stability result to $\Phi_0 \in H$.

□

Remark 1. *(i) A careful look at the proof of Theorem 2 leads us to claim that the choice of the constants $\lambda, \lambda_1, \lambda_2$ and γ should be in the following order: first, we choose λ small enough so that B_5 and B_6 are negative. Next, we choose λ_1 and λ_2 small satisfying* (2.33) *to ensure that B_1 and B_2 are negative. Finally, we choose γ as in* (2.37) *so that $B_3 + B_4$ and B_7 are negative.*
(ii) It is worth mentioning that, contrary to [33] *for the KdV equation, the stability result stated in Theorem 2 does not require any smallness condition on the length ℓ. This is expectable since the equation has a damping term.*
*(iii) The condition **A2** plays an important role in the previous study. We shall show later that such a requirement can be relaxed by means a perturbation argument but at the expense of a condition on the parameters η and ℓ and the spatial distribution b of the delayed term.*

Before stating our second stability result, we have the following estimate:

Proposition 3. *Suppose that the assumptions **A1**-**A2** hold. Then, the unique global solution u of (1.1) stemmed from $\Phi_0 \in H$ satisfies*

$$\|u\|_{L^2\left(0,T;H_0^2(\Omega)\right)} \leq C\|\Phi_0\|_H \left(1 + \|\Phi_0\|_H\right), \tag{2.39}$$

for some positive constant C.

Proof. Multiplying (1.1) by $e^{\lambda x}u$ and integrating the resulting equation over $\Omega\times[0,T]$, we obtain

$$\begin{aligned} 5\eta\lambda\lambda_1\int_0^T\int_\Omega e^{\lambda x}(\partial_x^2 u)^2\,dxdt &= -\eta\int_0^T(\partial_x^2u)^2(0,t)\,dt+(5\eta\lambda^2-3)\int_0^T\int_\Omega e^{\lambda x}(\partial_x u)^2\,dxdt\\ &\quad+\int_\Omega e^{\lambda x}\left(\alpha\lambda+\lambda^3-\eta\lambda^5-2a(x)\right)u^2\,dxdt\\ &\quad+\int_\Omega e^{\lambda x}\left(u_0^2(x)-u^2(x,T)\right)\,dx+\frac{2}{3}\lambda\int_0^T\int_\Omega e^{\lambda x}u^3\,dx\\ &\quad-2\int_0^T\int_\omega e^{\lambda x}b(x)u\partial_x u(x,t-\tau)\,dxdt. \end{aligned}\tag{2.40}$$

Owing to the embedding $H_0^1(I)\to L^\infty(I)$, Cauchy-Schwarz inequality, (2.4) and the fact that the energy $E(t)$ is decreasing (see (2.29), we have

$$\begin{aligned} \frac{2}{3}\lambda\int_0^T\int_\Omega e^{\lambda x}u^3\,dx &\leq \frac{2\sqrt{\ell}}{3}\lambda e^{\lambda\ell}\int_0^T\|u\|^2\|\partial_x u\|\,dt\\ &\leq \frac{2\sqrt{\ell}}{3}\lambda e^{\lambda\ell}\left(\|u_0\|^2+\|\widehat{z_0}^2(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))}\right)\int_0^T\|\partial_x u\|\,dt\\ &\leq \frac{2\sqrt{T}\lambda\ell^{3/2}e^{\lambda\ell}}{3\pi}\left(\|u_0\|^2+\|\widehat{z_0}^2(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))}\right)\|\partial_x^2u\|_{L^2(0,T;H_0^2(\Omega))}. \end{aligned}\tag{2.41}$$

In turn, by virtue of Young's inequality and (2.29), we get

$$-2\int_0^T\int_\omega e^{\lambda x}b(x)u\partial_x u(x,t-\tau)\,dxdt\leq e^{\lambda\ell}\|b\|_\infty T\left(\|u_0\|^2+\|\widehat{z_0}^2(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))}\right).\tag{2.42}$$

Inserting (2.41) and (2.42) in (2.40) and choosing λ small as for B_5 and B_6 (see (2.32)), it follows

$$\begin{aligned} 5\eta\lambda\lambda_1 e^{\lambda\ell}\|\partial_x^2u\|^2_{L^2(0,T;H_0^2(\Omega))} &\leq C_1T\left(\|u_0\|^2+\|\widehat{z_0}^2(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))}\right)\\ &\quad+C_2\sqrt{T}\left(\|u_0\|^2+\|\widehat{z_0}^2(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))}\right)\|\partial_x^2u\|_{L^2(0,T;H_0^2(\Omega))}. \end{aligned}\tag{2.43}$$

Now, it suffices to use Young's inequality to get

$$\begin{aligned} C_2\sqrt{T}\left(\|u_0\|^2+\|\widehat{z_0}^2(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))}\right)\|\partial_x^2u\|_{L^2(0,T;H_0^2(\Omega))} &\leq \frac{C_2^2T}{2\theta}\left(\|u_0\|^2+\|\widehat{z_0}^2(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))}\right)^2\\ &\quad+\frac{\theta}{2}\|\partial_x^2u\|^2_{L^2(0,T;H_0^2(\Omega))}, \end{aligned}$$

for any positive constant θ. Combining the latter with (2.43) yields

$$\begin{aligned} 5\eta\lambda\lambda_1 e^{\lambda\ell}\|\partial_x^2u\|^2_{L^2(0,T;H_0^2(\Omega))} &\leq C_1T\left(\|u_0\|^2+\|\widehat{z_0}^2(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))}\right)\\ &\quad+\frac{C_2^2T}{2\theta}\left(\|u_0\|^2+\|\widehat{z_0}^2(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))}\right)^2+\frac{\theta}{2}\|\partial_x^2u\|^2_{L^2(0,T;H_0^2(\Omega))}, \end{aligned}\tag{2.44}$$

which leads to the desired estimate (2.39) provided that θ is chosen sufficiently. □

Following the arguments of [27] and [29], we shall prove our second stability result:

Theorem 3. *(Semi-global stability) Assume that the conditions* ***A1-A2*** *hold. Then, for any positive constant $R>0$, there exist two positive constant M and κ depending on R such that the energy $E(t)$,* *along any solution of (1.1) stemmed from an initial data* $\Phi_0 = (u_0, \widehat{z_0}(\cdot, -\tau\cdot)) \in H$ *with* $\|\Phi_0\|_H \leq R$, *satisfies*

$$E(t) \leq Me^{-\kappa t}E(0), \quad \textit{for any } t > 0.$$

Proof. Using once again a density arguments, it suffices to prove the result for an initial condition $\Phi_0 = (u_0, \widehat{z_0}(\cdot, -\tau\cdot))$ in $\mathcal{D}(\mathcal{A})$ with $\|\Phi_0\|_H \leq R$. Thereafter, consider the energy $E(t)$, defined by (2.1), along the solution u of (1.1). In view of (2.29), we have:

$$\eta \left\|(\partial_x^2 u)^2(0,\cdot)\right\|_{L^2(0,T)} + \int_0^T \left(\int_\Omega a(x)u^2(x,t)\,dx + \int_\omega b(x)u^2(x,t-\tau)\,dx\right)dt \leq \frac{1}{K}\left(E(0) - E(T)\right). \tag{2.45}$$

In light of (2.29) and (2.45), we have

$$E(T) \leq (1 - KM_1)E(0),$$

provided that there exists a positive constant M_1 depending on R and T such that the following observability inequality holds:

$$\eta \left\|(\partial_x^2 u)^2(0,\cdot)\right\|_{L^2(0,T)} + \int_0^T \left(\int_\Omega a(x)u^2(x,t)\,dx + \int_\omega b(x)u^2(x,t-\tau)\,dx\right)dt \geq M_1 E(0). \tag{2.46}$$

Clearly, the desired result, namely, the exponential stability of $E(t)$ is a direct consequence of (2.29) and the semigroup property. Therefore, it amounts to proving (2.46). First, multiplying (1.1) by u and arguing as before, one can show that there exists a positive constant $C(T)$ such that

$$\begin{aligned} T\|u_0\|^2 \quad &\leq \quad \|u\|^2_{L^2(\Omega\times(0,T))} + \eta T\|\partial_x^2 u(0,\cdot)\|^2_{L^2(0,T)} \\ &\quad + C(T)\left(\int_0^T\int_\Omega a(x)u^2(x,t)\,dxdt + \int_0^T\int_\omega b(x)u^2(x,t-\tau)\,dxdt\right). \end{aligned} \tag{2.47}$$

Second, integrating (2.16) over $[0,T]$, we get

$$T\int_0^1\int_\omega \widehat{z_0}^2(x,-\rho\tau)\,dxd\rho \leq \int_0^T\int_0^1\int_\omega u^2(x,t-\rho\tau)\,dxd\rho dt + \frac{T}{\tau}\int_0^T\int_\omega u^2(x,1,t-\tau)\,dxdt.$$

In turn, we know that there exists a positive constant $C(T)$ such that [33]

$$\int_0^T\int_0^1\int_\omega u^2(x,t-\rho\tau)\,dxd\rho dt \leq C(T)\int_0^T\left(\int_\Omega a(x)u^2(x,t)\,dx + \int_\omega b(x)u^2(x,t-\tau)\,dx\right)dt,$$

for $T > \tau$. Combining the last two estimates and invoking **A1**-**A2**, we deduce the existence of another positive constant $C(T)$ such that

$$\int_0^1\int_\omega \widehat{z_0}^2(x,-\rho\tau)\,dxd\rho \leq C(T)\int_0^T\left(\int_\Omega a(x)u^2(x,t)\,dxdt + \int_\omega b(x)u^2(x,t-\tau)\,dx\right)dt. \tag{2.48}$$

Amalgamating (2.47) and (2.48), it follows that (2.46) holds whenever there exists a positive constant M_1 depending on R and T such that the solutions of (1.1) stemmed from Φ_0 with $\|\Phi_0\|_H \leq R$ satisfy

$$\eta \left\|(\partial_x^2 u)^2(0,\cdot)\right\|_{L^2(0,T)} + \int_0^T\int_\Omega a(x)u^2(x,t)\,dxdt + \int_0^T\int_\omega b(x)u^2(x,t-\tau)\,dxdt \geq M_1\|u\|^2_{L^2(\Omega\times(0,T))}. \tag{2.49}$$

The proof will be done by contradiction. In fact, if (2.49) were not true, then there exists a sequence of solutions of (1.1) $(u_n)_n$ in $\mathcal{M}$ stemmed from the initial condition $\Phi_0^n = (u_0^n, \widehat{z_0}^n(\cdot, -\tau\cdot)) \in H$ with $\|\Phi_0^n\|_H \leq R$ such that

$$\lim_{n\to\infty} \frac{\|u_n\|^2_{L^2(\Omega\times(0,T))}}{\eta \left\|(\partial_x^2 u_n)^2(0,\cdot)\right\|_{L^2(0,T)} + \int_0^T\int_\Omega a(x)|u_n(x,t)|^2\, dxdt + \int_0^T\int_\omega b(x)|u_n(x,t-\tau)|^2\, dxdt} = \infty.$$

Let $\lambda_n = \|u_n\|_{L^2(\Omega\times(0,T))}$, which is bounded thanks to (2.29) and the boundedness property $\|\Phi_0^n\|_H \leq R$. Moreover, let $v_n = \frac{u_n}{\lambda_n}$, which satisfies

$$\begin{aligned} &\partial_t v_n(x,t) + \partial_x^3 v_n(x,t) - \eta\partial_x^5 v_n(x,t) + \lambda_n v_n(x,t)\partial_x v_n(x,t) + \alpha\partial_x v_n(x,t) + a(x)v_n(x,t)\\ &\qquad + b(x)v_n(x,t-\tau) = 0, \qquad (x,t)\in\Omega\times(0,T), \end{aligned} \tag{2.50}$$

$$v_n(0,t) = v_n(\ell,t) = \partial_x v_n(0,t) = \partial_x v_n(\ell,t) = \partial_x^2 v_n(\ell,t) = 0, t>0, \tag{2.51}$$

$$\|v_n\|_{L^2(\Omega\times(0,T))} = 1, \tag{2.52}$$

$$\begin{aligned} &\lim_{n\to\infty}\left\{\eta\left\|(\partial_x^2 u_n)^2(0,\cdot)\right\|_{L^2(0,T)} + \int_0^T\int_\Omega a(x)^2|u_n(x,t)|^2\, dxdt + \int_0^T\int_\omega b(x)|u_n(x,t-\tau)|^2\, dxdt\right\}\\ &= 0. \end{aligned} \tag{2.53}$$

Multiplying (2.50) by $(T-t)v_n$, we obtain analogously to (2.18) (see also $(2.5)_5$), we get

$$\|v_n(\cdot,0)\|_{L^2(\Omega)} \leq C(T)\left(\|v_n\|_{0,T;L^2(\Omega)} + \left\|(\partial_x^2 v_n)^2(0,\cdot)\right\|_{L^2(0,T)} + \|v_n(\cdot,-\tau\cdot)\|_{L^2(\omega\times(0,1))}\right). \tag{2.54}$$

In view of a simple computation together with **A1**, one can readily verify that

$$\|v_n(\cdot,-\tau\cdot)\|^2_{L^2(\omega\times(0,1))} \leq \tau C\int_0^T\int_\omega b(x)|u_n(x,t-\tau)|^2\, dxdt, \quad \text{for } T>\tau,$$

which, together with (2.54) and (2.52)-(2.53), implies that $\left(\|v_n(\cdot,0)\|_{L^2(\Omega)}\right)_n$ is a bounded sequence. Keeping this fact in mind and since $(u_n)_n$ are solutions of (1.1), $\|\Phi_0^n\|_H \leq R$ and owing to (2.39), it follows that there exists a positive constant C independent of n such

$$\|v_n\|_{L^2(0,T;H_0^2(\Omega))} \leq C. \tag{2.55}$$

On the other hand, using (2.55) and the fact that v_n belongs to $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^2(\Omega))$, one can deduce that

$$\|v_n\partial_x v_n\|_{L^2(\omega\times(0,1))} \leq C,$$

for some positive constant C. Amalgamating the above properties of v_n and λ_n, one can claim from (2.50) that $(\partial_t v_n)_n$ is bounded in $L^2(0,T;H^{-3}(\Omega))$. This together with (2.55) and the compactness of the embedding $H_0^2(\Omega)\hookrightarrow L^2(\Omega)$, we conclude that $(v_n)_n$ is relatively compact in $L^2(0,T;L^2(\Omega))$ and hence we can extract a subsequence, still denoted by $(v_n)_n$, such that

$$\begin{cases} v_n \to v \quad \text{strongly in } L^2(0,T;L^2(\Omega)),\\ \qquad\qquad \|v\|_{L^2(0,T;\Omega)} = 1, \end{cases} \tag{2.56}$$

and

$$\begin{aligned} 0 &= \liminf_{n\to\infty}\left\{\eta\left\|(\partial_x^2 v_n)^2(0,\cdot)\right\|_{L^2(0,T)} + \int_0^T\int_\Omega a(x)|v_n(x,t)|^2\,dxdt + \int_0^T\int_\omega b(x)|v_n(x,t-\tau)|^2\,dxdt\right\} \\ &\geq \eta\left\|(\partial_x^2 v)^2(0,\cdot)\right\|_{L^2(0,T)} + \int_0^T\int_\Omega a(x)|v(x,t)|^2\,dxdt + \int_0^T\int_\omega b(x)|v(x,t-\tau)|^2\,dxdt. \end{aligned}$$

Clearly, the latter yields $a(x)v\equiv 0$ on $\Omega\times(0,T)$, $b(x)v(x,t-\tau)\equiv 0$ on $\omega\times(0,T)$ and $\partial_x^2 v(0,\cdot)\equiv 0$ on $(0,T)$, that is, $v\equiv 0$ on $\omega\times(-\tau,T)$ and $\partial_x^2 v(0,t)$ on $(0,T)$. In turn, $(\lambda_n)_n$ being bounded, one can extract a subsequence, also denoted by $(\lambda_n)_n$, such that $\lambda_n\to\lambda\geq 0$. Whereupon, the limit v belongs to $L^2(0,T;H_0^2(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ and satisfies

$$\text{(2.57)}\quad \begin{cases} \partial_t v(x,t)+\partial_x^3 v(x,t)-\eta\partial_x^5 v(x,t)+\lambda v(x,t)\partial_x v(x,t)+\alpha\partial_x v(x,t)=0, & (x,t)\in\Omega\times(0,T),\\ v(0,t)=v(\ell,t)=\partial_x v(0,t)=\partial_x v(\ell,t)=\partial_x^2 v(0,t)=\partial_x^2 v(\ell,t)=0, & t>0,\\ \|v\|_{L^2(\Omega\times(0,T))}=1, & \\ v(x,t)=0, & (x,t)\in\omega\times(-\tau,T). \end{cases}$$

Following the arguments in [27], a gain of regularity result is established in [34] (see Proposition 3.2, p. 112), which permits to use the Unique Continuation Principle [31] and conclude that v vanishes in $\Omega\times(0,T)$. This contradicts $\|v\|_{L^2(\Omega\times(0,T))}=1$. □

3. The problem (1.1) with $\omega\not\subseteq\widehat{\omega}$

We turn now to the interesting case, where a and b satisfy only the assumption **A1** but $\omega\not\subseteq\widehat{\omega}$ and hence the assumption **A2** does not hold. This implies that the dissipation property (2.29) of the system's energy is lost. To overcome this difficulty, we shall adopt, as in [33], a perturbation argument developed in [26] by considering the following auxiliary problem

$$\text{(3.1)}\quad \begin{cases} \partial_t u(x,t)+\partial_x^3 u(x,t)-\eta\partial_x^5 u(x,t)+u(x,t)\partial_x u(x,t)+\alpha\partial_x u(x,t) & \\ \qquad\qquad +(a(x)+\xi b(x))u(x,t)+b(x)u(x,t-\tau)=0, & (x,t)\in\Omega\times(0,\infty),\\ u(0,t)=u(\ell,t)=\partial_x u(0,t)=\partial_x u(\ell,t)=\partial_x^2 u(\ell,t)=0, & t>0,\\ u(x,0)=u_0(x), & x\in\Omega,\\ u(x,t)=z_0(x,t), & x\in\Omega,\ t\in\mathcal{T}, \end{cases}$$

where ξ is a positive constant. Then, we define the energy corresponding to (3.1) by

$$\text{(3.2)}\qquad F(t)=\int_\Omega u^2(x,t)\,dx+\tau\xi\int_\omega\int_0^1 b(x)(\partial_x u)^2(x,t-\tau\rho)\,d\rho dx.$$

Then, a formal computation shows that for any positive constant δ, we have:

$$\text{(3.3)}\quad F'(t)\leq -\eta(\partial_x^2 u)^2(0)-2\int_{\widehat{\omega}} a(x)u^2(x,t)\,dx+\left(\frac{1}{\delta}\int_\omega b(x)u^2(x,t)\,dx-\xi\right)+(\delta-\xi)\int_\omega b(x)u^2(x,t-\tau)\,dx,$$

which is nonnegative provided that $\xi>\max\{\delta,1/\delta\}$. For sake of simplicity, we take $\delta=1$ and hence $\xi>1$.

As in the previous section, the state space is $H := L^2(\Omega) \times L^2(\omega \times (0,1))$, which will be endowed with the inner product norm

$$\|(u,z)\|_H^2 = \|u\|^2 + \tau\xi \int_\omega \int_0^1 b(x) z^2(x,\rho)\, d\rho dx, \tag{3.4}$$

where $\xi > 1$.

3.1. **A linearized system associated to** (3.1)**.** Let us first deal with the following linearized system

$$\begin{cases} \partial_t u(x,t) + \partial_x^3 u(x,t) - \eta \partial_x^5 u(x,t) + \alpha \partial_x u(x,t) & \\ \qquad\qquad + (a(x) + \xi b(x)) u(x,t) + b(x) u(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(0,t) = u(\ell,t) = \partial_x u(0,t) = \partial_x u(\ell,t) = \partial_x^2 u(\ell,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = z_0(x,t), & x \in \Omega,\ t \in \mathcal{T}. \end{cases} \tag{3.5}$$

Taking $z(x,\rho,t) := \widehat{u}(x,t-\tau\rho)$, for $(x,\rho,t) \in \omega \times (0,1) \times (0,\infty)$, the problem (3.5) takes the following form

$$\begin{cases} \Phi_t(t) = \mathcal{B}\Phi(t), \\ \Phi(0) = \Phi_0, \end{cases} \tag{3.6}$$

in which $\Phi = (u,z)$, $\Phi_0 = \big(u_0, \widehat{z_0}(\cdot, \tau \cdot)\big)$ and $\mathcal{B}$ is the linear operator defined by:

(3.7)

$$\begin{aligned} \mathcal{D}(\mathcal{B}) &= \left\{ (u,z) \in H; u \in H^5(\Omega) \cap H_0^2(\Omega), z \in L^2\Big(\omega, H^1(\Omega)\Big); \partial_x^2 u(\ell) = 0, z(x,0) = \widehat{u}(x) \text{ in } \omega \right\}, \\ \mathcal{B}(u,z) &= \left(\eta \partial_x^5 u - \partial_x^3 u - \alpha \partial_x u - (a(\cdot) + \xi b(\cdot)) u - b(\cdot) \tilde{z}(\cdot, 1), -\frac{1}{\tau} \partial_\rho z \right), \quad \text{for any } (u,z) \in \mathcal{D}(\mathcal{B}). \end{aligned}$$

Thereafter, it is easy to verify that the operator $\mathcal{B}$ satisfies

$$\langle \mathcal{B}\Phi, \Phi \rangle_H \leq \frac{1-\xi}{2} \int_{\widehat{\omega}} a(x) u^2(x)\, dx + \frac{1-\xi}{2} \int_\omega b(x) z^2(x,1)\, dx - \frac{\eta}{2} (\partial_x^2 u)^2(0) \leq 0,$$

for any Φ in $\mathcal{D}(\mathcal{B})$ and hence $\mathcal{B}$ is dissipative in H. Additionally, the adjoint operator $\mathcal{B}^*$ is given by

$$\begin{aligned} \mathcal{D}(\mathcal{B}^*) &= \left\{ \begin{array}{l} (u,z) \in H;\ u \in H^5(\Omega) \cap H_0^2(\Omega), z \in L^2\Big(\omega, H^1(\Omega)\Big);\ \partial_x^2 u(0) = 0, \\ \xi z(x,1) + \widehat{u}(x) = 0 \text{ in } \omega \end{array} \right\}, \\ \mathcal{B}^*(u,z) &= \left(-\eta \partial_x^5 u + \partial_x^3 u + \alpha \partial_x u - (a(\cdot) + \xi b(\cdot)) u + \xi b(\cdot) \tilde{z}(\cdot, 0), \frac{1}{\tau} \partial_\rho z \right), \quad \text{for any } (u,z) \in \mathcal{D}(\mathcal{B}^*), \end{aligned}$$

and also satisfies the dissipativity

$$\langle \mathcal{B}^*\Phi, \Phi \rangle_H \leq -\int_{\widehat{\omega}} a(x) u^2(x)\, dx + \frac{1}{2}\left(\frac{1}{\xi} - \xi\right) \int_\omega b(x) u^2(x)\, dx - \frac{\eta}{2} (\partial_x^2 u)^2(\ell) \leq 0,$$

for any Φ in $\mathcal{D}(\mathcal{B}^*)$. Following the same lines as in Section 2.1.1, the operator $\mathcal{B}$ generates a C_0-semigroup of contractions $e^{t\mathcal{B}}$ and the linear problem (3.6) is well-posed in the sense of semigroups theory [28].

In order to prove the exponential stability of the linearized system (3.5), we first recall that $F(t)$ is defined by (3.2) and then consider the following functional

(3.8)

$$\mathcal{F}(t) = F(t) + \sigma_1 V_3(t) + \sigma_2 V_4(t),\ V_3(t) := \int_\Omega e^{\sigma x} u^2\, dx,\ V_4(t) := \tau \int_\omega \int_0^1 b(x) e^{-\delta\rho\tau} (\partial_x u)^2(x, t-\tau\rho)\, d\rho dx,$$

where σ_1, σ_2 and σ are positive constants to de determined and δ is an arbitrary positive constant. Using integrations by parts, Young's inequality as well as (2.4) and (3.3), we obtain, similarly to (2.32), that for any $\nu > 0$:

$$\begin{aligned}
\mathcal{F}'(t) + \nu \mathcal{F}(t) \leq\; & -\eta(1+\sigma_1)(\partial_x^2 u)^2(0,t) + \underbrace{\int_\omega b(x)\left(1-\xi+\sigma_1 e^{\sigma\ell}+\sigma_2 - 2\sigma_1\xi e^{\sigma x}\right) u^2\, dx}_{I_1} \\
& + \underbrace{\int_\omega b(x)\left(1-\xi+\sigma_1 e^{\sigma\ell}+\sigma_2-\sigma_2 e^{-\delta\tau}\right)(\partial_x u)^2(x,t-\tau)\, dx}_{I_2} \\
& + \underbrace{\sigma\sigma_1\left(\frac{\ell^4}{\pi^4}\left[\alpha+\sigma^2-\eta\sigma^4\right]e^{\sigma\ell}-5\eta\right)\|\partial_x^2 u\|^2}_{I_3} \\
& + \underbrace{\left(\sigma\sigma_1(5\eta\sigma^2-3)+\nu\frac{\ell^2}{\pi^2}\left[1+\sigma_1 e^{\sigma\ell}\right]\right)\|\partial_x u\|^2}_{I_4} \\
& + \underbrace{\tau\int_\omega\int_0^1 b(x)\left(\nu(\xi+\sigma_2)-\sigma_2\delta e^{-\delta\tau}\right)(\partial_x u)^2(x,t-\rho\tau)\, d\rho dx}_{I_5} \\
& -2\int_{\widehat{\omega}} a(x)\left(1+\sigma_1 e^{\sigma x}\right)u^2\, dx, \qquad (3.9)
\end{aligned}$$

where we picked up σ small so that $\alpha+\sigma^2-\eta\sigma^4>0$, that is, $\sigma < \sqrt{\frac{1+\sqrt{1+4\alpha\eta}}{2\eta}}$. This allows us to claim, on one hand, that $\alpha+\sigma^2-\eta\sigma^4 \leq \alpha + \frac{\beta}{4\eta}$, for any $\beta > 1$. On the other hand, the coefficient of I_3 (see (3.9)) can be made negative by taking

$$\sigma < \frac{1}{\ell}\ln\left(\frac{5\eta\pi^4}{\ell^4(\alpha+\beta/(4\eta))}\right),$$

provided that

$$\ell^4\left(\alpha+\frac{\beta}{4\eta}\right) < 5\eta\pi^4, \qquad (3.10)$$

where $\beta > 1$ is any arbitrary number.

With regard to the other terms of (3.9), we proceed as follows: we choose σ so that $5\eta\sigma^2-3<0$ and hence based on the above discussion we assume that

$$\sigma < \min\left\{\sqrt{\frac{3}{5\eta}}, \frac{1}{\ell}\ln\left(\frac{5\eta\pi^4}{\ell^4(\alpha+\beta/(4\eta))}\right), \sqrt{\frac{1+\sqrt{1+4\alpha\eta}}{2\eta}}\right\}.$$

Next, we take $\sigma_1 < (\xi-1)e^{-\sigma\ell}$ and $\sigma_2 < \xi - 1 - \sigma_1 e^{-\sigma\ell}$ and finally ν is chosen

$$\nu < \min\left\{\frac{\sigma_2\delta e^{-\delta\tau}}{\xi+\sigma_2}, \frac{\pi^2\sigma\sigma_1(3-5\eta\sigma^2)}{\ell^2\left(1+\sigma_1 e^{\sigma\ell}\right)}\right\}.$$

Thereby, all the terms $I_1 - I_5$ are negative and hence (3.9) gives the exponential stability of $\mathcal{F}$, which in turn implies the exponential stability of $F(t)$. This result is summarized in the following proposition:

Proposition 4. *Assume that the spatial distributions a and b satisfy **A1** while the parameters η and ℓ fulfill the condition* (3.10)*. Then, for every initial data $\Phi_0 = (u_0, \widehat{z_0}(\cdot, -\tau\cdot)) \in H$, the trivial solution of the problem (3.1) is uniformly exponentially stable in H and hence the energy $F(t)$ uniformly exponentially decays.*

Now, we are able to state our third exponential stability

Theorem 4. *Assume that the conditions **A1** and* (3.10) *hold. Moreover, we assume that there exists a positive constant μ such that $\|b\|_\infty \leq \mu$. Then, there exists a positive constant $r > 0$ such that for every initial data $\Phi_0 = (u_0, \widehat{z_0}(\cdot, -\tau\cdot)) \in H$ satisfying $\|\Phi_0\|_H \leq r$, the trivial solution of the problem (1.1) is uniformly exponentially stable in H and hence the corresponding energy $F(t)$ uniformly exponentially decays.*

Proof. We proceed by following several steps:

Step 1: First, we consider the linearized system (2.1) corresponding to (1.1), which can be written as a perturbation of the problem (3.1)

$$\begin{cases} \Phi_t(t) = (\mathcal{B} + \mathcal{P})\,\Phi(t), \\ \Phi(0) = \Phi_0, \end{cases} \tag{3.11}$$

where $\Phi = (u, z)$ and $\mathcal{B}$, defined by (3.7), generates a C_0-semigroup of contractions $e^{t\mathcal{B}}$, whereas $\mathcal{P}$ is a bounded linear operator on H defined by

$$\mathcal{P}\Phi(t) = (\xi b(\cdot)u, 0), \quad \text{for any } \Phi \in H.$$

In light of Proposition (4), there exist positive constants ς and ϑ such that $\left\|e^{t\mathcal{B}}\right\|_{\mathcal{L}(H)} \leq \varsigma e^{-\vartheta t}$, for any $t > 0$. These facts permit to use [28, Theorem 1.1, p. 76] and conclude that $\mathcal{B} + \mathcal{P}$ is also a generator of a C_0-semigroup of contractions $e^{t(\mathcal{B}+\mathcal{P})}$ on H satisfying $\left\|e^{t(\mathcal{B}+\mathcal{P})}\right\|_{\mathcal{L}(H)} \leq \varsigma e^{(\varsigma\xi\|b\|_\infty - \vartheta)t}$, for any $t > 0$. Taking $\|b\|_\infty < \mu := \frac{\vartheta}{\varsigma\xi}$ to get the exponential stability of the linearized system (2.1).

Step 2: In view of the outcomes of step 1 and arguing as in Sections 2, we can show that for each $\Phi_0 \in H$, the problem (1.1) has a unique global mild solution $u \in C\left(0, \infty; L^2(\Omega)\right) \cap L^2_{\text{loc}}\left(0, \infty; H^2_0(\Omega)\right)$ such that $\|u\|_{L^2\left(0,T;H^2_0(\Omega)\right)} \leq C\|\Phi_0\|^2_H \left(1 + \|\Phi_0\|^2_H\right)$ (see (2.44)).

Step 3: Now, consider an initial data $\Phi_0 = (u_0, \widehat{z_0}(\cdot, -\tau\cdot)) \in H$ with $\|\Phi_0\|_H \leq r$, where r is to be chosen later. Then, we argue as [29] (see also [7]) by writing the solution u of (1.1) as $u = u_1 + u_2$, where u_1 is solution of (2.1), while u_2 is solution of (2.19) with a source term $f(x,t) = -u(x,t)\partial_x u(x,t)$ and initial data $\Phi_0 = (0, 0)$. By virtue of the facts obtained in the previous steps, there exists a positive constant $\varpi < 1$ such that

$$\begin{aligned} \|(u(T), z(T))\|_H &\leq \|(u_1(T), z_1(T))\|_H + \|(u_2(T), z_2(T))\|_H \\ &\leq \varpi \|\Phi_0\|_H + C\|u\partial_x u\|_{L^1(0,T;H^1_0(\Omega))} \\ &\leq \varpi \|\Phi_0\|_H + C\|u\|^2_{L^2(0,T;H^2_0(\Omega))} \leq \|\Phi_0\|_H \left(\varpi + Cr(1 + r^3)\right). \end{aligned} \tag{3.12}$$

Given $\varpi < 1$, we choose r so that $r(1+r^3) < \frac{1-\varpi}{2C}$ and hence (3.12) yields $\|(u(T), z(T))\|_H \leq \frac{1+\varpi}{2} \|\Phi_0\|_H$. Repeating this argument on $[nT, (n+1)T]$, $n = 0, 1, \cdots$, the exponential stability follows. $\square$

4. Concluding Remarks

This article was concerned with the investigation of the well-posedness and stability of a nonlinear Kawahara equation subject to the effect of a localized interior delay damping. After showing the well-posedness of the problem, two stability results are show.

In a future work, it would be desirable to study the effect on the stability property of delay in the nonlinearity. We also aspire to investigate the impact of a memory term on the stability of Kawahara equation.

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