A NEW KIND OF THE VARIANT OF THE MODIFIED BERNSTEIN-KANTOROVICH OPERATORS DEFINED BY ÖZARSLAN AND DUMAN

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Abstract. In the present article, we define a new kind of the modified Bernstein-Kantorovich operators defined by Özarslan (https://doi.org/10.1080/01630563.2015.1079219) i.e. we introduce a new function ζ in the modified Bernstein-Kantorovich operators defined by Özarslan with the property $\zeta(\varkappa)$ is an infinitely differentiable function on $[0,1]$, $\zeta(0)=0$, $\zeta(1)=1$ and $\zeta'(\varkappa)>0\ \forall\ \varkappa\in[0,1]$. We substantiate an approximation theorem by using of the Bohman-Korovkins type theorem and scrutinize the rate of convergence with the aid of modulus of continuity, Lipschitz type functions for the our operators and the rate of convergence of functions by means of derivatives of bounded variation are also studied. We study an approximation theorem with the help of Bohman-Korovkins type theorem in $\mathcal{A}-$Statistical convergence.
Lastly, by means of a numerical example, we illustrate the convergence of these operators to certain functions through graphs with the help of MATHEMATICA and show that a careful choice of the function $\zeta(\varkappa)$ leads to a better approximation results as compared to the modified Bernstein-Kantorovich operators defined by Özarslan (https://doi.org/10.1080/01630563.2015.1079219).

Keywords: Modulus of continuity, asymptotic formula, Peetre's K-functional, local approximation, error estimate, rate of convergence, $\mathcal{A}-$Statistical convergence.

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1. Introduction

Let $\mathfrak{I}:=[0,1]$ and $\mathcal{C}(\mathfrak{I})$ be the space of all continuous functions on the interval $\mathfrak{I}$. In 1966, George [10] introduced classical Bernstein operators $\mathcal{B}_{\mathfrak{m}}(\varphi;\varkappa)$ as follows:

$$\mathcal{B}_{\mathfrak{m}}(\varphi;\varkappa)=\sum_{k=0}^{\mathfrak{m}}\mathfrak{p}_{\mathfrak{m},k}(\varkappa)\ \varphi\left(\frac{k}{\mathfrak{m}}\right),\quad \varkappa\in\mathfrak{I}, \tag{1.1}$$

where $\mathfrak{p}_{\mathfrak{m},k}(\varkappa)=\binom{\mathfrak{m}}{k}\ \varkappa^k(1-\varkappa)^{\mathfrak{m}-k}$.
Also, for $\mathbb{R}=[0,\infty)$ and $\varphi:\mathfrak{I}\to\mathbb{R}$ is an integrable function, the classical Bernstein-Kantorovich operators are defined by:

$$K_{\mathfrak{m}}(\varphi;\varkappa)=(\mathfrak{m}+1)\sum_{k=0}^{\mathfrak{m}}\mathfrak{p}_{\mathfrak{m},k}(\varkappa)\int_{\frac{k}{(\mathfrak{m}+1)}}^{\frac{(k+1)}{(\mathfrak{m}+1)}}\varphi(\mathfrak{t})\ d\mathfrak{t},\quad \varkappa\in\mathfrak{I},\ \mathfrak{m}\in\mathbb{N}. \tag{1.2}$$

The above operators $K_{\mathfrak{m}}(\varphi;\varkappa)$ can be also written as follows:

$$K_{\mathfrak{m}}(\varphi;\varkappa)=(\mathfrak{m}+1)\sum_{k=0}^{\mathfrak{m}}\mathfrak{p}_{\mathfrak{m},k}(\varkappa)\int_0^1\varphi\left(\frac{k+\mathfrak{t}}{\mathfrak{m}+1}\right)d\mathfrak{t}. \tag{1.3}$$

For $\alpha>0$, it very interesting to see that if we write $\mathfrak{t}$ with $\mathfrak{t}^\alpha$ in (1.3), then it does not effect the positivity or linearity of the operator $K_{\mathfrak{m}}(\varphi;\varkappa)$, it does originate a new sequence of non negative linear operators i.e. modified Bernstein-Kantorovich operators given by Özarslan and Duman [12] as follows:

$$K_{\mathfrak{m},\alpha}(\varphi;\varkappa)=\sum_{k=0}^{\mathfrak{m}}\mathfrak{p}_{\mathfrak{m},k}(\varkappa)\int_0^1\varphi\left(\frac{k+\mathfrak{t}^{\alpha}}{\mathfrak{m}+1}\right)d\mathfrak{t},\quad \varkappa\in\mathfrak{I}. \tag{1.4}$$

We can notice that, if we take the value of α equal to 1 in the above equation (1.4), then we obtained the classical Bernstein-Kantotrovich operators $K_{\mathfrak{m}}:=K_{\mathfrak{m},1}$ defined as in (1.2). Furthermore, the author showed that the order of approximation to a function by these operators is at least as good as that of ones classically used. They proved two direct approximation result with the help of usual second-order modulus of continuity and the second order modulus of continuity, respectively.
Motivated by the work of Özarslan and Duman [12], for $\varphi\in\mathcal{C}(\mathfrak{I})$, we introduce a variant of the operator (1.4) by means of an differentiable function $\zeta(\varkappa)$ on $\mathfrak{I}$ satisfying the properties $\zeta(0)=0,\ \zeta(1)=1$ and $\zeta'(\varkappa)>0$, for $\varkappa\in\mathfrak{I}$ as follows:

$$\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)=\sum_{k=0}^{\mathfrak{m}}\mathfrak{p}_{\mathfrak{m},k}(\zeta(\varkappa))\int_0^1(\varphi\circ\zeta^{-1})\left(\frac{k+\mathfrak{t}^{\alpha}}{\mathfrak{m}+1}\right)d\mathfrak{t},\qquad \varkappa\in\mathfrak{I}, \tag{1.5}$$

where $\mathfrak{p}_{\mathfrak{m},k}(\zeta(\varkappa))=\binom{\mathfrak{m}}{k}\ (\zeta(\varkappa))^k(1-\zeta(\varkappa))^{\mathfrak{m}-k}$ and we study some direct approximation theorems with the aid of the modulus of continuity and the Lipschitz type maximal function. A Voronovskaya type asymptotic theorem and the approximation of functions with derivatives of bounded variation are also investigated. In lastly, by drawing the graph with the help of MATHEMATICA software, we observed that by a suitable choice of the function $\zeta(\varkappa)$ a better order of convergence can be achieved by the our operators (1.5), than the operators (1.4). We show, by the suitable choice of $\zeta(\varkappa)$ operators (1.5) gives the better convergence and error estimate also.

2. Moments and central moments

Let $\mathfrak{e}_i(\mathfrak{t})=\mathfrak{t}^i,\ i=0,1,2,3,4$ is the basis function for the operator $K_{\mathfrak{m},\alpha}(\varphi;\varkappa)$.

Lemma 1. *For the operators $K_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ defined by (1.4), for each $\varkappa\in\mathfrak{I}$, we have*

(i) $K_{\mathfrak{m},\alpha}(1;\varkappa)=1$;

(ii) $K_{\mathfrak{m},\alpha}(\mathfrak{t};\varkappa)=\frac{\mathfrak{m}\varkappa}{\mathfrak{m}+1}+\frac{1}{(\alpha+1)(\mathfrak{m}+1)}$;

(iii) $K_{\mathfrak{m},\alpha}(\mathfrak{t}^2;\varkappa)=\frac{\mathfrak{m}\varkappa^2}{(\mathfrak{m}+1)^2}+\frac{\mathfrak{m}\varkappa(1-\varkappa)}{(\mathfrak{m}+1)^2}+\frac{2\mathfrak{m}\varkappa}{(\alpha+1)(\mathfrak{m}+1)^2}+\frac{1}{(2\alpha+1)(\mathfrak{m}+1)^2}$;

(iv) $K_{\mathfrak{m},\alpha}(\mathfrak{t}^3;\varkappa)=\frac{\mathfrak{m}(\mathfrak{m}-1)(\mathfrak{m}-2)\varkappa^3}{(\mathfrak{m}+1)^3}+\frac{3\mathfrak{m}(\mathfrak{m}-1)(\alpha+2)\varkappa^2}{(\alpha+1)(\mathfrak{m}+1)^3}+\frac{\mathfrak{m}(2\alpha^2+12\alpha+7)\varkappa}{(\alpha+1)(2\alpha+1)(\mathfrak{m}+1)^3}+\frac{1}{(3\alpha+1)(\mathfrak{m}+1)^3}$;

(v) $K_{\mathfrak{m},\alpha}(\mathfrak{t}^4;\varkappa)=\frac{\mathfrak{m}(\mathfrak{m}-1)(\mathfrak{m}-2)(\mathfrak{m}-3)\varkappa^4}{(\mathfrak{m}+1)^4}+\frac{2\mathfrak{m}(\mathfrak{m}-1)(3\alpha+5)\varkappa^3}{(\alpha+1)(\mathfrak{m}+1)^4)}+\frac{\mathfrak{m}(\mathfrak{m}-1)(14\alpha^2+51\alpha+25)\varkappa^2}{(\alpha+1)(2\alpha+1)(\mathfrak{m}+1)^4}+\frac{\mathfrak{m}\varkappa(6\alpha^3+61\alpha^2+62\alpha+15)}{(\alpha+1)(2\alpha+1)(3\alpha+1)(\mathfrak{m}+1)^4}$
$+\ \frac{1}{(4\alpha+1)(\mathfrak{m}+1)^4}$.

Let $\mathfrak{e}_{i,\zeta}(\mathfrak{t})=\zeta^i(\mathfrak{t}),\ i=0,1,2,3.$ is the basis function for the moments of operator $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\cdot;\varkappa)$.
As a consequence of the above lemma, we have

Lemma 2. *For the operators $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ given by (1.5), for each $\varkappa\in\mathfrak{I}$, we have*

(i) $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\mathfrak{e}_{0,\zeta}(\mathfrak{t});\varkappa)=1$;

(ii) $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\mathfrak{e}_{1,\zeta}(\mathfrak{t});\varkappa)=\frac{\mathfrak{m}\zeta(\varkappa)}{\mathfrak{m}+1}+\frac{1}{(\alpha+1)(\mathfrak{m}+1)}$;

(iii) $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\mathfrak{e}_{2,\zeta}(\mathfrak{t});\varkappa)=\frac{\mathfrak{m}\zeta^2(\varkappa)}{(\mathfrak{m}+1)^2}+\frac{\mathfrak{m}\zeta(\varkappa)(1-\zeta(\varkappa))}{(\mathfrak{m}+1)^2}+\frac{2\mathfrak{m}\zeta(\varkappa)}{(\alpha+1)(\mathfrak{m}+1)^2}+\frac{1}{(2\alpha+1)(\mathfrak{m}+1)^2}$;

(iv) $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\mathfrak{e}_{4,\zeta}(\mathfrak{t});\varkappa)=\frac{\mathfrak{m}(\mathfrak{m}-1)(\mathfrak{m}-2)\zeta^3(\varkappa)}{(\mathfrak{m}+1)^3}+\frac{3\mathfrak{m}(\mathfrak{m}-1)(\alpha+2)\zeta^2(\varkappa)}{(\alpha+1)(\mathfrak{m}+1)^3}+\frac{\mathfrak{m}(2\alpha^2+12\alpha+7)\zeta(\varkappa)}{(\alpha+1)(2\alpha+1)(\mathfrak{m}+1)^3}+\frac{1}{(3\alpha+1)(\mathfrak{m}+1)^3}$;

(v) $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\mathfrak{e}_{4,\zeta}(\mathfrak{t});\varkappa) = \frac{\mathfrak{m}(\mathfrak{m}-1)(\mathfrak{m}-2)(\mathfrak{m}-3)\zeta^4(\varkappa)}{(\mathfrak{m}+1)^4} + \frac{2\mathfrak{m}(\mathfrak{m}-1)(3\alpha+5)\zeta^3(\varkappa)}{(\alpha+1)(\mathfrak{m}+1)^4} + \frac{\mathfrak{m}(\mathfrak{m}-1)(14\alpha^2+51\alpha+25)\zeta^2(\varkappa)}{(\alpha+1)(2\alpha+1)(\mathfrak{m}+1)^4}$
$+ \frac{\mathfrak{m}(6\alpha^3+61\alpha^2+62\alpha+15)\zeta(\varkappa)}{(\alpha+1)(2\alpha+1)(3\alpha+1)(\mathfrak{m}+1)^4} + \frac{1}{(4\alpha+1)(\mathfrak{m}+1)^4}.$

Lemma 3. *For the operators $K_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ defined by (1.4), for each $\varkappa \in \mathfrak{I}$, we have*

(i) $K_{\mathfrak{m},\alpha}(\zeta(\varkappa)-\zeta(\mathfrak{t});\varkappa) = \frac{1}{(\mathfrak{m}+1)}\bigg(-\varkappa + \frac{1}{\alpha+1}\bigg);$

(ii) $K_{\mathfrak{m},\alpha}((\zeta(\varkappa)-\zeta(\mathfrak{t}))^2;\varkappa) = \frac{1}{(\mathfrak{m}+1)^2}\bigg(\varkappa^2 - \frac{2\varkappa}{\alpha+1} + \frac{1}{2\alpha} + \mathfrak{m}\varkappa(1-\varkappa)\bigg).$

As a consequence of the above lemma, we have

Lemma 4. *For the operators $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$, we have*

(i) $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\zeta(\mathfrak{t})-\zeta(\varkappa);\varkappa) = \frac{1}{(\mathfrak{m}+1)}\bigg[-\zeta(\varkappa) + \frac{1}{\alpha+1}\bigg];$

(ii) $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}((\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa) = \frac{1}{(\mathfrak{m}+1)^2}\bigg[\zeta^2(\varkappa) - \frac{2\zeta(\varkappa)}{\alpha+1} + \frac{1}{2\alpha+1} + \mathfrak{m}\zeta(\varkappa)(1-\zeta(\varkappa))\bigg].$

Let $\mathcal{C}(\mathfrak{I})$ be the space of all continuous functions defined on the interval $\mathfrak{I}$ and $\varphi \in \mathcal{C}(\mathfrak{I})$. The norm of function φ is defined by

$$\|\varphi\| = \sup_{\varkappa\in\mathfrak{I}} |\varphi(\varkappa)|.$$

Lemma 5. *For $\varphi \in \mathcal{C}(\mathfrak{I})$, we have*

$$|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)| \leq \|\varphi\|.$$

Proof. Applying the definition (1.5) and Lemma 2, we have

$$\begin{aligned} |\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)| &= \left|(\mathfrak{m}+1)\sum_{k=0}^{\mathfrak{m}} p^{(\alpha)}_{\mathfrak{m},k}(\zeta(\varkappa)) \int_0^1 (\varphi\circ\zeta^{-1})\left(\frac{k+\mathfrak{t}^\alpha}{\mathfrak{m}+1}\right)d\mathfrak{t}\right| \\ &\leq (\mathfrak{m}+1)\sum_{k=0}^{\mathfrak{m}} \left|p^{(\alpha)}_{\mathfrak{m},k}(\zeta(\varkappa))\right| \int_0^1 \left|(\varphi\circ\zeta^{-1})\left(\frac{k+\mathfrak{t}^\alpha}{\mathfrak{m}+1}\right)\right| d\mathfrak{t} \\ &\leq \|\varphi\|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(1;\varkappa) \\ &\leq \|\varphi\|, \ \textit{for all } \varkappa \in \mathfrak{I}. \end{aligned}$$

Hence $\quad \sup_{\varkappa\in\mathfrak{I}} |\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)| \leq \|\varphi\|,$

which completes the proof. ∎

Now, we discuss the bound for second order central moments of the operators $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$.

Lemma 6. *For $\alpha > 1$ and $\mathfrak{m} \in \mathbb{N}$, we have*

$$|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}((\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa)| \leq \frac{1}{(\mathfrak{m}+1)}(\gamma^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa))^2,$$

where

$$(\gamma^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa))^2 = \varphi^2_{\mathfrak{m},\zeta}(\varkappa) + \frac{1}{(\mathfrak{m}+1)(2\alpha+1)} \ \ \textit{and} \ \ \varphi^2_{\mathfrak{m},\zeta}(\varkappa) = \zeta(\varkappa)(1-\zeta(\varkappa)).$$

Proof. By using Lemma 4, we have

$$\begin{aligned} \mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}((\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa) &= \frac{1}{(\mathfrak{m}+1)^2}\left[\zeta^2(\varkappa) - \frac{2\zeta(\varkappa)}{\alpha+1} + \frac{1}{2\alpha+1} + \mathfrak{m}\zeta(\varkappa)(1-\zeta(\varkappa))\right] \\ &= \frac{1}{(\mathfrak{m}+1)^2}\left[\mathfrak{m}\zeta(\varkappa)(1-\zeta(\varkappa)) - \frac{2\zeta(\varkappa)}{\alpha+1} + \zeta^2(\varkappa)\right] + \frac{1}{(2\alpha+1)(\mathfrak{m}+1)^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(\mathfrak{m}+1)^2}\left[\mathfrak{m}\zeta(\varkappa)(1-\zeta(\varkappa))-\zeta(\varkappa)\left(\frac{2}{\alpha+1}-\zeta(\varkappa)\right)\right]+\frac{1}{(2\alpha+1)(\mathfrak{m}+1)^2}\\
&\leq \frac{1}{(\mathfrak{m}+1)^2}\left[\mathfrak{m}\zeta(\varkappa)(1-\zeta(\varkappa))-\zeta(\varkappa)(1-\zeta(\varkappa))\right]+\frac{1}{(2\alpha+1)(\mathfrak{m}+1)^2}\\
&\leq \frac{1}{(\mathfrak{m}+1)^2}\left(\zeta(\varkappa)(1-\zeta(\varkappa))(\mathfrak{m}-1)\right)+\frac{1}{(2\alpha+1)(\mathfrak{m}+1)^2}\\
&\leq \frac{1}{(\mathfrak{m}+1)^2}\left(\zeta(\varkappa)(1-\zeta(\varkappa))(\mathfrak{m}+1)\right)+\frac{1}{(2\alpha+1)(\mathfrak{m}+1)^2}\\
&\leq \frac{1}{(\mathfrak{m}+1)}\zeta(\varkappa)(1-\zeta(\varkappa))+\frac{1}{(2\alpha+1)(\mathfrak{m}+1)^2}\\
&\leq \frac{1}{(\mathfrak{m}+1)}\left[\zeta(\varkappa)(1-\zeta(\varkappa))+\frac{1}{(2\alpha+1)(\mathfrak{m}+1)}\right]\\
&\leq \frac{1}{(\mathfrak{m}+1)}\left[\varphi^2_{\mathfrak{m},\zeta}(\varkappa)+\frac{1}{(2\alpha+1)(\mathfrak{m}+1)}\right]\\
&\leq \frac{1}{(\mathfrak{m}+1)}(\gamma^{(\alpha}_{\mathfrak{m},\zeta}(\varkappa))^2,
\end{aligned}$$

we obtained the desired result. ☐

In the following lemma we gives the limiting value for the central moments of the operators $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$.

Remark 1. *By using Lemma 4, we get*

(i) $\lim_{\mathfrak{m}\to\infty}\mathfrak{m}\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}((\zeta(\mathfrak{t})-\zeta(\varkappa));\varkappa)=\frac{1}{\alpha+1}-\zeta(\varkappa)$;

(ii) $\lim_{\mathfrak{m}\to\infty}\mathfrak{m}\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}((\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa)=3\zeta(\varkappa)(1-\zeta(\varkappa))$;

(iii) $\lim_{\mathfrak{m}\to\infty}\mathfrak{m}^2\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}((\zeta(\mathfrak{t})-\zeta(\varkappa))^4;\varkappa)=(24\alpha-13)\zeta^4(\varkappa)-\frac{2\zeta^3(\varkappa)(14\alpha^3+25\alpha^2+21\alpha+10)}{(2\alpha+1)(\alpha+1)}-\frac{8\zeta^2(\varkappa)(3\alpha+2)}{(2\alpha+1)(\alpha+1)}$.

3. Basic Convergence Theorem

The following theorem shows that the operators $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(.;x)$ is an approximation process for continuous functions in $\varkappa\in\mathfrak{I}$.

Theorem 1. *Let* $\varphi\in\mathcal{C}(\mathfrak{I})$*. Then*

$$\lim_{\mathfrak{m}\to\infty}\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)=\varphi(\varkappa),$$

uniformly in $\mathfrak{I}$.

Proof. By Lemma, 2 $\lim_{\mathfrak{m}\to\infty}\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(1;\varkappa)=1$, $\lim_{\mathfrak{m}\to\infty}\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\zeta(\mathfrak{t});\varkappa)=\zeta(\varkappa)$ and $\lim_{\mathfrak{m}\to\infty}\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)=\zeta^2(\varkappa)$, uniformly in $\mathfrak{I}$. By well-known Bohman-Korovkin theorem it follows that $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)\to\varphi(\varkappa)$ as $\mathfrak{m}\to\infty$, uniformly in $\varkappa\in\mathfrak{I}$. ☐

4. Local Approximation

The Peetre's $K-$ functional is given by:

$$K_2(\varphi,\delta)=inf\{\|\varphi-g\|+\delta\|g^{''}\|:g\in W^2\},\ \delta>0,$$

where $W^2=\{g:g^{''}\in\mathcal{C}(\mathfrak{I})\}$ endowed with the norm $\|g\|_{w^2}=\|g\|+\|g^{'}\|+\|g^{''}\|$.
Following [4], there exists a positive constant $M>0$ such that

$$K_2(\varphi,\delta)\leq M\omega_2(\varphi,\sqrt{\delta}), \tag{4.1}$$

where the second order modulus of continuity for $\varphi \in \mathcal{C}(\mathfrak{I})$ is defined as

$$\omega_2(\varphi, \sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{\varkappa,\varkappa+2h\in I} |\varphi(\varkappa+2h) - 2\varphi(\varkappa+h) + \varphi(\varkappa)|.$$

We define the usual modulus of continuity for $\varphi \in \mathcal{C}(\mathfrak{I})$ as

$$\omega(\varphi, \delta) = \sup_{0<h\leq\delta} \sup_{\varkappa,\varkappa+h\in I} |\varphi(\varkappa+h) - \varphi(\varkappa)|.$$

The following result gives us the relation between error estimate $|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\cdot;\varkappa) - \varphi(\varkappa)|$ with second order modulus of continuity and usual modulus of continuity.

Theorem 2. *If $\varphi \in \mathcal{C}(\mathfrak{I})$, then for the operators $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\cdot;\varkappa)$, there exists a constant $M > 0$, such that*

$$|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa) - \varphi(\varkappa)| \leq M\ \omega_2\left(\varphi; \frac{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa)}{\sqrt{(\mathfrak{m}+1)}}\right) + \omega\left(\varphi; \left|\frac{1}{\mathfrak{m}+1}\left(\frac{1}{\alpha+1} - \zeta(\varkappa)\right)\right|\right).$$

Proof. We consider an auxiliary operators as follows:

$$\widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa) = \mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa) + \varphi(\varkappa) - (\varphi\circ\zeta^{-1})\left(\frac{\mathfrak{m}\zeta(\varkappa)}{\mathfrak{m}+1} + \frac{1}{(\mathfrak{m}+1)(\alpha+1)}\right). \tag{4.2}$$

Using Lemma 2, it is clear that, $\widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}(1;\varkappa) = 1$, $\widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}(\zeta(\mathfrak{t});\varkappa) = \zeta(\varkappa)$.
Let $g \in W^2$ and $\mathfrak{t} \in [0,1]$. By Taylor's expansion we have

$$g(\mathfrak{t}) = (g\circ\zeta^{-1})(\zeta(\varkappa)) + (\zeta(\mathfrak{t}) - \zeta(\varkappa))(g\circ\zeta^{-1})(\zeta(\varkappa)) + \int_{\zeta(\varkappa)}^{\zeta(\mathfrak{t})} (\zeta(\mathfrak{t}) - v)(g\circ\zeta^{-1})''(v)\,dv. \tag{4.3}$$

Operating $\widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}(\cdot;\varkappa)$ on the both sides of the above equation, we obtain

$$\begin{aligned}
\widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}(g(\mathfrak{t});\varkappa) &= \widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}\Big((g\circ\zeta^{-1})(\zeta(\varkappa));\varkappa\Big) + \widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}\Big((\zeta(\mathfrak{t}) - \zeta(\varkappa))(g\circ\zeta^{-1})(\zeta(\varkappa));\varkappa\Big)\\
&+ \widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}\left(\int_{\zeta(\varkappa)}^{\zeta(\mathfrak{t})} (\zeta(\mathfrak{t}) - v)(g\circ\zeta^{-1})''(v)\,dv;\varkappa\right)\\
&= \widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}\left(\int_{\zeta(\varkappa)}^{\zeta(\mathfrak{t})} (\zeta(\mathfrak{t}) - v)(g\circ\zeta^{-1})''(v)\,dv;\varkappa\right).
\end{aligned}$$

Following ([10], p.32)

$$(g\circ\zeta^{-1})''(u) = \frac{g''(\zeta^{-1}(u))}{(\zeta'(\zeta^{-1}(u)))^2} - \frac{g'(\zeta^{-1}(u))\zeta''(\zeta^{-1}(u))}{(\zeta'(\zeta^{-1}(u)))^3}. \tag{4.4}$$

Thus,

$$\begin{aligned}
|\widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}(g(\mathfrak{t});\varkappa) - g(\varkappa)| &= \left|\widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}\left(\int_{\zeta(\varkappa)}^{\zeta(\mathfrak{t})} (\zeta(\mathfrak{t}) - v)\left\{\frac{g''(\zeta^{-1}(u))}{(\zeta'(\zeta^{-1}(u)))^2} - \frac{g'(\zeta^{-1}(u))\zeta''(\zeta^{-1}(u))}{(\zeta'(\zeta^{-1}(u)))^3}\right\}dv;\varkappa\right)\right|\\
&= \left|\widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}\left(\int_{\zeta(\varkappa)}^{\zeta(\mathfrak{t})} (\zeta(\mathfrak{t}) - v)\frac{g''(\zeta^{-1}(u))}{(\zeta'(\zeta^{-1}(u)))^2}dv;\varkappa\right)\right.\\
&- \left.\widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}\left(\int_{\zeta(\varkappa)}^{\zeta(\mathfrak{t})} (\zeta(\mathfrak{t}) - v)\frac{g'(\zeta^{-1}(u))\zeta''(\zeta^{-1}(u))}{(\zeta'(\zeta^{-1}(u)))^3}dv;\varkappa\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}\Bigg(\int_{\zeta(\varkappa)}^{\zeta(\mathfrak{t})}(\zeta(\mathfrak{t})-v)\frac{g^{''}(\zeta^{-1}(u))}{(\zeta'(\zeta^{-1}(u)))^2}dv;\varkappa\Bigg)\\
&- \int_{\zeta(\varkappa)}^{\frac{1}{\mathfrak{m}+1}(\mathfrak{m}\zeta(\varkappa)+\frac{1}{\alpha+1})}\left[\frac{1}{\mathfrak{m}+1}\left\{\mathfrak{m}\zeta(\varkappa)+\frac{1}{\alpha+1}\right\}-v\right]\frac{g^{''}(\zeta^{-1}(u))}{(\zeta'(\zeta^{-1}(u)))^2}dv\\
&+ \Bigg|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}\Bigg(\int_{\zeta(\varkappa)}^{\zeta(\mathfrak{t})}(\zeta(\mathfrak{t})-v)\frac{g^{'}(\zeta^{-1}(u))\zeta"(\zeta^{-1}(u))}{(\zeta'(\zeta^{-1}(u)))^3}dv;\varkappa\Bigg)\\
&- \int_{\zeta(\varkappa)}^{\frac{1}{\mathfrak{m}+1}(\mathfrak{m}\zeta(\varkappa)+\frac{1}{\alpha+1})}\left[\frac{1}{\mathfrak{m}+1}\left\{\mathfrak{m}\zeta(\varkappa)+\frac{1}{\alpha+1}\right\}-v\right]\frac{g^{'}(\zeta^{-1}(u))\zeta"(\zeta^{-1}(u))}{(\zeta'(\zeta^{-1}(u)))^3}dv\Bigg|\\
&\leq \mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}\Bigg(\Bigg|\int_{\zeta(\varkappa)}^{\zeta(\mathfrak{t})}|\zeta(\mathfrak{t})-v|\left|\frac{g^{''}(\zeta^{-1}(u))}{(\zeta'(\zeta^{-1}(u)))^2}\right|dv\Bigg|;\varkappa\Bigg)\\
&+ \Bigg|\int_{\zeta(\varkappa)}^{\frac{1}{\mathfrak{m}+1}(\mathfrak{m}\zeta(\varkappa)+\frac{1}{\alpha+1})}\left|\frac{1}{\mathfrak{m}+1}\left\{\mathfrak{m}\zeta(\varkappa)+\frac{1}{\alpha+1}\right\}-v\right|\left|\frac{g^{''}(\zeta^{-1}(u))}{(\zeta'(\zeta^{-1}(u)))^2}\right|dv\Bigg|\\
&+ \mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}\Bigg(\Bigg|\int_{\zeta(\varkappa)}^{\zeta(\mathfrak{t})}|\zeta(\mathfrak{t})-v|\left|\frac{g^{'}(\zeta^{-1}(u))\zeta"(\zeta^{-1}(u))}{(\zeta'(\zeta^{-1}(u)))^3}\right|dv\Bigg|;\varkappa\Bigg)\\
&+ \Bigg|\int_{\zeta(\varkappa)}^{\frac{1}{\mathfrak{m}+1}(\mathfrak{m}\zeta(\varkappa)+\frac{1}{\alpha+1})}\left|\frac{1}{\mathfrak{m}+1}\left\{\mathfrak{m}\zeta(\varkappa)+\frac{1}{\alpha+1}\right\}-v\right|\left|\frac{g^{'}(\zeta^{-1}(u))\zeta"(\zeta^{-1}(u))}{(\zeta'(\zeta^{-1}(u)))^3}\right|\,dv\Bigg|.
\end{aligned}
$$

Since ζ is strictly increasing on the interval $\mathfrak{I}$, we have $\sup\limits_{\varkappa\in\mathfrak{I}}\zeta'(\varkappa)\geq a$, for some $a\in\mathbb{R}^+=(0,\infty)$, we get

$$
\begin{aligned}
|\widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}(g(\mathfrak{t});\varkappa)-g(\varkappa)| &\leq \mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}((\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa)\left\{\frac{\|g^{''}\|}{a^2}+\frac{\|g^{'}\|\,\|\zeta"\|}{a^3}\right\}\\
&+ \left(\frac{1}{\mathfrak{m}+1}\left\{\mathfrak{m}\zeta(\varkappa)+\frac{1}{\alpha+1}\right\}-\zeta(x)\right)^2\left\{\frac{\|g^{''}\|}{a^2}+\frac{\|g^{'}\|\,\|\zeta"\|}{a^3}\right\}\\
&\leq \left\{\frac{\|g^{''}\|}{a^2}+\frac{\|g^{'}\|\,\|\zeta"\|}{a^3}\right\}\left[\left(\frac{1}{\mathfrak{m}+1}\left(\frac{1}{\alpha+1}-\zeta(\varkappa)\right)\right)^2+\frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)}{\mathfrak{m}+1}\right]\\
&\leq \frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)}{a^2(\mathfrak{m}+1)}\|g^{''}\|+\frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)}{a^3(\mathfrak{m}+1)}\|g^{'}\|\,\|\zeta"\|. \qquad (4.5)
\end{aligned}
$$

In view of Lemma 5 and equation (4.2), we get

$$
\begin{aligned}
|\widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)| &\leq |\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)|+|\varphi(\varkappa)|+\left|(\varphi\circ\zeta^{-1})\left(\frac{\mathfrak{m}\zeta(\varkappa)(\alpha+1)+1}{(\mathfrak{m}+1)(\alpha+1)}\right)\right|\\
&\leq 3\|\varphi\|.
\end{aligned}
$$

Hence, for $\varphi\in$ and $g\in W^2$, using equation (4.2), we obtain

$$
\begin{aligned}
|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)-\varphi(\varkappa)| &\leq |\widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}(f-g;\varkappa)|+|\widetilde{\mathcal{L}}^{(\zeta)}_{\mathfrak{m},\alpha}(g;\varkappa)-g(\varkappa)|+|g(\varkappa)-\varphi(\varkappa)|\\
&+ \left|(\varphi\circ\zeta^{-1})\left(\frac{\mathfrak{m}\zeta(\varkappa)(\alpha+1)+1}{(\mathfrak{m}+1)(\alpha+1)}\right)-(\varphi\circ\zeta^{-1})(\zeta(\varkappa))\right|\\
&\leq 4\|f-g\|+\frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)}{a^2(\mathfrak{m}+1)}\|g^{''}\|+\frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)}{a^3(\mathfrak{m}+1)}\|g^{'}\|\ \ |\zeta"\|\\
&+ \omega\left(\varphi\circ\zeta^{-1};\ \left|\frac{1}{\mathfrak{m}+1}\left(\frac{1}{\alpha+1}-\zeta(\varkappa)\right)\right|\right).
\end{aligned}
$$

Let $C_1=\max\{4,\frac{1}{a^2},\frac{\|\zeta"\|}{a^3}\}$, then

$$\begin{aligned}|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)-\varphi(\varkappa)| &\leq C_1\left\{\|f-g\|+\frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)}{(\mathfrak{m}+1)}\|g^{''}\|+\frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)}{(\mathfrak{m}+1)}\|g^{'}\|+\frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)}{(\mathfrak{m}+1)}\|g\|\right\}\\ &\quad+\omega\left(\varphi\circ\zeta^{-1};\ \left|\frac{1}{\mathfrak{m}+1}\left(\frac{1}{\alpha+1}-\zeta(\varkappa)\right)\right|\right).\end{aligned}\tag{4.6}$$

We have

$$\begin{aligned}\omega(\varphi\circ\zeta^{-1};t) &= sup\{|\varphi(\zeta^{-1}(y))-\varphi(\zeta^{-1}(\varkappa))|:0\leq y-\varkappa\leq\mathfrak{t}\}\\ &=\sup\{|\varphi(\overline{y})-\varphi(\overline{\varkappa})|:0\leq\zeta(\overline{y})-\zeta(\overline{\varkappa})\leq\mathfrak{t}\}.\end{aligned}$$

If $0\leq\zeta(\overline{y})-\zeta(\overline{\varkappa})\leq\mathfrak{t}$, then $0\leq(\overline{y}-\overline{\varkappa})\zeta'(u)\leq\mathfrak{t}$ for some $u\in(\overline{\varkappa},\overline{y})$ i.e. $0\leq\overline{y}-\overline{\varkappa}\leq\frac{t}{\zeta'(u)}\leq\frac{t}{a}$, and so

$$\omega(\varphi\circ\zeta^{-1};t)\leq sup\{|\varphi(\overline{y})-\varphi(\overline{\varkappa})|:0\leq\overline{y}-\overline{\varkappa}\leq\frac{t}{a}\}=\omega(\varphi;\frac{t}{a}).\tag{4.7}$$

Using (4.7) in (4.6), we obtain

$$\begin{aligned}|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)-\varphi(\varkappa)| &= C_1\left\{\|f-g\|+\frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)\|g^{''}\|}{(\mathfrak{m}+1)}+\frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)\|g^{'}\|}{(\mathfrak{m}+1)}+\frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)\|g\|}{(\mathfrak{m}+1)}\right\}\\ &+\ \omega\left(\varphi;\left|\frac{1}{\mathfrak{m}+1}\left(\frac{1}{\alpha+1}-\zeta(\varkappa)\right)\right|\right).\end{aligned}$$

Now, taking the infimum on the right hand side over all $g\in W^2$, and using (4.1), we obtain

$$\begin{aligned}|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)-\varphi(\varkappa)| &\leq C_1K_2\left(\varphi;\frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)}{(\mathfrak{m}+1)}\right)+\omega\left(\varphi;\left|\frac{1}{\mathfrak{m}+1}\left(\frac{1}{\alpha+1}-\zeta(\varkappa)\right)\right|\right)\\ &\leq M\,\omega_2\left(\varphi;\frac{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa)}{\sqrt{(\mathfrak{m}+1)}}\right)+\omega\left(\varphi;\left|\frac{1}{\mathfrak{m}+1}\left(\frac{1}{\alpha+1}-\zeta(\varkappa)\right)\right|\right),\end{aligned}$$

which completes the proof. □

In the next result, we study the rate of approximation by the operators $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ for functions $\varphi(\varkappa)$ in a Lipschitz-type space.
Following [11], let us now consider the Lipschitz-type space:

$$Lip_M(\rho)=\left\{\varphi\in\mathcal{C}(\mathfrak{I}):|\varphi(\mathfrak{t})-\varphi(\varkappa)|\leq M\frac{|\mathfrak{t}-\varkappa|^{\rho}}{(\mathfrak{t}+\varkappa)^{\frac{\rho}{2}}}:\mathfrak{t}\in\mathfrak{I},\ \varkappa\in(0,1]\right\}.$$

Theorem 3. *Let $\varphi\in Lip_M(\rho)$. Then for all $\varkappa\in(0,1]$, we have*

$$|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)-\varphi(\varkappa)|\quad\leq\quad M\left(\frac{(\gamma^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa))^2}{(\mathfrak{m}+1)\zeta(\varkappa)}\right).$$

Proof. First, we show that result for the case $\rho=2$, we may write

$$\begin{aligned}|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)-\varphi(\varkappa)| &\leq \sum_{k=0}^{\mathfrak{m}}p^{(\alpha)}_{\mathfrak{m},k}(\zeta(\varkappa))\int_0^1\left|(\varphi\circ\zeta^{-1})\left(\frac{k+\mathfrak{t}^{\alpha}}{\mathfrak{m}+1}\right)-(\varphi\circ\zeta^{-1})(\zeta(\varkappa))\right|dt\\ &\leq M\sum_{k=0}^{\mathfrak{m}}p^{(\alpha)}_{\mathfrak{m},k}(\zeta(\varkappa))\int_0^1\frac{|\mathfrak{t}-\zeta(\varkappa)|^2}{\mathfrak{t}+\zeta(\varkappa)}d\mathfrak{t}.\end{aligned}$$

Using the fact, that $\frac{1}{\sqrt{t+\zeta(\varkappa)}} \leq \frac{1}{\sqrt{\zeta(\varkappa)}}$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}(\varphi;\varkappa)-\varphi(\varkappa)| &\leq \frac{M}{\zeta(\varkappa)}\sum_{k=0}^{\mathfrak{m}} p_{\mathfrak{m},k}^{(\alpha)}(\zeta(\varkappa))\int_0^1(\zeta(\mathfrak{t})-\zeta(\varkappa))^2 d\mathfrak{t} \\ &\leq \frac{M}{\zeta(\varkappa)}\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}((\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa) \\ &= \frac{M}{\zeta(\varkappa)(\mathfrak{m}+1)}(\gamma_{\mathfrak{m},\zeta}^{(\alpha)}(\varkappa))^2, \end{aligned}$$

hence the result is obtained for $\rho = 2$.
Now, we prove the theorem for the case $0 < \rho < 2$. Using Lemma 6 and applying the Hölder's inequality with $p = \frac{2}{\rho}$ and $q = \frac{2}{2-\rho}$, we get

$$\begin{aligned} |\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}(\varphi;\varkappa)-\varphi(\varkappa)| &\leq \sum_{k=0}^{\mathfrak{m}} p_{\mathfrak{m},k}^{(\alpha)}(\zeta(\varkappa))\int_0^1\left|(\varphi\circ\zeta^{-1})\left(\frac{k+\mathfrak{t}^\alpha}{\mathfrak{m}+1}\right)-(\varphi\circ\zeta^{-1})(\zeta(\varkappa))\right| d\mathfrak{t} \\ &\leq \left\{\sum_{k=0}^{\mathfrak{m}} p_{\mathfrak{m},k}^{(\alpha)}(\zeta(\varkappa))\left(\int_0^1\left|(\varphi\circ\zeta^{-1})\left(\frac{k+\mathfrak{t}^\alpha}{\mathfrak{m}+1}\right)-(\varphi\circ\zeta^{-1})(\zeta(\varkappa))\right|d\mathfrak{t}\right)^{\frac{2}{\rho}}\right\}^{\frac{\rho}{2}} \\ &\leq \left\{\sum_{k=0}^{\mathfrak{m}} p_{\mathfrak{m},k}^{(\alpha)}(\zeta(\varkappa))\int_0^1\left|(\varphi\circ\zeta^{-1})\left(\frac{k+\mathfrak{t}^\alpha}{\mathfrak{m}+1}\right)-(\varphi\circ\zeta^{-1})(\zeta(\varkappa))\right|^{\frac{2}{\rho}}d\mathfrak{t}\right\}^{\frac{\rho}{2}} \\ &\leq M\left\{\sum_{k=0}^{\mathfrak{m}} p_{\mathfrak{m},k}^{(\alpha)}(\zeta(\varkappa))\int_0^1\frac{(\mathfrak{t}-\zeta(\varkappa))^2}{t+\zeta(\varkappa)}d\mathfrak{t}\right\}^{\frac{\rho}{2}} \\ &\leq \frac{M}{(\zeta(\varkappa))^{\frac{\rho}{2}}}\left\{\sum_{k=0}^{\mathfrak{m}} p_{\mathfrak{m},k}^{(\alpha)}(\zeta(\varkappa))\int_0^1(\mathfrak{t}-\zeta(\varkappa))^2 d\mathfrak{t}\right\}^{\frac{\rho}{2}} \\ &\leq \frac{M}{(\zeta(\varkappa))^{\frac{\rho}{2}}}\,(\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}((\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa))^2 \\ &\leq M\left(\frac{\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}((\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa))^2}{\zeta(\varkappa)}\right)^{\frac{\rho}{2}} \\ &\leq \left(\frac{(\gamma_{\mathfrak{m},\zeta}^{(\alpha)}(\varkappa))^2}{(\mathfrak{m}+1)\zeta(\varkappa)}\right)^{\frac{\rho}{2}}. \end{aligned}$$

Thus, the proof is completed. $\square$

Next, we study a local direct estimate of the operators defined as in equation (1.5) in terms of the Lipschitz-type maximal function of order ξ given by Lenze [1] as

$$\tilde{\omega}_\xi(\varphi\circ\zeta^{-1};\varkappa) = \sup_{\mathfrak{t}\neq\varkappa,\ \mathfrak{t}\in\mathfrak{I}}\frac{|\varphi(\mathfrak{t})-\varphi(\varkappa)|}{|\mathfrak{t}-\varkappa|^\xi}, \quad \varkappa\in\mathfrak{I} \text{ and } \xi\in(0,1].$$

Theorem 4. *Let $\varphi \in \mathcal{C}(\mathfrak{I})$ and $0 < \xi \leq 1$, then for all $\varkappa \in \mathfrak{I}$, we have*

$$|\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}(\varphi;\varkappa)-\varphi(\varkappa)| \quad\leq\quad \tilde{\omega}_\xi(\varphi\circ\zeta^{-1};\varkappa)\left(\frac{(\gamma_{\mathfrak{m},\zeta}^{(\alpha)}(\varkappa))^2}{(\mathfrak{m}+1)}\right)^{\frac{\rho}{2}}.$$

Proof. We have

$$\begin{aligned} |\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}(\varphi;\varkappa)-\varphi(\varkappa)| &\leq \sum_{k=0}^{\mathfrak{m}} p_{\mathfrak{m},k}^{(\alpha)}(\zeta(\varkappa))\int_0^1\left|(\varphi\circ\zeta^{-1})\left(\frac{k+\mathfrak{t}^\alpha}{\mathfrak{m}+1}\right)-(\varphi\circ\zeta^{-1})(\zeta(\varkappa))\right| d\mathfrak{t} \\ &\leq \tilde{\omega}_\xi(\varphi\circ\zeta^{-1};\varkappa)\sum_{k=0}^{\mathfrak{m}} p_{\mathfrak{m},k}^{(\alpha)}(\zeta(\varkappa))\int_0^1|\mathfrak{t}-\zeta(\varkappa)|^\rho\, dx \end{aligned}$$

Now, applying Hölder inequality with $p = \frac{2}{\rho},\ q = 1 - \frac{2}{\rho}$, using Lemma (6), we obtain

$$\begin{aligned} |\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}(\varphi;\varkappa) - \varphi(\varkappa)| &\leq \tilde{\omega}_{\xi}(\varphi\circ\zeta^{-1};\varkappa)\left\{\sum_{k=0}^{\mathfrak{m}} p_{\mathfrak{m},k}^{(\alpha)}(\zeta(\varkappa))\int_0^1 (\mathfrak{t}-\zeta(\varkappa))^2\,dx\right\}^{\frac{\rho}{2}} \\ &\leq \tilde{\omega}_{\xi}(\varphi\circ\zeta^{-1};\varkappa)\left(\frac{(\gamma_{\mathfrak{m},\zeta}^{(\alpha)}(\varkappa))^2}{(\mathfrak{m}+1)}\right)^{\frac{\rho}{2}}. \end{aligned}$$

This complete the proof. □

Now we present a Voronovskaja type asymptotic formula for the operators $\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}(\varphi;\varkappa)$.

Theorem 5. *Let $\varphi \in \mathcal{C}(\mathfrak{I})$. If φ'' exists at a point $\varkappa \in \mathfrak{I}$, then we have*

$$\lim_{\mathfrak{m}\to\infty}\mathfrak{m}\left(\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}(\varphi;\varkappa)-\varphi(\varkappa)\right) = (\varphi\circ\zeta^{-1})'(\zeta(\varkappa))\left(\frac{1}{\alpha+1}-\zeta(\varkappa)\right)+\frac{(\varphi\circ\zeta^{-1})"(\zeta(\varkappa))}{2}\zeta(\varkappa)(1-\zeta(\varkappa)).$$

Proof. By Taylor formula, we can write

$$\begin{aligned} (\varphi\circ\zeta^{-1})(\zeta(\mathfrak{t})) &= (\varphi\circ\zeta^{-1})(\zeta(\varkappa)) + (\zeta(\mathfrak{t})-\zeta(\varkappa))(\varphi\circ\zeta^{-1})'(\zeta(\varkappa)) + \frac{1}{2}(\zeta(\mathfrak{t})-\zeta(\varkappa))^2(\varphi\circ\zeta^{-1})"(\zeta(\varkappa)) \\ &+ \xi(\zeta(\mathfrak{t}),\zeta(\varkappa))(\zeta(\mathfrak{t})-\zeta(\varkappa))^2, \end{aligned}$$

where $\xi(\zeta(\mathfrak{t}),\zeta(\varkappa)) \to 0$ as $\zeta(\mathfrak{t}) \to \zeta(\varkappa)$ and is a continuous function on $\mathfrak{I}$. Applying the operator $\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}$ on both sides of the above equation

$$\begin{aligned} \mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}(\varphi;\varkappa) &= \mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}\left((\varphi\circ\zeta^{-1})(\zeta(\varkappa));\varkappa\right) + \mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}\left((\zeta(\mathfrak{t})-\zeta(\varkappa))(\varphi\circ\zeta^{-1})'(\zeta(\varkappa));\varkappa\right) \\ &+ \frac{1}{2}\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}\left((\zeta(\mathfrak{t})-\zeta(\varkappa))^2(\varphi\circ\zeta^{-1})"(\zeta(\varkappa));\varkappa\right) + \mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}\left(\xi(\zeta(\mathfrak{t}),\zeta(\varkappa))(\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa\right) \\ &= \varphi(\varkappa) + (\varphi\circ\zeta^{-1})'(\zeta(\varkappa))\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}\left((\zeta(\mathfrak{t})-\zeta(\varkappa));\varkappa\right) \\ &+ \frac{1}{2}(\varphi\circ\zeta^{-1})"(\zeta(\varkappa))\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}\left((\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa\right) + \mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}\left(\xi(\zeta(\mathfrak{t}),\zeta(\varkappa))(\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa\right) \\ &= \varphi(\varkappa) + \frac{1}{\mathfrak{m}+1}(\varphi\circ\zeta^{-1})'(\zeta(\varkappa))\left(\frac{1}{\alpha+1}-\zeta(\varkappa)\right) \\ &+ \frac{1}{2(\mathfrak{m}+1)^2}(\varphi\circ\zeta^{-1})"(\zeta(\varkappa))\left[\zeta^2(\varkappa)-\frac{2\zeta(\varkappa)}{\alpha+1}+\frac{1}{2\alpha+1}+\mathfrak{m}\zeta(\varkappa)(1-\zeta(\varkappa))\right] \\ &+ \mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}\left(\xi(\zeta(\mathfrak{t}),\zeta(\varkappa))(\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa\right). \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{\mathfrak{m}\to\infty}\mathfrak{m}\,(\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}(\varphi;\varkappa)-\varphi(\varkappa)) &= \lim_{\mathfrak{m}\to\infty}\mathfrak{m}\,\frac{(\varphi\circ\zeta^{-1})'(\zeta(\varkappa))}{\mathfrak{m}+1}\left(\frac{1}{\alpha+1}-\zeta(\varkappa)\right) \\ &+ \lim_{\mathfrak{m}\to\infty}\mathfrak{m}\,\frac{(\varphi\circ\zeta^{-1})"(\zeta(\varkappa))}{2(\mathfrak{m}+1)^2}\left[\zeta^2(\varkappa)-\frac{2\zeta(\varkappa)}{\alpha+1}+\frac{1}{2\alpha+1}+\mathfrak{m}\zeta(\varkappa)(1-\zeta(\varkappa))\right] \\ &+ \lim_{\mathfrak{m}\to\infty}\mathfrak{m}\,\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}\left(\xi(\zeta(\mathfrak{t}),\zeta(\varkappa))(\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa\right) \\ &= (\varphi\circ\zeta^{-1})'(\zeta(\varkappa))\left(\frac{1}{\alpha+1}-\zeta(\varkappa)\right)+\frac{(\varphi\circ\zeta^{-1})"(\zeta(\varkappa))}{2}\zeta(\varkappa)(1-\zeta(\varkappa)) \\ &+ \lim_{\mathfrak{m}\to\infty}\mathfrak{m}\,\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}\left(\xi(\zeta(\mathfrak{t}),\zeta(\varkappa))(\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa\right), \end{aligned} \tag{4.8}$$

uniformly in $\varkappa \in \mathfrak{I}$. Applying Cauchy-Schwarz inequality and Remark 1, we get

$$\mathfrak{m}\,\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}\Bigg(\xi(\zeta(\mathfrak{t}),\zeta(\varkappa))(\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa\Bigg) \quad\leq\quad \mathfrak{m}\left\{\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\xi^2(\zeta(\mathfrak{t}),\zeta(\varkappa));\varkappa)\right\}^{\frac{1}{2}}\left\{\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}((\zeta(\mathfrak{t})-\zeta(\varkappa))^4;\varkappa)\right\}^{\frac{1}{2}}.$$

We observe that $\xi^2(\zeta(\mathfrak{t}),\zeta(\varkappa)) \in \mathcal{C}(\mathfrak{I})$ and $\xi^2(\zeta(\varkappa),\zeta(\varkappa)) = 0$, hence, by Theorem 1 we are led to

$$\lim_{\mathfrak{m}\to\infty}\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}\Bigg(\xi^2(\zeta(\mathfrak{t}),\zeta(\varkappa));\varkappa\Bigg) \quad=\quad \xi^2(\zeta(\varkappa),\zeta(\varkappa)) = 0,$$

uniformly with respect to $\varkappa \in \mathfrak{I}$.
Further, using Remark 1, $\lim\limits_{\mathfrak{m}\to\infty}\mathfrak{m}\sqrt{\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}((\zeta(\mathfrak{t})-\zeta(\varkappa))^4;\varkappa)}$ is finite and

$$\lim_{\mathfrak{m}\to\infty}\mathfrak{m}\,\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}\Bigg(\xi(\zeta(\mathfrak{t}),\zeta(\varkappa))(\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa\Bigg) = 0, \tag{4.9}$$

uniformly in $\varkappa \in \mathfrak{I}$.
Finally, consideration of (4.8) and (4.9) completes the proof. $\square$

5. Rate of Convergence

In mathematical analysis, a function of bounded variation, also known as BV function, is a real-valued function whose total variation is bounded (finite), the graph of a function having this property is well behaved in a precise sense. For a continuous function of a single variable, being of bounded variation means that the distance along the direction of the y-axis, neglecting the contribution of motion along $\varkappa$-axis, traveled by a point moving along the graph has a finite value.
We have the following chains of inclusions for continuous functions over a closed, bounded interval of the real line:
Continuously differentiable $\subseteq$ Lipschitz continuous $\subseteq$ absolutely continuous $\subseteq$ continuous and bounded variation $\subseteq$ differentiable almost everywhere.
DBV($\mathfrak{I}$) denotes the class of all absolute continuous function φ defined on $\mathfrak{I}$, having on $\mathfrak{I}$, a derivative φ' equivalent with a function of bounded variation on $\mathfrak{I}$. We notice that the functions $\varphi \in DVV(\mathfrak{I})$ possess a representation

$$\varphi(\varkappa) \quad=\quad \int_0^1 g(\mathfrak{t})d\mathfrak{t} + \varphi(0)$$

where $g \in BV(\mathfrak{I})$, i.e. g is a function of bounded variation on I.
The integral representation of the operator $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}$ is defined as

$$\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa) = \int_0^1 N^{(\alpha)}_{\mathfrak{m}}(\zeta,\varkappa,\mathfrak{t})(\varphi\circ\zeta^{-1})(\mathfrak{t})d\mathfrak{t}, \tag{5.1}$$

where the Kernal $N^{(\alpha)}_{\mathfrak{m}}(\zeta,\varkappa,\mathfrak{t})$ is given by

$$N^{(\alpha)}_{\mathfrak{m}}(\zeta,\varkappa,\mathfrak{t}) = \sum_{k=0}^{\mathfrak{m}} p^{(\alpha)}_{\mathfrak{m},k}(\zeta(\varkappa))\gamma_{\mathfrak{m},k}(\mathfrak{t}),$$

where $\gamma_{\mathfrak{m},k}(\mathfrak{t})$ is the characteristic function of the interval $[\frac{k}{\mathfrak{m}},\frac{k+1}{\mathfrak{m}}]$ with respect to $\mathfrak{I}$.

Lemma 7. *For a fixed $\varkappa \in (0,1)$ and sufficiently large $\mathfrak{m}$, we have*

(i) $\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,y) = \int_0^y N^{(\alpha)}_{\mathfrak{m}}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t} = \frac{(\gamma^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa))^2}{(\mathfrak{m}+1)(\zeta(\varkappa)-y)^2},\ 0 \leq y < \zeta(\varkappa)$

(ii) $1-\beta^{(\alpha)}_{\mathfrak{m},\zeta}(x,z) = \int_z^y N^{(\alpha)}_{\mathfrak{m}}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t} = \frac{(\gamma^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa))^2}{(z-\zeta(\varkappa))^2},\ \zeta(\varkappa) < z < 1,$

where $(\gamma^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa))^2 = \varphi^2_{\mathfrak{m},\zeta}(\varkappa) + \frac{1}{(\mathfrak{m}+1)(2\alpha+1)}$.

Proof. Using equation (5.1), we get

$$\begin{aligned}
\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa, y) &= \int_0^y N^{(\alpha)}_{\mathfrak{m}}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t}\\
&\le \left(\frac{\zeta(\varkappa)-\mathfrak{t}}{\zeta(\varkappa)-y}\right)^2 N^{(\alpha)}_{\mathfrak{m}}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t}\\
&\le \frac{1}{(\zeta(\varkappa)-y)^2}\int_0^y (\zeta(\varkappa)-\mathfrak{t})^2 N^{(\alpha)}_{\mathfrak{m}}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t}\\
&\le \frac{1}{(\zeta(\varkappa)-y)^2}\int_0^y (\mathfrak{t}-\zeta(\varkappa))^2 N^{(\alpha)}_{\mathfrak{m}}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t}
\end{aligned}$$

Now, Lemma 7, we obtained

$$\begin{aligned}
\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa, y) &\le \frac{1}{(\zeta(\varkappa)-y)^2}\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}((\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa)\\
&\le \frac{(\gamma^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa))^2}{(\mathfrak{m}+1)(\zeta(\varkappa)-y)^2},\ 0\le y<\zeta(\varkappa).
\end{aligned}$$

The proof of (ii) is similar hence the details are omitted. □

Theorem 6. *Let $\varphi\in DBV(\mathfrak{I})$. Then for every $\varkappa\in(0,1)$ and sufficiently large $\mathfrak{m}$, we have*

$$\begin{aligned}
|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)-\varphi(\varkappa)| &\le \frac{1}{2}\left|(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+))+(\varphi\circ\zeta^{-1})'(\zeta(\varkappa-))\right|\left\{\left|\frac{1}{\mathfrak{m}+1}\left(\frac{1}{\alpha+1}-\zeta(\varkappa)\right)\right|+\frac{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa)}{\sqrt{\mathfrak{m}+1}}\right\}\\
&+ \frac{(\gamma^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa))^2}{(\mathfrak{m}+1)}\left[\frac{1}{\zeta(\varkappa)}\sum_{k=1}^{[\sqrt{\mathfrak{m}}]}\left(\bigvee_{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{k}}^{\zeta(\varkappa)}(\varphi\circ\zeta^{-1})'_{\varkappa}\right)\right.\\
&+ \left.\frac{1}{(1-\zeta(\varkappa))}\sum_{k=1}^{[\sqrt{\mathfrak{m}}]}\left(\bigvee_{\zeta(\varkappa)}^{\zeta(\varkappa)+\frac{1-\zeta(\varkappa)}{k}}(\varphi\circ\zeta^{-1})'_{\varkappa}\right)\right]+\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}\left(\bigvee_{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}^{\zeta(\varkappa)}(\varphi\circ\zeta^{-1})'_{\varkappa}\right)\\
&+ \frac{1-\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}\left(\bigvee_{\zeta(\varkappa)}^{\zeta(\varkappa)+\frac{1-\zeta(\varkappa)}{k}}(\varphi\circ\zeta^{-1})'_{\varkappa}\right),
\end{aligned}$$

where $\bigvee_a^b(\varphi\circ\zeta^{-1})'_{\varkappa}$ denotes the total variation of $(\varphi\circ\zeta^{-1})'_{\varkappa}$ on $[a,b]$ and $(\varphi\circ\zeta^{-1})'_{\varkappa}$ is defined by

$$(\varphi\circ\zeta^{-1})'_{\varkappa}(\mathfrak{t})=\begin{cases}(\varphi\circ\zeta^{-1})'(\mathfrak{t})-(\varphi\circ\zeta^{-1})'(\zeta(\varkappa-)), & \text{if } 0\le\mathfrak{t}<\zeta(\varkappa)\\ 0, & \text{if } \mathfrak{t}=\zeta(\varkappa)\\ (\varphi\circ\zeta^{-1})'(\mathfrak{t})-(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+)), & \text{if } \zeta(\varkappa)<\mathfrak{t}<1.\end{cases}\tag{5.2}$$

Proof. Since $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(1;\varkappa)=1$ by applying (5.1), for every $\varkappa\in(0,1)$, we have

$$\begin{aligned}
\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)-\varphi(\varkappa) &= \mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)-\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(1;\varkappa)\cdot\varphi(\varkappa)\\
&= \mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi(\mathfrak{t});\varkappa)-\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi(\varkappa);\varkappa)\\
&= \mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}((\varphi(\mathfrak{t})-\varphi(\varkappa));\varkappa)\\
&= \int_0^1 N^{(\alpha)}_{\mathfrak{m}}(\zeta,\varkappa,\mathfrak{t})\Big((\varphi\circ\zeta^{-1})(\mathfrak{t})-(\varphi\circ\zeta^{-1})(\zeta(\varkappa))\Big)\,d\mathfrak{t}\\
&= \int_0^1\left(\int_{\zeta(\varkappa)}^{\mathfrak{t}}(\varphi\circ\zeta^{-1})(u)d\mathfrak{t}\right)N^{(\alpha)}_{\mathfrak{m}}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t}
\end{aligned}\tag{5.3}$$

For any $\varphi \in DBV(\mathfrak{I})$, from equation (5.2), we may write

$$\begin{aligned} (\varphi\circ\zeta^{-1})'(u) &= (\varphi\circ\zeta^{-1})'_{\varkappa}(u) + \frac{1}{2}\Big\{(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+)) + (\varphi\circ\zeta^{-1})'(\zeta(\varkappa-))\Big\} \\ &+ \frac{1}{2}\Big\{(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+)) - (\varphi\circ\zeta^{-1})'(\zeta(\varkappa-))\Big\} sgn(u-\zeta(\varkappa)) \\ &+ \delta_{\varkappa}(u)\Big[(\varphi\circ\zeta^{-1})'(u) - \frac{1}{2}\Big\{(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+)) + (\varphi\circ\zeta^{-1})'(\zeta(\varkappa-))\Big\}\Big], \end{aligned} \tag{5.4}$$

where

$$\delta_{\varkappa}(u) = \begin{cases} 1, & \text{if } u = \zeta(\varkappa) \\ 0, & \text{if } u \neq \zeta(\varkappa). \end{cases}$$

Obviously,

$$\int_0^1 \left(\int_{\zeta(\varkappa)}^{\mathfrak{t}} \left((\varphi\circ\zeta^{-1})'(u) - \frac{1}{2}\Big\{(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+)) + (\varphi\circ\zeta^{-1})'(\zeta(\varkappa-)\Big\}\right)\delta_{\varkappa}(u)du\right) N_{\mathfrak{m}}^{(\alpha)}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t} = 0. \tag{5.5}$$

Using (5.1), we have

$$\begin{aligned} &\int_0^1 \left(\int_{\zeta(\varkappa)}^{\mathfrak{t}} \left(\frac{1}{2}\Big\{(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+)) + (\varphi\circ\zeta^{-1})'(\zeta(\varkappa-)\Big\}\right)du\right) N_{\mathfrak{m}}^{(\alpha)}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t} \\ = &\ \frac{1}{2}\Big\{(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+)) + (\varphi\circ\zeta^{-1})'(\zeta(\varkappa-))\Big\}\int_0^1 (\mathfrak{t}-\zeta(\varkappa))N_{\mathfrak{m}}^{(\alpha)}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t} \\ = &\ \frac{1}{2}\Big\{(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+)) + (\varphi\circ\zeta^{-1})'(\zeta(\varkappa-))\Big\}\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}\Big(((\zeta(\mathfrak{t})-\zeta(\varkappa));\varkappa\Big). \end{aligned} \tag{5.6}$$

Applying Cauchy-Schwarz inequality

$$\begin{aligned} &\left| \int_0^1 N_{\mathfrak{m}}^{(\alpha)}(\zeta,\varkappa,\mathfrak{t})\left(\int_{\zeta(\varkappa)}^{\mathfrak{t}} \tfrac{1}{2}\Big\{(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+)) - (\varphi\circ\zeta^{-1})'(\zeta(\varkappa-))\Big\} sgn(u-\zeta(\varkappa))du\right)d\mathfrak{t}\right| \\ &\leq \Big|(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+)) - (\varphi\circ\zeta^{-1})'(\zeta(\varkappa-))\Big| \int_0^1 |\mathfrak{t}-\zeta(\varkappa)| N_{\mathfrak{m}}^{(\alpha)}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t} \\ &\leq \Big|(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+)) - (\varphi\circ\zeta^{-1})'(\zeta(\varkappa-))\Big| \mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}(|\zeta(\mathfrak{t})-\zeta(\varkappa)|;\varkappa) \\ &\leq \Big|(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+)) - (\varphi\circ\zeta^{-1})'(\zeta(\varkappa-))\Big| \Big(\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}((\zeta(\mathfrak{t})-\zeta(\varkappa))^2;\varkappa)\Big)^{\frac{1}{2}}. \end{aligned} \tag{5.7}$$

Applying Lemma 4, Lemma 7 and using (5.3)-(5.7), we get

$$\begin{aligned} |\mathcal{L}_{\mathfrak{m},\alpha}^{(\zeta)}(\varphi;\varkappa) - \varphi(\varkappa)| &\leq \frac{1}{2}\Big|(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+)) + (\varphi\circ\zeta^{-1})'(\zeta(\varkappa-))\Big| \left|\frac{1}{\mathfrak{m}+1}\Big(\frac{1}{\alpha+1} - \zeta(\varkappa)\Big)\right| \\ &+ \frac{1}{2}\Big|(\varphi\circ\zeta^{-1})'(\zeta(\varkappa+)) + (\varphi\circ\zeta^{-1})'(\zeta(\varkappa-))\Big| \frac{\gamma_{\mathfrak{m},\zeta}^{(\alpha)}(\varkappa)}{\sqrt{\mathfrak{m}+1}} \\ &+ \left|\int_0^{\zeta(\varkappa)}\left(\int_{\zeta(\varkappa)}^{\mathfrak{t}} (\varphi\circ\zeta^{-1})'_{\varkappa}(u)du\right)N_{\mathfrak{m}}^{(\alpha)}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t}\right| \\ &+ \left|\int_{\zeta(\varkappa)}^{1}\left(\int_{\zeta(\varkappa)}^{\mathfrak{t}} (\varphi\circ\zeta^{-1})'_{\varkappa}(u)du\right)N_{\mathfrak{m}}^{(\alpha)}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t}\right|. \end{aligned} \tag{5.8}$$

Let $H_{\mathfrak{m},\zeta}^{(\alpha)}((\varphi\circ\zeta^{-1})'_{\varkappa},\varkappa) = \int_0^{\zeta(\varkappa)}\left(\int_{\zeta(\varkappa)}^{\mathfrak{t}}(\varphi\circ\zeta^{-1})'_{\varkappa}(u)du\right)N_{\mathfrak{m}}^{(\alpha)}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t}$ and

$$G_{\mathfrak{m},\zeta}^{(\alpha)}((\varphi\circ\zeta^{-1})'_{\varkappa},\varkappa) = \int_{\zeta(\varkappa)}^1\left(\int_{\zeta(\varkappa)}^{\mathfrak{t}}(\varphi\circ\zeta^{-1})'_{\varkappa}(u)du\right)N_{\mathfrak{m}}^{(\alpha)}(\zeta,\varkappa,\mathfrak{t})d\mathfrak{t}.$$

To complete the proof, it is sufficient to estimate the terms $H^{(\alpha)}_{\mathfrak{m},\zeta}((\varphi\circ\zeta^{-1})'_{\varkappa},\varkappa)$ and $G^{(\alpha)}_{\mathfrak{m},\zeta}((\varphi\circ\zeta^{-1})'_{\varkappa},\varkappa)$. Since, $\int_a^b d_{\mathfrak{t}}\beta^{(\zeta)}_{n,r,s}(\varkappa,\mathfrak{t})\leq 1$ for all $[a,b]\subseteq\mathfrak{I}$, using integration by parts with $y=\zeta(\varkappa)-\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}$, we get

$$\begin{aligned}\left|H^{(\alpha)}_{\mathfrak{m},\zeta}((\varphi\circ\zeta^{-1})'_{\varkappa},\varkappa)\right| &= \left|\int_0^{\zeta(\varkappa)}\left(\int_{\zeta(\varkappa)}^{\mathfrak{t}}(\varphi\circ\zeta^{-1})'_{\varkappa}(u)du\right)d_{\mathfrak{t}}\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t})\right|\\ &= \left|\int_0^{\zeta(\varkappa)}\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t})(\varphi\circ\zeta^{-1})'_{\varkappa}(\mathfrak{t})d\mathfrak{t}\right|\\ &\leq \left(\int_0^{y}+\int_y^{\zeta(\varkappa)}\right)|(\varphi\circ\zeta^{-1})'_{\varkappa}(\mathfrak{t})|\;\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t})d\mathfrak{t}\\ &\leq \int_0^{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}|(\varphi\circ\zeta^{-1})'_{\varkappa}(\mathfrak{t})|\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t})d\mathfrak{t}+\int_{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}^{\zeta(\varkappa)}|(\varphi\circ\zeta^{-1})'_{\varkappa}(\mathfrak{t})|\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t})d\mathfrak{t}.\end{aligned}$$

Using Lemma 7 and considering $t=\zeta(\varkappa)-\frac{\zeta(\varkappa)}{u}$, we get

$$\begin{aligned}\int_0^{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}|(\varphi\circ\zeta^{-1})'_{\varkappa}(\mathfrak{t})|\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t})d\mathfrak{t} &\leq \frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)}{(\mathfrak{m}+1)}\int_0^{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}\left|(\varphi\circ\zeta^{-1})'_{\varkappa}(\mathfrak{t})-(\varphi\circ\zeta^{-1})'_{x}(\zeta(\varkappa))\right|\frac{d\mathfrak{t}}{(\zeta(\varkappa)-\mathfrak{t})^2}\\ &\leq \frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)}{(\mathfrak{m}+1)}\int_0^{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}\left(\bigvee_t^{\zeta(\varkappa)}(\varphi\circ\zeta^{-1})'_{\varkappa}\right)\frac{d\mathfrak{t}}{(\mathfrak{m}+1)}\\ &= \frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)}{(\mathfrak{m}+1)\zeta(\varkappa)}\int_1^{\sqrt{\mathfrak{m}}}\left(\bigvee_{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{u}}^{\zeta(\varkappa)}(\varphi\circ\zeta^{-1})'_{\varkappa}\right)d\mathfrak{t}\\ &\leq \frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)}{(\mathfrak{m}+1)\zeta(\varkappa)}\sum_{k=1}^{[\sqrt{\mathfrak{m}}]}\left(\bigvee_{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{k}}^{\zeta(\varkappa)}(\varphi\circ\zeta^{-1})'_{\varkappa}\right). \end{aligned}\tag{5.9}$$

Since $(\varphi\circ\zeta^{-1})'_{\varkappa}(\zeta(\varkappa))=0$ and $\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t})\leq 1$, then we have

$$\begin{aligned}\int_{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}^{\zeta(\varkappa)}|(\varphi\circ\zeta^{-1})'_{\varkappa}(\mathfrak{t})|\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t})d\mathfrak{t} &= \int_{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}^{\zeta(\varkappa)}\left|(\varphi\circ\zeta^{-1})'_{\varkappa}(\mathfrak{t})-(\varphi\circ\zeta^{-1})'_{\varkappa}(\zeta(\varkappa))\right|\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t})d\mathfrak{t}\\ &\leq \int_{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}^{\zeta(\varkappa)}\left(\bigvee_t^{\zeta(\varkappa)}(\varphi\circ\zeta^{-1})'_{\varkappa}\right)d\mathfrak{t}\\ &\leq \left(\bigvee_{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}^{\zeta(\varkappa)}(\varphi\circ\zeta^{-1})'_{\varkappa}\right)\int_{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}^{\zeta(\varkappa)}d\mathfrak{t}\\ &= \frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}\left(\bigvee_{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}^{\zeta(\varkappa)}(\varphi\circ\zeta^{-1})'_{\varkappa}\right),\end{aligned}$$

Thus,

$$\begin{aligned}|H^{(\alpha)}_{\mathfrak{m},\zeta}((\varphi\circ\zeta^{-1})'_{\varkappa},\varkappa)| &= \frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^2(\varkappa)}{(\mathfrak{m}+1)\zeta(\varkappa)}\sum_{k=1}^{[\sqrt{\mathfrak{m}}]}\left(\bigvee_{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{k}}^{\zeta(\varkappa)}(\varphi\circ\zeta^{-1})'_{\varkappa}\right)\\ &+ \frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}\left(\bigvee_{\zeta(\varkappa)-\frac{\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}^{\zeta(\varkappa)}(\varphi\circ\zeta^{-1})'_{\varkappa}\right).\end{aligned}\tag{5.10}$$

Also, using integration by part in $G^{(\alpha)}_{\mathfrak{m},\zeta}((\varphi\circ\zeta^{-1})'_{\varkappa},\varkappa)$ and applying Lemma 7 with $z=\zeta(\varkappa)+\frac{1-\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}$, we reached

$$\begin{aligned}
|G^{(\alpha)}_{\mathfrak{m},\zeta}((\varphi\circ\zeta^{-1})'_{\varkappa},\varkappa)| &= \left|\int_{\zeta(\varkappa)}^{1}\left(\int_{\zeta(\varkappa)}^{\mathfrak{t}}(\varphi\circ\zeta^{-1})'_{\varkappa}(u)du\right)N_{\mathfrak{m}}(\zeta,\varkappa,\mathfrak{t})^{(\alpha)}(\varkappa,\mathfrak{t})d\mathfrak{t}\right|\\
&= \left|\int_{\zeta(\varkappa)}^{z}\left(\int_{\zeta(\varkappa)}^{\mathfrak{t}}(\varphi\circ\zeta^{-1})'_{\varkappa}(u)du\right)d_{\mathfrak{t}}(1-\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t}))\right.\\
&+ \left.\int_{z}^{1}\left(\int_{\zeta(\varkappa)}^{\mathfrak{t}}(\varphi\circ\zeta^{-1})'_{\varkappa}(u)du\right)d_{\mathfrak{t}}(1-\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t}))\right|
\end{aligned}$$

$$\begin{aligned}
&= \left|\left[\left(\int_{\zeta(\varkappa)}^{\mathfrak{t}}(\varphi\circ\zeta^{-1})'_{\varkappa}(u)du\right)(1-\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t}))\right]_{\zeta(\varkappa)}^{z}-\int_{\zeta(\varkappa)}^{z}(\varphi\circ\zeta^{-1})'_{\varkappa}(\mathfrak{t})(1-\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t}))d\mathfrak{t}\right.\\
&+ \left.\left[\left(\int_{\zeta(\varkappa)}^{\mathfrak{t}}(\varphi\circ\zeta^{-1})'_{\varkappa}(u)du\right)(1-\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t}))\right]_{z}^{1}-\int_{z}^{1}(\varphi\circ\zeta^{-1})'_{\varkappa}(\mathfrak{t})(1-\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t}))d\mathfrak{t}\right|\\
&= \left|\int_{\zeta(\varkappa)}^{z}(\varphi\circ\zeta^{-1})'_{\varkappa}(\mathfrak{t})(1-\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t}))d\mathfrak{t}+\int_{z}^{1}(\varphi\circ\zeta^{-1})'_{\varkappa}(\mathfrak{t})(1-\beta^{(\alpha)}_{\mathfrak{m},\zeta}(\varkappa,\mathfrak{t}))d\mathfrak{t}\right|\\
&\leq \frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^{2}(\varkappa)}{\mathfrak{m}+1}\int_{z}^{1}\left(\bigvee_{\zeta(\varkappa)}^{\mathfrak{t}}(\varphi\circ\zeta^{-1})'_{\varkappa}\right)\frac{d\mathfrak{t}}{(\mathfrak{t}-\zeta(\varkappa))^{2}}+\int_{\zeta(\varkappa)}^{z}\bigvee_{\zeta(\varkappa)}^{\mathfrak{t}}(\varphi\circ\zeta^{-1})'_{\varkappa}d\mathfrak{t}\\
&\leq \frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^{2}(\varkappa)}{(\mathfrak{m}+1)}\int_{\zeta(\varkappa)+\frac{(1-\zeta(\varkappa))}{\sqrt{\mathfrak{m}}}}^{1}\left(\bigvee_{\zeta(\varkappa)}^{\mathfrak{t}}(\varphi\circ\zeta^{-1})'_{\varkappa}\right)\frac{d\mathfrak{t}}{(\mathfrak{t}-\zeta(\varkappa))^{2}}+\frac{1-\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}\left(\bigvee_{\zeta(\varkappa)}^{\zeta(\varkappa)+\frac{1-\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}(\varphi\circ\zeta^{-1})'_{\varkappa}\right).
\end{aligned}$$

By substituting $u=\frac{1-\zeta(\varkappa)}{\mathfrak{t}-\zeta(\varkappa)}$, we get

$$\begin{aligned}
G^{(\zeta)}_{n,r,s}((\varphi\circ\zeta^{-1})'_{\varkappa},\varkappa)| &\leq \frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^{2}(\varkappa)}{(\mathfrak{m}+1)}\int_{1}^{\sqrt{\mathfrak{m}}}\left(\bigvee_{\zeta(\varkappa)}^{\zeta(\varkappa)+\frac{1-\zeta(\varkappa)}{u}}(\varphi\circ\zeta^{-1})'_{\varkappa}\right)(1-\zeta(\varkappa))^{-1}du\\
&+ \frac{1-\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}\left(\bigvee_{\zeta(\varkappa)}^{\zeta(\varkappa)+\frac{1-\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}}(\varphi\circ\zeta^{-1})'_{\varkappa}\right)\\
&\leq \frac{{\gamma^{(\alpha)}_{\mathfrak{m},\zeta}}^{2}(\varkappa)}{(\mathfrak{m}+1)(1-\zeta(\varkappa))}\sum_{k=1}^{[\sqrt{\mathfrak{m}}]}\left(\bigvee_{\zeta(\varkappa)}^{\zeta(\varkappa)+\frac{1-\zeta(\varkappa)}{k}}(\varphi\circ\zeta^{-1})'_{\varkappa}\right)\\
&+ \frac{1-\zeta(\varkappa)}{\sqrt{\mathfrak{m}}}\left(\bigvee_{\zeta(\varkappa)}^{\zeta(\varkappa)+\frac{1-\zeta(\varkappa)}{k}}(\varphi\circ\zeta^{-1})'_{\varkappa}\right). \qquad (5.11)
\end{aligned}$$

Collecting the estimates (5.8)-(5.11), we get the required result. □

6. $\mathcal{A}$-Statistical approximation of Korovkin-type theorem

Let $\mathcal{A}=(b)_{i,j}$ be a positive infinite summability matrix with order $i\times j$. Let we have a sequence $\varkappa=(\varkappa)_i$, then the $\mathcal{A}-$transform $\varkappa$ denoted by $\mathcal{A}\varkappa=(\mathcal{A}\varkappa)_i$ is defined by as follows:

$$(\mathcal{A})_{i,j}=\sum_{j=0}^{\infty}a_{i,j}\varkappa_j$$

provided the above series converges for each i. The matrix $\mathcal{A}$ will be a regular matrix if $\lim_i(\mathcal{A}\varkappa)_i=l$, whenever $\lim_i(\varkappa)_i=l$. Then, the sequence $\varkappa=(\varkappa)_i$ is called $\mathcal{A}-$ statistically convergent to a number l i.e. $st_{\mathcal{A}}-\lim_i(\varkappa)_i=l$ if $\forall\epsilon>0,\ \Rightarrow\ \ \lim_i\sum_{j:|\varkappa_i-l|\geq\epsilon}a_{i,j}=0$. If we replace $\mathcal{A}$ by C_1 then $\mathcal{A}$ is Cesaro

matrix of order one and $\mathcal{A}-$ statistical convergence is change to the statistical convergence. In this way, if we take a particular case i.e. if $\mathcal{A}$ is an Identity matirix, then $\mathcal{A}-$statistical convergence is said to be ordinary convergence. In this direction, many researchers have studied for the statistical convergence properties for several sequences and classes of non-neagitive linear operators (cf. [1],[2],[3],[5]-[7] and [9] etc.).
For statistical convergence Gadjiev [8], proved the very famous Bohman-Korovkin type theorem.

Theorem 7. *Following* [8], *let* $\mathcal{L}_i : \mathcal{C}([c,d]) \to \mathcal{C}([c,d])$ *be the sequence of positive linear operators and satisfy the following conditions*

$$st_{\mathcal{A}} - \lim_i \|\mathcal{L}(e_i) - e_i\|_{\mathcal{C}([c,d])} = 0,$$

with $e_i(\mathfrak{t}) = \mathfrak{t}^i, i = 0,1,2,$ *then for any function* $\varphi \in \mathcal{C}([c,d])$, *we have*

$$st_{\mathcal{A}} - \lim_i \|\mathcal{L}_i(\varphi) - \varphi\|_{\mathcal{C}([c,d])} = 0.$$

It is very interesting to see that this result also work well for a $\mathcal{A}-$statistical convergence.

Theorem 8. *Let* $(a)_{i,j}$ *be a non-negative regular summability matrix. Then for any* $\varphi \in C(\mathfrak{I})$, *we have*

$$st_{\mathcal{A}} - \lim_i \|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi) - \varphi\|_{\mathcal{C}(\mathfrak{I})} = 0.$$

Proof. Let $e_i(\varkappa) = \varkappa^i,\ i = 0,1,2$. For $i = 0,1,2.$, it is sufficient to show that

$$st_{\mathcal{A}} - \lim_i \|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(e_i;\cdot) - e_i\|_{\mathcal{C}(\mathfrak{I})} = 0,$$

It is observed that

$$st_{\mathcal{A}} - \lim_i \|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(e_{0,\zeta};\cdot) - e_{0,\zeta}\|_{\mathcal{C}(\mathfrak{I})} = 0.$$

From Lemma 2, we have

$$\begin{aligned}
st_{\mathcal{A}} - \lim_i \|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(e_{1,\zeta};\cdot) - e_{1,\zeta}\|_{\mathcal{C}(\mathfrak{I})} &= \left|\frac{\mathfrak{m}\zeta(\varkappa)}{(\mathfrak{m}+1)} + \frac{1}{(\alpha+1)(\mathfrak{m}+1)} - \zeta(\varkappa)\right| \\
&\le \left|-\frac{\zeta(\varkappa)}{(\mathfrak{m}+1)} + \frac{1}{(\alpha+1)(\mathfrak{m}+1)}\right| \\
&\le \frac{1}{(\alpha+1)(\mathfrak{m}+1)}.
\end{aligned}$$

For $\epsilon >, 0$ let us define the following sets as follows $\mathcal{S}_1 = \{\mathfrak{m} \in \mathbb{N} : \|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(e_{1,\zeta};\cdot) - e_{1,\zeta}\| \ge \epsilon\}$ and $\mathcal{S}_2 = \{\mathfrak{m} \in \mathbb{N} : \frac{1}{(\alpha+1)(\mathfrak{m}+1)} \ge \epsilon\}$.
Then, we obtain $\mathcal{S}_1 \subset \mathcal{S}_2$ which gives that $\sum_{\mathfrak{m}\in\mathcal{S}_1} a_{i,j} \le \sum_{\mathfrak{m}\in\mathcal{S}_2} a_{i,j}$ hence

$$st_{\mathcal{A}} - \lim_i \|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(e_{1,\zeta};\cdot) - e_{1,\zeta}\|_{\mathcal{C}(\mathfrak{I})} = 0.$$

Consequently, we may write
$st_{\mathcal{A}} - \lim_i \|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(e_{2,\zeta};\cdot) - e_{2,\zeta}\|_{\mathcal{C}(\mathfrak{I})}$

$$\begin{aligned}
&\le \left|\frac{\mathfrak{m}\zeta^2(\varkappa)}{(\mathfrak{m}+1)^2} + \frac{\mathfrak{m}\zeta(\varkappa)(1-\zeta(\varkappa))}{(\mathfrak{m}+1)^2} + \frac{2\mathfrak{m}\zeta(\varkappa)}{(\alpha+1)(\mathfrak{m}+1)^2} + \frac{1}{(2\alpha+1)(\mathfrak{m}+1)^2} - \zeta^2(\varkappa)\right| \\
&\le \frac{1}{(\mathfrak{m}+1)^2}\left|\frac{2\mathfrak{m}\zeta(\varkappa)}{(\alpha+1)} + \frac{1}{(2\alpha+1)} - \left\{n^2\zeta^2(\varkappa) + \zeta^2(\varkappa) + 2\mathfrak{m}\zeta^2(\varkappa) - \mathfrak{m}\zeta(\varkappa)\right\}\right| \\
&\le \frac{2\mathfrak{m}}{(\mathfrak{m}+1)(\alpha+1)} + \frac{1}{(\mathfrak{m}+1)^2(2\alpha+1)}.
\end{aligned}$$

For $\epsilon > 0$ we define the following sets such that

$$\begin{aligned} G_1 &= \left\{ \mathfrak{m} \in \mathbb{N} : \|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(e_2;\cdot) - e_2\| \geq \epsilon \right\} \\ G_2 &= \left\{ \mathfrak{m} \in \mathbb{N} : \frac{2n}{(\mathfrak{m}+1)(\alpha+1)} \geq \frac{\epsilon}{2} \right\} \\ G_3 &= \left\{ \mathfrak{m} \in \mathbb{N} : \frac{1}{(\mathfrak{m}+1)^2(2\alpha+1)} \geq \frac{\epsilon}{2} \right\}. \end{aligned}$$

It is clear that, $G_1 \subseteq G_2 \cup G_3$ and which implies that $\sum_{\mathfrak{m}\in G_1} a_{i,j} \leq \sum_{\mathfrak{m}\in G_2} a_{i,j} + \sum_{\mathfrak{m}\in G_3} a_{i,j}$ and hence

$$st_{\mathcal{A}} - \lim_{i} \|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi) - \varphi\|_{\mathcal{C}(\mathfrak{I})} = 0,$$

we obtain the desired result. □

Finally, we show by the suitable choice of the function $\zeta(\varkappa)$ and the value of the parameters α the operators (1.5) gives the better convergence as well as error estimate also as compare to the operators (1.4).

7. Numerical Results

Example 1. *For $\mathfrak{m} = 5, 10, 20, 35, 45$ and $\alpha = 2$ the convergence of the operator $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ to the function $\varphi(\varkappa) = -6\varkappa^3 + 9\varkappa^2 - \frac{66}{25}\varkappa$ is illustrated in Figure 1 with the function $\zeta(\varkappa) = \frac{1}{2}(-1 + \sqrt{8\varkappa+1})$. Furthermore, It is observed that, the operator $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ converge to the function $\varphi(\varkappa)$ as the value of $\mathfrak{m}$ tend to ∞.*

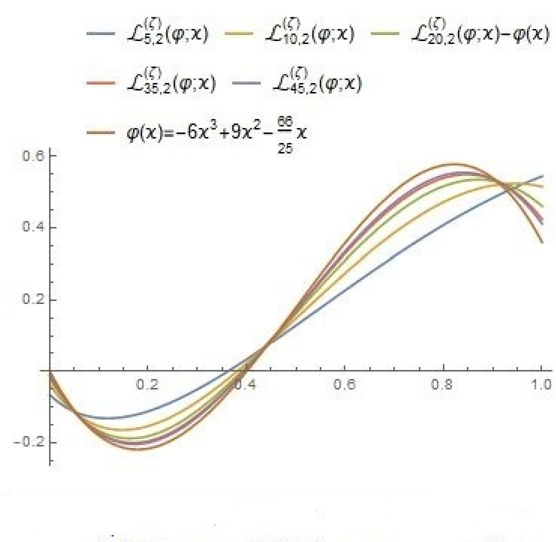

Figure 1. Convergence of operator $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ to the function $\varphi(\varkappa) = -6\varkappa^3 + 9\varkappa^2 - \frac{66}{25}\varkappa$.

Example 2. *For the choice of $\mathfrak{m} = 10, \alpha = 1$, and $\zeta(\varkappa) = \frac{3\varkappa - 4 + \sqrt{9\varkappa^2+16}}{4}$ the convergence of the two operators $K_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ and $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ to $\varphi(\varkappa) = \varkappa^4 - \varkappa^3 - \varkappa + 1$ is illustrated in Figure 2. In the Figure 2, we compare the rate of convergence of the operators $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ with $K_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ with the help of the certain functions $\varphi(\varkappa)$ by suitably choosing $\zeta(\varkappa)$, we notice that operator $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ gives better approximation to the functions $\varphi_1(\varkappa) = \varkappa^4 - \varkappa^3 - \varkappa + 1$ in Figure 2, than $K_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ in the interval* $[0.38, 1]$.

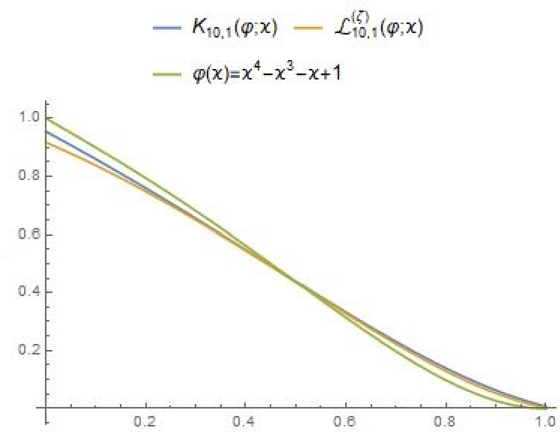

FIGURE 2. Convergence of the operators $K_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ and $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ to the function $\varkappa^4-\varkappa^3-\varkappa+1$.

Example 3. *For* $\mathfrak{m}=5,10,15,20$ *and* 25 *the error estimate* $|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)-\varphi(\varkappa)|$ *of the operators* $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ *with the function* $\varphi(\varkappa)=\varkappa^2+\varkappa-1$ *is illustrated in Figure 3, where* $\zeta(\varkappa)=\frac{3\varkappa-4+\sqrt{9\varkappa^2+6}}{4}$ *and* $\alpha=2$.
It is observed that, the error estimate $|\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)-\varphi(\varkappa)|$ *gives the better convergence as the value of* $\mathfrak{m}$ *is increased.*

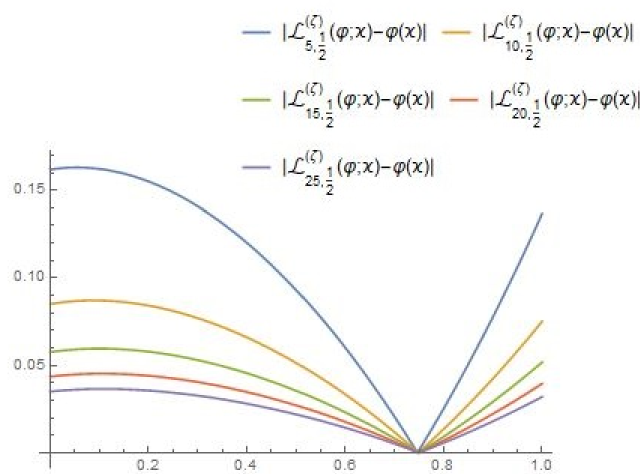

FIGURE 3. Error of approximation.

Example 4. *Figure 4,5 and Figure 6, for* $\zeta(\varkappa)=\frac{3\varkappa-4+\sqrt{9\varkappa^2+16}}{4}$, $\mathfrak{m}=15$ *and* $\alpha=2$, *show the comparison of error estimate between the operators* $K_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ *and* $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ *for the functions* $\varphi(\varkappa)=\varkappa^3+\varkappa^2+\varkappa-\frac{2}{\pi}+e^{-1},\varphi(\varkappa)=\varkappa^2+\varkappa-\frac{\pi}{2}$ *and* $\varphi(\varkappa)=\varkappa^3+9\varkappa^2-7\varkappa$, *is illustrated in Figure 4,5 and Figure 6, respectively. It is observed that for the suitable choice of* $\zeta(\varkappa)$ *and function* $\varphi(\varkappa)$ *the operator* $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ *gives a better error estimate as compare to the operator* $K_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ *when* $\mathfrak{m}=15$ *and* $\alpha=2$.

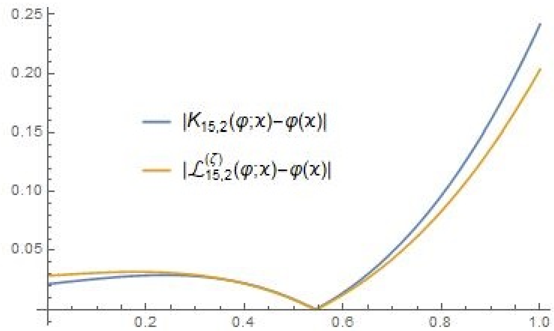

FIGURE 4. For $\mathfrak{m} = 15$, $\varphi(\varkappa) = \varkappa^3 + \varkappa^2 + \varkappa - \frac{2}{\pi} + e^{-1}$ and $\alpha = 2$, the error estimate for the operators $K_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ and $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$.

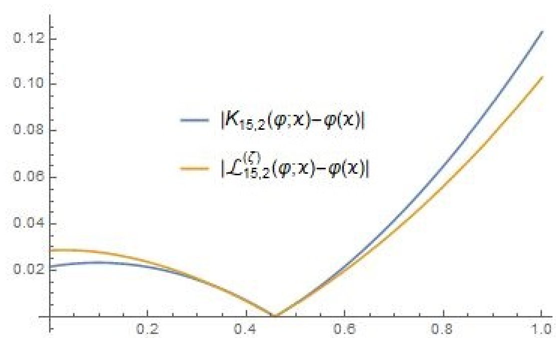

FIGURE 5. For $\mathfrak{m} = 15$, $\varphi(\varkappa) = \varkappa^2 + \varkappa - \frac{\pi}{2}$ and $\alpha = 2$, the error estimate for the operators $K_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ and $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$.

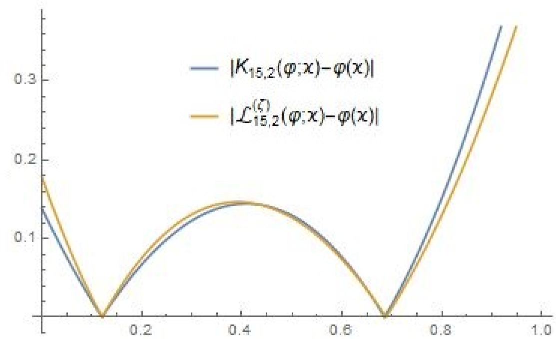

FIGURE 6. For $\mathfrak{m} = 15$, $\varphi(\varkappa) = \varkappa^3 + 9\varkappa^2 - 7\varkappa$ and $\alpha = 2$, the error estimate for the operators $K_{\mathfrak{m},\alpha}(\varphi;\varkappa)$ and $\mathcal{L}^{(\zeta)}_{\mathfrak{m},\alpha}(\varphi;\varkappa)$.

8. Conclusion

In this article, we consider a new kind of variant of the modified Bernstein-Kantorovich operators defined by Özarslan [12]. The operators (1.5) yields us to a better error estimation for a suitable choice of the different different functions as comparison to modified Bernstein-Kantorovich operators defined as in (1.4). The advantage of using a non-negative real parameter α and $\mathfrak{m}$ is that it provides exibility to the operators (1.5), so the results presented in this manuscript shows that depending on the value of the parameters α and $\mathfrak{m}$ an approximation to a function improves compared to modified Bernstein-Kantorovich operators defined as in (1.5).
Furthermore, the error approximation of the our operators (1.5) is better than modified Bernstein-Kantorovich operators defined as in (1.4).
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