A Spinor Model for Cascading Two-port Scattering Matrices In Conformal Geometric Algebra

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***Abstract*—Building on the work in [1], this paper shows how Conformal Geometric Algebra (CGA) can be used to model an arbitrary two-port scattering matrix as a rotation in four dimensional Minkowski space, known as a spinor. This spinor model plays the role of the wave-cascading matrix in conventional microwave network theory. Techniques to translate two-port scattering matrix in and out of spinor form are given. Once the translation is laid out, geometric interpretations are given to the physical properties of reciprocity, loss, and symmetry and some mathematical groups are identified. Methods to decompose a network into various sub-networks, are given. An example application of interpolating a 2-port network is provided demonstrating an advantage of the spinor model. Since rotations in four dimensional Minkowski space are Lorentz transformations, this model opens up the field of network theory to physicists familiar with relativity, and vice versa.**

Contents

I. Introduction

Two-port networks play an important role in microwave engineering, control theory, quantum mechanics, and several other disciplines. From a modeling standpoint, two-port networks can be thought of as operators or as quantities of interest. Microwave networks are traditionally represented by various matrix formats, such as the scattering (S), impedance (Z), and admittance (Y) matrices. Choosing a given format is equivalent to choosing a basis in which to frame a transformation. Because the different bases have different physical interpretations, one may be more natural for a given problem than another. For example, at high frequencies power is more easily measured than impedance so scattering matrices are generally used. When several two-port networks are cascaded together, matrices such as the wave-cascading (T) and (ABCD) matrix are used, which implements network cascading through matrix multiplication. The cascading matrix algebra has been very successful in impedance matching, filter theory [2], and calibration problems [3], [4].

An alternative to matrices is to use Geometric Algebra (GA) to model two-port networks as spinors. As shown in [1], the fundamental relations of transmission line theory become linearized by using a tool known as Conformal Geometric Algebra (CGA). This construction allows operations such as adding impedance and admittance, or changing line impedance to be implemented with rotations in a four dimensional minkowski space, otherwise known as Lorentz transformations. (The deeper physical reason for this we have not yet determined.) While the work in [1], gave some spinor representations for fundamental circuit elements, this paper presents a method to translate an arbitrary two-port s-matrix into a spinor. While essentially equivalent to the wave-cascading matrix, the CGA spinor approach provides unique geometric insight and basis invariance, neither of which are possible with linear algebra. Using spinors opens up the field of network theory to physicists familiar with Lorentz transformations, and allows techniques to be translated between the two disciplines.

A. Outline

The paper is divided into two main parts. Sections II and III deal with translating scattering matrix representation of two-port networks in and out of CGA spinor representation. This ability is required in order to interface existing infrastructure, and also provides a way to migrate one's understanding into the new CGA formalism. Section IV-B describes the geometry of some special cases of networks, while section V, demonstrates how to decompose two-port networks into sub-networks based on the physical properties of reciprocity, loss, and symmetry. Finally an example application of interpolation is given. In contrast with the conventional characterization based on matrix conditions, each decomposition is given

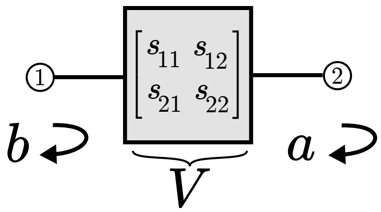


Figure 1: Circuit diagram of an arbitrary two-port highlighting the reflection a and b as seen looking into port 1. The symbol V is used to denote the spinor representation.

a concrete geometric interpretation. This paper does not present a direct geometric interpretation of the S, Z, or Y-matrix itself, nor show how such a model would be related to the wave cascading spinor model presented here, although this is certainly possible.

II. Reciprocal Networks

A. *Geometry of Reflectometry*

Our network spinors are geometric objects which represent the wave cascading matrix. The cascading matrix is useful because it models reciprocal networks as rotations, while the scattering matrix models lossless networks as rotations. Additionally, as it's name implies, cascading networks is accomplished by multiplication. There are several ways to setup the matrix-to-spinor translation. The most direct route is to translate the wave-cascading matrix directly into a spinor, and replace the matrix product with the geometric product. However, we choose to take an engineering approach and assemble our model from the s-matrix elements using physical arguments from reflectometry. While this may seem archaic to mathematicians, it was how our model was developed and may be easier for intuitive thinkers.

The basic problem in reflectometry is that some arbitrary two-port network V is located between a load of interest b and the observable measurement a, as shown in Figure 1. By measuring a set of known loads the two-port network can be sufficiently characterized, thereby allowing unknown loads to be measured. The S-matrix is a commonly used representation for the intervening two-port as it's elements are reflection and transmission coefficients and these quantities are more easily measured than impedance at microwave frequencies. Starting with the conventional formulation [2], the reflection coefficient a is transformed by a two-port network into b, according to the formula,

$$b = s_{11} + \frac{s_{12}s_{21}a}{1-s_{22}a}. \tag{1}$$

Where the s_{ij} are the various elements of the S-matrix for the two-port, and all variables are complex numbers. This formula can be re-arranged as,

$$b = s_{11} + s_{12}s_{21}\left(a^{-1}-s_{22}\right)^{-1}. \tag{2}$$

Writing the equation in this form illustrates how the relation can be broken up into a series of simpler functions, each of which is geometrically interpretable. In sequence: the function $\left(a^{-1}-s_{22}\right)^{-1}$ is known as a transversion, the $s_{12}s_{21}$ term affects a rotation/dilation, and finally, s_{11} performs a translation. Writing this as a sequence of operators,

$$V = T_{s_{11}} D_{|s_{21}s_{21}|} R_{\angle s_{12}s_{21}} K_{-s_{22}}. \tag{3}$$

Where V is the total transformation, and the elementary transformations are represented by a transversion K, rotation R, dilation D, and translation T. The subscripts of each operator correspond to the matrix element, or component thereof, which parameterizes it. For reciprocal networks the transmission coefficients are equal $s_{21} = s_{12}$ and so the dilation factor can be written as $|s_{21}|^2$ with a rotation angle is of $2\angle s_{21}$. The advantage of casting this relationship in operator form is that group theory can be put to use. The decomposition of eq 3 is identical to that of a general conformal transformation as given in [5]. Once this is recognized, much of network theory can be abstracted to group theory. In CGA, each operator is represented by a multivector which enacts a rotation, known as a *rotor*. An un-normalized rotor (meaning squares to something other than 1) is called a spinor. By finding the rotors for the elementary transformations in eq (3), any S-matrix can be converted into a CGA rotor, allowing two-port networks to be analyzed with the CGA framework. Since the rotors for all of the operations in eq (3) are well known and given in the literature, we could just write down the total rotor immediately. However, first we must we give an introduction to CGA to make the notation clear. More information can be found in [1].

B. *CGA*

This section provides a brief introduction to the CGA and notation employed in this paper. Start by representing reflection coefficient as a vector within a plane spanned by two orthonormal vectors e_1 and e_2 of positive signature. These can be thought of as the real and imaginary axes of the complex plane. Next, add a third dimension of positive signature (e_3) and a fourth of negative signature (e_4). The orthonormal vector basis for the conformal space is given by

$$e_1^2 = e_2^2 = e_3^2 = -e_4^2 = 1. \tag{4}$$

The basis generates a geometric algebra containing the following blades:

$$\underbrace{\alpha}_{1-\text{scalar}}, \quad \underbrace{e_i}_{4\text{-vectors}}, \quad \underbrace{e_{ij}}_{6-\text{bivectors}}, \quad \underbrace{e_{ijk}}_{4\text{-trivectors}}, \quad \underbrace{i}_{1\text{-pseudoscalar}} \tag{5}$$

Here the e_{12}-plane is identified as the original 2D space, and e_{34}-plane contains the added dimensions. Due to the signature of the added space, the e_{34}-plane is known as the *Minkowski plane,* which is commonly labeled E_0,

$$E_0 \equiv e_3 \wedge e_4. \tag{6}$$

It is convenient to further define a null basis.

$$e_o = \frac{1}{2}\left(e_4 - e_3\right) \tag{7}$$

$$e_\infty = e_4 + e_3 \tag{8}$$

These two vectors represent the points of infinity and zero, as their subscripts suggest. They have the properties,

$$e_o^2 = e_\infty^2 = 0 \tag{9}$$

$$e_\infty e_o = -1 + E_0. \tag{10}$$

In terms of the null basis, a vector x in the original space of e_{12} is mapped *upwards* to a conformal vector X, by the following.

$$X = \uparrow(x) = x + \frac{1}{2}x^2 e_\infty + e_o \tag{11}$$

The inverse, *downwards* map, is the made by normalizing the conformal vector then rejecting it from the Minkowski plane.

$$x = \downarrow(X) = \frac{X \wedge E_0}{-X \cdot e_\infty} E_0^{-1} \tag{12}$$

In the above formula and all others, we adhere to the convention that the inner and outer products take precedence over the geometric product. Now that the CGA has been laid out, rotors representations for the operations in eq (3) can be expressed.

C. S-Matrix to Rotor

Derived in [5], the CGA rotors for the operators in eq (3) as expressed in the basis defined above are,

$$T_x \equiv e^{\frac{1}{2}e_\infty x} \tag{13}$$

$$K_x \equiv e_2 e^{\frac{1}{2}e_o x} e_2 = e_{23} T_x e_{23} \tag{14}$$

$$D_\rho \equiv e^{-\ln\frac{\rho}{2}E_0} \tag{15}$$

$$R_\theta \equiv e^{-\frac{\theta}{2}e_{12}}. \tag{16}$$

Where the subscripts are either vectors in e_{12} or scalars which parameterize the rotor. The transversion rotor K is slightly different than that given in [5] because the complex inversions in eq (2) add additional reflections, so the K in eq (14) could be called a *complex transversion*. By defining these rotors, we have an explicit formula for translating a S-matrix into a CGA rotor, through eq (3). For implementation it is important to avoid any unnecessary multiplying or division of complex numbers. For example, the rotation operator

$$R_{\angle s_{12}s_{21}} = e^{\frac{\angle s_{12}s_{21}}{2}e_{12}}, \tag{17}$$

is better implemented as,

$$R_{\angle s_{12}s_{21}} = e^{(\angle s_{12} + \angle s_{21})e_{12}}. \tag{18}$$

The reverse procedure of translating a rotor into a S-matrix is discussed next.

D. Rotor to S-matrix

Translating a CGA rotor into a S-matrix is equivalent to finding the reflection and transmission coefficients for the two-port network when the network is terminated in matched impedances. A procedure to accomplish this for reciprocal networks can be developed by employing some concepts from microwave reflectometry. First, we express the relation in Figure (1) in the language of CGA,

$$b = V a \tilde{V}. \tag{19}$$

Where a and b are the up-projected null vectors of the reflection coefficients, and V is the rotor representing two-port network. In microwave terminology, we say that a load a is *embedded* in the twoport V. By definition, s_{11} is the reflection coefficient at port 1 when port 2 is matched, so this quantity is found by letting V act on a match, ie $a = e_o$.

$$s_{11} = \downarrow\left(V e_o \tilde{V}\right) \tag{20}$$

This determines T in (3). Geometrically this corresponds to determining how the origin is translated (also known as the *directivity*). From an operator perspective, determining T is possible because the origin is invariant to operations of K, D and R. It is also obvious by inspecting eq (2). Next, we can determine s_{22} in an identical way by *flipping* the network, and then terminating it with a match. The operation of physically flipping a network, which will exchange port indices's is denoted with an underbar and can be defined,

$$\underline{\mathrm{V}} \equiv e_{14} V^{-1} e_{14}. \tag{21}$$

A proof that this operator does permutes port 1 and 2 is given in Appendix (VIII-A). By using this flip operator, s_{22} is found in a similar way as s_{11},

$$s_{22} = \downarrow\left(\underline{\mathrm{V}} e_o \tilde{\underline{\mathrm{V}}}\right) \tag{22}$$

Which determines K in (3). Once T and K are found, they can be removed from V to leave only the rotation and dilation operators.

$$\tilde{T} V \tilde{K} = \tilde{T} T D R K \tilde{K} = DR \tag{23}$$

The parameters of DR can be found by taking the logarithm as described in [**?**]. Which completes the determination of the S-matrix.

E. Other Bases and Conversions

To convert the wave-cascading matrix to another format, such as ABCD-matrix, a basis transform must take place at some point. With the conventional treatment, two-port networks in different bases are related through a bilinear matrix equation such as ,

$$S = (Z - I)(Z + I)^{-1} \tag{24}$$

Where S and Z are complex NxN matrices containing the scattering and impedance parameters, and I is the identity matrix. While concise, this equation has no geometric interpretation. In contrast, CGA allows different basis representations to be related through simple, geometrically interpretable rotations as demonstrated in [1]. The basis rotors work in the same way for the cascading matrices, such as ABCD or T-matrices. For example, a rotor can be transformed from the reflection and wave-cascading spinor V into the ABCD spinor (A) with a simple $\frac{\pi}{2}$-rotation in the e_{13} plane.

$$A = e^{-\frac{\pi}{4}e_{13}} V e^{\frac{\pi}{4}e_{13}}. \tag{25}$$

The rotors are repeated below, and these can be used to transform a CGA versor to and from the desired basis.

$$R_{zs} \equiv e^{-\frac{\pi}{4}e_{13}} \tag{26}$$

$$R_{zy} \equiv e^{-\frac{\pi}{2}e_{23}} \tag{27}$$

$$R_{sy} = e^{-\frac{\pi}{2\sqrt{2}}(e_{23}+e_{21})} \tag{28}$$

However, since CGA provides a basis invariance such transformations are not as important. Next we extend the model to include non-reciprocal networks.

III. Non-reciprocal Networks

A. The Model

So far we have shown how reciprocal two-port networks can be modeled as rotations in Conformal Geometric Algebra. This sections extends that model to include non-reciprocal networks using whats known as a *duality spinor* [6]. Developing the theory in this way makes sense because; 1) reciprocal networks are far more common, 2) areciprocity has a geometrically distinct interpretation, and 3) because the areciprocity can be easily separated and removed. Any model for areciprocity must be imperceivable from the perspective of reflectometry because the transmission coefficients in eq 1 are not individually observable. As described in part 4 of [6], the transformation of any vector in the Dirac algebra of $G_{1,3}$ can be written,

$$p' = VpS \tag{29}$$

A Lorentz transformation preserves the length of vectors.

$$p'^2 = VpSVpS = p^2 \tag{30}$$

This will hold for any vector p only if SV either commutes or anti-commutes with p. The only grade of elements in $G_{1,3}$ which fulfill this property for any vector are scalars and pseudo-scalars, so we can write,

$$SV = \alpha + \beta i = \rho e^{\theta i}. \tag{31}$$

Where i is the pseudoscalar. Equation (31) implies

$$S = \rho e^{\theta i}\tilde{V}, \tag{32}$$

This form cleanly separates the reciprocal from the areciprocal; the areciprocal part is a duality rotation, while the reciprocal part is a bivector rotation. Using geometrically distinct objects for physically distinct network properties provides insight into the physics of 2-port networks that is difficult to attain with matrix representations. Reflecting on the proposed *duality spinor* model, we note that it has many of the required features of areciprocity; it inverts with the flip operation, it commutes when several two-ports are cascaded together, and it requires two-independent parameters. The duality spinor is the geometric representation of the determinant of the wave cascading matrix.

B. S-matrix Areciprocity to spinor

To convert a non-reciprocal s-matrix into a spinor, first compute V by way of equation (3). This requires the parameters s_{11}, s_{22}, and the product $s_{12}s_{21}$. Next, the duality spinor P is determined by the complex ratio of s_{12} to s_{21},

$$P = \sqrt{\rho}e^{\frac{\theta}{2}i}. \tag{33}$$

Where

$$\rho = |\frac{s_{12}}{s_{21}}| \tag{34}$$

$$\theta = \angle\frac{s_{12}}{s_{21}} \tag{35}$$

Once P is found the total non-reciprocal versor is created by multiplying V with P.

$$A = PV \tag{36}$$

The half-angle $\frac{\theta}{2}$ and $\sqrt{\rho}$ are used because we choose to implement the duality spinor in a double-sided formula. This way we just keep track of A, instead of P and V separately (but this would work too). Converting a duality spinor to a complex number is done by reversing this procedure. This requires a technique to separate the duality spinor from the total spinor, which is given in the next section.

C. Areciprocity spinor to S-matrix

Any versor which represents a non-reciprocal network can be broken up into reciprocal and areciprocal parts by separating the duality spinor from the bivector rotor. This is necessary since a bivector rotation in four dimensions will have scalar and psuedo-scalar components. A technique to accomplish this separation can be developed by exploiting each part's behavior in regard to reversion [6]. The spinor for a an arbitrary two-port network can be expressed,

$$A = PV. \tag{37}$$

Where P is an areciprocal duality spinor and V is a reciprocal rotor generated by the bivector U.

$$P = \sqrt{\rho} e^{\frac{\theta}{2} i} \qquad V = e^{U} \tag{38}$$

To separate A into P and V, first determine P and then remove it. Start by forming,

$$A\tilde{A} = PV\tilde{V}\tilde{P} = P^2 = \rho e^{\theta i}. \tag{39}$$

Once P is found, the rotor V can be found using,

$$V = P^{-1} A. \tag{40}$$

This section has given formulas needed to convert a s-matrix to and from a spinor representation in a four dimensional minkowski space. In the next section reciprocal networks are further decomposed based on the physical attributes of loss, symmetry and matched-ness.

IV. Special Cases and Groups

A. *Reciprocal Networks*

Since cascading any number of reciprocal networks yields a reciprocal network it is clear that they form a mathematical group which is a sub-group of all two-port networks. With CGA this group structure is represented geometrically by the fact that reciprocal networks are rotations, and non-reciprocal networks are spinors. Extending this logic, several other groups can be anticipated. For example, *lossless* networks form a group, as well as reflectionless or *matched* networks. Since two-port networks are modeled as rotations in CGA, we can analyze their group structure by identifying the planes of their rotations.

B. *Lossless Networks*

It is well known that lossless networks are represented by unitary scattering matrices, but there properties in cascading matrices or spinors is different. A mathematical representation for lossless networks can be made by considering the following physical argument. A lossless load has a reflection coefficient magnitude of unity, which can be visualized as a vector confined to the unit circle. In CGA, lines and circles are represented by tri-vectors and can be defined by taking the outer product of three null vectors which lay on the line/circle [7]. In this way, lines and circles become geometric objects in the algebra, as opposed to equations regarding their coordinates. Using this construction, the unit circle can be defined by the tri-vector,

$$\uparrow(e_1) \wedge \uparrow(-e_1) \wedge \uparrow(e_2) = e_{124}. \tag{41}$$

Cascading a lossless two-port network in front of a lossless load preserves the magnitude of the reflection coefficient, for there is no way for the power to dissipate. This is only true for lossless loads. Algebraically, this means that a lossless versor leaves e_{124} invariant.

$$V e_{124} \tilde{V} = e_{124} \tag{42}$$

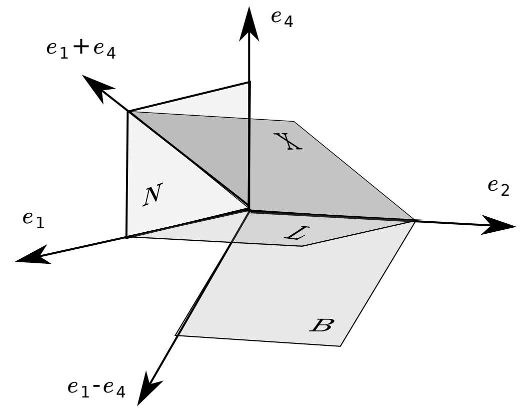


Figure 2: Lossless subspace of CGA with relevant vectors and bivector planes labeled.

Equivalently, we can say that a lossless versor commutes with e_{124}. The only generators which have this quality are those contained within the subspace defined by e_{124}, which can be interpreted as the lossless subspace. As shown in [1], the generators of the *discrete element group* which are contained within this subspace are those for reactance X, suscpetance B, and an impedance transformer N,

$$X \equiv e_{12} - e_{24} \tag{43}$$

$$B \equiv e_{12} + e_{24} \tag{44}$$

$$N \equiv e_{14} \tag{45}$$

These are by definition lossless elements. A figure illustrating the lossless subspace of CGA, with the bivector rotation planes labeled is shown in Figure 2. By summing infinitesimal rotations in both X and B equally, a rotation in the e_{12}-plane (labeled L) can be created, which extends the list of lossless elements to include matched transmission lines [1]. Mismatched transmission lines can be modeled as cascading impedance transformers on either side of a matched line, or by summing infinitesimal rotations in X and B unequally. This exact model for a lossless subspace was published in the 1950's by E.F. Bolinder [8], albeit through a different approach. Next we identify some special cases of reciprocal networks.

C. *Matched Networks*

Matched networks are defined by having no reflection coefficient at either port, meaning the diagonal elements of their s-matrix are zero ,

$$s_{11} = s_{22} = 0. \tag{46}$$

This special class of networks also forms a group, from the same physical argument given above. The matched condition reduces eq 3 to rotations and dilations, which are generated by rotations in e_{12} and e_{34} respectively, ie

$$V = e^{-\frac{\theta}{2} e_{12} - \frac{\ln \rho}{2} e_{34}}. \tag{47}$$

Property	Bivectors in Generator
Reciprocal	$e_{12}, e_{23}, e_{34}, e_{14}, e_{23}, e_{14}$
Symmetric	$e_{12}, e_{34}, e_{13}, e_{24}$
Asymmetric	$e_{12}, e_{23}, e_{34}, e_{14}$
Lossless	e_{12}, e_{24}, e_{14}
Non-propagating	e_{13}, e_{34}, e_{23}
Matched	e_{12}, e_{34}

Table I: List of network classifiers and the bivectors present in their generators. Groups are emboldened.

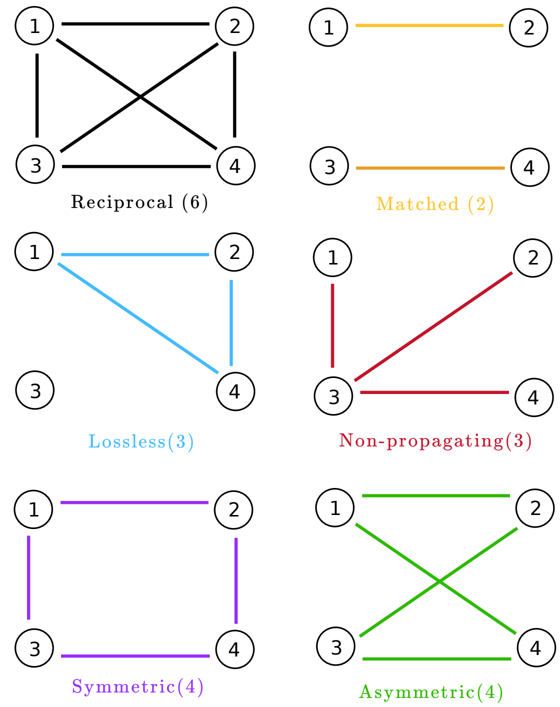


Figure 3: Graph illustrating bivector generators present in various network classifications. Combinations of classifications can be accomplished with the set theory intersection operation, between the group generators.

D. Symmetric Networks

Symmetric networks don't form a group because cascading two symmetric networks can produce a non-symmetric network. By definition, symmetric networks are invariant to a *flip,* which we can write,

$$V = \underline{V} = e_{14} V^{-1} e_{14}. \tag{48}$$

The only rotors which fulfill this property are those which don't contain e_{14} or e_{23}.

E. The Structure of Two-port Networks

Now that some special cases of networks have been identified by various means, it starts to become clear that two-port networks can be systematically classified based on the planes of rotation. This is most easily done by inspecting the bivectors present in their generators, as is done to classify Lie Groups in [9]. A list of different classifiers and the bivectors present in their generators is given in Table I. We find a graph more helpful to visualize this structure. The graph shown in Figure 3 represents vectors as nodes, and bivectors as connecting edges, with labels to indicate the group and degrees of freedom. The generators for various physical classifiers is visualized as subsets of edges. Different classifiers can be combined through the intersection operator of set-theory. For example, in reference to Figure 3, a symmetric, lossless network requires two parameters and it's generator contains e_{12} and e_{24}. Visually, this result can be determined by overlaying the two Cayley graphs and take the union.

V. Decomposition Methods

A. The Projective Split

A fundamental component in the Space Time Algebra (STA) is the concept of a projective split, where it represents the relationship between the Dirac and Pauli algebras [5], [6]. Since the geometry of CGA for two-port networks is identical to STA, we might suspect the split to be useful in network theory as well. It turns out that through a series of splits with various directions, reciprocal two-port networks can be decomposed based on their physical properties. Some of these physical properties form mathematical groups.

The concept is best illustrated with an example, so we revisit the lossless subgroup to demonstrate. As shown in section IV-B, lossless two-port networks for a 3-parameter group which can be identified as bivectors belonging to the e_{124}-subspace of CGA. This subgroup can also be generated by employing a projective split with e_3, in which case the bivectors of $G_{1,3}$ are mapped into vectors and bivectors in $G_{1,2}$. The bivectors not containing e_3 map into bivectors and form the group, while the bivectors containing e_3 map into vectors which do not form a group. The map can be defined as follows,

$$e_{3i} \to e_i \tag{49}$$

$$e_{ij} \to e_{ij}, \qquad i, j \neq 3. \tag{50}$$

In this way a lossy network can be decomposed into *lossless* and *phase-less* or *non-propagating* parts. The term *non-propagating*, although awkward, seems to be the most accurate description of the antithesis of lossless. If the non-propagating vector part is thought of as representing a translation and the lossless bivector part is thought of as a rotation, then two-port networks become a hyperbolic motion in a three-space. This means two-port's could be studied with a minkowskian motor algebra, an interesting idea.

The lossless group can be split once more, this time separating the group into symmetric and asymmetric parts.

By choosing e_2 as the splitting vector, the bivectors in $G_{1,2}$ are mapped into the vectors and bivectors of $G_{1,1}$ according to,

$$e_{2i} = e_i \tag{51}$$
$$e_{14} = e_{12}. \tag{52}$$

Here, the vectors in the minkowski plane represent the symmetric part, and the bivector represents the asymmetric part. As we have shown, employing the projective split decomposes networks by mapping bivectors into different grade objects in a sub-algebra. By choosing different splitting vectors, it is possible to achieve different decompositions and sub-algebras. An alternative approach to decomposition which does not leave $G_{1,3}$ is described next.

B. *Elemental Rotations*

Since CGA allows any two-port network to be modeled as a rotation in four dimensions, decomposing a network into simpler sub-networks can implemented as decomposing a rotation into a series of sub-rotations. Theorems about four dimensional rotations are well developed thanks to work done to characterize Lorentz transformations. From this we know that any rotation can be decomposed into a rotation which leaves a specified vector invariant, followed by a rotation in a plane containing that vector [10]. This fact is used in [6] to decompose a Lorentz transformation into time-like and space-like rotations. An exact translation of this decomposition in our basis amounts to choosing e_4 as the invariant vector. While possible, we have yet to find a use in network theory for such a decomposition. However, as we have shown in the last section, decomposing a rotation based on e_3 allows a network to be separated into lossless and non-propagating parts, so we revisit this dichotomy here as well.

The original derivation repeated here can be found in [6]. Start with a lossy rotor V and assume it can be broken up into a non-propagating part H and a lossless part U in cascade.

$$V = HU \tag{53}$$

Like V, both H and U are rotations, so

$$H\tilde{H} = U\tilde{U} = 1. \tag{54}$$

The lossless part will leave e_3 invariant,

$$U = e_3 U e_3, \tag{55}$$

and the non-propagating part will contain e_3,

$$H = e_3 \tilde{H} e_3. \tag{56}$$

Great. Next, form the quantity,

$$V e_3 \tilde{V} e_3, \tag{57}$$

Then express this in terms of H and U, and insert factors of e_3^2 strategically to find,

$$\begin{aligned} V e_3 \tilde{V} e_3 &= HU e_3 \tilde{U} \tilde{H} e_3 && (58)\\ &= H e_3 \left(e_3 U e_3\right) \tilde{U} e_3 \left(e_3 \tilde{H} e_3\right) && (59)\\ &= H e_3 U \tilde{U} e_3 H && (60)\\ &= H^2. && (61) \end{aligned}$$

So H can be found from V if the square root of H^2 can be computed. Since H is simple so is H^2, and the formula for the square root of a simple rotor is [10],

$$\sqrt{R} = \frac{(1+R)}{2\left(1+\langle R\rangle\right)}. \tag{62}$$

A direct formula for H in terms of V is thus,

$$H = \frac{\left(1 + V e_3 \tilde{V} e_3\right)}{2\left(1 + \left\langle V e_3 \tilde{V} e_3\right\rangle\right)}. \tag{63}$$

Once H is determined, U can be found from V by removing H.

$$U = \tilde{H} V \tag{64}$$

Which completes the determination of H and U from V. The same procedure can be used to further decompose the lossless rotation into symmetric and asymmetric parts by choosing e_2 as the invariant vector. Simply use U for V, and replace all e_3's with e_2.

VI. Application: Interpolation

A. *Algorithm*

As an example application of the spinor model presented here, we illustrate its use for interpolation. A two-port network is most commonly represented by an ordered list of S-matrices, each matrix being a discrete sample point. The independent variable that is changing across the samples is irrelevant, but could be frequency, time, temperature, etc. Inferring the network's response in between samples is the goal of interpolation. Some applications for interpolation include reducing the number of sample points required for simulation software to converge, or allowing for sparsely sampled measurement scenarios. Most conventional techniques operate on the elements of the s-matrix independently. One problem with this approach is that the interpolation between two lossless networks is not generally lossless. This is analogous to interpolating a rotation in 2D by acting on the real and imaginary parts of a complex number; the interpolated points don't lay on a circle.

Since the spinor model allows for reciprocal 2-port network to be represented as rotations, interpolation in some cases is more natural than for matrices. Recall that reciprocal networks are normalized spinors, also known as rotors, and we use the terms interchangeably. Interpolation of rotors can be done analogously to polar interpolation of a complex number. With this approach there are still several different methods which could be used [11], [12]. We have found that splitting the rotor into simple sub-rotors, based on a losslessness dichotomy, interpolating each simple rotor, and then re-assembly results works well. The results shown below are based on the following algorithm:

1) Convert the reciprocal S-matrix into a rotor (sec II-C)
2) Decompose rotor based on losslessness (sec V-B)
3) Interpolate each simple sub-rotor independently
4) Re-assemble sub-rotors back into single rotor
5) Convert rotor into S-matrix (sec II-D)

Step 3 can be done by using the methods described in [11], [12], but we repeat the idea. Any simple rotor can be written,

$$V_i = e^{G_i}, \tag{65}$$

Where G_i is the generating bivector. Given a list of rotors V_i, we can find a list of generators by taking the logarithm of V_i which is well defined for a simple rotor,

$$G_i = \log(V_i). \tag{66}$$

Then, the components of G_i can be interpolated component-wise, using standard techniques. We note that only recently has the solution to the general case rotor logarithm been solved [13], and this could make such interpolation algorithms work without decomposition in step 2 above.

B. Results

The interpolation method described above has been implemented in Python and compared to a conventional technique. The device under test was a lossy dielectric in halfspace, with uneven segments of air on each side as shown in figure 4. The s-parameters for this structure were generated over a frequency range of 1-10 GHz with 400 points using the Python module scikit-rf (scikit-rf.org). This 'true' response was then down-sampled to 10 points, i.e. a ~2% sampling rate, and the same down-sampled network data was then fed into each interpolation algorithm. The results of each algorithm along with the the true response are shown below in figure 5. The conventional interpolation, labeled 'Cartesian' is an element-wise, cubic spline interpolation on the real and imaginary components of the network's S-matrices. In the plots below, the spinor method clearly outperforms the conventional method. While the results may be hard to believe, we speculate that they are so accurate because this specific network is a very simple spinor, and it is the matrix representation which obfuscates this simplicity. A more exhaustive analysis of the spinor-based interpolation accuracy would be interesting to present in the future.

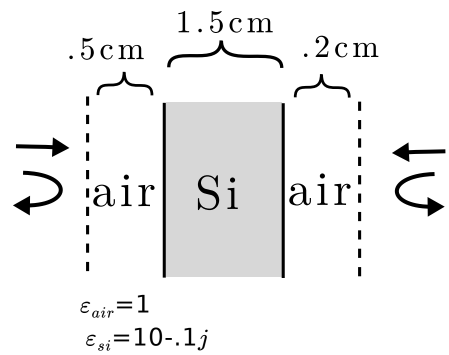
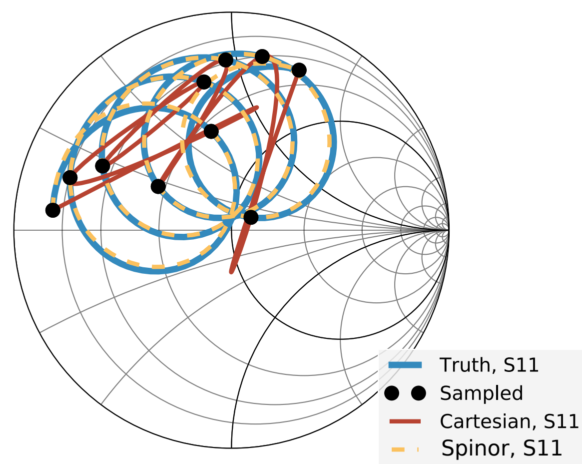
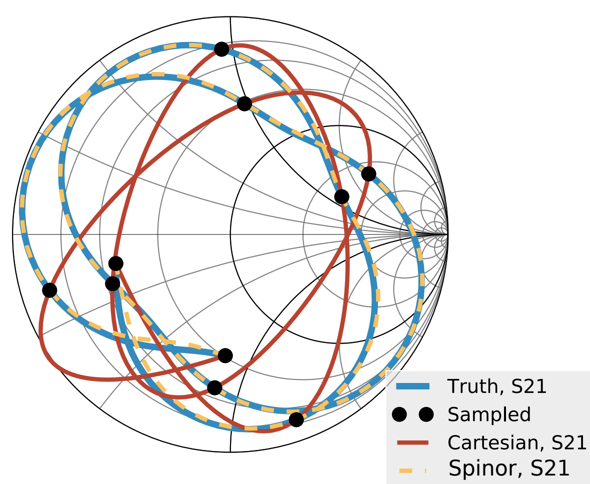


Figure 4: Physical model for a dielectric plate in half-space. This air sections on each side were designed to make the network asymmetric.



(a) Comparison of S_{11} for a dielectric in half-space.



(b) Comparison of S_{21} for a dielectric in half-space.

Figure 5: Comparison of S_{11} and S_{21} for a dielectric in half-space. Traces show true response, sampled response, Cartesian interpolation of Re/Im components of S-matrix, and the spinor-based model interpolation. (The spinor interpolation is so close to the true its hard to differentiate.)

VII. Conclusion

We have presented a spinor version of the wave-cascading matrix and developed methods for translating between the matrix and CGA spinor representations. Geometric approaches to two-port network decomposition based on the physical characteristics of reciprocity, loss, and symmetry were presented and associated rotation groups were identified. An arbitrary network can be decomposed into specific parts either with the projective split shown in Section V-A, or the elemental rotational decomposition given in Section V-B. An example application of interpolating a 2-port network is given in Section VI. The main advantage of using Geometric Algebra for modeling networks is the geometric *meaning* given to various physical attributes, and the ability to unify results with other fields.

VIII. Appendix

A. Proof of Flip Operator

Flipping a two-port network is defined as interchanging its ports. In regard to a s-matrix, this has the effect of swapping the following elements,

$$s_{12} \leftrightarrow s_{21} \tag{67}$$

$$s_{11} \leftrightarrow s_{22}. \tag{68}$$

We seek the geometrical equivalent of this operation. Given that nonreciprocal network can be expressed as the spinor,

$$A = PV. \tag{69}$$

Where P is an areciprocal duality spinor and V is a reciprocal rotor. The flip operation is

$$\underline{\mathrm{A}} \equiv e_{14}A^{-1}e_{14}. \tag{70}$$

a) Proof: The flip operation will effect the areciprocal and reciprocal parts of the network differently, so each is analyzed separately. Since a reciprocal network is a bivector rotor, inversion is equivalent to reversion, $V^{-1} = \tilde{V}$. Additionally, the areciprocal part P commutes with the e_{14}'s, which annihilate each other, so the flip operations reduces to inversion of P. Combining these facts, allows us to write,

$$\underline{\mathrm{A}} = e_{14}\left(PV\right)^{-1}e_{14} = P^{-1}e_{14}\tilde{V}e_{14}. \tag{71}$$

Given the relation of P to the s-matrix defined in the III-B, inverting P swaps s_{12} with s_{21}. Whats left is to swap the s_{11} and s_{22} elements. By inspecting eq (3), it can be seen that the flip operation exchanges the parameter of the transversion with that of the translation. The proof that this can be accomplished by reversion combined with a reflection in e_{14} is below,

$$\begin{aligned}
\underline{\mathrm{V}} &= e_{14}(\widetilde{T_t D_d R_r K_k})e_{14} && (72)\\
&= e_{14}\tilde{K}_k\tilde{R}_r\tilde{D}_d\tilde{T}_t e_{14} && (73)\\
&= \underbrace{e_{14}\tilde{K}_k e_{14}}_{-T_{-k}}\underbrace{e_{14}\tilde{R}_r e_{14}}_{R_r}\underbrace{e_{14}\tilde{D}_d e_{14}}_{D_d}\underbrace{e_{14}\tilde{T}_t e_{14}}_{-K_{-t}} && (74)\\
&= T_{-k}D_d R_r K_{-t} && (75)
\end{aligned}$$

Here we have used the fact that $e_{14}^2 = 1$ to insert pairs of e_{14} where convenient, and computed the result of reflecting each operator in e_{14}.

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