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RESEARCH ARTICLE

Controllability of Semilinear Neutral Differential Equations with Impulses and Nonlocal Conditions

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Abstract

When a real-life problem is mathematically modeled by differential equations or another type of equation, there are always intrinsic phenomena that are not taken into account and can affect the behavior of such a model. For example, external forces can abruptly change the model; impulses and delay can cause a breakdown of it. Considering these intrinsic phenomena in the mathematical model makes the difference between a simple differential equation and a differential equation with impulses, delay, and nonlocal conditions. So, in this work, we consider a semilinear nonautonomous neutral differential equation under the influence of impulses, delay, and nonlocal conditions. In this paper we study the controllability of these semilinear neutral differential equations with some of these intrinsic phenomena taking into consideration. Our aim is to prove that the controllability of the associated ordinary linear differential equation is preserved under certain conditions imposed on these new disturbances. In order to achieve our objective, we apply Rothe's fixed point Theorem to prove the exact controllability of the system. Finally, our method can be extended to the evolution equation in Hilbert spaces with applications to control systems governed by PDE's equations.

KEYWORDS:
Controllability of neutral equations; semilinear equations; impulses; nonlocal conditions; Rothe's fixed point theorem

1 | INTRODUCTION

This work is devoted to prove that, under certain conditions on the nonlinear terms, the controllability of the associated ordinary differential equation to a semilinear neutral differential equations with impulses, delay and nonlocal conditions is robust. To be more specific, in this paper we give a sufficient condition for the exact controllability of the following neutral differential equation with impulses, delay and nonlocal conditions

$$\begin{cases} \dfrac{d}{dt}\left[z(t) - f_{-1}(t, z_t)\right] = A_0(t)z(t) + B(t)u(t) + f_1(t, z_t, u(t)), t \neq t_k,\ t \in [0, \tau] \\ z(\theta) + h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(\theta) = \eta(\theta),\ \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)),\ k = 1, \dots, p, \end{cases} \tag{1}$$

where $z(t_k^+) = \lim_{t \to t_k^+} z(t)$, $z(t_k^-) = \lim_{t \to t_k^-} z(t)$, $A_0(t)$, and $B(t)$ are continuous matrices of dimension $n \times n$ and $n \times m$, respectively; the functions f_{-1}, f_1, and h are smooth enough; $0 < t_1 < t_2 < \cdots < t_p < \tau$, $0 < \tau_1 < \tau_2, \cdots < \tau_q < r < \tau$, and the control function u belongs to the space $C([0, \tau]; \mathbb{R}^m)$. Here, z_t stands as the function $z_t : [-r, 0] \longrightarrow \mathbb{R}^n$, $z_t(\theta) = z(t + \theta)$, and $\eta \in \mathcal{PW}_r$, where $\mathcal{PW}_r$ is a natural Banach space defined as

$$\mathcal{PW}_r = \Big\{ \eta : [-r, 0] \longrightarrow \mathbb{R}^n \ \Big|\ \eta \text{ is continuous except at the points } \theta_k,\ k = 1, \dots, p, \\ \text{where the one-sided limits } \eta(\theta_k^-),\ \eta(\theta_k^+) \text{ exist and } \eta(\theta_k^+) = \eta(\theta_k) \Big\},$$

and endowed with the norm

$$\|\eta\|_r = \sup_{\theta \in [-r, 0]} \|\eta(\theta)\|.$$

This work has been motivated by the fact that neutral equations can be seen as a perturbation on the derivative of an ordinary differential equation. It will confirm the conjecture that impulses or abrupt changes, delays, and nonlocal conditions on a system are intrinsic phenomena. This means that under certain conditions, they do not destroy some properties of the original system, such as controllability,[1,2,3] which is the objective of this work.

2 | PRELIMINARIES

In this section, we present some notations to be used through this work and define the Banach space where the solutions of problem (1) will take place. After that, we characterize the exact controllability for the associated ordinary linear system to the system (1).

We begin with the definition of a natural Banach spaces where this type of problems can be set.

$$\mathcal{PW}_{t_1..t_p} = \Big\{ z : [0, \tau] \to \mathbb{R}^n \ \Big|\ z \text{ is continuous except at the points } t_k,\ k = 1, \dots, p, \\ \text{where the one-sided limits } z(t_k^-), z(t_k^+) \text{ exist and } z(t_k^+) = z(t_k) \Big\}$$

equipped with the supremum norm, and

$$\mathcal{PW}_{\tau} = \left\{ \eta : [-r, \tau] \longrightarrow \mathbb{R}^n \;\middle|\; \eta|_{[-r,0]} \in \mathcal{PW}_r \text{ and } \eta|_{[0,\tau]} \in \mathcal{PW}_{t_1..t_p} \right\}$$

equipped with the norm

$$\|\eta\|_{\tau} = \sup_{\theta\in[-r,\tau]} \|\eta(\theta)\|_{\mathbb{R}^n}.$$

We will also consider

$$\mathbb{R}^{qn} = \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{q-\text{times}}$$

endowed with the norm

$$\|y\|_q = \sum_{i=1}^{q} \|y_i\|_{\mathbb{R}^n}.$$

Analogously, we define the Banach space

$$\mathcal{PW}_{rq} = \Big\{ \eta : [-r, 0] \longrightarrow \mathbb{R}^{qn} \;\Big|\; \eta \text{ is continuous except at the points } \theta_k,\ k = 1, \dots, p, \\ \text{where the one-sided limits } \eta(\theta_k^-), \eta(\theta_k^+) \text{ exist and } \eta(\theta_k^+) = \eta(\theta_k) \Big\}$$

endowed with the norm

$$\|\eta\|_{rq} = \sup_{\theta\in[-r,0]} \|\eta(\theta)\|_q = \sup_{\theta\in[-r,0]} \left(\sum_{i=1}^{q} \|\eta_i(\theta)\|_{\mathbb{R}^n} \right).$$

The functions involving system (1) are considered in the following spaces:

$$f_{-1} : [0, \tau] \times \mathcal{PW}_r \longrightarrow \mathbb{R}^n, \qquad f_1 : [0, \tau] \times \mathcal{PW}_r \times \mathbb{R}^m \longrightarrow \mathbb{R}^n,$$

$$h : \mathcal{PW}_{rq} \longrightarrow \mathcal{PW}_r, \qquad J_k : [0, \tau] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

To our knowledge, the study of the control system governed by non-autonomous semi-linear neutral equations is limited, unlike the autonomous linear neutral differential equations where there are several works. There is even an algebraic condition for the controllability of such systems that extends the well-known Kalman's condition for autonomous systems of linear ordinary differential equations.[4,5,6,7,8] However, for semilinear neutral equations, the literature is not broad. There are few works on the existence of solutions. Hernández and Pierri[9] studied an abstract neutral differential equations with state-dependent delay is , and Nieto and Tisdell[10] discussed the exact controllability of first-order impulsive differential equations. Recently, Malik and Kumar[3] investigated the controllability of neutral differential equation with impulses on time scales. As far as we know, this is the first time that the controllability of neutral equations with impulses and nonlocal conditions simultaneously has been studied. A recently work[11] proved the existence of solutions for impulsive time-varying neutral differential equations with impulses and nonlocal conditions. This reveals the novelty of our paper, which shows that neutral differential equations are just perturbations of ordinary differential equations from a controllability point of view.

2.1 | Characterization of the Linear System

Corresponding to the neutral semilinear system (1), we consider the linear system of ordinary differential equations

$$z'(t) = A_0(t)z(t) + B(t)u(t), \quad t \in (0, \tau]. \tag{2}$$

We suppose that the reader is familiar with the concept of exact controllability, however, for the sake of completeness, we will recall the adapted definition of exact controllability of (1).

Definition 1 (Exact controllability)**.** The system (1) is said to be exactly controllable on $[0, \tau]$ if for every $\eta \in \mathcal{PW}_r$ and $z^1 \in \mathbb{R}^n$, there exists a control $u \in C([0, \tau]; \mathbb{R}^m)$ such that the corresponding solution z of (1) satisfies

$$z(0) + h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) = \eta(0) \quad \text{and} \quad z(\tau) = z^1.$$

In order to state the well known characterizations of the controllability of system (2), we note that for all $z^0 \in \mathbb{R}^n$ and $u \in C([0, \tau]; \mathbb{R}^m)$ the initial value problem

$$\begin{cases} z'(t) = A_0(t)z(t) + B(t)u(t), \quad t \in (0, \tau], \\ z(0) = z^0 \end{cases} \tag{3}$$

admits only one solution given by

$$z(t) = E(t, 0)z^0 + \int_0^t E(t, \theta)B(\theta)d\theta, \quad t \in [0, \tau],$$

where $E(t, \theta) = \Phi(t)\Phi^{-1}(\theta)$ and Φ is the fundamental matrix of the uncontrolled linear system

$$z'(t) = A_0(t)z(t). \tag{4}$$

Since E is continuous in both variables, there exists $M \geq 1$ such that

$$\|E(t, \theta)\| \leq M, \quad \forall t, \theta \in [0, \tau]. \tag{5}$$

Definition 2. Corresponding with the linear system of ordinary differential equation (2), we define the controllability operator $\mathfrak{C} : L_2([0, \tau]; \mathbb{R}^m) \longrightarrow \mathbb{R}^n$ as follows

$$\mathfrak{C}u = \int_0^\tau E(\tau, \theta)B(\theta)u(\theta)d\theta. \tag{6}$$

The adjoint operator $\mathfrak{C}^* : \mathbb{R}^n \longrightarrow L_2([0, \tau]; \mathbb{R}^m)$ of the operator $\mathfrak{C}$ is given by

$$(\mathfrak{C}^* z)(s) = B^*(s)E^*(\tau, s)z, \quad \forall s \in [0, \tau], \quad \forall z \in \mathbb{R}^n, \tag{7}$$

Proposition 1. The systems (2) is controllable on $[0,\tau]$ if, and only if, $\mathrm{Ran}(\mathfrak{C}) = \mathbb{R}^n$.

Also, we define the Gramian operator as follows

$$\mathfrak{W}z = \mathfrak{C}\mathfrak{C}^* z = \int_0^\tau E(\tau,s)B(s)B^*(s)E^*(\tau,s)z\,ds. \tag{8}$$

Next, we shall use the following result from Curtain and Pritchard,[12] and Curtain and Zwart.[13]

Lemma 1. Let Y and Z be Hilbert space, $\mathcal{G} \in L(Y,Z)$ and $\mathcal{G}^* \in L(Z,Y)$ the adjoint operator. Then the following statements hold.

(i) $\mathrm{Ran}(\mathcal{G}) = Z \iff \exists \gamma > 0 : \ \|\mathcal{G}^* z\|_W \geq \gamma \|z\|_Z, \ \ z \in Z.$

(ii) $\overline{\mathrm{Ran}(\mathcal{G})} = Z \iff \ker(\mathcal{G}^*) = \{0\} \iff \mathcal{G}^* \text{ is } 1-1.$

Lemma 2. [14] The following statements are equivalent.

a) $\mathrm{Ran}(\mathfrak{C}) = \mathbb{R}^n$.

b) $\ker(\mathfrak{C}^*) = \{0\}$.

c) $\exists \gamma > 0 : \ \langle \mathfrak{C}\mathfrak{C}^* z, z\rangle > \gamma \|z\|^2$, $z \neq 0$ in $\mathbb{R}^n$.

d) $\exists \mathfrak{W}^{-1} \in L(\mathbb{R}^n)$ ($\mathfrak{W}^{-1}$ is bounded).

e) $B^*(s)E^*(\tau,s)z = 0, \quad \forall s \in [0,\tau] \Rightarrow z = 0.$

Therefore, the operator $\Upsilon : \mathbb{R}^n \to L_2([0,\tau];\mathbb{R}^m)$ defined by

$$\Upsilon z = B^*(\cdot)E^*(\tau,\cdot)\mathfrak{W}^{-1}z = \mathfrak{C}^*(\mathfrak{C}\mathfrak{C}^*)^{-1}z, \tag{9}$$

is called the steering operator and it is a right inverse of $\mathfrak{C}$, in the sense that

$$\mathfrak{C}\Upsilon = I. \tag{10}$$

Moreover,

$$\|\mathfrak{W}^{-1}z\| = \|(\mathfrak{C}\mathfrak{C}^*)^{-1}z\| \leq \gamma^{-1}\|z\|, \quad z \in \mathbb{R}^n, \tag{11}$$

and a control steering system (2) from initial state z^0 to a final state z^1 at time $\tau > 0$ is given by

$$u(t) = B^*(t)E^*(\tau,t)\mathfrak{W}^{-1}(z^1 - E(\tau,0)z^0) = \Upsilon(z^1 - E(\tau,0)z^0)(t), \quad t \in [0,\tau]. \tag{12}$$

Lemma 3. [1] Let S be any dense subspace of $L_2([0,\tau];\mathbb{R}^m)$. Then, system (2) is controllable with control $u \in L_2([0,\tau];\mathbb{R}^m)$ if, and only if, it is controllable with control $u \in S$. i.e.,

$$\mathrm{Ran}(\mathfrak{C}) = \mathbb{R}^n \iff \mathrm{Ran}(\mathfrak{C}|_S) = \mathbb{R}^n,$$

where $\mathfrak{C}|_S$ is the restriction of $\mathfrak{C}$ to S.

Remark 1. According to the previous Lemma, if the system is controllable, it is controllable with control functions in the following dense spaces of $L_2(0,\tau;\mathbb{R}^m)$:

$$S = C([0,\tau];\mathbb{R}^m), \quad \text{and} \quad S = C^\infty([0,\tau];\mathbb{R}^m).$$

Moreover, the operators $\mathfrak{C}$, $\mathfrak{W}$ and Υ are well defined in the space of continuous functions: $\mathfrak{C} : C([0,\tau];\mathbb{R}^m) \longrightarrow \mathbb{R}^n$ by

$$\mathfrak{C}u = \int_0^\tau E(\tau,s)B(s)u(s)ds, \tag{13}$$

and $\mathfrak{C}^* : \mathbb{R}^n \longrightarrow C([0,\tau];\mathbb{R}^m)$ by

$$(\mathfrak{C}^* z)(s) = B^*(s)E^*(\tau,s)z, \quad \forall s \in [0,\tau]. \quad \forall z \in \mathbb{R}^n. \tag{14}$$

Also, the Controllability Gramian operator still the same $\mathfrak{W} : \mathbb{R}^n \to \mathbb{R}^n$

$$\mathfrak{W}z = \mathfrak{C}\mathfrak{C}^* z = \int_0^\tau E(\tau,s)B(s)B^*(s)E^*(\tau,s)zds. \tag{15}$$

Finally, the operators $\Upsilon : \mathbb{R}^n \to C([0,\tau];\mathbb{R}^m)$ defined by

$$\Upsilon z = B^*(\cdot)E^*(\tau,\cdot)\mathfrak{W}^{-1}z = \mathfrak{C}^*(\mathfrak{C}\mathfrak{C}^*)^{-1}z, \tag{16}$$

is a right inverse of $\mathfrak{C}$, in the sense that

$$\mathfrak{C}\Upsilon = I. \tag{17}$$

To conclude this section, we shall state the Rothe's Fixed Point Theorem, which we will use to prove our main theorem

Theorem 1 (Rothe's Fixed Point Theorem[1,2,3,15,16])**.** Let E be a Banach space, and consider $B \subset E$ a convex closed subset containing the zero of E in its interior. Let $\Psi : B \longrightarrow E$ be a continuous function with $\Psi(B)$ relatively compact in E and $\Psi(\partial B) \subset B$. Then there exists $x^* \in B$ such that

$$\Psi(x^*) = x^*.$$

3 | MAIN HYPOTHESES

In this section, we will formulate our main hypotheses to be used in the proof of our principle result. The assumption that the linear control system of ordinary differential equations (2) is exactly controllable on $[0,\tau]$ will be required. We shall also consider the following hypotheses on the nonlinear terms that involve the semilinear system of time dependent neutral differential equations with impulses, delay and nonlocal conditions simultaneously:

(H1) The nonlinear terms are globally Lipschitz. i.e.,

$$\|h(z)-h(w)\|_r \leq L_g\|z-w\|_{rq},\ z,w\in\mathcal{PW}_{rq},$$

$$\left\|f_{-1}(t,\eta)-f_{-1}(t,\psi)\right\|_{\mathbb{R}^n} \leq L_{-1}\|\eta-\psi\|_r,\ \eta,\psi\in\mathcal{PW}_r,\ t\in[0,\tau],$$

$$\left\|f_1(t,\eta,u)-f_1(t,\psi,v)\right\|_{\mathbb{R}^n} \leq L_1\left\{\|\eta-\psi\|_r+\|u-v\|_{\mathbb{R}^m}\right\},\ \eta,\psi\in\mathcal{PW}_r,\ u,v\in\mathbb{R}^{\mathfrak{m}},\ t\in[0,\tau],$$

$$\left\|J_k(t,z)-J_k(t,w)\right\|_{\mathbb{R}^n} \leq d_k\|z-w\|_{\mathbb{R}^n},\ z,w\in\mathbb{R}^{\mathfrak{n}},\ t\in[0,\tau].$$

For all bounded set $\mathfrak{B}$ in $\mathcal{PW}_\tau$ there exists a continuous function $\rho:[0,\tau]\longrightarrow\mathbb{R}_+$ depending on $\mathfrak{B}$ such that $\rho(0)=0$, and for all $z\in\mathfrak{B}$, and $t_2,t_1\in[0,\tau]$ we have that

$$\left\|f_{-1}(t_2,z_{t_2})-f_{-1}(t_1,z_{t_1})\right\|_{\mathbb{R}^n} \leq \rho\left(|t_2-t_1|\right)\|z\|_\tau$$

$$\left\|h(z)(t_2)-h(z)(t_1)\right\|_{\mathbb{R}^n} \leq \rho\left(|t_2-t_1|\right)\|z\|_{rq}.$$

(H2)

$$\left\|f_1(t,\eta,u)\right\|_{\mathbb{R}^n} \leq a_0\|\eta(-r)\|_{\mathbb{R}^n}^{\alpha_0}+\|u\|_{\mathbb{R}^m}^{\beta_0}+c_0,\quad \eta\in\mathcal{PW}_r,\quad t\in[0,\tau]$$

(H3)

$$\left\|J_k(t,z)\right\|_{\mathbb{R}^n} \leq a_k\|z\|_{\mathbb{R}^n}^{\alpha_k}+c_k,\quad k=1,2,\dots,p,\quad z\in\mathbb{R}^n,\quad t\in[0,\tau]$$

(H4)

$$\|h(z)\|_r \leq e\|z\|_{rq}^{\eta_1},\quad z\in\mathcal{PW}_{rq},$$

$$\|f_{-1}(t,\eta)\|_{\mathbb{R}^n} \leq \|\eta(-r)\|_{\mathbb{R}^n}^{\omega_1},\quad \eta\in\mathcal{PW}_r,\quad t\in[0,\tau],$$

where $0\leq\alpha_k<1$, $k=0,1,2,3,\dots,p$, $0\leq\beta_0<1$, $0\leq\eta_1<1$, and $0\leq\omega_1<1$.

Remark 2. Obviously, every bounded and globally Lipschitz function chosen conveniently, satisfies the hypotheses **(H1)**-**(H4)**.

4 | MAIN THEOREM

In this section, we shall prove that the neutral differential equation with impulses and nonlocal conditions is exactly controllable if the system of ordinary differential equations (2) is controllable and the hypotheses **(H1)-(H4)** are satisfied.

Under the above hypotheses, for all $\eta \in \mathcal{PW}_r$ and $u \in C([0,\tau];\mathbb{R}^m)$ the system (1) admits one solution $z(t) = z(t,\eta,u)$ given by[3,11]

$$z(t) = \begin{cases} E(t,0)\big[\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\big] \\ +f_{-1}(t,z_t) + \displaystyle\int_0^t E(t,\theta)\left[A_0(\theta)f_{-1}(\theta,z_\theta) + f_1(\theta,z_\theta,u(\theta))\right] d\theta \\ + \displaystyle\int_0^t E(t,\theta)B(\theta)u(\theta)d\theta + \sum_{0<t_k<t} E(t,t_k)J_k(t_k,z(t_k)), \quad t \in [0,\tau], \\ \eta(t) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(t), \quad t \in [-r,0]. \end{cases} \tag{18}$$

Now, let us suppose for a moment that system (1) is exactly controllable. That is to say, for all $\eta \in \mathcal{PW}_r$ and $z^1 \in \mathbb{R}^n$ there exists $u \in C([0,\tau];\mathbb{R}^m)$ such that the corresponding solution of (1), $z(t) = z(t,\eta,u)$ satisfies

$$z(0) + h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) = \eta(0), \quad z(\tau) = z^1,$$

i.e.,

$$\begin{aligned} z^1 = {} & E(\tau,0)\left[\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\right] \\ & + f_{-1}(\tau,z_\tau) + \int_0^\tau E(\tau,s)\left[A_0(s)f_{-1}(s,z_s) + f_1(s,z_s,u(s))\right] ds \\ & + \int_0^\tau E(\tau,s)B(s)u(s)ds + \sum_{k=1}^q E(\tau,t_k)J_k(t_k,z(t_k)). \end{aligned}$$

Hence

$$\begin{aligned} \mathfrak{C}u = {} & z^1 - E(\tau,0)\left[\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\right] \\ & - f_{-1}(\tau,z_\tau) - \int_0^\tau E(\tau,\theta)\left[A_0(\theta)f_{-1}(\theta,z_\theta) + f_1(\theta,z_\theta,u(\theta))\right] d\theta \\ & - \sum_{k=1}^q E(\tau,t_k)J_k(t_k,z(t_k)). \end{aligned}$$

Then

$$u(t) = B^*(t)E^*(\tau,t)\mathfrak{W}^{-1}\mathscr{L}(z,u),$$

where $\mathscr{L} : \mathcal{PW}_\tau \times C([0,\tau];\mathbb{R}^m) \longrightarrow \mathbb{R}^n$ is given by the following formula

$$\mathscr{L}(z,u) = z^1 - f_{-1}(\tau, z_\tau) - \int_0^\tau E(\tau,\theta)\left[A_0(\theta)f_{-1}(\theta,z_\theta) + f_1(\theta,z_\theta,u(\theta))\right] d\theta -$$
$$E(\tau,0)\left[\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\right]$$
$$-\sum_{k=1}^{q} E(\tau,t_k)J_k(t_k, z(t_k)).$$

Next, we consider the operator $\Omega : \mathcal{PW}_\tau \times C([0,\tau];\mathbb{R}^m) \longrightarrow \mathcal{PW}_\tau \times C([0,\tau];\mathbb{R}^m)$ defined as follows

$$\Omega(z,u) = \left(\Omega_1(z,u), \Omega_2(z,u)\right) = (y, v),$$

where $\Omega_1 : \mathcal{PW}_\tau \times C([0,\tau];\mathbb{R}^m) \longrightarrow \mathcal{PW}_\tau$ and $\Omega_2 : \mathcal{PW}_\tau \times C([0,\tau];\mathbb{R}^m) \longrightarrow C([0,\tau];\mathbb{R}^m)$ are operators defined by:

$$\Omega_1(z,u)(t) = y(t) = \begin{cases} E(t,0)\big[\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) \\ \qquad -f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\big] \\ + \displaystyle\int_0^t E(t,\theta)\left[A_0(\theta)f_{-1}(\theta,z_\theta) + f_1(\theta,z_\theta,u(\theta))\right] d\theta \\ + \displaystyle\int_0^t E(t,\theta)B(\theta)u(\theta)d\theta + f_{-1}(t,z_t) \\ + \displaystyle\sum_{0<t_k<t} E(t,t_k)J_k(t_k, z(t_k)), \quad t \in [0,\tau], \\ \eta(t) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(t), \quad t \in [-r, 0]. \end{cases}$$

and

$$\Omega_2(z,u)(t) = v(t) = B^*(t)E^*(\tau,t)\mathfrak{W}^{-1}\mathscr{L}(z,u), \quad t \in [0,\tau],$$

respectively.

Taking into account the above discussion, the following proposition is now obvious.

Proposition 2. System (1) is controllable if, and only if, the operator Ω has a fixed point, i.e.,

$$\exists (z,u) \in \mathcal{PW}_\tau \times C([0,\tau];\mathbb{R}^m) : \Omega(z,u) = (z,u).$$

Now we are in position to present the main theorem of this paper.

Theorem 2. Suppose conditions **(H1)-(H4)** hold and the linear system (2) is controllable on $[0,\tau]$. Then, the semilinear neutral differential equation (1) is also controllable on $[0,\tau]$. Moreover, for $\eta \in \mathcal{PW}_r$ and $z^1 \in \mathbb{R}^n$ there exists $u \in C([0,\tau];\mathbb{R}^m)$ such that the corresponding solution $z(t) = z(t, \eta, u)$ of (1) satisfies

$$\begin{aligned} z^1 = z(\tau) =& E(\tau, 0)\left[\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\right] \\ &+ f_{-1}(\tau, z_\tau) + \int_0^\tau E(\tau,\theta)\left[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta, u(\theta))\right] d\theta \\ &+ \int_0^\tau E(\tau,\theta)B(\theta)u(\theta)d\theta + \sum_{k=1}^{q} E(\tau, t_k) J_k(t_k, z(t_k)) \end{aligned}$$

and

$$u(t) = B^*(t)E^*(\tau, t)\mathfrak{W}^{-1}\mathscr{L}(z, u), \quad t \in [0, \tau].$$

Proof. The proof of this Theorem will be given by asseverations. After that, we will confirm our main statement.

Asseveration 1. The operator Ω is continuous.

It is enough to prove that the operators Ω_1 and Ω_2 are continuous. On the one hand, we prove the continuity of Ω_1. To this end, we proceed as follows:

For $t \in [0, \tau]$, we get that

$$\left\|\Omega_1(z, u)(t) - \Omega_1(w, v)(t)\right\| \le K_1 \|z - w\| + K_2 \|u - v\|,$$

where

$$\begin{aligned} K_1 &= M\left[L_g + L_{-1}L_g + L_{-1} + \tau L_{-1}\left\|A_0\right\| + L_1\tau + d\right] \\ K_2 &= M\tau\left[L_1 + \|B\|\right] \end{aligned}$$

with $d = \sum_{k=1}^{q} d_k$, $\|B\| = \sup_{\theta\in[0,\tau]} \|B(\theta)\|$, and $\left\|A_0\right\| = \sup_{\theta\in[0,\tau]} \left\|A_0(\theta)\right\|$.

For $t \in [-r, 0]$ we have that

$$\left\|\Omega_1(z, u)(t) - \Omega_1(w, v)(t)\right\| \le L_g \|z - w\|.$$

These two inequalities imply the continuity of Ω_1.

On the other hand the continuity of Ω_2 follows from the continuity of B, E, and $\mathscr{L}$.

Asseveration 2. The operator Ω maps bounded sets of $\mathcal{PW}_\tau \times C([0, \tau]; \mathbb{R}^m)$ into equicontinuous sets of $\mathcal{PW}_\tau \times C([0, \tau]; \mathbb{R}^m)$.

In fact, let D be a bounded set of $\mathcal{PW}_\tau \times C([0,\tau];\mathbb{R}^m)$, and consider the following inequalities:

For $0 < t_1 < t_2 < \tau$ and $(z,u) \in D$, we have that

$$\begin{aligned}
\left\|\Omega_1(z,u)(t_2) - \Omega_1(z,u)(t_1)\right\| \leq & \left\|E(t_2,0) - E(t_1,0)\right\| \left[\|\eta(0)\| + \left\|h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})\right\|\right. \\
& \left. + \left\|f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\right\|\right] \\
& + \int_0^{t_1} \left\|E(t_2,\theta) - E(t_1,\theta)\right\| \|B(\theta)\| \|u(\theta)\| d\theta \\
& + \int_{t_1}^{t_2} \left\|E(t_2,\theta)\right\| \|B(\theta)\| \|u(\theta)\| d\theta \\
& + \rho\left(|t_2 - t_1|\right) \|z\| + \int_0^{t_1} \left\|E(t_2,\theta) - E(t_1,\theta)\right\| \\
& \times \left(\left\|A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta, u(\theta))\right\| d\theta\right) \\
& + \int_{t_1}^{t_2} \left\|E(t_2,\theta)\right\| \left\|A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta, u(\theta))\right\| d\theta \\
& + \sum_{0 < t_k < t_1} \left\|E(t_2,t_k) - E(t_1,t_k)\right\| \left\|J_k(t_k, z(t_k))\right\| \\
& + \sum_{t_1 < t_k < t_2} \left\|E(t_2,t_k)\right\| \left\|J_k(t_k, z(t_k))\right\|.
\end{aligned}$$

For $-r < t_1 < t_2 < 0$, we have that

$$\begin{aligned}
\left\|\Omega_1(z,u)(t_2) - \Omega_1(z,u)(t_1)\right\| \leq & \left\|\eta(t_2) - \eta(t_1)\right\| + \\
& \left\|h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(t_2) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(t_1)\right\| \\
\leq & \left\|\eta(t_2) - \eta(t_1)\right\| + \rho\left(|t_2 - t_1|\right) \|z\|_{\mathcal{PW}_p}.
\end{aligned}$$

Since $\left\|E(t_2,\theta) - E(t_1,\theta)\right\| \to 0, \rho\left(|t_2 - t_1|\right) \to 0$ as $t_1 \to t_2$ and the above inequalities, we obtain that $\Omega_1(D)$ is equicontinuous.

On the other hand, for $0 < t_1 < t_2 < \tau$ and $(z,u) \in D$, the following estimate holds

$$\left\|\Omega_2(z,u)(t_2) - \Omega_2(z,u)(t_1)\right\| \leq \left\|\mathfrak{W}^{-1}\mathscr{L}(z,u)\right\| \left\|B^*(t_2)E^*(\tau,t_2) - B^*(t_1)E^*(\tau,t_1)\right\|$$

Analogously, since $\left\|B^*(t_2)E^*(\tau,t_2) - B^*(t_1)E^*(\tau,t_1)\right\| \to 0$ as $t_2 \to t_1$ and $\mathscr{L}(z,u)$ is bounded in D, we get that $\Omega_2(D)$ is equicontinuous.

Asseveration 3. The set $\Omega(D)$ is relatively compact on $\mathcal{PW}_\tau \times C([0,\tau];\mathbb{R}^m)$.

Indeed, since the functions f_{-1}, f_1, h, and J_k are smooth enough, there exist positive constants M_1, M_2, M_3, M_4, and M_{-1} such that for all $(z,u) \in D$ and all $t \in [-r,\tau]$ we have that

$$\begin{aligned}
\left\| f_{-1}(t, z_t) \right\| &\leq M_{-1} \\
\left\| f_1(t, z_t, u(t)) \right\| &\leq M_1 \\
\left\| \mathfrak{W}^{-1} \mathscr{L}(z,u) \right\| &\leq M_2 \\
\| h(z) \| &\leq M_3 \\
\left\| J_k(t, z(t)) \right\| &\leq M_4.
\end{aligned}$$

Hence $\Omega(D)$ is bounded.

Now, let $\left\{ \varphi_i = (\varphi_{i1}, \varphi_{i2}) : i \in \mathbb{N} \right\}$ be a sequence in $\Omega(D) \subset \mathcal{PW}_\tau \times C([0,\tau];\mathbb{R}^m)$. Since $\left\{ \varphi_{i2} \right\}_{i\in\mathbb{N}}$ is a sequence in $\Omega_2(D) \subset C([0,\tau];\mathbb{R}^m)$, which is uniformly bounded and equicontinuous, we can apply the Arzelà-Ascoli theorem directly to ensure the existence of a convergent subsequence of $\left\{ \varphi_{i2} \right\}_{i\in\mathbb{N}}$ that, without loss of generality, we can keep calling $\left\{ \varphi_{i2} \right\}_{i\in\mathbb{N}}$.

On the other hand, we consider the sequence $\left\{ \varphi_{i1} \right\}_{i\in\mathbb{N}}$, which is in $\Omega_1(D) \subset \mathcal{PW}_\tau$. Since $\Omega_1(D)$ is a uniformly bounded and equicontinuous family, on $[-r, t_1]$, there exists a convergent subsequence $\left\{ \varphi_{i1}^1 \right\}_{i\in\mathbb{N}} \subset \left\{ \varphi_{i1} \right\}_{i\in\mathbb{N}}$ by applying the Arzelà-Ascoli theorem again. Now, consider $\left\{ \varphi_{i1}^1 \right\}_{i\in\mathbb{N}}$ on $[t_1, t_2]$. Then $\left\{ \varphi_{i1}^1 \right\}_{i\in\mathbb{N}}$ has a convergent subsequence $\left\{ \varphi_{i1}^2 \right\}_{i\in\mathbb{N}}$ on $[t_1, t_2]$. Continuing with this process the subsequence $\left\{ \varphi_{i1}^{p+1} \right\}_{i\in\mathbb{N}}$ converges uniformly on each interval $[-r, t_1], [t_1, t_2], \dots, [t_p, \tau]$. Therefore, the subsequence $\left\{ \varphi_i^{p+1} = (\varphi_{i1}^{p+1}, \varphi_{i2}^{p+1}) : i \in \mathbb{N} \right\}$ of $\left\{ \varphi_i \right\}_{i\in\mathbb{N}}$ is uniformly convergent. Hence $\overline{\Omega(D)}$ is compact, i.e., $\Omega(D)$ is relatively compact.

Asseveration 4. The operator Ω satisfies the following condition.

$$\lim_{\|(z,u)\| \to \infty} \frac{|||\Omega(z,u)|||}{|||(z,u)|||} = 0,$$

where

$$|||(z,u)||| = \|z\| + \|u\| = \|z\|_{\mathcal{PW}_\tau} + \|u\|_0,$$

is the norm in the Banach space $\mathcal{PW}_\tau \times C([0,\tau];\mathbb{R}^m)$, with

$$\|u\|_0 = \sup_{t\in[0,\tau]} \|u(t)\|_{\mathbb{R}^m}.$$

From the definition of $\mathscr{L}$, we have that

$$\begin{aligned}
\|\mathscr{L}(z,u)\| \leq & \|z^1\| + \|E(\tau,0)\|\|\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) \\
& - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\| + \|f_{-1}(\tau, z_\tau)\| \\
& + \int_0^\tau \|E(\tau,\theta)\|\|A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta, u(\theta))\| d\theta \\
& + \sum_{k=1}^{q} \|E(\tau, t_k)\|\|J_k(t_k, z(t_k))\|.
\end{aligned}$$

Hypotheses **(H1)**-**(H4)** imply that

$$\begin{aligned}
\|\mathscr{L}(z,u)\| \leq & \|z^1\| + M\|\eta(0)\| + M\left[e\|z\|^{\eta_1} + 2^{\omega_1}\|\eta\|^{\omega_1} + e^{\omega_1}\|z\|^{\omega_1\eta_1}\right] \\
& + \|z\|^{\omega_1} + M\tau\left[\|A_0\|\|z\|^{\omega_1} + a_0\|z\|^{\alpha_0} + \|u\|^{\beta_0} + c_0\right] \\
& + M\sum_{k=1}^{q}\left[a_k\|z\|^{\alpha_k} + c_k\right] \\
\leq & K + M\left[e\|z\|^{\eta_1} + 2^{\omega_1}e^{\omega_1}\|z\|^{\omega_1\eta_1}\right] + \|z\|^{\omega_1} \\
& + M\tau\left[\|A_0\|\|z\|^{\omega_1} + a_0\|z\|^{\alpha_0} + \|u\|^{\beta_0}\right] + M\sum_{k=1}^{q}\left[a_k\|z\|^{\alpha_k}\right],
\end{aligned}$$

where $K = \|z^1\| + M\left[\|\eta(0)\| + 2^{\omega_1}\|\eta\|^{\omega_1} + \tau c_0 + \sum_{k=1}^{q} c_k\right]$. Now, as consequence of (11), we obtain that

$$\left\|\Omega_2(z,u)\right\| \leq \|B^*(t)\|\|E^*(\tau,t)\|\mathfrak{W}^{-1}\mathscr{L}(z,u)\| \leq \|B(t)\|\|E(\tau,t)\|\gamma^{-1}\|\mathscr{L}(z,u)\|.$$

Hence,

$$\begin{aligned}
\left\|\Omega_2(z,u)\right\| \leq & \|B\|M\gamma^{-1}K + \|B\|M^2\gamma^{-1}\left[e\|z\|^{\eta_1} + 2^{\omega_1}e^{\omega_1}\|z\|^{\omega_1\eta_1}\right] + \|B\|M\gamma^{-1}\|z\|^{\omega_1} \\
& + \|B\|M^2\gamma^{-1}\tau\left[\|A_0\|\|z\|^{\omega_1} + a_0\|z\|^{\alpha_0} + \|u\|^{\beta_0}\right] \\
& + \|B\|M^2\gamma^{-1}\sum_{k=1}^{q} a_k\|z\|^{\alpha_k}.
\end{aligned} \tag{19}$$

Likewise,

$$\begin{aligned}
\left\|\Omega_1(z,u)\right\| \leq & M\|\eta(0)\| + M\left[e\|z\|^{\eta_1} + 2^{\omega_1}\|\eta\|^{\omega_1} + 2^{\omega_1}e^{\omega_1}\|z\|^{\omega_1\eta_1}\right] \\
& + \|z\|^{\omega_1} + M\tau\left[\|A_0\|\|z\|^{\omega_1} + a_0\|z\|^{\alpha_0} + \|u\|^{\beta_0} + c_0\right] \\
& + M^2\tau\|B\|^2\gamma^{-1}\|\mathscr{L}(z,u)\| + M\sum_{k=1}^{q}\left[a_k\|z\|^{\alpha_k} + c_k\right] \\
\leq & K_0 + K_1\big(M\|\eta(0)\| + M\left[e\|z\|^{\eta_1} + 2^{\omega_1}\|\eta\|^{\omega_1} + 2^{\omega_1}e^{\omega_1}\|z\|^{\omega_1\eta_1}\right] + \|z\|^{\omega_1} \\
& + M\tau\left[\|A_0\|\|z\|^{\omega_1} + a_0\|z\|^{\alpha_0} + \|u\|^{\beta_0} + c_0\right] + M\sum_{k=1}^{q}\left[a_k\|z\|^{\alpha_k} + c_k\right]\big),
\end{aligned} \tag{20}$$

where $K_0 = M^2\tau\|B\|^2\gamma^{-1}\|z^1\|$ and $K_1 = M^2\tau\|B\|^2\gamma^{-1} + 1$. Let $K_2 = K_1 + \|B\|M\gamma^{-1}$. Then, by (19) and (20),

$$\begin{aligned}
|||\Omega(z,u)||| =& \left\|\Omega_1(z,u)\right\| + \left\|\Omega_2(z,u)\right\| \\
\leq& K_3 + K_4\|z\|^{\omega_1} + K_5\|z\|^{\omega_1\eta_1} + K_6\|z\|^{\eta_1} + \\
& K_7\|z\|^{\alpha_0} + K_8\|u\|^{\beta_0} + K_9\sum_{k=1}^{q} a_k\|z\|^{\alpha_k},
\end{aligned}$$

where

$$K_3 = K_0 + M\left[K_1\left(\|\eta(0)\| + \tau c_0 + \sum_{k=1}^{q} c_k + 2^{\omega_1}\|\eta\|^{\omega_1}\right) + \|B\|\gamma^{-1}K\right],$$

$$K_4 = K_1 + \|B\|M\gamma^{-1} + K_1M\tau\|A_0\| + \|B\|M^2\gamma^{-1}\tau\|A_0\|, \quad K_5 = M2^{\omega_1}e^{\omega_1}K_2,$$

and

$$K_6 = MeK_2, \quad K_7 = M\tau a_0K_2, \quad K_8 = M\tau K_2, \quad K_9 = MK_2.$$

Consequently,

$$\begin{aligned}
\frac{|||\Omega(z,u)|||}{|||(z,u)|||} =& \frac{\left\|\Omega_1(z,u)\right\| + \left\|\Omega_2(z,u)\right\|}{\|z\| + \|u\|} \\
\leq& \frac{K_3}{\|z\| + \|u\|} + K_4\|z\|^{\omega_1-1} + K_5\|z\|^{\omega_1\eta_1-1} + K_6\|z\|^{\eta_1-1} + \\
& K_7\|z\|^{\alpha_0-1} + K_8\|u\|^{\beta_0-1} + K_9\sum_{k=1}^{q} a_k\|z\|^{\alpha_k-1},
\end{aligned}$$

whence

$$\lim_{|||(z,u)|||\to\infty} \frac{|||\Omega(z,u)|||}{|||(z,u)|||} = 0.$$

Asseveration 5. The operator Ω has at least one fixed point.

Actually, by the previous lemma we have that for $0 < \rho < 1$ there exists $R > 0$ such that

$$\frac{|||\Omega(z,u)|||}{|||(z,u)|||} < \rho \quad \text{if} \quad |||(z,u)||| \geq R.$$

Therefore, if $|||(z,u)||| = R$, then $|||\Omega(z,u)||| \leq \rho|||(z,u)||| \leq \rho R < R$. This implies that

$$\Omega\left(\partial B(0,R)\right) \subset B(0,R),$$

where $B(0,R)$ is the closed ball of radius R centered at zero. The foregoing Asseverations 1, 2, 3, and 4 together with the Rothe's fixed theorem 1 allow us to conclude that there exists $(z,u) \in \mathcal{PW}_\tau \times C([0,\tau];\mathbb{R}^m)$ such that

$$\Omega(z,u) = (z,u).$$

By Proposition 2 and Asseveration 5, the system (1) is exactly controllable on $[0, \tau]$. Furthermore,

$$u(t) = B^*(t)E^*(\tau, t)\mathfrak{W}^{-1}\mathscr{L}(z, u)$$

and

$$\begin{aligned}
z^1 =& E(\tau, 0)\left[\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\right] \\
&+ f_{-1}(\tau, z_\tau) + \int_0^\tau E(\tau, \theta)\left[A_0(\theta)f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta, u(\theta))\right] d\theta \\
&+ \int_0^\tau E(\tau, \theta)B(\theta)u(\theta)d\theta + \sum_{k=1}^{q} E(\tau, t_k)J_k(t_k, z(t_k)).
\end{aligned}$$

□

5 | AN EXAMPLE

As an application, in this section we will illustrate our result with an example where Theorem 2 can be applied. In this regard, we consider the following semilinear time dependent neutral control system with impulses, delay and nonlocal condition

$$\begin{cases} \dfrac{d}{dt}\left[z(t) - f_{-1}(t, z_t)\right] = A_0(t)z(t) + B(t)u(t) + f_1(t, z_t, u(t)), t \neq t_k,\ t \in [0, \tau] \\ z(\theta) + h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(\theta) = \eta(\theta),\ \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)),\ k = 1, \dots, p, \end{cases} \tag{21}$$

where $A_0(t) = a(t)A$ and $B(t) = b(t)B$ with A_0 and B, $n \times n$ and $n \times m$ constant matrices, respectively. Here, $a \in L^1[0, \tau]$, $b \in C[0, \tau]$ satisfy

$$\int_0^\tau a(s)ds \neq 0, \qquad b(t) \neq 0, \qquad t \in [0, \tau]$$

From Leiva and Zambrano[17], if the following rank condition holds

$$Rank[B; A_0B; \cdots; A_0^{n-1}B] = n,$$

then the time dependent linear system given by

$$z'(t) = A_0(t)z(t) + B(t)u(t), \qquad t \in [0, \tau]$$

is exactly controllable on $[0, \tau]$. The nonlinear terms and the impulsive functions are given as follows.

- $$\begin{aligned} f_1 \,:\, [0,\tau]\times\mathcal{PW}_r([-r,0];\mathbb{R}^n)\times\mathbb{R}^m &\longrightarrow \mathbb{R}^n \\ (t,\phi,u) &\longmapsto f_1(t,\phi,u) = \begin{pmatrix} \sqrt[3]{\|u\|+1} + \sqrt[3]{\phi_1(-r)} \\ \sqrt[3]{\|u\|+1} + \sqrt[3]{\phi_2(-r)} \\ \vdots \\ \sqrt[3]{\|u\|+1} + \sqrt[3]{\phi_n(-r)} \end{pmatrix}, \end{aligned}$$

- $$\begin{aligned} f_{-1} \,:\, [0,\tau]\times\mathcal{PW}_r([-r,0];\mathbb{R}^n) &\longrightarrow \mathbb{R}^n \\ (t,\phi) &\longmapsto f_{-1}(t,\phi) = \begin{pmatrix} \sqrt[3]{\phi_1(-r)} \\ \sqrt[3]{\phi_2(-r)} \\ \vdots \\ \sqrt[3]{\phi_n(-r)} \end{pmatrix}, \end{aligned}$$

- $$\begin{aligned} h \,:\, \mathcal{PW}_{rq}([-r,0];(\mathbb{R}^n)^q) &\longrightarrow \mathcal{PW}_r([-r,0];\mathbb{R}^n) \\ (\phi_1,\phi_2,\cdots,\phi_q) &\longmapsto h(\phi_1,\phi_2,\cdots,\phi_q) = \sum_{i=1}^{q} \begin{pmatrix} \sin\big(\phi_{i1}\big) \\ \sin\big(\phi_{i2}\big) \\ \vdots \\ \sin\big(\phi_{in}\big) \end{pmatrix}, \end{aligned}$$

- For $k = 1,\dots,p$,

$$\begin{aligned} J_k \,:\, [0,\tau]\times\mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (z,u) &\longmapsto J_k(z,u) = \cos\Big(\sqrt{\|u\|+1}\Big) \begin{pmatrix} \sin\big(z_1^k\big) \\ \sin\big(z_2^k\big) \\ \vdots \\ \sin\big(z_n^k\big) \end{pmatrix}. \end{aligned}$$

Then

$$\|f_1(t,\phi,u)\| \leq \sqrt{n}\|\phi(-r)\|^{1/3} + \sqrt{n}\|u\|^{1/3} + \sqrt{n},$$

$$\|f_{-1}(t,\phi)\| \leq \sqrt{n}\|\phi(-r)\|^{1/3},$$

and since h and J_k, $k = 1,2,\cdots,p$ are bounded, the conditions of Theorem 2 are satisfied. Hence, the system (21) is exactly controllable on $[0,\tau]$.

6 | CONCLUSION AND FINAL REMARK

In this work, we proved that under certain conditions the semi-linear control system of nonautonomous neutral differential equations with impulses and nonlocal conditions is exactly controllable if the associated linear control system of nonautonomous ordinary differential equations is exactly controllable, which was achieved using the uniform continuity of the evolution operator and Rothe's fixed point theorem. In fact, the uniform continuity of the evolution operator helped us to prove the equicontinuity and the uniform boundedness of a family of functions in the cartesian product space of the solutions space and the controls space. By contrast, in infinite-dimensional Banach spaces, the uniform continuity far away from zero of the evolution operator is achieved assuming compactness of the evolution family. This implies that the linear control system governed by the ordinary evolution equation cannot be exactly controllable anymore, only approximately controllable. Hence, only the approximate controllability can be studied. We believe Rothe's fixed point theorem could be used in this case as well.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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