REGULARITY OF WEAK SOLUTIONS TO ELLIPTIC PROBLEM WITH IRREGULAR DATA

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ABSTRACT. This paper is concerned with the study of the nonlinear elliptic equations in a bounded subset $\Omega \subset \mathbb{R}^N$

$$Au = f,$$

where A is an operator of Leray-Lions type acted from the space $W_0^{1,p(\cdot)}(\Omega)$ into its dual. when the second term f belongs to $L^{m(\cdot)}$, with $m(\cdot) > 1$ being small. we prove existence and regularity of weak solutions for this class of problems $p(x)$-growth conditions. The functional framework involves Sobolev spaces with variable exponents as well as Lebesgue spaces with variable exponents.

1. INTRODUCTION

The purpose of this article is to study the existence and regularity of weak solutions for a class of nonlinear elliptic equations with variable exponents. A prototype example is

$$(P) \qquad \begin{cases} -\operatorname{div}\left(|Du|^{p(\cdot)-2} Du\right) = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipchitz boundary $\partial\Omega$, the right-hand side $f \in L^{m(\cdot)}(\Omega)$, $m(\cdot)$ as in (1.6).

The equation (P) can be viewed as a generalization of the classical p-Laplace equation where the constant $p \in (1, +\infty)$.

Instead of (P) we will consider more general nonlinear elliptic equations with variable exponents of the form

$$\begin{cases} -\operatorname{div}\left(\widehat{a}(x, Du)\right) = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

Recall that a LerayLions type operator is a Caratheodory function $\widehat{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, satisfying, a.e $x \in \Omega$ and for all $\xi, \xi' \in \mathbb{R}^N$, the following:

$$\widehat{a}(x,\xi)\xi \geq \alpha|\xi|^{p(\cdot)}, \quad \widehat{a}(x,\xi) = (a_1, \ldots, a_N) \tag{1.2}$$

$$|\widehat{a}(x,\xi)| \leq \beta\left(h + |\xi|^{p(\cdot)-1}\right) \tag{1.3}$$

$$(\widehat{a}(x,\xi) - \widehat{a}(x,\xi'))(\xi - \xi') > 0, \ \xi \neq \xi', \tag{1.4}$$

where α, β are strictly positive real numbers, h is a given positive function in $L^{p'(\cdot)}(\Omega)$ where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$, while $m(\cdot) : \overline{\Omega} \to (1, +\infty)$ and the variable exponent $p(\cdot) : \overline{\Omega} \to (1, +\infty)$ are continuous functions such that:

$$1 + \frac{1}{m^+} - \frac{1}{N} < p(x) < N, \quad \text{for all } x \in \overline{\Omega}, \quad m^+ = \max_{x \in \overline{\Omega}} m(x). \tag{1.5}$$

where

$$1 < m(x) < \frac{Np(x)}{Np(x) - N + p(x)}, \quad \nabla m \in L^\infty(\Omega), \quad \text{for all } x \in \overline{\Omega}. \tag{1.6}$$

Key words and phrases. Nonlinear elliptic problem; Leray-Lions operator; Variable exponents; Weak solution; Irregular Data.

As another prototype example we consider the model problem

$$\begin{cases} -\mathrm{div}\left(|Du|^{p(\cdot)-2}\,Du\right) = \delta & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \tag{1.7}$$

where δ is the Dirac measure at the origin, $p(\cdot)$ as in (1.5) , and

$$B = \{x \in \mathbb{R}^N \mid |x| < 1\}.$$

Variable exponent Lebesgue-Sobolev spaces have been intensively studied during the last years. These spaces of functions provide a useful tool for the study of both elliptic and parabolic equations with variable exponents, In addition, It involves today in various branches of applied science. In some cases, they provide realistic models for the study of natural phenomena in electro-rheological fluids and an important applications are related to image processing. We refer the reader to [5] and the references therein. Clearly, the nonlinearity of (1.7) is more complicated than nonlinearity of the p-Laplacian. As the exponent which appear in (1.7) depends on the variable x, the functional setting involves Lebesgue and Sobolev spaces with variable exponent $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, we refer to [4],[6] and [8] for further properties of variable exponent LebesgueSobolev spaces. In the constant case $2 - \frac{1}{N} < p(\cdot) = p$ with $Au = -\mathrm{div}\,(\widehat{a}(x,u,Du))$, the existence of a distributional solution u of (1.1) in the space $W_0^{1,q}(\Omega)$ for all $q \in \left[1; \frac{N(p-1)}{(N-1)}\right)$ has been proved in [3]. Therefore, the study of problem (1.1) is a new and interesting topic. Inspired by [2], [10] and [11], we prove the existence of weak solution for the problem (1.1) with right-hand side in $L^{m(\cdot)}(\Omega)$ where $m(\cdot)$ and the variable exponent $p(\cdot)$ are restricted as in (1.5)-(1.6), similar results can be found in [1], [10], [12] and [13]. The main steps of the proof consist of obtaining uniform estimate for suitable approximate problems and then passing to the limit.
Throughout this paper, we denote by C or $C_i, i = 1, 2, \cdots$, some generic positive constants independent of n.

2. Lebesgue-Sobolev Spaces with variable exponents

In this section we recall some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. We refer to [4],[6] and [8] for further properties of variable exponent Lebesgue-Sobolev spaces.

Let $p : \overline{\Omega} \to [1, \infty)$ be a continuous function. We denote by $L^{p(\cdot)}(\Omega)$ the space of measurable function $f(x)$ on Ω such that

$$\rho_{p(.)}(f) = \int_\Omega |f(x)|^{p(x)} dx < +\infty.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm

$$\|f\|_{p(.)} := \|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 \mid \rho_{p(.)}(f/\lambda) \le 1\right\}$$

becomes a Banach space. Moreover, if $p^- = \inf_{x\in\overline{\Omega}} p(x) > 1$, then $L^{p(\cdot)}(\Omega)$ is reflexive and the dual of $L^{p(\cdot)}(\Omega)$ can be identified with $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the Hölder type inequality:

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{p(.)}\|v\|_{p'(.)} \le 2\|u\|_{p(.)}\|v\|_{p'(.)}.$$

holds true.

We define also the Banach space $W_0^{1,p(x)}(\Omega)$ by

$$W_0^{1,p(.)}(\Omega) = \left\{f \in L^{p(\cdot)}(\Omega),\, |Df| \in L^{p(\cdot)}(\Omega) \text{ and } f = 0 \text{ on } \partial\Omega\right\}$$

endowed with the norm $\|f\|_{W_0^{1,p(.)}(\Omega)} = \|Df\|_{p(.)}$. The space $W_0^{1,p(\cdot)}(\Omega)$ is separable and reflexive provided that with $1 < p^- \le p^+ < \infty$. The smooth functions are in general not dense in

$W_0^{1,p(.)}(\Omega)$, but if the exponent variable $p(x) > 1$ is logarithmic Hölder continuous, that is

$$|p(x) - p(y)| \leq -\frac{M}{\ln(|x-y|)} \quad \forall x, y \in \Omega \text{ such that } |x-y| \leq 1/2, \tag{2.1}$$

then the smooth functions are dense in $W_0^{1,p(\cdot)}(\Omega)$.

For $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in C(\overline{\Omega}, [1, +\infty))$, the Poincaré inequality holds

$$\|u\|_{p(.)} \leq C\|Du\|_{p(.)}, \tag{2.2}$$

for some constant C which depends on Ω and the function p.
An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result

Lemma 2.1 ([8]). *If (u_n), $u \in L^{p(\cdot)}(\Omega)$, then the following relations hold*

- $\|u\|_{p(.)} < 1(> 1; = 1) \Leftrightarrow \rho_{p(.)}(u) < 1(> 1; = 1)$,
- $\min\left(\rho_{p(.)}(u)^{\frac{1}{p^+}}; \rho_{p(.)}(u)^{\frac{1}{p^-}}\right) < \|u\|_{p(.)} < \max\left(\rho_{p(.)}(u)^{\frac{1}{p^+}}; \rho_{p(.)}(u)^{\frac{1}{p^-}}\right)$.
- $\|u\|_{p(.)} \leq \rho_{p(.)}(u) + 1$.
- $\|u_n - u\|_{p(.)} \to 0 \Leftrightarrow \rho_{p(.)}(u_n - u) \to 0$,
  since $p^+ < \infty$.

An important embedding as follows

Lemma 2.2 ([7]). *Let $\Omega \in \mathbb{R}^N$ be an open bounded set, with Lipschitz boundary, and let $p : \overline{\Omega} \to (1, N)$ satisfy the log-Hölder continuity condition* (2.1). *Then we have the following continuous embedding:*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^\star(\cdot)}(\Omega),$$

where $p^\star(\cdot) = \frac{Np(\cdot)}{N-p(\cdot)}$.

3. Main results

Definition 3.1. *A function u is a weak solution of problem* (1.1) *if*

$$u \in W_0^{1,1}(\Omega), \quad \widehat{a}(x, Du) \in (L^1(\Omega))^N,$$

and

$$\int_\Omega \widehat{a}(x, Du)D\varphi\, dx = \int_\Omega f\varphi\, dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Our main results are the following:

Theorem 3.1. *Let $f \in L^{m(\cdot)}(\Omega)$, $m(\cdot) = m^+$ and assume that $p(\cdot)$ and m^+ are restricted as in* (1.5)-(1.6). *Let $\hat{a}$ be a Carathéodory function satisfying (1.2)-(1.4). Then the problem (1.1) has at least one weak solution $u \in W_0^{1,q(\cdot)}(\Omega)$ where $q(\cdot)$ is a continuous function on $\overline{\Omega}$ satisfying*

$$1 \leq q(x) < \frac{Nm^+(p(x)-1)}{N-m^+} \quad \textit{for all} \quad x \in \overline{\Omega}. \tag{3.1}$$

Theorem 3.2 ([11]). *Let $f \in L^{m(\cdot)}(\Omega)$ and assume that $p(\cdot)$ and $m(\cdot)$ are restricted as in* (1.5)-(1.6). *Let $\hat{a}$ be a Carathéodory function satisfying (1.2)-(1.4). Then the problem (1.1) has at least one weak solution $u \in W_0^{1,p(\cdot)}(\Omega)$*

Proof of Theorem 3.1. The proof needs three steps.
Step 1: Approximate problem
By the density property, we can choose a sequence $(f_n)_n \subset C_0^\infty(\Omega)$

$$f_n \longrightarrow f \quad \text{strongly in } L^{m^+}(\Omega), \text{as } n \longrightarrow \infty.$$

such that

$$\|f_n\|_{L^{m^+}(\Omega)} \leq \|f\|_{L^{m^+}(\Omega)}, \quad \forall n \geq 1. \tag{3.2}$$

For $u \in W_0^{1,p(\cdot)}(\Omega)$, we put

$$Au = -\mathrm{div}\,(\widehat{a}(x, Du)).$$

The operator A maps $W_0^{1,p(\cdot)}(\Omega)$ into $\left(W_0^{1,p(\cdot)}(\Omega)\right)'$, thanks (1.4) A is monotone. The growth condition (1.3) implies that A is hemicontinuous.
i.e., for all u, v, $w \in W_0^{1,p(\cdot)}(\Omega)$, the mapping $\mathbb{R} \ni \lambda \mapsto \langle A(u + \lambda v), w\rangle$ is continuous.
By (1.2) and Lemma 2.2 [6], we can write

$$\begin{aligned}\frac{\langle Au, u\rangle}{\|u\|_{W_0^{1,p(\cdot)}(\Omega)}} &\geq \alpha \frac{\rho_{p(\cdot)}(Du)}{\|u\|_{W_0^{1,p(\cdot)}(\Omega)}} \\ &\geq \alpha \frac{\min\left\{\|u\|^{p^+}_{W_0^{1,p(\cdot)}(\Omega)}, \|u\|^{p^-}_{W_0^{1,p(\cdot)}(\Omega)}\right\}}{\|u\|_{W_0^{1,p(\cdot)}(\Omega)}},\end{aligned}$$

this prove that A is coercive. By (1.3), we get the operator A is bounded.
Thus, we get the desired result.
Consequently, there exists at least one weak solution $(u_n)_{n\in\mathbb{N}} \subset W_0^{1,p(\cdot)}(\Omega)$ (cf. J.L. Lions [9] , Theorem 2.7, page 180) satisfying

$$\int_\Omega \hat{a}(x, Du_n) D\varphi\, dx = \int_\Omega f_n \varphi\, dx, \quad \forall \varphi \in W_0^{1,p(\cdot)}(\Omega). \tag{3.3}$$

Step 2: Uniform estimates
We prove the following estimates:

Lemma 3.1. *Let $p(\cdot)$ as in (1.5), and $m(\cdot) = m^+$ as in (1.6). Then, for any constant $0 < \delta < 1$, there exists a constant C_δ independent of n such that*

$$\int_\Omega \frac{|Du_n|^{p(x)}}{(1 + |u_n|)^\delta}\, dx \leq C_\delta \left(1 + \left(\int_\Omega (1 + |u_n|)^{(1-\delta)\frac{m^+}{m^+-1}}\, dx\right)^{1-\frac{1}{m^+}}\right) \tag{3.4}$$

Proof of Lemma 3.1. For any given $0 < \delta < 1$, we define the function $\psi_\delta : \mathbb{R} \to \mathbb{R}$ by

$$\psi_\delta(t) = \int_0^t \frac{dt}{(1 + |t|)^\delta}.$$

Note that ψ_δ is a continuous function satisfies $\psi_\delta(0) = 0$, and $|\psi_\delta^{'}(\cdot)| \leq 1$, we take $\psi_\delta(u_n)$ as a test function in (3.3), we obtain

$$\int_\Omega \hat{a}(x, Du_n) D\psi_\delta(u_n)\, dx = \int_\Omega f_n \psi_\delta(u_n)\, dx.$$

Since for any $0 < \delta < 1$, $|\psi_\delta(t)| = \frac{1}{1-\delta}\left(\frac{1}{(1+|t|)^{\delta-1}} - 1\right)$, by (1.2), Hölder's inequality and

$$(a_1 + a_2)^r \leq \max\{1, 2^{r-1}\}\left(a_1^r + a_2^r\right), \quad a_i \geq 0, \quad r > 0,$$

Which yields (3.4). □

Lemma 3.2. *Let $p(\cdot)$ as in (1.5), and $m(\cdot) = m^+$ as in (1.6), and $f \in L^{m^+}(\Omega)$. Then there exists a constant C_1 such that*

$$\|u_n\|_{W_0^{1,q(\cdot)}(\Omega)} \leq C_1,$$

for all continuous functions $q(\cdot)$ as in (3.1).

Remark 3.1. *Note that the result given in Lemma 3.2 also holds for any measurable function $q : \overline{\Omega} \to \mathbb{R}$ such that*

$$ess \inf_{x\in\overline{\Omega}} \left(\frac{Nm^+(p(x) - 1)}{N - m^+} - q(x)\right) > 0.$$

Indeed, in both cases there exists a continuous function $s : \overline{\Omega} \to \mathbb{R}$ such that for almost every $x \in \overline{\Omega}$:

$$q(x) \leq s(x) \leq \frac{Nm^+(p(x)-1)}{N-m^+}.$$

From Lemma 3.2, we deduce, in both cases, that $(u_n)_n$ is bounded in $W_0^{1,s(\cdot)}(\Omega)$. Finally, by the continuous embedding $W_0^{1,s(\cdot)}(\Omega) \hookrightarrow W_0^{1,q(\cdot)}(\Omega)$, we have the desert result.

Proof of Lemma 3.2. Firstly, note that since $m^+ > 1$ and $p(\cdot)$ is defined as in (1.5), we get

$$1 < \frac{Nm^+(p(x)-1)}{N-m^+}, \quad \text{for all} \quad x \in \overline{\Omega}.$$

Now, consider the following cases:

Case (a): Let q^+ be a constant satisfying

$$q^+ < \frac{Nm^+(p^- - 1)}{N-m^+}. \tag{3.5}$$

Note that the assumption (1.6) implies that

$$\frac{Nm^+(p^- - 1)}{N-m^+} < p^-. \tag{3.6}$$

Using Hölder's inequality with (3.4), we obtain

$$\begin{aligned}
&\int_\Omega |Du_n|^{q^+}\,dx = \int_\Omega \frac{|Du_n|^{q^+}}{(1+|u_n|)^{\delta\frac{q^+}{p^-}}}(1+|u_n|)^{\delta\frac{q^+}{p^-}}\,dx \\
&\leq C_2\left(1+\left(\int_\Omega (1+|u_n|)^{(1-\delta)\frac{m^+}{m^+-1}}\,dx\right)\right)^{(1-\frac{1}{m^+})\frac{q^+}{p^-}} \cdot \left(1+\left(\int_\Omega (1+|u_n|)^{\delta\frac{q^+}{p^--q^+}}\,dx\right)\right)^{1-\frac{q^+}{p^-}},
\end{aligned} \tag{3.7}$$

By (3.5) and (3.6) , we get

$$1-\left(\frac{Nq^+}{N-q^+}\right)\left(\frac{m^+-1}{m^+}\right) < \frac{m^+(p^- - q^+)}{(m^+-1)q^+ + m^+(p^- - q^+)} < 1. \tag{3.8}$$

Now, choose $\delta \in (0,1)$ such that

$$\frac{\delta q^+}{p^- - q^+} < \frac{m^+(1-\delta)}{m^+-1} < q^{+\star} = \frac{Nq^+}{N-q^+}. \tag{3.9}$$

Notice that (3.8) and (3.9) are respectively equivalent to

$$1-\left(\frac{Nq^+}{N-q^+}\right)\left(\frac{m^+-1}{m^+}\right) < \delta < \frac{m^+(p^- - q^+)}{(m^+-1)q^+ + m^+(p^- - q^+)} < 1. \tag{3.10}$$

Therefore, by (3.7), (3.9) and using Sobolev inequality with $q^{+\star}$, we obtain

$$\begin{aligned}
\int_\Omega |Du_n|^{q^+}\,dx &\leq C_3\left(1+\int_\Omega |u_n|^{\frac{m(1-\delta)}{m^+-1}}\,dx\right)^{1-\frac{q^+}{m^+p^-}} \\
&\leq C_4\left(1+\int_\Omega |u_n|^{q^{+\star}}\,dx\right)^{1-\frac{q^+}{m^+p^-}} \\
&\leq C_5\left(1+\int_\Omega |Du_n|^{q^+}\,dx\right)^{(\frac{N}{N-q^+})(1-\frac{q^+}{m^+p^-})} \\
&\leq C_6 + C_6\left(\int_\Omega |Du_n|^{q^+}\,dx\right)^{(\frac{N}{N-q^+})(1-\frac{q^+}{m^+p^-})},
\end{aligned} \tag{3.11}$$

By the fact that

$$m^+ < \frac{Np^-}{Np^- - N + p^-} < \frac{N}{p^-}, \tag{3.12}$$

together with the assumption (3.5), this implies that

$$q^+ < m^+ p^- \text{ and } 0 < \Big(\frac{N}{N-q^+}\Big)\Big(1 - \frac{q^+}{m^+p^-}\Big) < 1.$$

Hence, the estimate (3.11) imply that (Du_n) is bounded in $L^{q^+}(\Omega)$.
Since $|Du_n|^{q(\cdot)} \leq |Du_n|^{q^+} + 1$, we obtain that (u_n) is bounded in $W_0^{1,q(\cdot)}(\Omega)$. This completes the proof in Case (a).

Case (b): Let q be a continuous function satisfying (3.1) and

$$q^+ \geq \frac{Nm^+(p^- - 1)}{N - m^+}.$$

By the continuity of $p(\cdot)$ and $q(\cdot)$ on $\overline{\Omega}$, there exists a constant $\eta > 0$ such that

$$\max_{y \in B(x,\eta)\cap\Omega} q(y) < \min_{y \in B(x,\eta)\cap\Omega} \frac{Nm^+(p(y)-1)}{N-m^+} \text{ for all } x \in \overline{\Omega}. \tag{3.13}$$

Note that $\overline{\Omega}$ is compact and therefore we can cover it with a finite number of balls $(B_i)_{i=1,\dots,k}$. Moreover, there exists a constant $\rho > 0$ such that

$$|\Omega_i| = \text{meas}(\Omega_i) > \rho,\ \Omega_i := B_i \cap \Omega, \quad \text{for all } i = 1, \dots, k. \tag{3.14}$$

We denote by q_i^+ the local maximum of q on $\overline{\Omega_i}$ (respectively p_i^- the local minimum of p on $\overline{\Omega_i}$), such that

$$q_i^+ < \frac{Nm^+(p_i^- - 1)}{N - m^+} \text{ for all } i = 1, \dots, k. \tag{3.15}$$

Using the same arguments as before locally, we obtain the similar estimate as in (3.11)

$$\int_{\Omega_i} |Du_n|^{q_i^+}\, dx \leq C_7 \left(1 + \int_{\Omega_i} |u_n|^{q_i^{+\star}}\, dx\right)^{1 - \frac{q_i^+}{m^+p_i^-}}, \text{ for all } i = 1, \dots, k. \tag{3.16}$$

On the other hand, the Poincaré-Wirtinger inequality gives

$$\|u_n - \widetilde{u_n}\|_{L^{q_i^{+\star}}(\Omega_i)} \leq C_8 \|Du_n\|_{L^{q_i^+}(\Omega_i)}, \tag{3.17}$$

$$\text{where} \qquad \widetilde{u_n} = \frac{1}{|\Omega_i|}\int_{\Omega_i} u_n(x)\, dx, \quad q_i^{+\star} = \frac{Nq_i^+}{N - q_i^+}.$$

Moreover, note that the sequence $(u_n)_n$ is bounded in $L^1(\Omega)$. So, from (3.14), we have

$$\|\widetilde{u_n}\|_{L^1(\Omega)} \leq C_8,$$

Therefore, by (3.17), we deduce that

$$\begin{aligned}\|u_n\|_{L^{q_i^{+\star}}(\Omega_i)} &\leq \|u_n - \widetilde{u_n}\|_{L^{q_i^{+\star}}(\Omega_i)} + \|\widetilde{u_n}\|_{L^{q_i^+}(\Omega_i)} \\ &\leq C_8 \|Du_n\|_{L^{q_i^+}(\Omega_i)} + C_9, \quad \text{for all} \quad i = 1, \cdots, k.\end{aligned}$$

Thus, using (3.16), we obtain

$$\int_{\Omega_i} |Du_n|^{q_i^+}\, dx \leq C_{10} + C_{10}\left(\int_{\Omega_i} |Du_n|^{q_i^+}\, dx\right)^{\left(\frac{N}{N-q_i^+}\right)\left(1 - \frac{q_i^+}{m^+p_i^-}\right)},$$

By (3.15) and arguing locally as in (3.12), we deduce

$$0 < \Big(\frac{N}{N - q_i^+}\Big)\Big(1 - \frac{q_i^+}{m^+p_i^-}\Big) < 1,$$

so that

$$\int_{\Omega_i} |Du_n|^{q_i^+}\,dx \le C_{11}, \text{ for all } i = 1,\dots,k.$$

Recall that

$$q(x) \le q_i^+, \qquad \text{for all } x \in \Omega_i \text{ and for all } i = 1,\dots,k.$$

So, we get

$$\int_{\Omega_i} |Du_n|^{q(x)}\,dx \le \int_{\Omega_i} |Du_n|^{q_i^+}\,dx + |\Omega_i| \le C_{12}.$$

Since $\Omega \subset \bigcup_{i=1}^{N} \Omega_i,$ for all $i = 1,\dots,k.$ we deduce that

$$\int_{\Omega} |Du_n|^{q(x)}\,dx \le \sum_{i=1}^{k} \int_{\Omega_i} |Du_n|^{q(x)}\,dx \le C_{13}.$$

This finishes the proof of the Case(b). □

Remark 3.2. *Remark that in the constant case and $f \in L^{m^+}(\Omega)$, we choose in* (3.7)

$$\delta = \frac{pN - m^+p - m^+Np + m^+N}{N - m^+p} \in (0,1),$$

to obtain

$$q = \frac{m^+N(p-1)}{N-m^+} \Longrightarrow (1-\delta)\frac{m^+}{m^+-1} = \frac{\delta q}{p-q} = \frac{Nq}{N-q},$$

It is easy to check that, instead of the global estimate (3.11)*, we find*

$$\int_{\Omega} |Du_n|^q\,dx \le C + C\left(\int_{\Omega} |Du_n|^q\,dx\right)^{\left(\frac{N}{N-q}\right)\left(1-\frac{q}{m^+p}\right)},$$

where $0 < \left(\frac{N}{N-q}\right)\left(1 - \frac{q}{m^+p}\right) < 1$*. Then* (1.1) *has at least one weak solution u, possesses the regularity $u \in W_0^{1,q}(\Omega)$ far all $q = \frac{Nm^+(p-1)}{N-m^+}$. For the nonconstant case, it remains an open problem to show that*

$$u \in W_0^{1,q(\cdot)}(\Omega), \quad \textit{with} \quad q(\cdot) = \frac{Nm^+(p(\cdot)-1)}{N-m^+}.$$

Step 3: Passage to the limit

From Lemma 3.2 together with the continuous embedding $W_0^{1,q(\cdot)}(\Omega) \hookrightarrow W_0^{1,q^-}(\Omega)$, we have a subsequence (still denoted $(u_n)_n$) such that

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W_0^{1,q^-}(\Omega), \tag{3.18}$$

$$u_n \to u \quad \text{strongly in} \quad L^{q^-}(\Omega) \tag{3.19}$$

$$u_n \to u \quad \text{a.e in} \quad \Omega. \tag{3.20}$$

To complete the proof, we need the following lemmas:

Lemma 3.3. *We have*

$$Du_n \to Du \quad \textit{a.e in} \quad \Omega, \tag{3.21}$$

Proof. In order to prove this lemma it is sufficient to show that:

$$Du_n \to Du \quad \text{in measure}.$$

By (3.19),(3.18),(1.2),(1.3), (3.1) and using Lebesgue's dominated convergence theorem, we get the convergence of (Du_n) to (Du) in measure, which proves the Lemma 3.3. □

Lemma 3.4. *We have*

$$\widehat{a}(x, Du_n) \to \widehat{a}(x, Du) \quad \text{strongly in} \quad L^{q(\cdot)}(\Omega), \tag{3.22}$$

for some continuous function $q(\cdot) : \overline{\Omega} \to [1, \frac{Nm^+}{N-m^+})$*, where* m *is a defined in* (1.6).

Proof. To prove (3.22), we apply Vitali's theorem with taking in consideration Lemma 3.2, (3.20), (3.21), (1.3) and (1.5). □

Finally, for $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \hat{a}(x, Du_n) D\varphi \, dx = \int_\Omega f_n \varphi \, dx. \tag{3.23}$$

Using (3.22), we can pass to the limit for $n \to +\infty$ in the weak formulation (3.23), we obtain that u is a weak solution for (1.1). □

Proof of Theorem 3.2. The proof of of Theorem 3.2 is similar to the proof of Theorem 3.4. in [10]. □

Remark 3.3. *Under the assumption* $f \in L^{m^+}(\Omega)$ *in Theorem 3.1, we can deduce that* f *is never in the dual space* $\left(W_0^{1,p(\cdot)}(\Omega)\right)'$*, so that the result of this paper deals with irregular data. If* m^+ *tends to be* 1*, then* $q(\cdot) = \frac{Nm^+(p(\cdot)-1)}{N-m^+}$ *tends to be* $\frac{N(p(\cdot)-1)}{N-1}$.

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