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RESEARCH ARTICLE

Discrete Sturm-Liouville equations with point interaction

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Scattering solutions and several properties of scattering function of a discrete Sturm-Liouville boundary value problem with point interaction (PBVP) are derived. Moreover, resolvent operator, continuous and discrete spectrum of this PBVP are investigated. An asymptotic equation is utilized to get the properties of eigenvalues. An example illustrating the main results is given.

KEYWORDS:
point interaction, discrete Sturm-Liouville equation, scattering function, resolvent, eigenvalue

1 | INTRODUCTION

Difference equations are equations on the one hand about solving discrete, as a practical mathematical model of many events that occur in their own interests. Applications of difference equations are very widespread and rich in applied mathematics, biology, genetics, economics, control theory, theoretical physics, industry, engineering and many more. On the other hand, one can produce a discrete boundary value problem with a difference equation and a boundary condition together. In the theory of difference equations the theory of discrete boundary value problems with point interaction have a very large place [1-5]. In these problems, there is a discrete equation and one or more points in which equation has discontinuities. To deal with these discontinuities, some conditions are necessary. These conditions are called point interactions, impulsive conditions, transmission conditions, jump conditions or interface conditions [6-11]. These conditions create instantaneous state changes in the problem with point interaction, certain intervals occur in the equation and the general solution of the relations between the solutions in sub-intervals. Problems with point interaction appear in boundary value problems by adding a different feature to the problem and has received considerable attention due to its potential applications in the study of population models, finance, economics, epidemic models, heat transfer etc. see, for example [12-14].

Recently, discrete boundary value problems with point interaction have been extensively studied by many researchers in the way of scattering and spectral analysis [15-22]. But the results all in these studies are given on upper half complex plane or on a subset of it. Because they consist spectral parameter as a trigonometric function's form of z. But in this work, we will get results on unit disk for $\lambda = z + z^{-1}$ (λ is a spectral parameter). As a result of this, we will consider a discrete Sturm-Liouville boundary value problem with point interaction (PBVP) given by

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N} \setminus \left\{m_0 - 1, m_0, m_0 + 1\right\} \tag{1}$$

$$y_0 = 0 \tag{2}$$

[0]**Abbreviations:** PBVP, point boundary value problem

$$y_{m_0+1} = \gamma_1 y_{m_0-1},$$
$$\triangle y_{m_0+1} = \gamma_2 \bigtriangledown y_{m_0-1}, \quad \gamma_1\gamma_2 \neq 0, \quad \gamma_1, \gamma_2 \in \mathbb{R}, \tag{3}$$

where $\lambda = z + z^{-1}$ is a spectral parameter, $\left\{a_n\right\}_{n\in\mathbb{N}\cup\{0\}}$ and $\left\{b_n\right\}_{n\in\mathbb{N}}$ are real sequences satisfying the condition

$$\sum_{n=1}^{\infty} n\left(\left|1-a_n\right| + \left|b_n\right|\right) < \infty, \tag{4}$$

$\bigtriangledown$ is the backward and $\triangle$ is the forward difference operators, i.e.,

$$\bigtriangledown y_n(z) = y_{n+1}(z) - y_n(z)$$
$$\triangle y_n(z) = y_n(z) - y_{n-1}(z).$$

The remaining part of this paper is organized as follows: Section 2 consists some basic notations and definitions. In Section 3, we constitute the scattering and Jost function of PBVP (1)-(3) by using scattering solutions. In Section 4, we give resolvent operator by the help of the unbounded solution and we find continuous and discrete spectrum of PBVP (1)-(3). We also investigate the properties of eigenvalues by using an asymptotic equation. Finally, we give an example to illustrate our main results in Section 5.

2 | PRELIMINARIES

First, we will define three regions:

$$D_0 := \{z : |z| = 1\},$$

$$D_1 := \{z : |z| < 1\} \setminus \{0\}$$

and

$$D_2 := \{z : |z| \leq 1\} \setminus \{0\}.$$

Throughout the paper, we use the fundamental solutions $P_n(z)$ and $Q_n(z)$, $n \in \mathbb{N}_0$ of (1) for $z \in D_2$ and $\lambda = z + z^{-1}$ which satisfy the following initial conditions

$$P_0(z) = 0, \quad P_1(z) = 1$$
$$Q_0(z) = \frac{1}{a_0}, \quad Q_1(z) = 0,$$

respectively. For each $n \geq 0$, $P_n(z)$ is a polynomial of degree $(n-1)$. and $Q_n(z)$ is a polynomial of degree $(n-2)$. The Wronskian of two solutions $y = \left\{y_n(z)\right\}$ and $u = \left\{u_n(z)\right\}$ of the equation (1) defined by

$$W[y, u] := a_n \left\{y_n(z)u_{n+1}(z) - y_{n+1}(z)u_n(z)\right\}$$

for $n \in \mathbb{N}_0$. It is easy to see that the Wronskian is independently of n. Furthermore, $P_n(z)$ and $Q_n(z)$ are linear independent solutions of (1), because $W[P, Q] = -1$ for all $z \in \mathbb{C}$ and these solutions are entire functions with respect to z. On the other hand, equation (1) has a bounded solution $e(z) := \left\{e_n(z)\right\}$ satisfying the condition

$$\lim_{n\to\infty} e_n(z)z^{-n} = 1, \quad z \in D_0 \tag{5}$$

for $\lambda = z + z^{-1}$ and $n = m_0 + 1, m_0 + 2, \ldots$. The solution $e_n(z)$ is represented by

$$e_n(z) = \rho_n(z)z^n \left(1 + \sum_{m=1}^{\infty} A_{nm} z^m\right)$$

in[23], where ρ_n and A_{nm} are expressed in terms of the sequences $\left\{a_n\right\}$ and $\left\{b_n\right\}$ as

$$\rho_n := \prod_{k=n}^{\infty} [a_k]^{-1},$$

$$A_{n1} := -\sum_{k=n+1}^{\infty} b_k,$$

$$A_{n2} := \sum_{k=n+1}^{\infty} \left\{ 1 - a_k^2 + b_k \sum_{p=k+1}^{\infty} b_p \right\},$$

$$A_{n,m+2} := A_{n+1,m} + \sum_{k=n+1}^{\infty} \left\{ (1 - a_k^2) A_{k+1,m} - b_k A_{k,m+1} \right\}$$

for $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$. The function $e(z)$ is analytic with respect to z in $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ and continuous in $\overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \mathrm{Im} z \geq 0\}$. Also, $e(z)$ has analytic continuation from D_0 to D_1 [26].

From definition of Wronskian and equation (5), we easily find that

$$W[e_n(z), e_n(-z)] = z^{-1} - z, \quad z \in D_0. \tag{6}$$

Now, we define the following solution of (1)-(3)

$$E_n(z) := \begin{cases} \alpha(z) P_n(z) + \beta(z) Q_n(z), & n = 0, 1, \ldots, m_0 - 1 \\ \\ e_n(z), & n = m_0 + 1, m_0 + 2, \ldots \end{cases} \tag{7}$$

for $z \in D_2$. If we use point interactions (3) for this solution, we can write

$$E_{m_0-1}(z) = \frac{1}{\gamma_1} E_{m_0+1}$$

$$\nabla E_{m_0-1}(z) = \frac{1}{\gamma_2} \triangle E_{m_0+1}(z)$$

and

$$\frac{1}{\gamma_1} e_{m_0+1}(z) = \alpha(z) P_{m_0-1}(z) + \beta(z) Q_{m_0-1}(z)$$

$$\frac{1}{\gamma_2} \triangle e_{m_0+1}(z) = \alpha(z) \nabla P_{m_0-1}(z) + \beta(z) \nabla Q_{m_0-1}(z). \tag{8}$$

From the definition of Wronskian and equations (8), we obtain the coefficients $\alpha(z)$ and $\beta(z)$ for $z \in D_2$ as

$$\alpha(z) = \frac{a_{m_0-2}}{\gamma_1 \gamma_2} \left\{ \gamma_1 \triangle e_{m_0+1}(z) Q_{m_0-1}(z) - \gamma_2 e_{m_0+1}(z) \nabla Q_{m_0-1}(z) \right\}$$

and

$$\beta(z) = -\frac{a_{m_0-2}}{\gamma_1 \gamma_2} \left\{ \gamma_1 \triangle e_{m_0+1}(z) P_{m_0-1}(z) - \gamma_2 e_{m_0+1}(z) \nabla P_{m_0-1}(z) \right\}. \tag{9}$$

The function $E(z) = \left\{ E_n(z) \right\}$ is the Jost solution of PBVP (1)-(3). Next, we think other solution $F(z) = \left\{ F_n(z) \right\}$ of (1)-(3) by

$$F_n(z) := \begin{cases} P_n(z), & n = 0, 1, \ldots, m_0 - 1 \\ \\ c(z) e_n(z) + d(z) e_n(-z), & n = m_0 + 1, m_0 + 2, \ldots \end{cases} \tag{10}$$

for $z \in D_0$. By using the condition (3) and equation (6), we get

$$c(z) = -\frac{a_{m_0+1}}{z - z^{-1}} \left\{ \gamma_1 \triangle e_{m_0+1}(-z) P_{m_0-1}(z) - \gamma_2 e_{m_0+1}(-z) \nabla P_{m_0-1}(z) \right\}$$

and

$$d(z) = \frac{a_{m_0+1}}{z - z^{-1}} \left\{ \gamma_1 \triangle e_{m_0+1}(z) P_{m_0-1}(z) - \gamma_2 e_{m_0+1}(z) \nabla P_{m_0-1}(z) \right\}$$

for $z \in D_0$.

Corollary 1. Since $P_n(z)$ is a polynomial of degree $(n-1)$., the coefficients $c(z)$ and $d(z)$ satisfy the following equations for $z \in D_0$

$$i. \quad If \quad n \quad is \quad an \quad odd \quad number, \quad then \quad P(-z) = P(z) \quad and \quad d(z) = c(-z)$$

$$ii. \quad If \quad n \quad is \quad an \quad even \quad number, \quad then \quad P(-z) = -P(z) \quad and \quad d(z) = -c(-z).$$

Lemma 1. For $z \in D_0$, the coefficients $\beta(z)$ and $d(z)$ have the following relation

$$d(z) = -\frac{a_{m_0+1}}{z - z^{-1}} \frac{\gamma_1 \gamma_2}{a_{m_0-2}} \beta(z).$$

Theorem 1. For all $z \in D_0$, $\beta(z) \neq 0$.

Proof. Assume that $\beta(z_0) = 0$ for a $z_0 \in D_0$. By using Corollary 1 and Lemma 2, we can write

$$\beta(z_0) = d(z_0) = 0.$$

It follows from that $F_n(z_0) = 0$ for all $n \in \mathbb{N} \cup \{0\}$, but this is a contradiction, i.e., $\beta(z) \neq 0$ for all $z \in D_0$. □

Lemma 2. The Wronskian of the solutions $E(z)$ and $F(z)$ is given by

$$W[E(z), F(z)] = \begin{cases} \beta(z), & n = 0, 1, \dots, m_0 - 1 \\ \\ \dfrac{a_{m_0+1}}{a_{m_0-2}} \gamma_1 \gamma_2 \beta(z), & n = m_0 + 1, m_0 + 2, \dots \end{cases}$$

for $z \in D_0$.

Proof. Using the definition of Wronskian, we write the Wronskian of $E(z)$ and $F(z)$ for $n \in \left\{0, 1, \dots, m_0 - 1\right\}$ as

$$\begin{aligned} W[E(z), F(z)] &= a_0 \left\{ E_0(z) F_1(z) - E_1(z) F_0(z) \right\} \\ &= a_0 \left\{ \alpha(z) P_0(z) + \beta(z) Q_0(z) \right\} P_1(z) \\ &\quad - a_0 \left\{ \alpha(z) P_1(z) + \beta(z) Q_1(z) \right\} P_0(z). \end{aligned}$$

Since $P_0(z) = 0$, $P_1(z) = 1$, $Q_0(z) = \frac{1}{a_0}$ and $Q_1(z) = 0$, we obtain the Wronskian of these two solutions as $\beta(z)$ for $n \in \left\{0, 1, \dots, m_0 - 1\right\}$ by using the last equation. On the other hand it is easy writing

$$W[E(z), F(z)] = a_{m_0+1} d(z) \left\{ e_{m_0+1}(z) e_{m_0+2}(-z) - e_{m_0+2}(z) e_{m_0+1}(-z) \right\}$$

for $n \in \left\{m_0 + 1, m_0 + 2, \dots\right\}$. It follows from (6) and Lemma 2 that

$$W[E(z), F(z)] = \frac{a_{m_0+1}}{a_{m_0-2}} \gamma_1 \gamma_2 \beta(z)$$

for $n \in \left\{m_0 + 1, m_0 + 2, \dots\right\}$. It completes the proof. □

3 | JOST SOLUTION AND SCATTERING FUNCTION

Now, we will define the Jost function $E_0(z)$ of PBVP (1)-(3) by using boundary equation (2) and Jost solution $E(z)$ of (1)-(3) as

$$\begin{aligned} E_0(z) &= \alpha(z) P_0(z) + \beta(z) Q_0(z) \\ &= \frac{\beta(z)}{a_0}. \end{aligned} \tag{11}$$

Similarly to the Sturm-Liouville equation, the function $E_0(z)$ is analytic in D_1 and continuous in D_2.

Definition 1. The function

$$S(z) := \frac{\overline{E_0(z)}}{E_0(z)}, \quad z \in D_0$$

is called the scattering function of PBVP (1)-(3).

It is trivial from equation (11) and Definition 5 that the scattering function can be also expressed by means of coefficient $\beta(z)$

$$S(z) = \frac{\overline{E_0(z)}}{E_0(z)} = \frac{\overline{\beta(z)}}{\beta(z)} = \frac{\beta(z^{-1})}{\beta(z)} \tag{12}$$

for all $z \in D_0$.

Theorem 2. The function $S(z)$ satisfies

$$S^{-1}(z) = \overline{S(z)} = S(z^{-1}) \quad \textit{and} \quad |S(z)| = 1$$

for all $z \in D_0$.

Proof. It is clear from (12) that

$$S(z^{-1}) = \frac{\beta(z)}{\beta(z^{-1})} \quad and \quad S^{-1}(z) = \frac{\beta(z)}{\beta(z^{-1})} = \overline{S(z)}$$

for all $z \in D_0$. Also, since $|S(z)|^2 = \overline{S(z)}S(z)$, then from (12), we obtain

$$|S(z)| = \frac{\beta(z)}{\beta(z^{-1})}\frac{\beta(z^{-1})}{\beta(z)} = 1$$

for all $z \in D_0$. It completes the proof of Theorem 6. □

4 | RESOLVENT OPERATOR, CONTINUOUS SPECTRUM AND DISCRETE SPECTRUM OF PBVP

For all $z \in D_2$, we will define the following unbounded solution of (1)

$$G_n(z) := \begin{cases} P_n(z), & n = 0, 1, \dots, m_0 - 1 \\ q(z)e_n(z) + k(z)\hat{e}_n(z), & n = m_0 + 1, m_0 + 2, \dots \end{cases} \tag{13}$$

where $\hat{e}_n(z)$ represents the unbounded solution of (1) for $n \in \{m_0 + 1, m_0 + 2, \dots\}$ and satisfies the condition

$$\lim_{n\to\infty} \hat{e}_n(z)z^n = 1, \quad z \in D_2.$$

It easy to seen that $W[e_n(z), \hat{e}_n(z)] = z^{-1} - z$ for $n \in \{m_0 + 1, m_0 + 2, \dots\}$ and $z \in D_2$. Analogously the solution $F_n(z)$, using the condition (3), we get

$$q(z) = -\frac{a_{m_0+1}}{z - z^{-1}}\left\{\gamma_1 \triangle \hat{e}_{m_0+1}(z)P_{m_0-1}(z) - \gamma_2\hat{e}_{m_0+1}(z)\bigtriangledown P_{m_0-1}(z)\right\}$$

and

$$k(z) = \frac{a_{m_0+1}}{z - z^{-1}}\left\{\gamma_1 \triangle e_{m_0+1}(z)P_{m_0-1}(z) - \gamma_2 e_{m_0+1}(z)\bigtriangledown P_{m_0-1}(z)\right\}$$

for all $z \in D_2$. Note that, $k(z) = d(z)$ for all $z \in D_0$. Also using (7) and (13), we get

$$W[E(z), G(z)] = \begin{cases} \beta(z), & n = 0, 1, \dots, m_0 - 1 \\ \dfrac{a_{m_0+1}}{a_{m_0-2}}\gamma_1\gamma_2\beta(z), & n = m_0 + 1, m_0 + 2, \dots \end{cases} \tag{14}$$

for $z \in D_2$.

Corollary 2. It is trivial from Lemma 4 and (14) that for all $z \in D_0$

$$W[E(z), F(z)] = W[E(z), G(z)].$$

Theorem 3. The resolvent operator of PBVP (1)-(3) is defined by

$$R_\lambda g_n := \sum_{k=1}^{\infty} R_{nk}(z)g_k, \quad k \neq m_0$$

where

$$R_{nk}(z) = \begin{cases} -\dfrac{G_k E_n}{W[E,G]}, & k \leq n \\ -\dfrac{G_n E_k}{W[E,G]}, & k > n \end{cases}$$

is the Green function of (1)-(3) for $z \in D_2$, $\beta(z) \neq 0$ and $k, n \neq m_0$.

Proof. We must solve the following equation to find the resolvent operator of PBVP (1)-(3)

$$\bigtriangledown(a_n\bigtriangledown y_n) + h_n y_n - \lambda y_n = g_n, \quad g_n \in \ell_2(\mathbb{N}). \tag{15}$$

We can write the general solution $y = y_n(z)$ of (15) by using the fundamental solutions of PBVP (1)-(3) as

$$y_n(z) = m_n E_n(z) + t_n G_n(z), \tag{16}$$

where m_n, t_n are coefficients and they are different from zero. Using the method of variation of parameters for $k \neq m_0$, we obtain m_n and t_n by

$$m_n = -\sum_{k=1}^{n} \frac{G_k g_k}{W[E,G]} \tag{17}$$

$$t_n = -\sum_{k=n+1}^{\infty} \frac{E_k g_k}{W[E,G]}. \tag{18}$$

It follows from (16), (17) and (18) that the Green function of (1)-(3) is $R_{nk}(z)$ defined in Theorem 8. Also, we obtain the resolvent operator of (1)-(3) by the help of the Green function $R_{nk}(z)$. □

In the following, we will introduce the set of eigenvalues of PBVP (1)-(3) by using the definition of eigenvalues[27] and Theorem 8 as

$$\sigma_d := \left\{ \lambda \in \mathbb{C} : \lambda = z + z^{-1}, \quad z \in D_1, \quad \beta(z) = 0 \right\}.$$

Theorem 4. Assume (4). Then $\beta(z)$ satisfies the following asymptotic equation for $z \in D_2$

$$\beta(z) = z^{2m_0}\left[A + o(1)\right], \quad |z| \to \infty, \quad A \neq 0 \tag{19}$$

where

$$A = -\frac{a_{m_0-2\rho(m_0+2)}}{\gamma_2 a_1 \dots a_{m_0-2}}. \tag{20}$$

Proof. Since the $P_n(z)$ is polynomial of degree $(n-1)$. with respect to λ, we can obtain that for $n = 1, 2, \dots, m_0 - 1$

$$\lim_{|z|\to\infty} \left\{ P_n(z) z^{-(n-1)} \right\} = \frac{1}{a_1 \dots a_{n-1}} + o(1), \quad z \in D_2 \tag{21}$$

and

$$\lim_{|z|\to\infty} \left\{ e_n(z) z^{-n} \right\} = \rho_n, \quad z \in D_2 \tag{22}$$

by using (5), where $\rho_n := \left(\prod_{k=n}^{\infty} a_k\right)^{-1}$. It follows from (9), (21) and (22) that

$$\begin{aligned}
\beta(z) = -\frac{a_{m_0-2}}{\gamma_1\gamma_2} \{ & \gamma_1 e_{m_0+2}(z) z^{-(m_0+2)} P_{m_0-1}(z) z^{-(m_0-2)} z^{2m_0} \\
& + \gamma_1 e_{m_0+1}(z) z^{-(m_0+1)} P_{m_0-1}(z) z^{-(m_0-2)} z^{2m_0-1} \\
& - \gamma_2 e_{m_0+1}(z) z^{-(m_0+1)} P_{m_0-1}(z) z^{-(m_0-2)} z^{2m_0-1} \\
& + \gamma_2 e_{m_0+1}(z) z^{-(m_0+1)} P_{m_0-2}(z) z^{-(m_0-3)} z^{2m_0-2} \}
\end{aligned}$$

and

$$\begin{aligned}
\beta(z) z^{-2m_0} = & -\frac{a_{m_0-2}}{\gamma_1\gamma_2} \left[\gamma_1 e_{m_0+2}(z) z^{-(m_0+2)} P_{m_0-1}(z) z^{-(m_0-2)} \right] \\
& + \frac{a_{m_0-2}}{z\gamma_1\gamma_2} \left[\gamma_1 e_{m_0+1}(z) z^{-(m_0+1)} P_{m_0-1}(z) z^{-(m_0-2)} \right. \\
& \qquad \left. - \gamma_2 e_{m_0+1}(z) z^{-(m_0+1)} P_{m_0-1}(z) z^{-(m_0-2)} \right] \\
& + \frac{a_{m_0-2}}{z^2\gamma_1\gamma_2} \left[\gamma_2 e_{m_0+1}(z) z^{-(m_0+1)} P_{m_0-2}(z) z^{-(m_0-3)} \right].
\end{aligned}$$

If we write last equation in limit form, we find

$$\lim_{|z|\to\infty} \left\{ \beta(z) z^{-2m_0} \right\} = -\frac{a_{m_0-2}}{\gamma_1\gamma_2} \left\{ \gamma_1 \rho_{m_0+2} \frac{1}{a_1 \dots a_{m_0-2}} \right\}.$$

Last equation gives that

$$\lim_{|z|\to\infty} \left\{ \beta(z) z^{-2m_0} \right\} = A,$$

where

$$A = -\frac{a_{m_0-2}}{\gamma_2} \frac{\rho_{m_0+2}}{a_1 \dots a_{m_0-2}}$$

for all $z \in D_2$ and it completes the proof of Theorem 9. □

Theorem 9 is important for us because it satisfies to say that the set of eigenvalues of PBVP (1)-(3) denoted by σ_d is bounded under the condition (4). If we denote the continuous spectrum of PVBP (1)-(3) by σ_c, we can give the following theorem.

Theorem 5. Under the condition (4), the continuous spectrum of the operator L generated by PBVP (1)-(3) is $[-2, 2]$, i.e., $\sigma_c(L) = [-2, 2]$.

Proof. Let us introduce the operators, L_1 and L_2 generated in $\ell_2(\mathbb{N})$ by the following difference expressions

$$(L_1 y)_n = y_{n-1} + y_{n+1}, \quad n \in \mathbb{N} \setminus \left\{m_0 - 1, m_0 + 1\right\}$$

and

$$(L_2 y)_n = (a_{n-1} - 1)y_{n-1} + b_n y_n + (a_n - 1)y_{n+1}, \quad n \in \mathbb{N} \setminus \left\{m_0 - 1, m_0, m_0 + 1\right\}$$

with the boundary condition (2), respectively. It is clear from that $L = L_1 + L_2$ and L_1 is a self-adjoint operator with $\sigma_c(L_1) = [-2, 2]$. Because L_1 has no eigenvalues and the spectrum of the self-adjoint operator L_1 consists only of its continuous spectrum (see [24]). On the other hand, L_2 is a compact operator in $\ell_2(\mathbb{N})$ under the condition (4) (see [24]). By using Weyl Theorem [25] of a compact perturbation, we find that

$$\sigma_c(L_1) = \sigma_c(L) = [-2, 2].$$

□

Furthermore, we can write

$$\sigma_d(L) \subset (-\infty, -2) \cup (2, \infty). \tag{23}$$

from the definition of eigenvalues of PBVP (1)-(3) and we also write the $\sigma_d(L)$ as

$$\sigma_d(L) = \left\{\lambda \in \mathbb{C} : \lambda = z + z^{-1}, \quad z \in (-1, 0) \cup (0, 1), \quad \beta(z) = 0\right\}. \tag{24}$$

Theorem 6. The operator L has a finite number of real eigenvalues under the assumption (4).

Proof. Since $\left\{a_n\right\}$ and $\left\{b_n\right\}$ are real sequences, the operator L is selfadjoint and from the operator theory, its eigenvalues are real. To complete the proof of Theorem 11, we have to show that $\beta(z)$ has finitely many zeros in D_2. Using (23), we get that the limit points of the set of all eigenvalues of (1)-(3) or of L could not be different from $\pm 2, \pm\infty$. Since $\lambda = z + z^{-1}$, the limit points of the set of all eigenvalues of L could be $\pm\infty$ only in the case of $z = 0$. But it contradicts the fact that the operator L is bounded, so we cannot consider 0 as a zero of the function $\beta(z)$. On the other hand, equation (24) implies that the limit points of the set of all eigenvalues of L could be ± 2 for $z = \pm 1$. But from the operator theory and Theorem 10, the eigenvalues of selfadjoint operators cannot be elements of its continuous spectrum. Because of this reason, we also cannot consider $z = \pm 1$ as zeros of $\beta(z)$, i.e., the set of all eigenvalues of the operator L has not any limit points. It gives from the Bolzano-Weierstrass Theorem that the set of zeros of $\beta(z)$ in D_2 is finite. □

5 | EXAMPLE

In this section, we will define an unperturbated problem generated by following difference equation, boundary condition and point interaction

$$y_{n-1}(z) + y_{n+1}(z) = (z + z^{-1})y_n(z), \quad n \in \mathbb{N} \setminus \{2, 3, 4\}$$

$$\begin{aligned} y_0(z) &= 0 \\ y_4(z) &= \gamma_1 y_2(z) \\ \triangle y_4(z) &= \gamma_2 \nabla y_2(z), \end{aligned} \tag{25}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$ and $\gamma_1\gamma_2 \neq 0$. It is evident that the problem (25) is a special case of PBVP (1)-(3). We obtain the problem (25) assuming $a_n \equiv 0$, $b_n \equiv 0$ for all $n \in \mathbb{N}$, $m_0 = 3$ in PVBP (1)-(3). We will discuss our main results on this example. This special case provides some advantages for readers to understand main results clearly. In this example, the solution $e_n(z)$ turns into z^n and the fundamental solution $P_n(z)$ of (1)-(3) has the following values for $n = 0, 1, 2$

$$P_0(z) = 0 \qquad P_1(z) = 1 \qquad P_2(z) = \lambda.$$

It follows from (7) and (9) that $\beta(z)$ and Jost solution of this problem are obtained by

$$\beta(z) = -\frac{a_1}{\gamma_1\gamma_2} z^3 \left\{\gamma_2(z^2 - z + 1) - \gamma_1(z^3 - z^2 + z - 1)\right\} \tag{26}$$

and

$$E_n(z) = \begin{cases} \alpha(z)P_n(z) + \beta(z)Q_n(z), & n \in \{0, 1, 2\} \\ \\ z^n, & n \in \{4, 5, \dots\}, \end{cases}$$

respectively. By using (26), we find the scattering function of (25)

$$S(z) = z^{-9} \frac{\left\{\gamma_2 z(z^2 - z + 1) + \gamma_1(z^3 - z^2 + z + 1)\right\}}{\left\{\gamma_2(z^2 - z + 1) - \gamma_1(z^3 - z^2 + z - 1)\right\}}.$$

Also, the continuous spectrum of the problem (25) is found $[-2, 2]$ by Theorem 10. To get the eigenvalues of the problem (25), it is necessary for us to find the zeros of $\beta(z)$ for $z \in D_1$. Because it is written in the following form of this problem,

$$\sigma_d = \left\{\lambda = z + z^{-1} : \beta(z) = 0, z \in D_1\right\},$$

where $\beta(z)$ is defined by (26). If $\beta(z) = 0$, then, we write

$$\frac{\gamma_2}{\gamma_1} = \frac{z^3 - z^2 + z - 1}{z^2 - z + 1} \tag{27}$$

for $z \in D_1$. Let us assume $\gamma_2 = B\gamma_1$, $B \in \mathbb{R}$. By using (27), we obtain

$$z^3 - (B + 1)z^2 + (B + 1)z - (B + 1) = 0. \tag{28}$$

Note that, it is clear from (28) that B can be never -1. If $B = -1$, it gives a contradiction. Because, we get $z = 0$ in this case. If $z = 0$, λ is undefined. So, $B \neq -1$.

Case 1: For $B = 1$ in (28), we get

$$z_1 \cong 1,544$$

$$z_2 \cong 0,228 + 1,115i$$

$$z_3 \cong 0,228 - 1,115i.$$

Since z_1, z_2 and z_3 do not belong to D_1, the problem (25) has not eigenvalues in this case.

Case 2: If we solve (28) for $B = 2$, we find

$$z_1 \cong 2,260$$
$$z_2 \cong 0,370 + 1901i$$
$$z_3 \cong 0,370 - 1901i.$$

In Case 2, the problem (25) again has not eigenvalues because similar to Case 1, z_1, z_2, z_3 are not in D_1 in this case, too.

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