Received xx xxxx 20xx; | Revised x xxxx 20xx; | Accepted x xxxx 20xx

DOI: xxx/xxxx

ARTICLE TYPE

Fractional differential inclusions with multi-point boundary conditions involving Hilfer-Hadamard derivative

Berrabah Fatima Zohra[1] | Hedia Benaouda*[2] | Gisèle Mophou[3]

[1]Faculty of Exact and Applied Sciences, University of Oran 1, PO BOX 1524, El Mnaouer, Oran 31000, Algeria
[2]Laboratory of mathematics and informatics, University of Tiaret, PO BOX 78 Tiaret, Algeria
[3]Department of Mathematics and computers sciences, University of Antilles, 97159, France
German Research Chair at A.I.M.S,, Limbe, Cameroon

Correspondence
*Hedia Benaouda, Laboratory of mathematics and computers sciences, University of Tiaret, PO BOX 78 Tiaret, Algeria. Email: b_hedia@univ-tiaret.dz

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Summary

In this paper, we studied the existence and uniqueness result of solutions for boundary value problems for Hilfer-Hadamard type fractional differential inclusions with multi-point boundary conditions, in the first approach we deal with a non-convex valued right hand side and in the second approach we consider the Carathéodory case. Finally the compactness of solution sets is also obtained.

KEYWORDS:
Differential inclusions, mixed Hilfer-Hadamard fractional derivative, fixed point theorem, existence solution, topological structure, compactness.

1 | INTRODUCTION AND STATEMENT OF THE PROBLEM

The fractional differential equations give proofs of the more appropriate models for various areas of engineering, physics, bioengineering and other applied sciences.

For some fundamental results in the theory of fractional calculus and fractional differential equations see Hedia *et al.*[8,9,11,12] and the papers[2,7,13].

However, there are a few related works on Hilfer fractional derivatives, for the so-called Hilfer fractional derivatives, one can see[17]. It seems that Hilfer *et al.*[21,22] have initially proposed linear differential equations with the new fractional operator, Hilfer fractional derivative and applied operational calculus to solve such simple fractional differential equations. Thereafter, Furati *et al.*[17] extended to study nonlinear problems and presented the existence, nonexistence and stability results for initial value problems of nonlinear fractional differential equations with Hilfer fractional derivative in a suitable weighted space of continuous functions. The fractional derivative due to Hadamard, introduced in 1892[19], differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral can be found in[23]. In the paper,[4] the authors considered the following problem,

$$ {}_{H}D^{\alpha,\beta} y(t) + f(t, y(t)) = 0,\ t \in J := [1, e],\ 1 \le \alpha \le 2,\ 0 \le \beta \le 1 \tag{1}$$

$$ y(1+\epsilon) = 0,\ {}_{H}D^{1,1} y(e) = \nu\ {}_{H}D^{1,1} y(\tau). \tag{2}$$

[0]**Abbreviations:** ANA, anti-nuclear antibodies; APC, antigen-presenting cells; IRF, interferon regulatory factor

They used some theorems of fixed point for studying the results of existence and uniqueness for Hilfer-Hadamard-type fractional differential equations. In the paper,[5] the authors deal with the existence and uniqueness of nonlocal boundary conditions for Hilfer Hadamard type fractional differential equation,

$$_{H}D^{\alpha,\beta}y(t) + f(t, y(t)) = 0,\ t \in J := [1, e],\ 1 \le \alpha \le 2,\ 0 \le \beta \le 1, \tag{3}$$

$$y(1+\epsilon) = \sum_{1}^{n-2} v_i y(\xi_i),\ _{H}D^{1,1}y(e) = \sum_{1}^{n-2} \sigma_i\ _{H}D^{1,1}y(\zeta_i). \tag{4}$$

where $_{H}D^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $1 < \alpha \le 2$ and type $\beta \in [0,1]$, $f : J \times R \to \mathbb{R}$ is a continuous function, $0 < \epsilon < 1$, $\zeta_i \in (1, e)$, $\xi_i \in (1, e)$ $v_i, \sigma_i \in \mathbb{R}$, for all $i = 1, \dots n-2$, $\zeta_1 < \zeta_2 < \dots < \zeta_n - 2$ and $\xi_1 < \xi_2 < \dots < \xi_n - 2$ $_{H}D^{1,1} = t\frac{d}{dt}$. In the paper[1] the authors considered the following coupled system of implicit Hilfer-Hadamard fractional differential equations:

$$\begin{cases} _{H}D_1^{\alpha,\beta}x_1(t) = f_1(t, x_1(t), x_2(t), ^{H}D_1^{\alpha,\beta}x_1(t), ^{H}D_1^{\alpha,\beta}x_2(t)), \\ _{H}D_1^{\alpha,\beta}x_2(t) = f_2(t, x_1(t), x_2(t), ^{H}D_1^{\alpha,\beta}x_1(t), ^{H}D_1^{\alpha,\beta}x_2(t)),\ t \in I, \end{cases}$$

with the initial conditions

$$\begin{cases} ^{H}I^{1-\gamma}x_1(t) = \phi_1, \\ ^{H}I^{1-\gamma}x_2(t) = \phi_2, \end{cases}$$

where $I := [1, T], T > 1, \alpha \in (0, 1), \beta \in [0, 1], \gamma = \alpha + \beta - \alpha\beta, \phi_i, i = 1, 2 \in E$ and $f_i, i = 1, 2$ are given continuous functions, E is a real (or complex) Banach space with norm $\|.\|_E$ and dual E^*, such that E is the dual of a weakly compactly generated Banach space X, $^{H}I^{1-\gamma}$is the left-sided mixed Hadamard integral of order $1-\gamma$, and $_{H}D_1^{\alpha,\beta}$is the Hilfer-Hadamard fractional derivative of order α and type β.

The authors proved the existence of weak solutions for a coupled system of implicit fractional differential equations of Hilfer-Hadamard type. In the paper[26] the authors considered the Hilfer-Hadamard-type IDE with nonlocal condition of the form :

$$_{H}D_1^{\alpha,\beta}x(t) = f(t, x(t), ^{H}D_1^{\alpha,\beta}x(t)),\ 0 < \alpha < 1,\ 0 \le \beta \le 1,\ t \in J = [1, b]. \tag{5}$$

$$^{H}I^{1-\gamma}x_1(t) = \sum c_i x(\tau_i), \alpha \le \gamma = \alpha + \beta - \alpha\beta \le 1,\ \tau_i \in [1, b], \tag{6}$$

where $_{H}D_1^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order α and type β. X be a Banach space, $: J \times X \times X \to X$ is a given continuous function and $^{H}I^{1-\gamma}$ is the left-sided mixed Hadamard integral of order $1-\gamma$. They make use SchaeferŠs fixed-point theorem to investigate the existence of solutions to nonlocal initial value problems for implicit differential equations with Hilfer-Hadamard fractional derivative. Then the Ulam stability result is obtained by using Banach contraction principle. In[20] the author deal with a class of semi-linear Hilfer fractional differential equation with nonlocal conditions,

$$D_{0^+}^{\alpha,\beta}x(t) = Ax(t) + f(t, x(t)),\quad t \in (0, b], \tag{7}$$

$$I_{0^+}^{1-\gamma}x(t) = \sum_{i=1}^{m} \lambda_i x(\tau_i),\ \alpha \le \gamma = \alpha + \beta - \alpha\beta,\ \tau_i \in (0, b], \tag{8}$$

where the two parameter family of fractional derivative $D^{\alpha,\beta}$ denote the left-sided Hilfer fractional derivative introduced in[21], $0 < \alpha \le 1, 0 \le \beta \le 1$. the state $x(.)$ takes value in a Banach space E with norm $\|.\|$, A is the infinitesimal generator of semigroup of bounded linear operators (i.e. C_0 semigroup) $T(t)_{t\ge 0}$ in Banach space E. The operator $I_{0^+}^{1-\gamma}$ denotes the left-sided Riemann-Liouville fractional integral, $f : (0, b] \times E \to E$ will be specified in later sections. τ_i, $i = 1, 2, \dots, m$ are pre-fixed points satisfying $0 < \tau_1 \le \dots \le \tau_m < b$ and $\Gamma(\gamma) \ne \sum_{i=1}^{m} \lambda_i \tau_i$ where $\Gamma(\gamma) = \int_0^{+\infty} x^{1-\gamma}e^{-x}dx$. The author has given a new result concerning the existence of solution of (7)-(8) by using measure of non-compactness combined with condensing map in Banach space. In 1890, Peano[25] proved that the Cauchy problem for ordinary differential equations has local solutions although the uniqueness property does not hold in general. For the case where the uniqueness does not hold, Kneser[24] proved in 1923 that the solution set is a continuum, i.e. closed and connected. In 1942, Aronszajn[6] improved this result for differential inclusions in the sense that he showed that the solution set is compact and acyclic, and he specified this continuum to be an R_δ-set. An analogous result has been obtained for differential inclusions with u.s.c. convex valued nonlinearities by DeBelasi and Myjak[16].

Very recently, Topological structure of the solution set for ordinary differential equations and inclusions is developed recently by Browder and Gupta in [14]. We bearing in the mind that the application of nonlocal condition

$$ {}_H D^{1,1} y(e) = \sum_{i=1}^{n} v_i \ {}_H D^{1,1} y(\eta_i), $$

in physical problems yields better effect than the initial condition

$$ I_{0^+}^{1-\gamma} x(1) = y_0. $$

To the best of our knowledge, the study of the structure of the solution set for fractional differential inclusions with Hilfer-Hadamard-Type derivative is untreated problem, and this fact, is the main motivation of this paper. Our aim is to study the existence and uniqueness result of solutions and topological structure of solution sets for boundary value problems with Hilfer-Hadamard-Type fractional differential inclusions of the form:

$$ {}_H D^{\alpha,\beta} y(t) \in F(t, y(t)),\ t \in J := [1, e],\ 1 \leq \alpha \leq 2,\ 0 \leq \beta \leq 1 \tag{9} $$

$$ y(1+\epsilon) = 0,\ {}_H D^{1,1} y(e) = \sum_{i=1}^{n} v_i \ {}_H D^{1,1} y(\eta_i), \tag{10} $$

where ${}_H D^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $1 \leq \alpha \leq 2$,
$0 \leq \beta \leq 1,\ \eta_i \in (1, e),\ 0 < \epsilon < 1\ {}_H D^{1,1} = t\frac{d}{dt},\ J = [1, e]\ F : J \times R \to P(R)$ is a multivalued map (multimap for short).
This paper is organized as follows; In section 2 we introduce some preliminary results needed in the following sections. In section 3 we present an existence results for the problem (9)-(10) when the right hand side is nonconvex as well as convex valued. The first result relies on the fixed point theorem for contraction multivalued maps due to Covitz and Nadler , while the second result is based upon the nonlinear alternative of Leray-Schauder type. Section 4 we present the topological structure of solution sets.

2 | PRELIMINARIES

In this section, we introduce notations, definitions and preliminary facts that will be used in the remainder of this paper.
Let $C(J, \mathbb{R})$ be a Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$ \|y\|_\infty = \sup\{|y(t)|;\ \ t \in J\}. $$

$L^1(J, \mathbb{R})$ denote the Banach space of functions $y : J \to \mathbb{R}$ that are Lebesgue integrable with norm

$$ \|y\|_{L^1} = \int_J |y(t)| dt. $$

$AC(J, \mathbb{R})$ is the space of function $y : J \to \mathbb{R}$ which are absolutely continuous.
$L^\infty(J)$ be the Banach space of measurable function $y : J \to \mathbb{R}$ which are essentially bounded equipped with the norm

$$ \|y\|_{L^\infty} = \inf\{c > 0,\ \ |y(t)| \leq c;\ \ a.e\ \ t \in J\}. $$

Let $(X; \|.\|)$ be a Banach space.
$P_{cl}(X) = \{Y \in P(X) :\ \ Y$ is closed$\}$; $P_b(X) = \{Y \in P(X) :\ \ Y$ is bounded$\}$; $P_{cp}(X) = \{Y \in P(X) :\ \ Y$ is compact$\}$; $P_{cp,c}(X) = \{Y \in P(X) :\ \ Y$ is compact and convex$\}$.
A multivalued map $G : X \to P(X)$ is convex (closed) valued if $G(X)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{X \subset B} G(X)$ is bounded in X for all $B \subset P_b(X)$ (i.e, $\sup_{x \in B}\{\sup\{|y|;\ \ y \in G(x)\}\} < \infty$).
G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in\subset P_b(X)$.
If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_\star$, $y_n \to y_\star$, $y_n \in G(x_n)$ imply $y_\star \in G(x_\star)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

Let (X,d) be a metric space induced from the normed space $(X,\|.\|)$. Consider $H_d : P(X)\times P(X) \to \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized (complete) metric space. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y\in\mathbb{R}$, the function,

$$t \to d(y, G(t)) = \inf\{|y-z| : \quad z\in G(t)\},$$

is measurable.

Definition 1. A multivalued operator $G : X \to P_{cl}(X)$ is called

a) $\gamma-$Lipschitz if there exists $\gamma > 0$ such that

$$H_d(G(x),G(y)) \leq \gamma d(x,y) \quad \text{for each } \ x,y\in X$$

b) a contraction if it is $\gamma-$Lipschitz with $\gamma < 1$.

Definition 2. The multivalued map $F : J\times X \to P(X)$ is said to be L^1Carathéodory if

i) $t \to F(t,u)$ is measurable for each $u \in X$,

ii) $u \to F(t,u)$ is upper semicontinuous on X for almost all $t\in J$;

iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J,\mathbb{R}_+)$ such that

$$\|F(t,u)\|_{P(X)} = \sup\{|v| : \quad v\in F(t,u)\} \leq \varphi_\rho(t) \quad \forall \|u\| \leq \rho \quad \text{and for a.e. } \ t\in J.$$

Theorem 1. (Covitz-Nadler)
Let (X,d) be a complete metric space. If $G : X \to P_{cl}(X)$ is a contraction, then $FixG \neq \emptyset$.

Proposition 1. [15]
If Π_1 and Π_2 are compact valued measurable multifunctions then the multifunction $t \mapsto \Pi_1(t)\cap\overline{\Pi_2(t)}$ is measurable. If (Π_n) is a sequence of compact valued measurable multifunctions then $t\mapsto \cap\Pi_n(t)$ is measurable, and if $\overline{\cup\Pi_n(t)}$ is compact, $\overline{t\mapsto \cup\Pi_n(t)}$ is measurable.

Theorem 2. (nonlinear alternative [18])
Let X be a Banach space with $C \subset X$ closed and convex. Assume U is a relatively open subset of C with $0\in C$ and $G : U \to C$ is upper semicontinuous and completely continuous multivalued map. Then either,

i) G has a fixed point in U or

ii) there is a point $u\in\partial U$ and $\lambda\in(0,1)$ with $u\in\lambda G(u)$.

For further reading and details on multivalued analysis, we refer the reader to the books [10,15]. We introduce now some notations and definitions of fractional calculus and present preliminary results needed in our proofs later.

Definition 3. [23] Let $h \in L^1([a,b],\mathbb{R})$. The Riemann-Liouville fractional integral of order $\alpha > 0$ of the function h is defined almost everywhere in $[a,b]$ by

$${}^{RL}I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function. When $a=0$, we write ${}^{RL}I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t>0$ and $\varphi_\alpha(t) = 0$ for $t\leq 0$. the equality holds everywhere if $h\in C([a,b];\mathbb{R})$.

Definition 4. [23] Let $\alpha > 0$ and n be the smallest integer greater than or equal to α and $h : [a,b] \to \mathbb{R}$ be a function such that ${}^{RL}I_a^{n-\alpha} h \in AC^n([a,b],\mathbb{R})$. Then the Riemann-Liouville fractional derivative of order α of the function h is defined almost every where in $[a, b]$ by

$$
\begin{aligned}
{}^{RL}D^{\alpha}_{a+}h(t) &= \frac{d^n}{dt^n}\,{}^{RL}I^{n-\alpha}_{a+}h(t) \\
&= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}h(s)ds.
\end{aligned}
$$

Definition 5. [23] Let $t \in [0, +\infty)$ and $\alpha > 0$. The Hadamard fractional integral of order α, applied to the function $h \in L^p[a, b]$, $1 \le p < +\infty$, $0 < a < b < \infty$, for $t \in [a, b]$, is defined as

$$
{}_H I^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{h(\tau)}{\tau}d\tau, \quad t > 1,
$$

where $\log(.) = \log_e(.)$.

Definition 6. (Hadamard fractional integral)
Let $\delta = t\frac{d}{dt}$, $\alpha > 0$ and $n = [\alpha] + 1$, where $[\alpha]$ is the integer part of α. The Hadamard fractional derivative of order a applied to the function $h \in AC^n_{\delta}[a, b]$, $0 = a < b < \infty$, is defined as

$$
{}_H D^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\int_a^t (\ln\frac{t}{\tau})^{n-\alpha-1}\frac{h(\tau)}{\tau}d\tau = \delta^n({}_H I^{n-\alpha}h)(t)
$$

Property 1. If $Re(\alpha), Re(\beta) > 0$, and $0 < a < b < \infty$, then

$$
\left({}_H I^{\alpha}_{a^+}\left(\log\frac{\tau}{a}\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{t}{a}\right)^{\beta+\alpha-1}
$$

$$
\left({}_H D^{\alpha}_{a^+}\left(\log\frac{\tau}{a}\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{t}{a}\right)^{\beta-\alpha-1}
$$

Property 2. [23](page 114) Let $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ be such that $Re(\alpha) > Re(\beta) > 0$

- If $0 < a < b < \infty$ and $1 \le p \le \infty$, then for $\varphi \in L^p(a, b)$

$$
{}_H D^{\beta}_{a^+}\,{}_H I^{\alpha}_{a^+}\varphi = {}_H I^{\alpha-\beta}_{a^+}\varphi \quad and \quad {}_H D^{\beta}_{b^-}\,{}_H I^{\alpha}_{b^-}\varphi = {}_H I^{\alpha-\beta}_{b^-}\varphi
$$

- In Particular if $\beta = m \in \mathbb{N}$, then

$$
{}_H D^{m}_{a^+}\,{}_H I^{\alpha}_{a^+}\varphi = {}_H I^{\alpha-m}_{a^+}\varphi \quad and \quad {}_H D^{m}_{b^-}\,{}_H I^{\alpha}_{b^-}\varphi = {}_H I^{\alpha-m}_{b^-}\varphi.
$$

Theorem 3. [23](page 116) Let $Re(\alpha) > 0$, $n = [\alpha] + 1$ and $0 < a < b < \infty$. Also let $\left({}_H I^{n-\alpha}_{a^+}\varphi\right)(t)$ be the Hadamard type fractional integral of the form

$$
\left({}_H I^{n-\alpha}_{a^+}\varphi\right)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{n-\alpha-1}\frac{\varphi(\tau)}{\tau}d\tau, \quad 0 < a < t < b < \infty
$$

If $\varphi(t) \in L(a, b)$ and $\left({}_H I^{n-\alpha}_{a^+}\varphi\right)(t) \in AC^n_{\delta}[a, b]$ where

$$
AC^n_{\delta}[a, b] = \left\{\varphi : [a, b] \to \mathbb{C} : \quad \delta^{n-1}\varphi \in AC[a, b], \ \ \delta = t\frac{d}{dt}\right\}
$$

then

$$
\left({}_H I^{\alpha}_{a^+}\,{}_H D^{\alpha}_{a^+}\varphi\right)(t) = \varphi(t) - \sum_{k=1}^{n}\frac{\left(\delta^{n-k}\left({}_H I^{n-\alpha}_{a^+}\varphi\right)\right)(a)}{\Gamma(\alpha-k+1)}\left(\log\frac{t}{a}\right)^{\alpha-k}.
$$

Definition 7. (Hilfer fractional derivative)
Let $n-1<\alpha<n$, $0\leq\beta\leq 1$, $\varphi\in L^1(a,b)$. The Hilfer fractional derivative $D^{\alpha,\beta}$ of order α and type β of φ is defined as

$$\begin{aligned}(D^{\alpha,\beta}\varphi)(t) &= \left(I^{\beta(n-\alpha)}\left(\frac{d}{dt}\right)^n I^{(n-\alpha)(1-\beta)}\varphi\right)(t)\\ &= \left(I^{\beta(n-\alpha)}\left(\frac{d}{dt}\right)^n I^{n-\gamma}\varphi\right)(t),\quad \gamma=n\beta+\alpha-\alpha\beta\\ &= \left(I^{\beta(n-\alpha)}D^{\gamma}\varphi\right)(t),\end{aligned}$$

where $I^{(.)}$ and $D^{(.)}$ is the Riemann-Liouvill fractional integral and derivative respectively.

Definition 8. (Hilfer-Hadamard fractional derivative)[4]
Let $n-1<\alpha<n$, $0\leq\beta\leq 1$, $\varphi\in L^1(a,b)$. The Hilfer-Hadamard fractional derivative ${}_HD^{\alpha,\beta}$ of order α and type β of φ is defined as

$$\begin{aligned}({}_HD^{\alpha,\beta}\varphi)(t) &= \left({}_HI^{\beta(n-\alpha)}\delta^n\ {}_HI^{(n-\alpha)(1-\beta)}\varphi\right)(t)\\ &= \left({}_HI^{\beta(n-\alpha)}\delta^n\ {}_HI^{n-\gamma}\varphi\right)(t),\quad \gamma=n\beta+\alpha-\alpha\beta\\ &= \left({}_HI^{\beta(n-\alpha)}\ {}_HD^{\gamma}\varphi\right)(t),\end{aligned}$$

where ${}_HI^{(.)}$, ${}_HD^{(.)}$ is the Hadamard fractional integral and derivative respectively.

3 | EXISTENCE RESULTS

Recall that $C(J,\mathbb{R})$ is a Banach space of all continuous functions from J into $\mathbb{R}$ endowed with the norm

$$\|y\|=\sup_{t\in J]}|y(t)|.$$

Definition 9. A function $y\in AC^2(J,\mathbb{R})$ is said to be a solution of (9)-(10) if there exists a function $v\in L^1(J,\mathbb{R})$ with $v(t)\in F(t,y(t))$ for a.e $t\in J$ such that ${}_HD^{\alpha,\beta}y(t)=v(t)$; $1\leq\alpha\leq 2$, $0\leq\beta\leq 1$ a.e $t\in J$ and $y(1+\epsilon)=0$; ${}_HD^{1,1}y(e)=\sum_{i=1}^{n}\nu_i\ {}_HD^{1,1}y(\eta_i)$.

To prove the existence of solutions to (9)-(10), we need the following auxiliary lemmas.

Lemma 1. [3] Let $\alpha>0$ and $y\in C[1,+\infty)\cap L^1[1,+\infty)$. Then the solution of Hadamard fractional differential equation $\left({}_HD^{\alpha}y(t)\right)=0$ is given by

$$y(t)=\sum_{i=1}^{n}c_i\,(\ln t)^{\alpha-i},$$

where $c_i\in\mathbb{R}$, $i=1,\dots,n$ are arbitrary constants and $n-1<\alpha<n$.

Lemma 2. Let $Re(\alpha)>0$, $0\leq\beta\leq 1$, $\gamma=\alpha+n\beta-\alpha\beta$ then $n-1<\gamma\leq n$, $n=[Re(\alpha)]+1$, $0<a<b<\infty$ if $\varphi\in L^1(a,b)$ and $\left({}_HI^{n-\gamma}\varphi\right)(t)\in AC^n_\delta[a,b]$ then

$$\begin{aligned}{}_HI^{\alpha}_{a^+}\left({}_HD^{\alpha,\beta}_{a^+}\varphi\right)(t) &= {}_HI^{\alpha}_{a^+}\left({}_HI^{\beta(n-\alpha)}\ {}_HD^{\gamma}\varphi\right)(t)\\ &= \left({}_HI^{\alpha+\beta n-\beta\alpha}\ {}_HD^{\gamma}\varphi\right)(t)\\ &= \left({}_HI^{\gamma}\ {}_HD^{\gamma}\varphi\right)(t)\\ &= \varphi(t)-\sum_{k=0}^{n-1}\frac{\left(\delta^{n-k-1}\left({}_HI^{n-\gamma}_{a^+}\varphi\right)\right)(a)}{\Gamma(\gamma-k)}\left(\log\frac{t}{a}\right)^{\gamma-k-1}.\end{aligned}$$

As a consequence of Lemma (1) and Lemma (2) we have the following result which is useful in what follows.

Lemma 3. For $1<\alpha\leq 2$, $0\leq\beta\leq 1$ and $h\in C(J,\mathbb{R})$, $\gamma=\alpha+2\beta-\alpha\beta\Rightarrow\gamma\in(1,2]$. The problem

$$ {}_HD^{\alpha,\beta}y(t)=\varphi(t),\ t\in J:=J,\ 1\leq\alpha\leq 2,\ 0\leq\beta\leq 1 \tag{11}$$

$$y(1+\epsilon)=0,\ {}_H D^{1,1} y(e)=\sum_{i=1}^{n} \nu_i\ {}_H D^{1,1} y(\eta_i), \tag{12}$$

has a unique solution it giving in the formulae

$$\begin{aligned} y(t) = &\ {}_H I^{\alpha}\varphi(t) \\ &+ \ {}_H I^{\alpha}\varphi(1+\epsilon)\,(\log t)^{\gamma-2}\,\frac{(\gamma-1)\mu_1-(\gamma-2)\mu_2\log t}{\Delta} \\ &+ (\log(1+\epsilon))^{\gamma-2}\left[\sum_{i=1}^{n}\nu_i\ {}_H I^{\alpha-1}\varphi(\eta_i) - \ {}_H I^{\alpha-1}\varphi(e)\right]\frac{\log(1+\epsilon)\,(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta} \end{aligned} \tag{13}$$

where

$$\begin{aligned} \Delta &= (\gamma-2)\mu_2\left[\log(1+\epsilon)\right]^{\gamma-1}-(\gamma-1)\mu_1\left[\log(1+\epsilon)\right]^{\gamma-2} \\ \mu_1 &= 1-\sum_{i=1}^{n}\nu_i\left(\log\eta_i\right)^{\gamma-2} \\ \mu_2 &= 1-\sum_{i=1}^{n}\nu_i\left(\log\eta_i\right)^{\gamma-3}, \end{aligned}$$

with

$$(\gamma-2)\mu_2\log(1+\epsilon)-(\gamma-1)\mu_1\neq 0.$$

Proof.
Assume y satisfies (11)-(12), lemma (2) implies

$$\begin{aligned} {}_H I^{\alpha}\left({}_H D^{\alpha,\beta} y\right)(t) &= \ {}_H I^{\alpha}\varphi(t) \\ &= y(t)-\sum_{j=0}^{1}\frac{\left(\delta^{2-j-1}({}_H I^{2-\gamma} y)\right)(1)}{\Gamma(\gamma-j)}\left(\log\frac{t}{1}\right)^{\gamma-j-1}, \\ &= y(t)-c_0\,(\log t)^{\gamma-1}-c_1\,(\log t)^{\gamma-2}. \end{aligned}$$

Then

$$y(t)=\ {}_H I^{\alpha}\varphi(t)+c_0\,(\log t)^{\gamma-1}+c_1\,(\log t)^{\gamma-2}, \tag{14}$$

and

$$\begin{aligned} y(1+\epsilon)=0 \Rightarrow\ {}_H I^{\alpha}\varphi(1+\epsilon)+c_0\,(\log(1+\epsilon))^{\gamma-1}+c_1\,(\log(1+\epsilon))^{\gamma-2}=0 \\ c_0=\frac{-1}{(\log(1+\epsilon))^{\gamma-1}}\ {}_H I^{\alpha}\varphi(1+\epsilon)-\frac{c_1}{\log(1+\epsilon)}, \end{aligned}$$

by Property (1), (2)$\left({}_H I^{\beta(n-\alpha)}\ {}_H D^{\gamma}\varphi\right)$

$$\begin{aligned} {}_H D^{1,1} y(t) &= \ {}_H D^{1,1}\left[{}_H I^{\alpha}\varphi(t)+c_0\,(\log t)^{\gamma-1}+c_1\,(\log t)^{\gamma-2}\right] \\ &= \ {}_H D^{1,1}\ {}_H I^{\alpha}\varphi(t)+c_0\ {}_H D^{1,1}\,(\log t)^{\gamma-1}+c_1\ {}_H D^{1,1}\,(\log t)^{\gamma-2} \\ &= \left({}_H I^{1(2-1)}\ {}_H D^{2}\ {}_H I^{\alpha}\varphi(t)\right)+c_0\left({}_H I^{1(2-1)}\ {}_H D^{2}\,(\log t)^{\gamma-1}\right) \\ &+ c_1\left({}_H I^{1(2-1)}\ {}_H D^{2}\,(\log t)^{\gamma-2}\right) \\ &= \left({}_H I^{1}\ {}_H I^{\alpha-2}\varphi(t)\right)+c_0\left({}_H I^{1}\frac{\Gamma(\gamma)}{\Gamma(\gamma-2)}\,(\log t)^{\gamma-2-1}\right) \\ &+ c_1\left({}_H I^{1}\frac{\Gamma(\gamma-1)}{\Gamma(\gamma-1-2)}\,(\log t)^{\gamma-1-2-1}\right) \\ &= \ {}_H I^{\alpha-1}\varphi(t)+c_0\frac{\Gamma(\gamma)}{\Gamma(\gamma-1)}\,(\log t)^{\gamma-2}+c_1\frac{\Gamma(\gamma-1)}{\Gamma(\gamma-2)}\,(\log t)^{\gamma-3} \\ &= \ {}_H I^{\alpha-1}\varphi(t)+c_0\frac{(\gamma-1)\Gamma(\gamma-1)}{\Gamma(\gamma-1)}\,(\log t)^{\gamma-2}+c_1\frac{(\gamma-2)\Gamma(\gamma-2)}{\Gamma(\gamma-2)}\,(\log t)^{\gamma-3} \\ &= \ {}_H I^{\alpha-1}\varphi(t)+c_0(\gamma-1)\,(\log t)^{\gamma-2}+c_1(\gamma-2)\,(\log t)^{\gamma-3}. \\ {}_H D^{1,1} y(e) &= \ {}_H I^{\alpha-1}\varphi(e)+c_0(\gamma-1)+c_1(\gamma-2) \end{aligned}$$

$$\sum_{i=1}^{n} v_i \ {}_HD^{1,1} y(\eta_i) = \sum_{i=1}^{n} v_i \ {}_HI^{\alpha-1}\varphi(\eta_i) + c_0(\gamma-1)\sum_{i=1}^{n} v_i \left(\log \eta_i\right)^{\gamma-2}$$
$$+ \ c_1(\gamma-2)\sum_{i=1}^{n} v_i \left(\log \eta_i\right)^{\gamma-3}$$
$$= \ {}_HI^{\alpha-1}\varphi(e) + c_0(\gamma-1) + c_1(\gamma-2).$$

$$c_1\left[(\gamma-2)\left[1-\sum_{i=1}^{n} v_i \left(\log \eta_i\right)^{\gamma-3}\right]\right] =$$
$$\sum_{i=1}^{n} v_i \ {}_HI^{\alpha-1}\varphi(\eta_i) - \ {}_HI^{\alpha-1}\varphi(e) - c_0\left[(\gamma-1)\left[1-\sum_{i=1}^{n} v_i \left(\log \eta_i\right)^{\gamma-2}\right]\right].$$

Let

$$\mu_1 = 1-\sum_{i=1}^{n} v_i \left(\log \eta_i\right)^{\gamma-2} \quad \mu_2 = 1-\sum_{i=1}^{n} v_i \left(\log \eta_i\right)^{\gamma-3},$$

then

$$c_1 = \frac{1}{(\gamma-2)\mu_2}\left[\sum_{i=1}^{n} v_i \ {}_HI^{\alpha-1}\varphi(\eta_i) - \ {}_HI^{\alpha-1}\varphi(e)\right] - c_0\frac{(\gamma-1)\mu_1}{(\gamma-2)\mu_2}.$$

$$c_0 = \frac{-1}{(\log(1+\epsilon))^{\gamma-1}} \ {}_HI^{\gamma}\varphi(1+\epsilon) - \frac{1}{(\gamma-2)\mu_2\log(1+\epsilon)}\left[\sum_{i=1}^{n} v_i \ {}_HI^{\alpha-1}\varphi(\eta_i) - \ {}_HI^{\alpha-1}\varphi(e)\right]$$
$$+ c_0\frac{(\gamma-1)\mu_1}{(\gamma-2)\mu_2\log(1+\epsilon)}.$$

$$c_0 = -\frac{1}{\Delta}\left[(\gamma-2)\mu_2 \ {}_HI^{\alpha}\varphi(1+\epsilon) + (\log(1+\epsilon))^{\gamma-2}\left(\sum_{i=1}^{n} v_i \ {}_HI^{\alpha-1}\varphi(\eta_i) - \ {}_HI^{\alpha-1}\varphi(e)\right)\right],$$

then

$$c_1 = \frac{1}{\Delta}\left[\log(1+\epsilon)^{\gamma-1}\left(\sum_{i=1}^{n} v_i \ {}_HI^{\alpha-1}\varphi(\eta_i) - \ {}_HI^{\alpha-1}\varphi(e)\right) + (\gamma-1)\mu_1 \ {}_HI^{\alpha}\varphi(1+\epsilon)\right].$$

Now substituting the values of c_0 and c_1 in(14) we get(13).

3.1 | The Lipschitz case

We prove the existence of solutions for the problem (9)-(10) with a nonconvex valued right hand side. Our proof is based on the fixed point theorem for multi-valued map due to Covitz and Nadler that is theorem (1).

Theorem 4. Assume that the following hypothesis holds:

- (H_1) $F : J \times \mathbb{R} \to P_{cp}(\mathbb{R})$, such that $F(.,y) : J \to P_{cp}(\mathbb{R})$ is measurable for each $y \in \mathbb{R}$.
- (H_2) There exists $l \in L^1(J,\mathbb{R})$ such that

$$H_d(F(t,y), F(t,\bar{y})) \le l(t)|y-\bar{y}| \quad \text{for every} \ \ y, \bar{y} \in \mathbb{R},$$

  and

$$d(0, F(t,0)) \le l(t) \quad \text{for almost all} \ \ t \in J.$$

- (H_3) If Φ satisfied the condition $\|l\|_\infty \Phi < 1$ where

$$\Phi = \left\{ \frac{1-\left[\log(1+\epsilon)\right]^{3-\alpha-\gamma}}{\Gamma(\alpha+1)}\left|\frac{(\gamma-1)\mu_1-(\gamma-2)\mu_2}{(\gamma-2)\mu_2\log(1+\epsilon)-(\gamma-1)\mu_1}\right| \right.$$
$$\left. + \frac{1+\sum_{i=1}^{n}|v_i|}{\Gamma(\alpha)}\frac{1+\log(1+\epsilon)}{\left|(\gamma-2)\mu_2\log(1+\epsilon)-(\gamma-1)\mu_1\right|}\right\}.$$

Then the problem of boundary value (9)-(10) has at least one solution on J.

Proof. Transform the problem (9)-(10) into a fixed point problem. Consider the multivalued operator,

$$N(y)(t)=\left\{\begin{array}{l} h\in C(J,\mathbb{R}):\\ h(t)={}_HI^{\alpha}v(t)\\ +{}_HI^{\alpha}v(1+\epsilon)\,(\log t)^{\gamma-2}\,\frac{(\gamma-1)\mu_1-(\gamma-2)\mu_2\log t}{\Delta}\\ +(\log(1+\epsilon))^{\gamma-2}\left[\sum_{i=1}^{n}\nu_i\,{}_HI^{\alpha-1}v(\eta_i)-{}_HI^{\alpha-1}v(e)\right]\frac{\log(1+\epsilon)(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta}, \end{array}\right\} \tag{15}$$

where

$$\begin{aligned} &v\in S_{F,y}=\{v\in L^1(J,R)|v(t)\in F(t,y(t))\text{ for a.e } t\in J\}\\ &\mu_1=1-\sum_{i=1}^{n}\nu_i\left(\log\eta_i\right)^{\gamma-2}\qquad \mu_2=1-\sum_{i=1}^{n}\nu_i\left(\log\eta_i\right)^{\gamma-3}, \end{aligned}$$

with

$$(\gamma-2)\mu_2\log(1+\epsilon)-(\gamma-1)\mu_1\neq 0.$$

Remark 1. For each $y\in C(J,\mathbb{R})$, the set $S_{F,y}$ is nonempty since by (H1), F has a measurable selection (see[15], Theorem III.6).

We shall prove that N satisfies the assumptions of Theorem (1).
The proof will be given in two steps.
Step 1. $N(y)\in P_{cl}(C(J,\mathbb{R}))$ for each $y\in C(J,\mathbb{R})$.
Indeed, let $(y_n)_{n\in\mathbb{N}}\in N(y)$ such that $y_n\to\bar{y}$ in $C(J,\mathbb{R})$; then $\bar{y}\in C(J,\mathbb{R})$ and there exists $v_n\in S_{F,y}$ such that for each $t\in J$,

$$\begin{aligned} y_n(t)=&\ {}_HI^{\alpha}v_n(t)\\ &+{}_HI^{\alpha}v_n(1+\epsilon)\,(\log t)^{\gamma-2}\,\frac{(\gamma-1)\mu_1-(\gamma-2)\mu_2\log t}{\Delta}\\ &+(\log(1+\epsilon))^{\gamma-2}\left[\sum_{i=1}^{n}\nu_i\,{}_HI^{\alpha-1}v_n(\eta_i)-{}_HI^{\alpha-1}v_n(e)\right]\frac{\log(1+\epsilon)\,(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta}, \end{aligned}$$

$$\begin{aligned} =&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{v_n(\tau)}{\tau}d\tau\\ &+\frac{\frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta}}{\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{v_n(\tau)}{\tau}d\tau\\ &+(\log(1+\epsilon))^{\gamma-2}\left[\sum_{i=1}^{n}\nu_i\frac{1}{\Gamma(\alpha-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{v_n(\tau)}{\tau}d\tau-\frac{1}{\Gamma(\alpha-1)}\int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{v_n(\tau)}{\tau}d\tau\right]\\ &\times\frac{\log(1+\epsilon)(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta}. \end{aligned}$$

Using the fact that F has compact values and from (H2) we pass into a subsequence to obtain that v_n converges to v in $L^1(J,\mathbb{R})$. Thus $v\in S_{F,y}$ and for each $t\in J$ we have

$$y_n(t)\to\bar{y}(t)$$

$$\begin{aligned} \bar{y}(t)=&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{v(\tau)}{\tau}d\tau\\ &+\frac{\frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta}}{\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{v(\tau)}{\tau}d\tau\\ &+\frac{(\log(1+\epsilon))^{\gamma-2}}{\Gamma(\alpha-1)}\frac{\log(1+\epsilon)(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta} \end{aligned}$$

$$\left[\sum_{i=1}^{n} \nu_i \int_1^{\eta_i} \left(\log \frac{\eta_i}{\tau}\right)^{\alpha-2} \frac{\upsilon(\tau)}{\tau} d\tau - \int_1^{e} \left(\log \frac{e}{\tau}\right)^{\alpha-2} \frac{\upsilon(\tau)}{\tau} d\tau\right]$$

So $\bar{y} \in N(y)$.

Step 2. We show that there exists $\delta < 1$ such that

$$H_d(N(y), N(\bar{y})) \leq \delta \|y - \bar{y}\|,$$

for each $y, \bar{y} \in C(J, \mathbb{R})$.

Let $y, \bar{y} \in C(J, \mathbb{R})$ and $h_1 \in N(y)$, then there exists $\upsilon_1 \in F(t, y(t))$ such that for each $t \in J$,

$$\begin{aligned}
h_1(t) &= \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{\upsilon_1(\tau)}{\tau} d\tau \\
&+ \frac{\frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta}}{\Gamma(\alpha)} \int_1^{1+\epsilon} \left(\log \frac{1+\epsilon}{\tau}\right)^{\alpha-1} \frac{\upsilon_1(\tau)}{\tau} d\tau \\
&+ \frac{(\log(1+\epsilon))^{\gamma-2}}{\Gamma(\alpha-1)} \left[\sum_{i=1}^{n} \nu_i \int_1^{\eta_i} \left(\log \frac{\eta_i}{\tau}\right)^{\alpha-2} \frac{\upsilon_1(\tau)}{\tau} d\tau - \int_1^{e} \left(\log \frac{e}{\tau}\right)^{\alpha-2} \frac{\upsilon_1(\tau)}{\tau} d\tau\right] \\
&\frac{\log(1+\epsilon)(\log t)^{\gamma-2} - (\log t)^{\gamma-1}}{\Delta}.
\end{aligned}$$

By (H2), we have

$$H_d(F(t, y(t)), F(t, \bar{y}(t))) \leq l(t)|y(t) - \bar{y}(t)|.$$

So, there exists $w \in F(t, \bar{y}(t))$ such that

$$|\upsilon_1(t) - w(t)| \leq l(t)|y(t) - \bar{y}(t)| \quad t \in J.$$

Consider $U : J \to P(\mathbb{R})$ given by

$$U(t) = \left\{w \in \mathbb{R} : \ |\upsilon_1(t) - w(t)| \leq l(t)|y(t) - \bar{y}(t)|\right\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{y}(t))$ is measurable by proposition (1)[15], there exists a function $\upsilon_2(t)$ which is measurable selection for $U \cap F(t, \bar{y})$, $\upsilon_2(t) \in F(t, \bar{y}(t))$ and for each $t \in J$.

$$|\upsilon_1(t) - \upsilon_2(t)| \leq l(t)|y(t) - \bar{y}(t)| \quad t \in J.$$

Let us define for each $t \in J$,

$$\begin{aligned}
h_2(t) &= \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{\upsilon_2(\tau)}{\tau} d\tau \\
&+ \frac{\frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta}}{\Gamma(\alpha)} \int_1^{1+\epsilon} \left(\log \frac{1+\epsilon}{\tau}\right)^{\alpha-1} \frac{\upsilon_2(\tau)}{\tau} d\tau \\
&+ \frac{\log(1+\epsilon)^{\gamma-1}(\log t)^{\gamma-2} - \log(1+\epsilon)^{\gamma-2}(\log t)^{\gamma-1}}{\Delta\Gamma(\alpha-1)} \\
&\left[\sum_{i=1}^{n} \nu_i \int_1^{\eta_i} \left(\log \frac{\eta_i}{\tau}\right)^{\alpha-2} \frac{\upsilon_2(\tau)}{\tau} d\tau - \int_1^{e} \left(\log \frac{e}{\tau}\right)^{\alpha-2} \frac{\upsilon_2(\tau)}{\tau} d\tau\right].
\end{aligned}$$

Thus

$$
\begin{aligned}
|h_1(t)-h_2(t)| \le{}& \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{1}{\tau}|v_1(\tau)-v_2(\tau)|d\tau\\
&+\left|\frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta}\right|\frac{1}{\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{1}{\tau}|v_1(\tau)-v_2(\tau)|d\tau\\
&+(\log(1+\epsilon))^{\gamma-2}\left[\sum_{i=1}^{n}|\nu_i|\frac{1}{\Gamma(\alpha-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{1}{\tau}|v_1(\tau)-v_2(\tau)|d\tau\right.\\
&\left.-\frac{1}{\Gamma(\alpha-1)}\int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{1}{\tau}|v_1(\tau)-v_2(\tau)|d\tau\right]\left|\frac{\log(1+\epsilon)(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta}\right|\\
\le{}& \frac{1}{\alpha\Gamma(\alpha)}(\log t)^{\alpha}\,|l(t)|\,|y(t)-\bar{y}(t)|+\frac{1}{\alpha\Gamma(\alpha)}\left[\log(1+\epsilon)\right]^{\alpha}|l(t)|\,|y(t)-\bar{y}(t)|\\
&.\left|\frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta}\right|\\
&+(\log(1+\epsilon))^{\gamma-2}\left[\sum_{i=1}^{n}|\nu_i|\frac{1}{(\alpha-1)\Gamma(\alpha-1)}\left(\log\eta_i\right)^{\alpha-1}+\frac{1}{(\alpha-1)\Gamma(\alpha-1)}(\log e)^{\alpha-1}\right]\\
&.\,|l(t)|\,|y(t)-\bar{y}(t)|\left|\frac{\log(1+\epsilon)(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta}\right|\\
\le{}& \|l\|\|y-\bar{y}\|\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{\left[\log(1+\epsilon)\right]^{\alpha}}{\Gamma(\alpha+1)}\left|\frac{(\gamma-1)\mu_1+(\gamma-2)\mu_2}{\Delta}\right|\right.\\
&\left.+(\log(1+\epsilon))^{\gamma-2}\frac{1+\ \sum_{i=1}^{n}|\nu_i|}{\Gamma(\alpha)}\frac{1+\log(1+\epsilon)}{|\Delta|}\right\}\\
\le{}& \ \|l\|\|y-\bar{y}\|\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{\left[\log(1+\epsilon)\right]^{\alpha}}{\Gamma(\alpha+1)}\left|\frac{(\gamma-1)\mu_1+(\gamma-2)\mu_2}{\Delta}\right|\right.\\
&\left.+(\log(1+\epsilon))^{\gamma-2}\frac{1+\sum_{i=1}^{n}|\nu_i|}{\Gamma(\alpha)}\frac{1+\log(1+\epsilon)}{|\Delta|}\right\}.
\end{aligned}
$$

For an analogous relation, obtained by interchanging the roles of y and $\bar{y}$

$$
\begin{aligned}
H_d(N(y),N(\bar{y})) \le \|l\|&\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{\left[\log(1+\epsilon)\right]^{\alpha}}{\Gamma(\alpha+1)}\left|\frac{(\gamma-1)\mu_1+(\gamma-2)\mu_2}{\Delta}\right|\right.\\
&\left.+\ (\log(1+\epsilon))^{\gamma-2}\frac{1+\sum_{i=1}^{n}|\nu_i|}{\Gamma(\alpha)}\frac{1+\log(1+\epsilon)}{|\Delta|}\right\}\|y-\bar{y}\|.
\end{aligned}
$$

Since N is a contraction by (H3) and thus by theorem (1), N has a fixed point y which is a solution to (9)-(10).

3.2 | The Carathéodory case

We consider the case when N has convex values, our approach is based on the nonlinear alternative Leray-Schauder type fore multivalued map.

Let us impose the conditions below for convenience.

- (A_1) $F : J \times \mathbb{R} \to P_{cp,c}(\mathbb{R})$ is a Carathéodory multivalued map.

- (A_2) There exist $p \in C(J, \mathbb{R}_+)$ and $\psi : [0, \infty) \to (0, \infty)$ continuous and non decreasing such that

$$\|F(t, y)\|_P = \sup\{|v| : v(t) \in F(t, y)\} \le p(t)\psi(\|y\|)$$

for $t \in J$ and each $y \in C(J, \mathbb{R})$.

- (A_3) There exists a number $M > 0$ such that

$$\frac{M}{\|p\|\psi(\|M\|)\left\{\frac{1}{\Gamma(\alpha+1)} + m_{1,\epsilon} + m_{2,\epsilon}\right\}} > 1,$$

where

$$m_{1,\epsilon} = \frac{[\log(1+\epsilon)]^{\alpha}}{\Gamma(\alpha+1)}\left|\frac{(\gamma-1)\mu_1 + (\gamma-2)\mu_2}{\Delta}\right|,$$

$$m_{2,\epsilon} = (\log(1+\epsilon))^{\gamma-2}\,\frac{1+\sum_{i=1}^{n}|v_i|}{\Gamma(\alpha)}\,\frac{1+\log(1+\epsilon)}{|\Delta|}.$$

Theorem 5. Assume that the conditions (A_1)-(A_3) and (H_2) hold then the problem (9)-(10) has at least one solution.

Proof. Consider the operator $N : C(J, \mathbb{R}) \to P(C(J, \mathbb{R}))$ defined by (15). We will show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type.
The proof will be given in several steps.
Step 1. $N(y)$ is convex for each $y \in C(J, \mathbb{R})$.
Indeed, if h_1, h_2 belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$, we

$$\begin{aligned}
h_j(t) = {} & \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{v_j(\tau)}{\tau}d\tau + \frac{(\gamma-1)\mu_1(\log t)^{\gamma-2} - (\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta} \\
& \frac{1}{\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{v_j(\tau)}{\tau}d\tau \\
& + \frac{(\log(1+\epsilon))^{\gamma-2}}{\Gamma(\alpha-1)}\left[\sum_{i=1}^{n} v_i \int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{v_j(\tau)}{\tau}d\tau - \int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{v_j(\tau)}{\tau}d\tau\right] \\
& \times \frac{\log(1+\epsilon)(\log t)^{\gamma-2} - (\log t)^{\gamma-1}}{\Delta} \quad j = 1, 2.
\end{aligned}$$

Let $0 \le \delta \le 1$, Then for each $t \in J$ we have

$$\begin{aligned}
& [\delta h_1 + (1-\delta)h_2](t) \\
& = \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{1}{\tau}[\delta v_1(\tau) + (1-\delta)v_2(\tau)]d\tau \\
& + \frac{1}{\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{1}{\tau}[\delta v_1(\tau) + (1-\delta)v_2(\tau)]d\tau \\
& \frac{(\gamma-1)\mu_1(\log t)^{\gamma-2} - (\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta} \\
& + \frac{(\log(1+\epsilon))^{\gamma-2}}{\Gamma(\alpha-1)} \times \frac{\log(1+\epsilon)(\log t)^{\gamma-2} - (\log t)^{\gamma-1}}{\Delta} \\
& \times\left[\sum_{i=1}^{n} v_i\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{1}{\tau}[\delta v_1(\tau) + (1-\delta)v_2(\tau)]d\tau - \int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{1}{\tau}[\delta v_1(\tau) + (1-\delta)v_2(\tau)]d\tau\right]
\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$\delta h_1 + (1-\delta)h_2 \in N(y).$$

Step 2. N maps bounded sets into bounded sets in $C(J,\mathbb{R})$.
Let $B_r = \{y \in C(J,\mathbb{R}):\ \ \|y\| \le r\}$ be a bounded sets in $C(J,\mathbb{R})$ and $y \in B_r$; then for each $h \in N(y)$ there exists $v \in S_{F,y}$ such that for each $t \in J$, it follows by using (A_2)

$$\begin{aligned}
h(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{v(\tau)}{\tau}d\tau \\
&+ \frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{v(\tau)}{\tau}d\tau \\
&+ (\log(1+\epsilon))^{\gamma-2}\left[\sum_{i=1}^n \nu_i\frac{1}{\Gamma(\alpha-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{v(\tau)}{\tau}d\tau - \frac{1}{\Gamma(\alpha-1)}\int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{v(\tau)}{\tau}d\tau\right] \\
&\frac{\log(1+\epsilon)(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta} \\
&\le \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{|v(\tau)|}{\tau}d\tau + \frac{1}{\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{|v(\tau)|}{\tau}d\tau \\
&\cdot\left|\frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta}\right| \\
&+ (\log(1+\epsilon))^{\gamma-2}\left[\left|\sum_{i=1}^n \nu_i\frac{1}{\Gamma(\alpha-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{v(\tau)}{\tau}d\tau\right|\right. \\
&\left.+\left|\frac{1}{\Gamma(\alpha-1)}\int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{v(\tau)}{\tau}d\tau\right|\right]\left|\frac{\log(1+\epsilon)(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta}\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{p(\tau)\psi(\|y\|)}{\tau}d\tau + \frac{\left|\frac{(\gamma-1)\mu_1-(\gamma-2)\mu_2}{\Delta}\right|}{\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{p(\tau)\psi(\|y\|)}{\tau}d\tau \\
&+ (\log(1+\epsilon))^{\gamma-2}\left[\left|\sum_{i=1}^n \nu_i\frac{1}{\Gamma(\alpha-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{p(\tau)\psi(\|y\|)}{\tau}d\tau\right|\right. \\
&\left.+\left|\frac{1}{\Gamma(\alpha-1)}\int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{p(\tau)\psi(\|y\|)}{\tau}d\tau\right|\right]\left|\frac{\log(1+\epsilon)(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta}\right| \\
&\le \frac{1}{\Gamma(\alpha+1)}\|p\|\psi(\|y\|) + \frac{\left[\log(1+\epsilon)\right]^\alpha}{\Gamma(\alpha+1)}\left|\frac{(\gamma-1)\mu_1+(\gamma-2)\mu_2}{\Delta}\right|\|p\|\psi(\|y\|) \\
&+ (\log(1+\epsilon))^{\gamma-2}\left[\frac{\sum_{i=1}^n|\nu_i|}{\Gamma(\alpha)}\|p\|\psi(\|y\|) + \frac{1}{\Gamma(\alpha)}\|p\|\psi(\|y\|)\right]\frac{1+\log(1+\epsilon)}{|\Delta|} \\
&\le \|p\|\psi(r)\left\{\frac{1}{\Gamma(\alpha+1)} + \frac{\left[\log(1+\epsilon)\right]^\alpha}{\Gamma(\alpha+1)}\left|\frac{(\gamma-1)\mu_1+(\gamma-2)\mu_2}{\Delta}\right|\right. \\
&\left.+(\log(1+\epsilon))^{\gamma-2}\frac{1+\sum_{i=1}^n|\nu_i|}{\Gamma(\alpha)}\frac{1+\log(1+\epsilon)}{|\Delta|}\right\} := l.
\end{aligned}$$

Step 3. N maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.
Let $t_1, t_2 \in J$, $t_1 < t_2$ and B_r be a bounded set of $C(J,\mathbb{R})$ as in Step 2.

Let $y \in B_r$ and $h \in N(y)$, using condition (A_2) one has

$$
\begin{aligned}
&\left|h(t_2)-h(t_1)\right| = \\
&\Bigg|\frac{1}{\Gamma(\alpha)}\int_1^{t_2}\left(\log\frac{t_2}{\tau}\right)^{\alpha-1}\frac{\upsilon(\tau)}{\tau}d\tau+\frac{1}{\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{\upsilon(\tau)}{\tau}d\tau \\
&\times\frac{(\gamma-1)\mu_1(\log t_2)^{\gamma-2}-(\gamma-2)\mu_2(\log t_2)^{\gamma-1}}{\Delta} \\
&+(\log(1+\epsilon))^{\gamma-2}\left[\sum_{i=1}^{n}\nu_i\frac{1}{\Gamma(\alpha-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{\upsilon(\tau)}{\tau}d\tau-\frac{1}{\Gamma(\alpha-1)}\int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{\upsilon(\tau)}{\tau}d\tau\right] \\
&\frac{\log(1+\epsilon)(\log t_2)^{\gamma-2}-(\log t_2)^{\gamma-1}}{\Delta}-\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left(\log\frac{t_1}{\tau}\right)^{\alpha-1}\frac{\upsilon(\tau)}{\tau}d\tau \\
&-\frac{(\gamma-1)\mu_1(\log t_1)^{\gamma-2}-(\gamma-2)\mu_2(\log t_1)^{\gamma-1}}{\Delta\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{\upsilon(\tau)}{\tau}d\tau \\
&-(\log(1+\epsilon))^{\gamma-2}\left[\sum_{i=1}^{n}\nu_i\frac{1}{\Gamma(\alpha-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{\upsilon(\tau)}{\tau}d\tau-\frac{1}{\Gamma(\alpha-1)}\int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{\upsilon(\tau)}{\tau}d\tau\right] \\
&\times\frac{\log(1+\epsilon)(\log t_1)^{\gamma-2}-(\log t_1)^{\gamma-1}}{\Delta}\Bigg| \\
&=\Bigg|\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{\tau}\right)^{\alpha-1}-\left(\log\frac{t_1}{\tau}\right)^{\alpha-1}\right]\frac{\upsilon(\tau)}{\tau}d\tau+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{\tau}\right)^{\alpha-1}\frac{\upsilon(\tau)}{\tau}d\tau \\
&.+\frac{(\gamma-1)\mu_1\left[\left(\log t_2\right)^{\gamma-2}-\left(\log t_1\right)^{\gamma-2}\right]-(\gamma-2)\mu_2\left[\left(\log t_2\right)^{\gamma-1}-\left(\log t_1\right)^{\gamma-1}\right]}{\Delta} \\
&\times\frac{1}{\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{\upsilon(\tau)}{\tau}d\tau \\
&+(\log(1+\epsilon))^{\gamma-2}\left[\sum_{i=1}^{n}\nu_i\frac{1}{\Gamma(\alpha-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{\upsilon(\tau)}{\tau}d\tau-\frac{1}{\Gamma(\alpha-1)}\int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{\upsilon(\tau)}{\tau}d\tau\right] \\
&.\times\frac{\log(1+\epsilon)\left[(\log t_2)^{\gamma-2}-(\log t_1)^{\gamma-2}\right]-\left[(\log t_2)^{\gamma-1}-(\log t_1)^{\gamma-1}\right]}{\Delta}\Bigg| \\
&\leq\frac{\|p\|\psi(r)}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{\tau}\right)^{\alpha-1}-\left(\log\frac{t_1}{\tau}\right)^{\alpha-1}\right]\frac{d\tau}{\tau}+\frac{\|p\|\psi(r)}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} \\
&+\frac{\|p\|\psi(r)}{\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} \\
&.\times\left|\frac{(\gamma-1)\mu_1\left[\left(\log t_2\right)^{\gamma-2}-\left(\log t_1\right)^{\gamma-2}\right]-(\gamma-2)\mu_2\left[\left(\log t_2\right)^{\gamma-1}-\left(\log t_1\right)^{\gamma-1}\right]}{\Delta}\right| \\
&+(\log(1+\epsilon))^{\gamma-2}\left[\sum_{i=1}^{n}\nu_i\frac{\|p\|\psi(r)}{\Gamma(\alpha-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{d\tau}{\tau}+\frac{\|p\|\psi(r)}{\Gamma(\alpha-1)}\int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{d\tau}{\tau}\right] \\
&.\times\frac{\log(1+\epsilon)\left|(\log t_2)^{\gamma-2}-(\log t_1)^{\gamma-2}\right|+\left|(\log t_2)^{\gamma-1}-(\log t_1)^{\gamma-1}\right|}{|\Delta|}.
\end{aligned}
$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero.
As a consequence of Step 1 to 3 therefore it follows by Ascoli-Arzela theorem that $N : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is completely continuous.
Step 4. N has a closed graph.
Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$, then we need to show that $h_* \in N(y_*)$ associated with $h_n \in N(y_n)$, there exists $v_n \in S_{F,y_n}$ such that for each $t \in J$,

$$\begin{aligned} h_n(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{v_n(\tau)}{\tau} d\tau \\ &+ \frac{(\gamma-1)\mu_1(\log t)^{\gamma-2} - (\gamma-2)\mu_2(\log t)^{\gamma-1}}{\left((\gamma-2)\mu_2 \left[\log(1+\epsilon)\right]^{\gamma-1} - (\gamma-1)\mu_1 \left[\log(1+\epsilon)\right]^{\gamma-2}\right)\Gamma(\alpha)} \\ &\int_1^{1+\epsilon} \left(\log \frac{1+\epsilon}{\tau}\right)^{\alpha-1} \frac{v_n(\tau)}{\tau} d\tau \\ &+ \left[\sum_{i=1}^n v_i \frac{1}{\Gamma(\alpha-1)} \int_1^{\eta_i} \left(\log \frac{\eta_i}{\tau}\right)^{\alpha-2} \frac{v_n(\tau)}{\tau} d\tau - \frac{1}{\Gamma(\alpha-1)} \int_1^{e} \left(\log \frac{e}{\tau}\right)^{\alpha-2} \frac{v_n(\tau)}{\tau} d\tau\right] \\ &\times \frac{\log(1+\epsilon)(\log t)^{\gamma-2} - (\log t)^{\gamma-1}}{(\gamma-2)\mu_2 \log(1+\epsilon) - (\gamma-1)\mu_1}. \end{aligned} \tag{16}$$

Thus we must show that there exists $v_* \in S_{F,y_*}$ such that for each $t \in J$

$$\begin{aligned} h_*(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{v_*(\tau)}{\tau} d\tau + \frac{1}{\Gamma(\alpha)} \int_1^{1+\epsilon} \left(\log \frac{1+\epsilon}{\tau}\right)^{\alpha-1} \frac{v_*(\tau)}{\tau} d\tau \\ &\times \frac{(\gamma-1)\mu_1(\log t)^{\gamma-2} - (\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta} \\ &+ (\log(1+\epsilon))^{\gamma-2} \frac{\log(1+\epsilon)(\log t)^{\gamma-2} - (\log t)^{\gamma-1}}{\Delta}. \\ &\times \left[\sum_{i=1}^n v_i \frac{1}{\Gamma(\alpha-1)} \int_1^{\eta_i} \left(\log \frac{\eta_i}{\tau}\right)^{\alpha-2} \frac{v_*(\tau)}{\tau} d\tau - \frac{1}{\Gamma(\alpha-1)} \int_1^{e} \left(\log \frac{e}{\tau}\right)^{\alpha-2} \frac{v_*(\tau)}{\tau} d\tau\right] \end{aligned}$$

Since $F(t,.)$ is upper semicontinuous by (A_1), then for every $\varepsilon > 0$, there exist $n_0(\varepsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \varepsilon B(0,1) \quad a.e.\ t \in J.$$

Since $F(.,.)$ has compact values then there exists a subsequence $v_{n_k}(.)$ such that

$$v_{n_k}(.) \longrightarrow v_*(.) \quad as \;\; k \to \infty,$$

$$v_*(t) \in F(t, y_*(t)) \quad a.e. \;\; t \in J.$$

For every $w \in F(t, y_*(t))$, we have

$$|v_{n_m}(t) - v_*(t)| \leq |v_{n_m}(t) - w| + |w - v_*(t)|.$$

Then

$$|v_{n_m}(t) - v_*(t)| \leq d(v_{n_m}(t), F(t, y_*(t))).$$

By an analogous relation, obtained by interchanging the roles of v_{n_m} and v_* and using condition (H_2), it follows that

$$|v_{n_m}(t) - v_*(t)| \leq H_d(F(t, y_n(t)), F(t, y_*(t))) \leq l(t)\|y_n - y_*\|_\infty.$$

Then

$$
\begin{aligned}
|h_n(t)-h_*(t)| \le{}& \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{|v_{n_m}(\tau)-v^*(\tau)|}{\tau}d\tau \\
&+\frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{|v_{n_m}(\tau)-v^*(\tau)|}{\tau}d\tau \\
&+\frac{(\log(1+\epsilon))^{\gamma-2}}{\Gamma(\alpha-1)}\frac{\log(1+\epsilon)(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta} \\
&\left[\sum_{i=1}^n \nu_i\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{|v_{n_m}(\tau)-v^*(\tau)|}{\tau}d\tau-\int_1^e\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{|v_{n_m}(\tau)-v^*(\tau)|}{\tau}d\tau\right] \\
\le{}& \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{\|y_{n_m}-y_*\|_\infty}{\tau}d\tau \\
&+\frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{\|y_{n_m}-y_*\|_\infty\|}{\tau}d\tau \\
&+\frac{\|y_{n_m}-y_*\|_\infty}{\Gamma(\alpha-1)}(\log(1+\epsilon))^{\gamma-2}\left[\sum_{i=1}^n \nu_i\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{1}{\tau}d\tau-\int_1^e\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{1}{\tau}d\tau\right] \\
&\frac{\log(1+\epsilon)(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta}.
\end{aligned}
$$

Since $\|y_{n_m}-y_*\|\to 0$ as $m\to 0$, one has $\|h_n-h_*\|\to 0$ as $n\to 0$.

Step 5. A priori bounds on solutions.

Let y be such that $y\in\lambda N(y)$ with $\lambda\in(0,1)$ then there exists $v\in S_{F,y}$ such that for each $t\in J$ and taking account (A_2) we have

$$
\begin{aligned}
y(t) ={}& \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{v(\tau)}{\tau}d\tau \\
&+\frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{v(\tau)}{\tau}d\tau \\
&+(\log(1+\epsilon))^{\gamma-2}\left[\frac{\sum_{i=1}^n \nu_i}{\Gamma(\alpha-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{v(\tau)}{\tau}d\tau-\frac{1}{\Gamma(\alpha-1)}\int_1^e\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{v(\tau)}{\tau}d\tau\right] \\
&\frac{\log(1+\epsilon)(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta} \\
\le{}& \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{|v(\tau)|}{\tau}d\tau+\frac{1}{\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{|v(\tau)|}{\tau}d\tau \\
&\cdot\left|\frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta}\right| \\
&+\frac{\log(1+\epsilon)+1}{|\Delta|}\frac{(\log(1+\epsilon))^{\gamma-2}}{\Gamma(\alpha-1)}\left[\sum_{i=1}^n|\nu_i|\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{|v(\tau)|}{\tau}d\tau+\int_1^e\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{|v(\tau)|}{\tau}d\tau\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\|p\|\psi(\|y\|)}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{1}{\tau}d\tau + \frac{\|p\|\psi(\|y\|)}{\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{1}{\tau}d\tau \\
&\frac{|(\gamma-1)\mu_1|+|(\gamma-2)\mu_2|}{|\Delta|} \\
&+(\log(1+\epsilon))^{\gamma-2}\left[\sum_{i=1}^n |\nu_i|\frac{\|p\|\psi(\|y\|)}{\Gamma(\alpha-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{1}{\tau}d\tau + \frac{\|p\|\psi(\|y\|)}{\Gamma(\alpha-1)}\int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{1}{\tau}d\tau\right] \\
&\frac{\log(1+\epsilon)+1}{|\Delta|} \\
&\leq \frac{\|p\|\psi(\|y\|)}{\Gamma(\alpha+1)} + \frac{\|p\|\psi(\|y\|)}{\Gamma(\alpha+1)}\left[\log(1+\epsilon)\right]^{\alpha} \\
&.\frac{\left|(\gamma-1)\mu_1\right|+\left|(\gamma-2)\mu_2\right|}{|\Delta|} + (\log(1+\epsilon))^{\gamma-2}\frac{\log(1+\epsilon)+1}{|\Delta|}\left[\sum_{i=1}^n|\nu_i|\frac{\|p\|\psi(\|y\|)}{\Gamma(\alpha)} + \frac{\|p\|\psi(\|y\|)}{\Gamma(\alpha)}\right] \\
&\leq \|p\|\psi(\|y\|)\left\{\frac{1}{\Gamma(\alpha+1)} + \frac{\left[\log(1+\epsilon)\right]^{\alpha}}{\Gamma(\alpha+1)}\frac{|(\gamma-1)\mu_1|+|(\gamma-2)\mu_2|}{|\Delta|}\right. \\
&\left.+(\log(1+\epsilon))^{\gamma-2}\frac{\sum_{i=1}^n|\nu_i|+1}{\Gamma(\alpha)}\left|\frac{\log(1+\epsilon)+1}{\Delta}\right|\right\},
\end{aligned}
$$

which implies that

$$
\frac{\|y\|}{\|p\|\psi(\|y\|)\left\{\frac{1}{\Gamma(\alpha+1)} + m_{1,\epsilon} + m_{2,\epsilon}\right\}} \leq 1.
$$

In view of (A3), there exists M such that $\|y\| \neq M$.

Let us set $U = \{y \in C(J,\mathbb{R}) : \ \|y\| < M\}$. The operator $N : \bar{U} \longrightarrow P(C(J,\mathbb{R}))$ is upper semicontinuous and completely continuous; From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder, we deduce that N has a fixed point $y \in \bar{U}$ which is a solution of the problem (9)-(10).

4 | TOPOLOGICAL STRUCTURE OF THE SOLUTION SET

Below we shall concentrate our considerations on the topological structure of the set of fixed points of (9)-(10). Let us consider the hypothesis below

(B) There exists $p \in C(J,\mathbb{R})$ such that

$$
\|F(t,y)\|_P \leq p(t) \quad for \ \ t \in J \ \ and \ \ y \in \mathbb{R}.
$$

Theorem 6. Assume that the conditions (A_1) and (B) hold. Then the solution set of (9)-(10) is nonempty and compact in $C(J,\mathbb{R})$.

Proof .

Let

$$
S = \{y \in C(J,\mathbb{R}); \ \ y \text{ is solution of (9)-(10)}\}.
$$

From Theorem (5) $S \neq \emptyset$.

Now, we prove that S is compact.

Let $(y_n)_{n\in\mathbb{N}} \in S$, then there exists $v_n \in S_{F,y_n}$ and $t \in J$ such that

$$
y_n(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{v_n(\tau)}{\tau}d\tau
$$

$$+\frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{\upsilon_n(\tau)}{\tau}d\tau$$
$$+\frac{(\log(1+\epsilon))^{\gamma-2}}{\Gamma(\alpha-1)}\times\frac{\log(1+\epsilon)(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta}$$
$$\times\left[\sum_{i=1}^{n}\nu_i\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{\upsilon_n(\tau)}{\tau}d\tau-\int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{\upsilon_n(\tau)}{\tau}d\tau\right]$$

From (B) we can prove that there exists an $M_1>0$ such that $\|y_n\|\le M_1$ for every $n\ge 1$. As in Step 3 in Theorem (5), we can easily show that the set $\{y_n;\ \ n\ge 1\}$ is equicontinuous in $C(J,\mathbb{R})$, hence by Arzela-Ascoli theorem we can conclued that there exists a subsequence (denoted by $\{y_n\}$) of $\{y_n\}$ converging to y in $C(J,\mathbb{R})$.
We shall show that there exist $\upsilon(.)\in F(.,y(.))$ such that

$$y(t)=\frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{\upsilon(\tau)}{\tau}d\tau$$
$$+\frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{\upsilon(\tau)}{\tau}d\tau$$
$$+\frac{(\log(1+\epsilon))^{\gamma-2}}{\Gamma(\alpha-1)}\times\frac{\log(1+\epsilon)(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta}$$
$$\times\left[\sum_{i=1}^{n}\nu_i\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{\upsilon(\tau)}{\tau}d\tau-\int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{\upsilon(\tau)}{\tau}d\tau\right]$$

Since $F(t,.)$ is upper continuous, for every $\varepsilon>0$ there exists $n_0(\varepsilon)\ge 0$ such that for every $n\ge n_0$ we have

$$\upsilon_n(t)\in F(t,y_n(t))\subset F(t,y(t))+\varepsilon B(0,1)\ \ a.e.\ t\in J,$$

Since $F(.,.)$ has compact values, there exists a subsequence υ_{n_m} such that

$$\upsilon_{n_m}(.)\to\upsilon(.)\ \ \ as\ \ m\to\infty,\ \upsilon(t)\in F(t,y(t))\ \ \ a.e.\ t\in J.$$

It is clear that the subsequence $\upsilon_{n_m}(t)$ is integrally bounded.
By Lebesgue dominated convergence theorem, yields $\upsilon\in L^1(J,\mathbb{R})$ which implies that $\upsilon\in S_{F,y}$. Thus for $t\in J$, we have

$$y(t)=\frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{\upsilon(\tau)}{\tau}d\tau$$
$$+\frac{(\gamma-1)\mu_1(\log t)^{\gamma-2}-(\gamma-2)\mu_2(\log t)^{\gamma-1}}{\Delta\Gamma(\alpha)}\int_1^{1+\epsilon}\left(\log\frac{1+\epsilon}{\tau}\right)^{\alpha-1}\frac{\upsilon(\tau)}{\tau}d\tau$$
$$+\frac{(\log(1+\epsilon))^{\gamma-2}}{\Gamma(\alpha-1)}\times\frac{\log(1+\epsilon)(\log t)^{\gamma-2}-(\log t)^{\gamma-1}}{\Delta}$$
$$\times\left[\sum_{i=1}^{n}\nu_i\int_1^{\eta_i}\left(\log\frac{\eta_i}{\tau}\right)^{\alpha-2}\frac{\upsilon(\tau)}{\tau}d\tau-\int_1^{e}\left(\log\frac{e}{\tau}\right)^{\alpha-2}\frac{\upsilon(\tau)}{\tau}d\tau\right]$$

Then $S\in P_{cp}(C(J,\mathbb{R}))$.

5 | CONCLUSION

In this work, we deal with the problem concerning existence of solution sets and its topological structure for Hilfer-Hadamard-Type fractional differential inclusions modeled by inclusion (9)-(10) with Hilfer-Hadamard-Type fractional derivative and multi-point boundary conditions. The nonconvex valued right hand side in the first leg forced us to make use Banach contraction, in the second leg we apply the nonlinear alternative of Leray-Schauder to give our existence result we ended by giving topological structure of the solution sets when it is nonempty.

Acknowledgements

The authors would like to express their thanks to the editor and anonymous referees for his her suggestions and comments that improved the quality of the paper.

References

1. S. Abbas, M. Benchohra, N. Hamidi and Y. Zhou, implicit oupled Hilfer-Hadamard fractional differential systems under weak topologies, *Adv. Differ. equ.* **328**, 2018.
2. R. P. Agarwal, M. Benchohra and S.Hamani, Boundary value problems for differential inclusions with fractional order, *Adv. Stud. Contemp. Math.* **12** (2), (2008) 181-196.
3. B. Ahmad, S. K. Ntouyas, A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations, *Fract. Calc. Appl. Anal.* **17**, (2014) 348-360.
4. Y. Ahmed, A. Salamooni and D. D. Pawar, Existence and uniqueness of boundary value problems for Hilfer-Hadamard-type fractional differential equations, *arXiv:1801.10400v1[math.AP]* 31 Jan 2018.
5. Y. Ahmed, A. Salamooni and D. D. Pawar, Existence and uniqueness of nonlocal boundary conditions for Hilfer-Hadamard-type fractional differential equations, *arXiv:1802.04262v2 [math.AP]*13 Mar 2018.
6. N. Aronszajn, Le correspondant topologique de l'unicité dans la théorie des équations différentielles, *Ann. of Math.* **43**, (1942) 730-738.
7. M. Beddani and B. Hedia, Solution sets for fractional differential inclusions, *J. Fractional Calc. Appl.* **10** (2) July (2019) 273-289.
8. M.Beddani and B.Hedia, Existence result for a fractional differential equation involvinga special derivative, *Moroccan J. of Pure and Appl. Anal.* **8(1)** (2022) 67-77.
9. M.Beddani and B.Hedia, Existence result for fractional differential equation on unbounded domain, *Kragujev. J. Math* **48** (5), (2024)755-766.
10. M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation **2**, New York, 2006.
11. F.Z. Berrabah, M. Boukrouche and B. Hedia, Fractional Derivatives with Respect to Time for Non-Classical Heat Problem, *J. Math. Phys. Anal. Geom.* **17**,(2021) 30Ű53.
12. F. Z. Berrabah, B. Hedia and J. Henderson, Fully Hadamard and Erdélyi-Kober-type integral boundary value problem of a coupled system of implicit differential equations, *Turk. J. Math.* **43**,(2019) 1308-1329.
13. F.Z. Berrabah, B. Hedia and M. Kirane, Lyapunov- and HartmanŰWintner-type inequalities for a nonlinear fractional BVP with generalized ψ-Hilfer derivative, *Math Meth Appl Sci.* **44**, (2021)2637Ű2649.

14. F. E. Browder and G. P. Gupta, Topological degree and nonlinear mappings of analytic type in Banach spaces, *J. Math. Anal. Appl.* **26**,(1969)390-402.

15. C. Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions*. Lecture Notes in Mathematics, **580**, Springer-Verlag, Berlin-Heidelberg-New York 1977.

16. F.S. DeBelasi, J. Myjak, on the solutions set for differential inclusions, *Bull. Pol. Acad. Sci. Math.* **12**,(1985)17-23.

17. K.M. Furati, M.D. Kassim and N.E. Tatar, Non-existence of global solutions for a differential equation involving Hilfer fractional derivative, *Electron. J. Diff. Eq.* **235**,(2013)1-10.

18. A. Granas, J. Dugundji, *Fixed Point Theory*, Springer Monographs in Mathematics, Springer, New York, 2003.

19. J. Hadamard, Essai sur l'etude des fonctions donnees par leur developpment de Taylor, *J. Mat. Pure Appl. Ser.* **8**, (1892) 101-186.

20. B. Hedia, *Nonlocal conditions for semilinear fractional differential equations with Hilfer derivative*, Fractional Calculus: ICFDA 2018, Amman, Jordan, July 16-18 pages 69-84.

21. R. Hilfer, Experimental evidence for fractional time evolution in glass materials, *Chem. Physics* **284**, (2002) 399-408.

22. R. Hilfer, Y. Luchko, Z. Tomovski, Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives, *Fract. Calc. Appl. Anal.* **12**, (2009) 289-318.

23. A. A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math. Soc.* **38**, (2001) 1191-1204.

24. A. Kneser, Unteruchung und asymptotische darstellung der integrale gewisser differentialgleichungen bei grossen werthen des arguments, *J. Reine Angew. Math.* **116**,(1896) 178-212.

25. G. Peano, Démonstration de l'intgrabilité des équations différentielles ordinaires, *Math. Ann.* **37**, (1890)182-238.

26. D. Viveka , K. Kanagarajana, E. M. Elsayedb, Nonlocal initial value problems for implicit differential equations with HilferŰHadamard fractional derivative, *nonlinear anal. model.* **23**,(2018) 341-360.

How to cite this article: Williams K., B. Hoskins, R. Lee, G. Masato, and T. Woollings (2016), A regime analysis of Atlantic winter jet variability applied to evaluate HadGEM3-GC2, *Q.J.R. Meteorol. Soc.*, *2017;00:1–6*.