# A numerical study on the non-smooth solutions of the nonlinear weakly singular fractional integro-differential equations

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**Abstract**

The solutions of weakly singular fractional integro-differential equations involving the Caputo derivative have singularity at the lower bound of the domain of integration. In this paper, we design an algorithm to prevail on this non-smooth behaviour of solutions of the nonlinear fractional integro-differential equations with a weakly singular kernel. The convergence of the proposed method is investigated. The proposed scheme is employed to solve four numerical examples in order to test its efficiency and accuracy.

**Keywords:** Weakly singular kernel, Fractional integro-differential equation, Chebyshev polynomials, Convergence analysis, Error estimate.

**Mathematics Subject Classification [2010]:** 26A33, 47G20, 45Gxx.

## 1 Introduction

The fractional calculus is a natural generalization of calculus that studies integral and derivative operators (the differential operator may be of a fractional order) [8, 16]. Fractional calculus can be used for modeling many phenomena in science and engineering. The field of viscoelasticity looks to be the most pervasive utilization of fractional differential and integral operators [21]. In recent years, due to the many applications of fractional calculus in modeling natural phenomena, much attentions have been paid to the numerical solution of fractional equations (see e.g. [1, 5, 6, 9, 14, 20, 23–25]). There are phenomena in the theory of relaxation of dynamical systems [21], polymer physics and rheology and biophysics [8], the radiative equilibrium [11] and heat conduction problem [22] that are formulated as fractional order integro-differential equations with weakly singular kernels. For further applications of fractional order integro-differential equations with weakly singular kernels in physical problems, refer to [10]. In this study, we consider the following nonlinear weakly singular fractional integro-differential equation

$$ {}_0^C D_t^{\alpha} u(t) = g(t) + p(t)u(t) + \int_0^t \frac{1}{(t-s)^{\beta}} u^n(s)ds, \ 0 \leq \beta < 1, \ \alpha > 0, \ t \in I(T), \tag{1.1}$$

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with boundary value conditions

$$u^{(i)}(0) = u_0^{(i)},\ i = 0(1)\lceil\alpha\rceil - 1,$$

where $u(t)$ is a solution which should be determined, $g(t)$ and $p(t)$ are known and continuous on $I(T) := [0, T]$, ${}_0^C D_t^\alpha$ is the Caputo fractional differential operator, $n > 0$ is a integer number as power of $u$, $u_0^i (i = 0(1)\lceil\alpha\rceil - 1)$ are given real numbers where $\lceil\alpha\rceil$ is the ceiling function of $\alpha$.

Uniqueness and existence of solution of fractional integro-differential equations investigated in [18] and [17], respectively. The Eq.(1.1) has been solved in [4, 19] by a stable least residue method and a modification of hat functions(MHFs), respectively.

As we already know, the solutions of (1.1) have singularity at the lower bound of the domain of integration. This non-smooth behaviour of solutions has rarely been considered in the previous researches and filling this gap is the motivation for our work.

One of the most powerful ways to deal with poorly behaved integrands is product integration [13]. Here, we will rewrite Eq.(1.1) into another form by using some properties of fractional operators and design an algorithm based on the product integration and Nyström methods and the first kind of Chebyshev polynomials for solving the rewritten equation. We will observe that the proposed method, without need to smoothing, can overcome the non-smooth behaviour of solutions.

Since the Caputo fractional derivative is non-local, so it means that nearly all numerical methods for solving the Eq.(1.1) will be very time-consuming and the low computing time of methods for solving these equations is very important, so we have reported computing time of method in the numerical examples.

In Section 2, we recall some basic contents that are used in the numerical and theoretical parts. In Section 3, we explain how to implement the method. In Section 4, we investigate the convergence of the proposed method. In Section 5, to reveal that the numerical results verify the validity of convergence analysis, four numerical examples with smooth and non-smooth solutions are prepared.

## 2 Preliminaries

In this section, firstly, in the sub section 2.1 (Part 1), we recall some properties and definitions from [21], which, are used to rewrite the Eq.(1.1) to another form. Next, in the sub section 2.2 (Part 2), we recall some basic concepts, useful inequalities, and lemmas from [2, 3, 12], which, are used in convergence analysis.

### 2.1 Part 1

**Definition 1.** For $\alpha > 0$, the Riemann-Liouville integral operator $\mathfrak{I}_t^\alpha$ is given as [21]

$$\mathfrak{I}_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} u(s) ds. \tag{2.1}$$

**Definition 2.** Let $\alpha \in \mathbb{R}$, $n - 1 < \alpha < n$, $n \in \mathbb{M}$ and $u(t)$ be a real valued continuous function defined on $[0, \infty)$, then the Caputo fractional derivative of order $\alpha > 0$ is defined by [21]

$${}_a^c D_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} u(\tau) d\tau. \tag{2.2}$$

The operators $\mathfrak{I}_t^{\alpha}$ and ${}_0^C D_t^{\alpha}$ are satisfy in the following properties

$$\mathfrak{I}_t^{\alpha}(\mathfrak{I}_t^{\beta}u(t)) = \mathfrak{I}_t^{\beta}(\mathfrak{I}_t^{\alpha}u(t)) = \mathfrak{I}_t^{\alpha+\beta}u(t), \tag{2.3}$$

$$\mathfrak{I}_t^{\alpha}({}_0^C D_t^{\alpha}u(t)) = u(t) - \sum_{i=0}^{m-1} u^{(i)}(0)\frac{t^i}{t!},\ m-1<\alpha\leq m,\ t>0. \tag{2.4}$$

where $\alpha,\beta>0$ and $\alpha,\beta\notin\mathbb{M}$.

### 2.2 Part 2

**Lemma 1.** *[3, 12] Let*

$$\mathcal{W}z = \int_{-1}^{\eta}(\eta-\zeta)^{-\alpha}K(\eta,\zeta)z(\zeta)d\zeta,$$

*then, for any function $z\in C([-1,1])$, there exists a constant $C>0$, so that*

$$\|\mathcal{W}z\|_{0,h}\leq C\|z\|_{\infty},\quad 0<h<1-\alpha,$$

*where*

$$\|\mathcal{W}z\|_{s,h} = \max_{0\leq h\leq s}\max_{x\in[-1,1]}|\partial_x^h(\mathcal{W}z(x))| + \max_{0\leq h\leq s}\sup_{\substack{x,y\in[-1,1],\\ x\neq y}}\frac{|\partial_x^h(\mathcal{W}z(x))|-|\partial_x^h(\mathcal{W}z(y))|}{|x-y|^h}.$$

The following norm is known as the Sobolev norm [2]

$$\|z\|_{H^m_{\omega^C}(-1,1)} = \left(\sum_{j=0}^{m}\|z^{(j)}\|^2_{L^2_{\omega^C}(-1,1)}\right)^{\frac{1}{2}},\quad \textit{for } m\geq 0, \tag{2.5}$$

where the Hilbert space $(H^m_{\omega^C}(-1,1))$ is defined as follows

$$H^m_{\omega^C}(-1,1) = \{\ v\in\ L^2_{\omega^C}(-1,1):\ \textit{for } 0\leq j\leq m,\ z^{(j)}\in L^2_{\omega^C}(-1,1)\ \},$$

with the following inner product:

$$(z,v)_{m,\omega^C} = \sum_{j=0}^{m}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}z^{(j)}v^{(j)}dx.$$

Notice that, in the continuation, we use the notation $\mathbb{P}_M$, which denotes the space of all polynomials of degree not exceeding $M$.

**Lemma 2.** *[2] If $z\in H^m_{\omega^C}(-1,1)$ for some $m\geq 1$, then*

$$\|I_M^C z - z\|_{L^2_{\omega^C}(-1,1)}\leq CM^{-m}|z|_{H^{m;M}_{\omega^C}(-1,1)}, \tag{2.6}$$

*where $I_M^C z\in\mathbb{P}_M$ represent the interpolant of $z$ at Chebyshev Gauss points and*

$$|z|_{H^{m,M}_{w^C}(-1,1)} = \left(\sum_{j=\min(m,M+1)}^{m}\|z^{(j)}\|^2_{L^2_{w^C}(-1,1)}\right)^{\frac{1}{2}}, \tag{2.7}$$

*for $m\geq 0$.*

**Lemma 3.** *[2] If $z \in H^m_{\omega^C}(-1,1)$ for some $m \geq 1$ and $I^C_M z \in \mathbb{P}_M$ represent the interpolant of $z$ at Chebyshev Gauss points, then*

$$\|I^C_M z - z\|_{L^\infty(-1,1)} \leq CM^{-m+\frac{1}{2}}|z|_{H^{m;M}_{\omega^C}(-1,1)}. \tag{2.8}$$

**Lemma 4.** *[2] Let $\{l_j(x)\}_{j=0}^M$ be the $M$-th Lagrange interpolation polynomials associated with the Gauss points of the Chebyshev polynomials. Then*

$$\|I^C_M\|_{L^\infty} = \mathcal{O}(\log M)$$

## 3 Description of method

This section is dedicated to introduce the first kind of Chebyshev polynomials and the used trick to rewrite the equation before implementing the method. Then, using the Gauss quadrature and defining the appropriate weights and ideas from Nyström and product integration methods, we present an algorithm for solving the Eq.(1.1).

The first kind of Chebyshev polynomials $T_M(x)$ defined by [15]

$$T_M(x) = \cos(M\cos^{-1}(x)).$$

The interval of orthogonality of these polynomials with respect to the weight function $w(x) = (1-x^2)^{-1/2}$, is $[-1, 1]$.
We can generates all the polynomials $T_M(x)$ by the following recursion relation

$$T_M(x) = 2xT_{M-1}(x) - T_{M-2}(x), \qquad M = 2, 3, ....,$$

with

$$T_0(x) = 1, \qquad T_1(x) = x.$$

The Gauss quadrature formula [7]

$$\int_{-1}^{1} f(x)w(x)dx \approx f(x_0)w_0 + f(x_1)w_1 + \ldots + f(x_M)w_M,$$

is exact for any polynomial of degree $\leq 2M+1$.

For $\alpha > 0$ and $\alpha \notin \mathbb{M}$, applying the Riemann-Liouville integral operatore $\mathfrak{I}^\alpha_t$ to both sides of (1.1) and using (2.1),(2.4) and boundary value conditions of (1.1), we get

$$u(t) = \sum_{i=0}^{\lceil\alpha\rceil-1} u_0^{(i)}\frac{t^i}{i!} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds +$$
$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)u(s)ds + \mathfrak{I}^\alpha_t\left(\int_0^t (t-s)^{-\beta}u^n(s)ds\right). \tag{3.1}$$

Now, we denote the last term of (3.1) as $N(u(t))$. Thus, using (2.1) and (2.3), we have

$$N(u(t)) = \mathfrak{I}^\alpha_t\left(\frac{\Gamma(1-\beta)}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}u^n(s)ds\right) = \Gamma(1-\beta)\mathfrak{I}^\alpha_t\mathfrak{I}^{1-\beta}_t u^n(t)$$
$$= \Gamma(1-\beta)\mathfrak{I}^{\alpha-\beta+1}_t u^n(t) = \frac{\Gamma(1-\beta)}{\Gamma(\alpha-\beta+1)}\int_0^t (t-s)^{\alpha-\beta}u^n(s)ds. \tag{3.2}$$

Substituting, (3.2) into (3.1), the Eq. (1.1) becomes

$$u(t)=\sum_{i=0}^{\lceil\alpha\rceil-1}u_0^{(i)}\frac{t^i}{i!}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p(s)u(s)ds$$
$$+\frac{\Gamma(1-\beta)}{\Gamma(\alpha-\beta+1)}\int_0^t(t-s)^{\alpha-\beta}u^n(s)ds. \tag{3.3}$$

Now, for the sake of applying the theory of orthogonal Chebyshev polynomials, using the following change of variables

$$\begin{cases} s=\frac{T}{2}\zeta+\frac{T}{2} & -1\le\zeta\le\eta,\\ t=\frac{T}{2}\eta+\frac{T}{2} & -1\le\eta\le 1,\end{cases}$$

the Eq. (3.3) becomes

$$\tilde{u}(\eta)=\tilde{h}_1(\eta)+\tilde{h}_2(\eta)+\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-1}\tilde{p}(\zeta)\tilde{u}(\zeta)d\zeta+\gamma\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-\beta}\tilde{u}^n(\zeta)d\zeta, \tag{3.4}$$

where

$$\begin{cases} \tilde{u}(\eta)=u(\frac{T}{2}\eta+\frac{T}{2}),\\ \tilde{h}_1(\eta)=\sum_{i=0}^{\lceil\alpha\rceil-1}u_0^{(i)}\frac{\left(\frac{T}{2}\eta+\frac{T}{2}\right)^i}{i!},\\ \tilde{h}_2(\eta)=\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-1}\tilde{g}(\zeta)d\zeta,\\ \tilde{g}(\zeta)=\frac{1}{\Gamma(\alpha)}(\frac{T}{2})^{\alpha}g(\frac{T}{2}\zeta+\frac{T}{2}),\\ \tilde{p}(\zeta)=\frac{1}{\Gamma(\alpha)}(\frac{T}{2})^{\alpha}p(\frac{T}{2}\zeta+\frac{T}{2}),\\ \gamma=\frac{\Gamma(1-\beta)}{\Gamma(\alpha-\beta+1)}(\frac{T}{2})^{\alpha-\beta+1},\\ \tilde{u}^n(\zeta)=u^n(\frac{T}{2}\zeta+\frac{T}{2}).\end{cases}$$

Using the Lagrange interpolating polynomial, we can approximate $\tilde{u}(\zeta)$ as

$$I_M^C(\tilde{u};\zeta)=\sum_{j=0}^{M}l_j(\zeta)\tilde{u}(x_j), \tag{3.5}$$

where

$$l_i(\eta)=\prod_{j=0,j\neq i}^{M}\frac{\eta-x_j}{x_i-x_j},\ i=0(1)M.$$

and $\left\{x_j=-\cos\left(\frac{(2j+1)\pi}{2M+2}\right)\right\}_{j=0}^{M}$ are the Chebyshev Gauss quadrature points.

Now, we define

$$V_{ij}=\int_{-1}^{x_i}(x_i-\zeta)^{\alpha-1}l_j(\zeta)d\zeta,\ i,j=0(1)M,$$

$$W_{ij}=\int_{-1}^{x_i}(x_i-\zeta)^{\alpha-\beta}l_j(\zeta)d\zeta,\ i,j=0(1)M.$$

To approximate the integral terms of (3.4), we use

$$\tilde{u}(\eta)=\tilde{h}_1(\eta)+\tilde{h}_2(\eta)+\sum_{j=0}^{M}V_{ij}\tilde{p}(x_j)\tilde{u}(x_j)+\gamma\sum_{j=0}^{M}W_{ij}\tilde{u}^n(x_j). \tag{3.6}$$

Now, similar to the Nyström method on grid points $x_i,\ i = 0(1)M$, we have

$$\tilde{u}(x_i) = \tilde{h}_1(x_i) + \tilde{h}_2(x_i) + \sum_{j=0}^{M} V_{ij}\tilde{p}(x_j)\tilde{u}(x_j) + \gamma \sum_{j=0}^{M} W_{ij}\tilde{u}^n(x_j). \tag{3.7}$$

The above system, is a system of $(M+1)$ nonlinear equations with $(M+1)$ number of unknowns. Now, by solving this system by Newton method, we get the values of $\tilde{u}(x_j)$. Substituting these values into the Eq.(3.6), for all $\eta \in [-1,1]$, the values of $\tilde{u}(\eta)$ are obtained.

## 4 Over estimating error for convergence

This section devoted to determine over estimate for $\|e_1\|_{L^2_{\omega^C}(-1,1)}$ using the considered contents in the section 2.

**Theorem 1.** *Consider the weakly singular nonlinear fractional integro-differential equation (1.1) and (3.4). Let $e_1(\eta) = I_M^C(\tilde{u}(\eta)) - \tilde{u}(\eta)$, then the following estimates are hold:*

$$\begin{aligned}
\|e_1\|_{L^2_{\omega^C}(-1,1)} &\le CM^{-m}\left(|\tilde{h}_1|_{H^{m;M}_{\omega^C}(-1,1)} + |\tilde{h}_2|_{H^{m;M}_{\omega^C}(-1,1)} + |\tilde{u}|_{H^{m;M}_{\omega^C}(-1,1)} + |\tilde{u}^n|_{H^{m;M}_{\omega^C}(-1,1)}\right)\\
&+ CM^{-k-m+\frac{1}{2}}\log(M+1)\left(|\tilde{u}|_{H^{m;M}_{\omega^C}(-1,1)} + |\tilde{u}^n|_{H^{m;M}_{\omega^C}(-1,1)}\right)\\
&+ CM^{-k}\log(M+1)\left(\|\tilde{u}\|_{L^\infty} + \|\tilde{u}^n\|_{L^\infty}\right)\\
&+ CM^{\frac{1}{2}-m}\log M\mathcal{H}_{\tilde{u}}.
\end{aligned} \tag{4.1}$$

*provided that $M$ is sufficiently large and*

$$\mathcal{H}_{\tilde{u}} = \max_{0\le i\le M}\left\{(x_i+1)^\alpha B(\alpha,1)\left(|\tilde{p}(\zeta)\tilde{u}(\zeta)|_{H^{m;M}_{\omega^C}(-1,1)} + \|\tilde{p}(\zeta)\|_{L^\infty}|\tilde{u}|_{H^{m;M}_{\omega^C}(-1,1)}\right)\right\}. \tag{4.2}$$

*Proof.* The Eq.(3.7) can be rewritten as follows

$$\begin{aligned}
\tilde{u}(x_i) &= \tilde{h}_1(x_i) + \tilde{h}_2(x_i) + \int_{-1}^{x_i}(x_i-\zeta)^{\alpha-1}\tilde{p}(\zeta)e_1(\zeta)d\zeta\\
&+ \gamma\int_{-1}^{x_i}(x_i-\zeta)^{\alpha-\beta}e_n(\zeta)d\zeta + \int_{-1}^{x_i}(x_i-\zeta)^{\alpha-1}\tilde{p}(\zeta)\tilde{u}(\zeta)d\zeta\\
&+ \gamma\int_{-1}^{x_i}(x_i-\zeta)^{\alpha-\beta}\tilde{u}^n(\zeta)d\zeta + M_1(x_i) + M_2(x_i).
\end{aligned} \tag{4.3}$$

where

$$M_1(x_i) = \sum_{j=0}^{M} V_{ij}\tilde{p}(x_j)\tilde{u}(x_j) - \int_{-1}^{x_i}(x_i-\zeta)^{\alpha-1}\tilde{p}(\zeta)I_M^C(\tilde{u}(\zeta))d\zeta, \tag{4.4}$$

$$M_2(x_i) = \gamma\sum_{j=0}^{M} W_{ij}\tilde{u}^n(x_j) - \gamma\int_{-1}^{x_i}(x_i-\zeta)^{\alpha-\beta}I_M^C(\tilde{u}^n(\zeta))d\zeta, \tag{4.5}$$

such that from (3.5), we have $M_2(x_i) = 0$.

Multiplying (4.3) by $l_j(\eta)$ and sum up from 0 to $M$ and considering $e_n(\eta) = I_M^C(\tilde{u}^n(\eta)) - \tilde{u}^n(\eta)$, we get

$$\begin{aligned} I_M^C\left(\tilde{u}(\eta)\right) &= I_M^C\left(\tilde{h}_1(\eta)\right) + I_M^C\left(\tilde{h}_2(\eta)\right) + I_M^C\left(\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-1}\tilde{p}(\zeta)e_1(\zeta)d\zeta\right) \\ &+ I_M^C\left(\gamma\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-\beta}e_n(\zeta)d\zeta\right) + I_M^C\left(\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-1}\tilde{p}(\zeta)\tilde{u}(\zeta)d\zeta\right) \\ &+ I_M^C\left(\gamma\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-\beta}\tilde{u}^n(\zeta)d\zeta\right) + I_M^C\left(M_1(\eta)\right). \end{aligned} \tag{4.6}$$

Subtracting (4.6) from the Eq.(3.4), we have

$$\begin{aligned} e_1(\eta) &= D_1 + D_2 + I_M^C\left(\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-1}\tilde{p}(\zeta)e_1(\zeta)d\zeta\right) + I_M^C\left(\gamma\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-\beta}e_n(\zeta)d\zeta\right) \\ &+ D_3 + D_4 + I_M^C\left(M_1(\eta)\right). \end{aligned} \tag{4.7}$$

where

$$\begin{cases} D_1 = I_M^C\left(\tilde{h}_1(\eta)\right) - \tilde{h}_1(\eta), \\ D_2 = I_M^C\left(\tilde{h}_2(\eta)\right) - \tilde{h}_2(\eta), \\ D_3 = I_M^C\left(\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-1}\tilde{p}(\zeta)\tilde{u}(\zeta)d\zeta\right) - \int_{-1}^{\eta}(\eta-\zeta)^{\alpha-1}\tilde{p}(\zeta)\tilde{u}(\zeta)d\zeta, \\ D_4 = I_M^C\left(\gamma\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-\beta}\tilde{u}^n(\zeta)d\zeta\right) - \gamma\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-\beta}\tilde{u}^n(\zeta)d\zeta. \end{cases} \tag{4.8}$$

The Eq.(4.7) can be rewritten as follows

$$\begin{aligned} e_1(\eta) &= D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + \int_{-1}^{\eta}(\eta-\zeta)^{\alpha-1}\tilde{p}(\zeta)e_1(\zeta)d\zeta \\ &+ \gamma\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-\beta}e_n(\zeta)d\zeta + I_M^C\left(M_1(\eta)\right). \end{aligned} \tag{4.9}$$

where

$$D_5 = I_M^C\left(\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-1}\tilde{p}(\zeta)e_1(\zeta)d\zeta\right) - \int_{-1}^{\eta}(\eta-\zeta)^{\alpha-1}\tilde{p}(\zeta)e_1(\zeta)d\zeta, \tag{4.10}$$

$$D_6 = I_M^C\left(\gamma\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-\beta}e_n(\zeta)d\zeta\right) - \gamma\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-\beta}e_n(\zeta)d\zeta \tag{4.11}$$

In the following, in order to simplify the expressions, we use $\|.\|$ instead of $\|.\|_{L^2_{\omega^C}(-1,1)}$. Now, from the Eq.(4.9), we have

$$\begin{aligned} &\|e_1(\eta)\| \le \|D_1\| + \|D_2\| + \|D_3\| + \|D_4\| + \|D_5\| + \|D_6\| \\ &+ \|\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-1}\tilde{p}(\zeta)e_1(\zeta)d\zeta\| + \|\gamma\int_{-1}^{\eta}(\eta-\zeta)^{\alpha-\beta}e_n(\zeta)d\zeta\| \\ &+ \|I_M^C\left(M_1(\eta)\right)\|. \end{aligned} \tag{4.12}$$

In order to determine over estimate for $\|e_1(\eta)\|$, it,s sufficient that we obtain the over estimate for terms of the right hand side of the inequality (4.12).

Using inequality (2.6) for $\|D_1\|$, we have

$$\|D_1\| \le CM^{-m}|\tilde{h}_1|_{H^{m;M}_{\omega^C}(-1,1)}, \tag{4.13}$$

and similarly

$$\|D_2\| \leq CM^{-m}|\tilde{h}_2|_{H^{m;M}_{\omega^C}(-1,1)}. \tag{4.14}$$

Using generalized Hardy's inequality (Lemma (5) from [12]) and (2.6), we have

$$\| \int_{-1}^{\eta} (\eta-\zeta)^{\alpha-1}\tilde{p}(\zeta)e_1(\zeta)d\zeta\| \leq C\|\tilde{p}(\zeta)e_1(\zeta)\| \leq C\|\tilde{p}(\zeta)\|\|e_1(\zeta)\| \leq CM^{-m}|\tilde{u}|_{H^{m;M}_{\omega^C}(-1,1)}, \tag{4.15}$$

and

$$\|\gamma \int_{-1}^{\eta} (\eta-\zeta)^{\alpha-\beta}e_n(\zeta)d\zeta\| \leq C\|e_n(\zeta)\| \leq CM^{-m}|\tilde{u}^n|_{H^{m;M}_{\omega^C}(-1,1)}. \tag{4.16}$$

In this position, we need to recall and setting some notations. The notation $C^{r,k}([0,T])$ indicate the space of functions whose $r$-th derivatives are Holder continuous with power $k$, possessed with the norm $\|\cdot\|_{r,k}$ (defined in Lemma 1) for $r \geq 0$ and $k \in [0,1]$.

We know that $I^C_M$ is the interpolation operator, so

$$I^C_M\mathcal{Y}(z) = \mathcal{Y}(z), \tag{4.17}$$

where $\mathcal{Y}(z) \in \mathbb{P}_M$.

Considering $\tau_M$:$C^{r,k}([0,T]) \mapsto \mathbb{P}_M$ as a linear operator and setting $\mathcal{W}e_1 = \int_{-1}^{\eta}(\eta-\zeta)^{\alpha-1}\tilde{p}(\zeta)e_1(\zeta)d\zeta$, we have

$$\begin{aligned}
&\|D_5\| = \|I^C_M\mathcal{W}e_1 - \mathcal{W}e_1\| = \|I^C_M\mathcal{W}e_1 - I^C_M(\tau_M\mathcal{W}e_1) + \tau_M\mathcal{W}e_1 - \mathcal{W}e_1\| \\
&\leq \|I^C_M\|\|\mathcal{W}e_1 - \tau_M\mathcal{W}e_1\| + \|\tau_M\mathcal{W}e_1 - \mathcal{W}e_1\| \\
&\leq (\|I^C_M\|_{L^\infty} + 1)\|\mathcal{W}e_1 - \tau_M\mathcal{W}e_1\|_{L^\infty},
\end{aligned} \tag{4.18}$$

Using Lemma 4 and inequality (16) from [12] for $r = 0$ and Lemmas 1 and 3, we get the following bound for (4.18)

$$\begin{aligned}
&\|D_5\| \leq C_{0,k}(\log M + 1)M^{-k}\|\mathcal{W}e_1\|_{0,k} \leq C\log(M+1)M^{-k}\|e_1\|_{L^\infty} \\
&\leq C\log(M+1)M^{-k+\frac{1}{2}-m}|\tilde{u}|_{H^{m;M}_{\omega^C}(-1,1)}.
\end{aligned} \tag{4.19}$$

Using a similar process to get (4.19), we get

$$\|D_3\| \leq C_{0,k}(\log M + 1)M^{-k}\|\mathcal{W}\tilde{u}\|_{0,k} \leq C\log(M+1)M^{-k}\|\tilde{u}\|_{L^\infty}. \tag{4.20}$$

Also, similarly, considering $\mathcal{V}e_n = \int_{-1}^{\eta}(\eta-\zeta)^{\alpha-\beta}e_n(\zeta)d\zeta$ and $\mathcal{V}\tilde{u}^n = \int_{-1}^{\eta}(\eta-\zeta)^{\alpha-\beta}\tilde{u}^n(\zeta)d\zeta$, respectively, we get

$$\begin{aligned}
&\|D_6\| \leq C_{0,k}(\log M + 1)\gamma M^{-k}\|\mathcal{V}e_n\|_{0,k} \\
&\leq C\log(M+1)M^{-k}\|e_n\|_{L^\infty} \\
&\leq C\log(M+1)M^{-k+\frac{1}{2}-m}|\tilde{u}^n|_{H^{m;M}_{\omega^C}(-1,1)},
\end{aligned} \tag{4.21}$$

and

$$\|D_4\| \leq C_{0,k}\log(M+1)\gamma M^{-k}\|\mathcal{V}\tilde{u}^n\|_{0,k}$$

$$C\log(M+1)M^{-k}\|\tilde{u}^n\|_{L^\infty}. \tag{4.22}$$

Now, using Lemma 4, we estimate $I_M^C(M_1(\eta))$ as

$$\|I_M^C(M_1(\eta))\| \le \max_{0\le i\le M}|M_1(x_i))|\|I_M^C\|_{L^\infty} \le \max_{0\le i\le M}|M_1(x_i))|\log M, \tag{4.23}$$

On the other hand

$$\begin{aligned}
&|M_1(x_i)| = |\int_{-1}^{x_i}(x_i-\zeta)^{\alpha-1}I_M^C(\tilde{p}(\zeta)\tilde{u}(\zeta))d\zeta - \int_{-1}^{x_i}(x_i-\zeta)^{\alpha-1}\tilde{p}(\zeta)I_M^C(\tilde{u}(\zeta))d\zeta| \\
&= |\int_{-1}^{x_i}(x_i-\zeta)^{\alpha-1}\left(I_M^C(\tilde{p}(\zeta)\tilde{u}(\zeta)) - \tilde{p}(\zeta)\tilde{u}(\zeta) + \tilde{p}(\zeta)\tilde{u}(\zeta) - \tilde{p}(\zeta)I_M^C(\tilde{u}(\zeta))\right)d\zeta| \\
&\le \int_{-1}^{x_i}|(x_i-\zeta)^{\alpha-1}\left(I_M^C(\tilde{p}(\zeta)\tilde{u}(\zeta)) - \tilde{p}(\zeta)\tilde{u}(\zeta) + \tilde{p}(\zeta)\tilde{u}(\zeta) - \tilde{p}(\zeta)I_M^C(\tilde{u}(\zeta))\right)|d\zeta \\
&\le \max_{0\le\zeta\le x_i}|I_M^C(\tilde{p}(\zeta)\tilde{u}(\zeta)) - \tilde{p}(\zeta)\tilde{u}(\zeta) + \tilde{p}(\zeta)\tilde{u}(\zeta) - \tilde{p}(\zeta)I_M^C(\tilde{u}(\zeta))|\,|\int_{-1}^{x_i}(x_i-\zeta)^{\alpha-1}d\zeta|,
\end{aligned} \tag{4.24}$$

where

$$\int_{-1}^{x_i}(x_i-\zeta)^{\alpha-1}d\zeta = (x_i+1)^\alpha B(\alpha,1), \tag{4.25}$$

with

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt, \qquad Re(a)>0, Re(b)>0. \tag{4.26}$$

Using inequality (2.8)

$$\begin{aligned}
&|M_1(x_i)| \le (x_i+1)^\alpha B(\alpha,1)\Big\{\|I_M^C(\tilde{p}(\zeta)\tilde{u}(\zeta)) - \tilde{p}(\zeta)\tilde{u}(\zeta)\|_{L^\infty} + \\
&\|\tilde{p}(\zeta)\left(\tilde{u}(\zeta) - I_M(\tilde{u}(\zeta))\right)\|_{L^\infty}\Big\} \\
&\le (x_i+1)^\alpha B(\alpha,1)\left(CM^{\frac{1}{2}-m}\left(|\tilde{p}(\zeta)\tilde{u}(\zeta)|_{H^{m;M}_{\omega^C}(-1,1)} + \|\tilde{p}(\zeta)\|_{L^\infty}|\tilde{u}|_{H^{m;M}_{\omega^C}(-1,1)}\right)\right).
\end{aligned} \tag{4.27}$$

Thus, we have

$$\|I_M^C(M_1(\eta))\| \le CM^{\frac{1}{2}-m}\log M\mathcal{H}_{\tilde{u}} \tag{4.28}$$

where

$$\mathcal{H}_{\tilde{u}} = \max_{0\le i\le M}\Big\{(x_i+1)^\alpha B(\alpha,1)\left(|\tilde{p}(\zeta)\tilde{u}(\zeta)|_{H^{m;M}_{\omega^C}(-1,1)} + \|\tilde{p}(\zeta)\|_{L^\infty}|\tilde{u}|_{H^{m;M}_{\omega^C}(-1,1)}\right)\Big\}. \tag{4.29}$$

Combining the above estimates and (4.12), the intended error estimate (4.1) is achieved. □

# 5 Numerical examples

We now present some numerical examples to exhibit the validity of the proposed numerical method. We use the presented method based on the shifted Chebyshev-Gauss quadrature points as the grid points on $[0,1]$. We used Wolfram Mathematica 12.1 software to get the errors which have been done on a laptop with the following specifications:

$$\begin{cases} Processor: \ Intel(R) \ Core(TM)i7-2640M \ CPU \ @ \ 2.80 \ GHz, \\ Installed \ Memory \ (RAM): \ 8GB, \\ System \ Type: \ 64-bit \ Operating \ System, \ x64-based \ processor. \end{cases} \tag{5.1}$$

In order to test the efficiency and accuracy of the presented method, the following examples have been prepared.

**Example 1.** Consider the following equation

$$ {}_0^C D_t^{\alpha} u(t) = g(t) + p(t)u(t) + \int_0^t \frac{1}{(t-s)^{\beta}} u^n(s) ds, \ t \in [0,1],$$

where

$$\begin{cases} n = 1, \\ \alpha = \frac{1}{3}, \\ \beta = \frac{1}{2}, \frac{1}{4}, \\ p(t) = \frac{-32}{35} t^{\frac{1}{2}}, \end{cases}$$

with initial value $u(0) = 0$ and $g(t)$ is chosen such that the exact solution is $u(t) = \sinh(t^{2/3})$.

**Example 2.** Consider the following equation

$$ {}_0^C D_t^{\alpha} u(t) = g(t) + p(t)u(t) + \int_0^t \frac{1}{(t-s)^{\beta}} u^n(s) ds, \ t \in [0,1],$$

where

$$\begin{cases} n = 2, \\ \alpha = 1, \\ \beta = \frac{1}{2}, \frac{8}{10}, \\ p(t) = 0, \end{cases}$$

with initial value $u(0) = 0$ and $g(t)$ is chosen such that the exact solution is $u(t) = t^3$.

**Example 3.** Consider the following equation

$$ {}_0^C D_t^{\alpha} u(t) = g(t) + p(t)u(t) + \int_0^t \frac{1}{(t-s)^{\beta}} u^n(s) ds, \ t \in [0,1],$$

where

$$\begin{cases} n = 2, \\ \alpha = \frac{2}{3}, \\ \beta = \frac{1}{2}, \\ p(t) = t, \end{cases}$$

with initial value $u(0) = 0$ and $g(t)$ is chosen such that the exact solution is $u(t) = t^{\alpha}, t^{\beta}$.

**Example 4.** Consider the following equation

$$ {}_0^C D_t^{\alpha} u(t) = g(t) + p(t)u(t) + \int_0^t \frac{1}{(t-s)^{\beta}} u^n(s)ds, \ t \in [0,1], $$

where

$$ \begin{cases} n = 1, \\ \alpha = \frac{3}{5}, \\ \beta = \frac{1}{2}, \frac{6}{10}, \\ p(t) = t^4, \end{cases} $$

with initial value $u(0) = 2$ and $g(t)$ is chosen such that the exact solution is $u(t) = \exp(t) + 1$.

Table 1 contains the obtained maximum errors by the presented method and their computing time (in seconds) in Example 1 with $\beta = 1/2, \beta = 1/4$. Table 2 incorporates the obtained maximum errors by the proposed method and their computing time in Example 2 with $\beta = 1/2, \beta = 8/10$. Table 3 contains the obtained maximum errors by the presented method and their computing time in Example 3 for $u(t) = t^{2/3}, t^{1/2}$. Table 4 involves the obtained maximum errors by the presented method and their computing time in Example 4 for $\beta = 1/2, 6/10$.

Figs. 1(a) and 1(b) have been plotted for Example 1 to show the behavior of errors for the calculated solutions by the presented method, with $\beta = 1/2, 1/4$, respectively, for $M = 10$. Figs. 2(a) and 2(b) represent the behavior of errors for the calculated solutions by the presented method, for Example 2, with $\beta = 1/2, 8/10$, respectively, for $M = 10$. Also, about Example 3, for $M = 10$ we plotted Figs. 3(a) and 3(b) with $u(t) = t^{2/3}, t^{1/2}$, respectively. Figs. 4(a) and 4(b) are correspond to the Example 4 with $\beta = 1/2, 6/10$, respectively, for $M = 10$.

Table 1: The obtained maximum errors $||u - I_M^C(u)||_\infty$ by the presented method and their computing time (in seconds) in Example 1 with $\beta = 1/2, 1/4$.

| $M$ | $\|\|u - I_M^C(u)\|\|_\infty$ *for* $\beta = 1/2$ | *CPU time(s)* | $\|\|u - I_M^C(u)\|\|_\infty$ *for* $\beta = 1/4$ | *CPU time(s)* |
|---|---|---|---|---|
| 2 | $7.0e-03$ | 0.671875 | $3.4e-03$ | 0.671875 |
| $2^2$ | $1.7e-03$ | 1.20313 | $7.7e-04$ | 1.20313 |
| $2^3$ | $3.0e-04$ | 2.60938 | $1.1e-04$ | 2.625 |
| $2^4$ | $4.4e-05$ | 8.89063 | $1.4e-05$ | 8.78125 |
| $2^5$ | $6.0e-06$ | 41.3906 | $1.6e-06$ | 41.9531 |

Table 2: The obtained maximum errors $||u - I_M^C(u)||_\infty$ by the proposed method and their computing time (in seconds) in Example 2 with $\beta = 1/2, 8/10$.

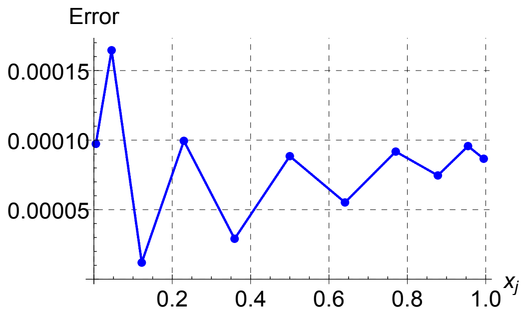
| $M$ | $\|\|u - I_M^C(u)\|\|_\infty$ *for* $\beta = 1/2$ | *CPU time(s)* | $\|\|u - I_M^C(u)\|\|_\infty$ *for* $\beta = 8/10$ | *CPU time(s)* |
|---|---|---|---|---|
| 2 | $1.6e-02$ | 0.15625 | $8.5e-02$ | 0.125 |
| $2^2$ | $3.5e-04$ | 0.203125 | $2.1e-03$ | 0.296875 |
| $2^3$ | $1.11022e-16$ | 0.8125 | $3.33067e-16$ | 0.96875 |
| $2^4$ | $1.11022e-16$ | 3.20313 | $3.33067e-16$ | 3.625 |
| $2^5$ | $2.22045e-16$ | 14.9219 | $3.33067e-16$ | 15.6875 |

Table 3: The obtained maximum errors $||u - I_M^C(u)||_\infty$ by the proposed method and their computing time (in seconds) in Example 3 for $u(t) = t^{2/3}, t^{1/2}$.

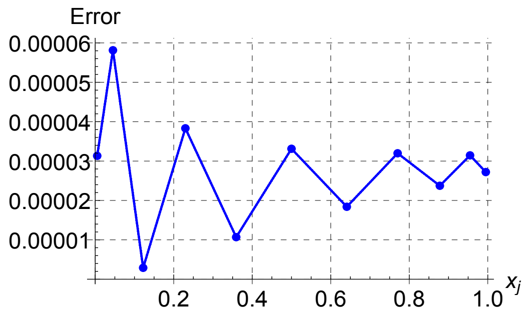
| $M$ | $\|\|u - I_M^C(u)\|\|_\infty$ *for* $u(t) = t^{2/3}$ | *CPU time(s)* | $\|\|u - I_M^C(u)\|\|_\infty$ *for* $u(t) = t^{1/2}$ | *CPU time(s)* |
|---|---|---|---|---|
| 2 | $2.3e-02$ | 0.203125 | $1.2e-02$ | 0.203125 |
| $2^2$ | $1.5e-03$ | 0.40625 | $8.6e-04$ | 0.375 |
| $2^3$ | $8.1e-05$ | 1.35938 | $3.6e-05$ | 1.75 |
| $2^4$ | $3.87e-06$ | 6.79688 | $1.43e-06$ | 6.54688 |
| $2^5$ | $1.67e-07$ | 38.0313 | $5.2e-08$ | 38.4219 |

Table 4: The obtained maximum errors $||u - I_M^C(u)||_\infty$ by the presented method and their computing time (in seconds) in Example 4 for $\beta = 1/2, 6/10$.

| $M$ | $\|\|u - I_M^C(u)\|\|_\infty$ *with* $\beta = 1/2$ | *CPU time(s)* | $\|\|u - I_M^C(u)\|\|_\infty$ *with* $\beta = 6/10$ | *CPU time(s)* |
|---|---|---|---|---|
| 2 | $1.2e-01$ | 0.296875 | $2.1e-01$ | 0.359375 |
| $2^2$ | $2.6e-03$ | 0.828125 | $3.5e-03$ | 0.46875 |
| $2^3$ | $2.5e-08$ | 2.40625 | $3.1e-08$ | 1.98438 |
| $2^4$ | $3.9e-15$ | 7.15625 | $5.3e-15$ | 5.375 |
| $2^5$ | $5.3e-15$ | 33.125 | $5.3e-15$ | 29.7031 |

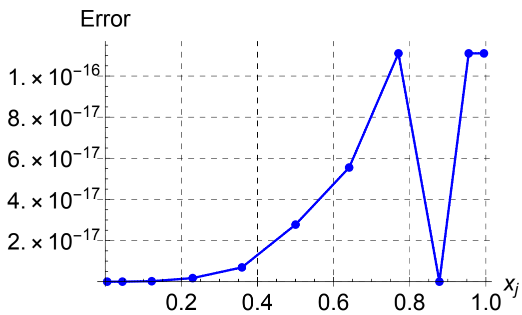



(a) The associated graph with $\beta = 1/2$.

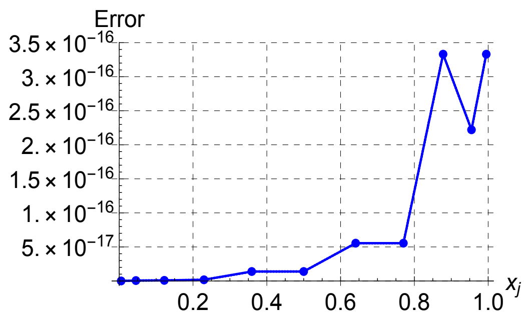



(b) The associated graph with $\beta = 1/4$.

Figure 1: The error behaviors of the presented method for $M = 10$ in Example 1 with $\beta = 1/2, 1/4$.

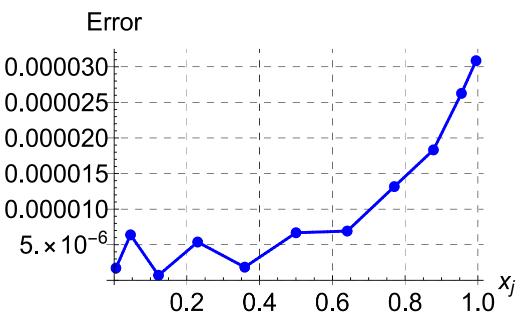



(a) The associated graph with $\beta = 1/2$.

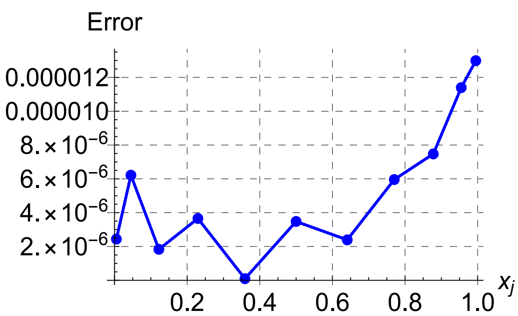



(b) The associated graph with $\beta = 8/10$.

Figure 2: The error behaviors of the presented method for $M = 10$ in Example 2 with $\beta = 1/2, 8/10$.

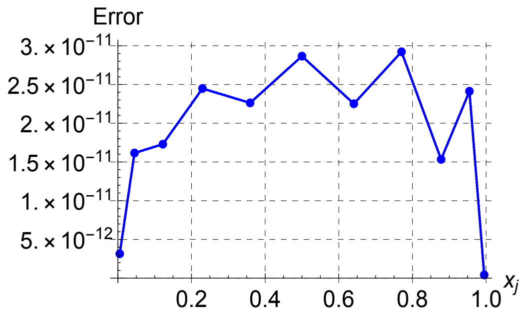



(a) The associated graph with $u(t) = t^{2/3}$.




(b) The associated graph with $u(t) = t^{1/2}$.

Figure 3: The error behaviors of the presented method for $M = 10$ in Example 3 with $u(t) = t^{2/3}, t^{1/2}$.




(a) The associated graph with $\beta = 1/2$.


(b) The associated graph with $\beta = 6/10$.

Figure 4: The error behaviors of the presented method for $M = 10$ in Example 4 with $\beta = 1/2, 6/10$.

# 6 Conclusion

In this paper, we proposed an algorithm to numerical solution of the nonlinear fractional integro-differential equation with a weakly singular kernel and studied the non-smooth behavior of solutions of these equations. Convergence of the method has been investigated by obtaining the over estimate error and an upper bound of error has been provided. The reported numerical results as several tables and figures, show the accuracy and good agreement of the approximate solutions and low CPU time (in seconds) of method. Also, we observed that the presented method is efficient for solving the weakly singular fractional non linear integro-differential equations with smooth and non-smooth solutions. Therefore, the sixth-kind Chebyshev polynomials can be used to numerically solve similar equations.

# 7 Declarations

Funding: This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.
Conflicts of interest/Competing interests: None.

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