Convergence Analysis of Legendre wavelets in numerical solution of linear weakly singular Volterra integral equation for union of some intervals with application in heat conduction

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Abstract

In this paper we apply the Legendre wavelets basis to solve the linear weakly singular Volterra integral equation of the second kind. The basis is defined on $[0, 1)$, and in this work we extend this interval to $[0, n)$ for some positive integer n. For this aim we solve the problem on $[0, 1)$; then we apply the Legendre wavelets on $[1, 2)$ and use the lag solution on $[0, 1)$ to obtain the solution on $[0, 2)$ and continue this procedure. Convergence analysis of Legendre wavelets on $[n, n + 1)$, is considered in Section2. We give a convergence analysis for the proposed method, established on compactness of operators. In numerical results we give two sample problems from heat conduction. For this purpose, in Section 6 we give an equivalent theorem between the proposed heat conduction problem and an integral equation. Then we solve the equivalent integral equation by the proposed method on union of some interval and obtain the solution of the heat conduction problem. As Tables and Figures of two and three dimensional plots show, accuracy of the method is reasonable and there is not any propagation of error from lag intervals. The convergence analysis and these sample problems demonstrate the accuracy and applicability of the method.

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1 Introduction

Wavelets are a powerful techniques to solve the linear and nonlinear integral equations, but most authors consider these techniques on a bounded interval solution such as $[0, 1)$ [9, 26]. We are going to solve the problem in the larger interval $[0, n)$ for some positive integer n. In this paper, we consider the linear weakly singular Volterra integral equation as follows:

$$\phi(t) - \int_0^t \phi(\tau)p(t, \tau)K(t, \tau)d\tau = r(t), \quad t > 0, \tag{1}$$

where $K(t, \tau)$ is a smooth kernel and is known, $r(t)$ is a known function in $L^2(0, \infty)$, $p(t, \tau)$ is a weakly singular kernel such as mentioned in Theorem 3 and ϕ is the unknown function. There are some methods for solving (1) such as product integration method [4, 5, 6, 7, 8, 13, 22], collocation methods [10] and so on. The product integration method is used for short intervals, and for a larger interval we must continue with another method [22]. Volterra integral equations with weakly singular kernels have solutions whose derivatives are unbounded at the left endpoint of the interval of integration. Due to this

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singular behaviour, the optimal global and local convergence results for collocation solutions in piecewise polynomial spaces on uniform meshes will no longer be valid. The use of appropriately graded meshes, or of non-polynomial collocation spaces on uniform meshes, are two of the possible alternative approaches for dealing with this order reduction problem, and these cause the complexity of problem [10]. Hence, in this paper we suggest the proposed method, and show that the method is convergent and applicable for union of some intervals.

One of the most important applications of equation (1) is in the heat conduction, which arises in physical and mechanical phenomena. For this purpose, we consider the following heat conduction problem in one spatial dimension

$$u_t = u_{xx}, \qquad 0 < x < \infty, \quad 0 < t, \tag{2}$$

$$u(x,0) = f(x), \quad 0 < x < \infty, \tag{3}$$

$$u_x(0,t) + \alpha(t)u(0,t) + \int_0^t F(t,\tau)u(0,\tau)d\tau = g(t), \qquad 0 < t, \tag{4}$$

and

$$|u(x,t)| \leq C_1 \exp\left\{C_2 x^2\right\}. \tag{5}$$

Here $u(x,t)$ is the temperature and is unknown, $C_i, i = 1,2,$ are positive constants, and the known functions $f, \alpha, g,$ and F are explained in Theorem 7. We convert this problem to a weakly singular integral equation. Both applicability and accuracy of the method are illustrated by some benchmark sample problems from this system. The organization of this paper is as follows:

The Legendre wavelets is introduced, in Section 2, and extended to a larger interval. In Section 3, numerical solution of weakly singular Volterra integral equation is illustrated by a test equation. In Section 4, algorithm of the Legendre wavelet is explained on $[0, n_0)$, for some positive integer n_0. In Section 5, we give a convergence analysis for the proposed method, established on compactness of operators in Banach spaces. In Section 6, an equivalent integral equation associated with (2)-(5) is obtained. This section shows the application of the method in the heat transfer. Finally, in Section 7, numerical results of two sample problems originated from heat transfer solved by the proposed method are reported.

2 Legendre Wavelets on union of intervals

Wavelets are powerful tools in approximation theory and numerical analysis of the Hilbert space $L^2(\mathbb{R})$ [14]. There are several bases for wavelets, such as Haar wavelet, Daubechies wavelets, Chebyshev wavelets, Legendre wavelets, and so on [11, 15, 3, 21, 23, 25]. In this paper, we consider the Legendre wavelets, which are an orthonormal set of functions with respect to the weight function $w(t) = 1$, on the interval $[0,1)$, as follows:

$$\psi_{nm}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}2^{k/2}P_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}} \\ 0, & otherwise \end{cases}, \tag{6}$$

where $n = 1, ..., 2^{k-1}$, k is an integer, m is the degree of Legendre polynomial P_m, $m = 0, 1, ..., M-1$, for some positive integer M. A function $f \in L^2(0,1)$, can be represented as series of Legendre wavelets

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} f_{nm}\psi_{nm}(t), \tag{7}$$

where $f_{nm} =< f, \psi_{nm} >= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(\tau)\psi_{nm}(\tau)d\tau$, is the inner product of f and ψ_{nm} in the Hilbert space $L^2(0,1)$. Suppose

$$V_{k-1}^M = \left\{\psi_{nm} : n = 1, ..., 2^{k-1}, m = 0, 1, ..., M-1\right\}, \quad P_{k-1}^M\left(f(t)\right) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} f_{nm}\psi_{nm}(t),$$

then, we have the following theorem about the error of the approximated solution in subspaces V_{k-1}^M [19].

Theorem 1. *Let* $f \in C^M[0,1]$ *and* $P_{k-1}^M(f(t)) \in V_{k-1}^M$, *then*

$$\left|f(t) - P_{k-1}^M(f(t))\right| \leq M_1 2^{-M(k+1)} \max_{\xi \in [0,1]} \left|f^{(M)}(\xi)\right|,$$

where M_1,*is a constant.*

Proof. See Theorem 2.4 of [19]. □

Now suppose $n_0 \in \mathbb{N}$, and let $f \in L^2(0, n_0)$. Since $[0, n_0) = \bigcup_{l=0}^{n_0-1}[l, l+1)$, we put $t \in I_l := [l, l+1)$ for $l \in \{0, 1, ..., n_0 - 1\}$, and we are ready to establish the idea in the interval $[0, n_0)$. In this paper, we assume, left hand side:=right hand side, which means that "left hand side" is defined as the known "right hand side" quantity. Similarly left hand side=:right hand side, which means that "right hand side" is defined as the known "left hand side" quantity.

Lemma 1. *Let* $\{p_i\}_{i=1}^N$ *is a sequence of orthogonal polynomials on* $[0,1]$ *with respect to the weight function* $\omega(t)$, *then* $\{q_i\}_{i=1}^N$ *is a sequence of orthogonal polynomials on* $[a,b]$ *with respect to the weight function* $\widetilde{\omega}(t)$ *where*

$$q_i(t) = p_i\left(\frac{t-a}{b-a}\right), \qquad \widetilde{\omega}(t) = \omega\left(\frac{t-a}{b-a}\right), \qquad t \in [a,b].$$

Proof. Put $x = \frac{t-a}{b-a}$, then for $i, j \in \{1, ..., N\}$ and $i \neq j$

$$\int_a^b q_i(t)q_j(t)\widetilde{\omega}(t)dt = (b-a)\int_0^1 p_i(x)p_j(x)\omega(x)dx = 0.$$

□

Theorem 2. *Let* $f \in C^M[l, l+1]$, $V_{k-1,l}^M := \left\{\psi_{nm}(t-l) : n = 1, ..., 2^{k-1}, m = 0, 1, ..., M-1\right\}$, *and* $P_{k-1,l}^M(f(t)) := \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} f_{nm}^{(l)}\psi_{nm}(t-l) \in V_{k-1,l}^M$,*where* $f_{nm}^{(l)} = \int_{l+\frac{n-1}{2^{k-1}}}^{l+\frac{n}{2^{k-1}}} f(\tau)\psi_{nm}(\tau - l)d\tau$, *then*

$$\left|f(t) - P_{k-1,l}^M(f(t))\right| \leq M_1 2^{-M(k+1)} \max_{\xi \in [l,l+1]} \left|f^{(M)}(\xi)\right|,$$

where M_1,*is a constant.*

Proof. A multiresolution analysis framework developed by Alpert [1], Mallat [18], Meyer [20], and discussed at length by Daubechies [16] shows that $\overline{V^M} = L^2[0,1]$, where $V^M := \cup_{k=1}^{\infty} V_{k-1}^M$. Similar analysis shows that $\overline{V^{M,l}} = L^2[l, l+1]$, where $V^{M,l} := \cup_{k=1}^{\infty} V_{k-1,l}^M$. Application of Theorem 1 for $\overline{V_{k-1,l}^M}$ insted of V_{k-1}^M forces the statement. □

For the recent function f, we obtain

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} f_{nm}^{(l)}\psi_{nm}(t-l), \quad t \in [l, l+1). \tag{8}$$

In the numerical process, we consider the following approximation

$$f(t) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} f_{nm}^{(l)}\psi_{nm}(t-l) = F_l^T\Psi(t-l), \quad t \in [l, l+1), 0 \leq l < n_0, \tag{9}$$

where

$$F_l = \left(f_{10}^{(l)}, f_{11}^{(l)}, ..., f_{1,M-1}^{(l)}, f_{20}^{(l)}, ..., f_{2,M-1}^{(l)}, ..., f_{2^{k-1},0}^{(l)}, ..., f_{2^{k-1},M-1}^{(l)}\right)^T$$
$$= \left(f_1^{(l)}, f_2^{(l)}, ..., f_M^{(l)}, f_{M+1}^{(l)}, ..., f_{2^{k-1}M}^{(l)}\right)^T, \quad (10)$$

$$\Psi(t-l) = \left(\psi_{10}(t-l), \psi_{11}(t-l), ..., \psi_{1,M-1}(t-l), \psi_{20}(t-l), ..., \psi_{2,M-1}(t-l), ..., \psi_{2^{k-1},0}(t-l), ..., \psi_{2^{k-1},M-1}(t-l)\right)^T$$
$$= \left(\psi_1(t-l), \psi_2(t-l), ..., \psi_M(t-l), \psi_{M+1}(t-l), ..., \psi_{2^{k-1}M}(t-l)\right)^T.$$

For the simplicity of numerical evaluations, we rearrange indices in the second representation of vectors.

3 Application of Legendre wavelets by means the Lag terms on Volterra integral equation

We consider the following test equation to illustrate the technique

$$\phi(t) - \int_0^t \phi(\tau)p(t,\tau)d\tau = r(t), \quad t \in [0, n_0), \quad (11)$$

where ϕ is unknown function, r is the right hand side function and is known, and $p(t,\tau)$ is the weakly singular kernel and in most problems is defined by $p(t,\tau) = |t-\tau|^{-\alpha}, 0 < \alpha < 1$ or $p(t,\tau) = \log|t-\tau|$. Since the data of Theorem 7 are piecewise-continuous and wavelets span such solutions, we apply wavelets. Using Eq. (9) with $l = 0$, for approximate $\phi(t)$ and $r(t)$ in Eq. (11) forces

$$\left(\Phi^{(0)}\right)^T \Psi(t) - \int_0^t \left(\Phi^{(0)}\right)^T \Psi(\tau)p(t,\tau)d\tau = \left(R^{(0)}\right)^T \Psi(t), \quad t \in [0, 1), \quad (12)$$

where $\left(\Phi^{(0)}\right)^T = \left[\phi_1^{(0)}, \phi_2^{(0)}, ..., \phi_{2^{k-1}M}^{(0)}\right]^T$, is unknown vector. Let $v^{(0)}(t) = \int_0^t \Psi(\tau)p(t,\tau)d\tau$, then from Eq. (9) we obtain $v^{(0)}(t) \simeq \left(V^{(0)}\right)^T \Psi(t)$, where $V^{(0)}$ is a $2^{k-1}M \times 2^{k-1}M$ matrix. Substitution of these quantities in (12) yields, $\left(\Phi^{(0)}\right)^T \left(I - V^{(0)}\right)^T \Psi(t) = \left(R^{(0)}\right)^T \Psi(t)$. Hence the linear system

$$\left(I - V^{(0)}\right) \Phi^{(0)} = R^{(0)}, \quad (13)$$

must be solved, to obtain the solution on $[0, 1)$. For $l \geq 1$ we rewrite (11) on $[l, l+1)$

$$\phi(t) - \int_l^t \phi(\tau)p(t,\tau)d\tau = r(t) + \sum_{s=0}^{l-1} \int_s^{s+1} \phi(\tau)p(t,\tau)d\tau, \quad t \in [l, l+1). \quad (14)$$

We approximate the ϕ in the right hand side by the known lag term $\phi(\tau) \simeq \left(\Phi^{(s)}\right)^T \Psi(\tau-s), s \leq \tau \leq s+1$, ϕ in the left hand side is approximated by $\phi(\tau) \simeq \left(\Phi^{(l)}\right)^T \Psi(\tau-l)$, and results in

$$\left(\Phi^{(l)}\right)^T \Psi(t-l) - \left(\Phi^{(l)}\right)^T \int_l^t \Psi(\tau-l)p(t,\tau)d\tau$$
$$= r(t) + \sum_{s=0}^{l-1} \left(\Phi^{(s)}\right)^T \int_s^{s+1} \Psi(\tau-s)p(t,\tau)d\tau, t \in [l, l+1). \quad (15)$$

Let $v^{(l)}(t) = \int_l^t \Psi(\tau-l)p(t,\tau)d\tau \simeq \left(V^{(l)}\right)^T \Psi(t-l)$, $w^{(s)}(t) = \int_s^{s+1} \Psi(\tau-s)p(t,\tau)d\tau \simeq \left(W^{(s)}\right)^T \Psi(t-l)$, $r(t) \simeq \left(R^{(l)}\right)^T \Psi(t-l)$, and substituting these quantities in (15) forces

$$\left(\Phi^{(l)}\right)^T\left(I-\left(V^{(l)}\right)^T\right)\Psi(t-l)=\left\{\left(R^{(l)}\right)^T+\sum_{s=0}^{l-1}\left(\Phi^{(s)}\right)^T\left(W^{(s)}\right)^T\right\}\Psi(t-l),t\in[l,l+1). \qquad (16)$$

Hence,

$$\left(I-V^{(l)}\right)\Phi^{(l)}=\left\{R^{(l)}+\sum_{s=0}^{l-1}W^{(s)}\Phi^{(s)}\right\}, \qquad (17)$$

must be solved recursively for $l=1,...,n_0-1$, which is the initial $\Phi^{(0)}$ obtained from (13).

4 Algorithm

In this section we give an algorithm for the above discussion. This algorithm can be written by any mathematical Programming software such as Mathematica, MathLab, C, Pascal, and so on. In this work we apply the Mathematica Programming software.

Remark 1. *In this paper, an $N-$column vector V with ith component $v_i, i=1,...,N$ is denoted by $V=[v_i : i=1,...,N]$ and an $N\times N$ Matrix A with (i,j)th component a_{ij} is denoted by $A=[a_{ij} : i=1,...,N, j=1,...,N]$. These notations are similar to Mathematica programing.*

- Step1 Take the initial integer data k, M, n_0, and the known right hand side function $r(t)$. Put $N=2^{k-1}M$, $\alpha_{l,i}=l+\left[\frac{i-1}{M}\right]/2^{k-1}, l=0,1,...,n_0-1, i=1,...,N$, $lagv=[0,...,0]^T$, an $N-$column vector with zero components, $\phi=[\ \ ]$, a null vector, and define $\psi_{mn}(t)$ as the same as (6), and let

$$\Psi(t)=\left[\psi_{\left[\frac{i-1}{M}\right]+1,i-M\left[\frac{i-1}{M}\right]-1}(t) : i=1,...,N\right]=:\left[\psi_i(t) : i=1,...,N\right];$$

  For $i=0,1,2,...,\quad j=1,2,3,...,\quad i<j$, set

$$\begin{aligned} w(t,i,j) &= \left[\begin{cases}\int_{\alpha_{i,i_0}}^{\alpha_{i,i_0}+\frac{1}{2^{k-1}}} p(t,\tau)\psi_{i_0}(\tau-i)d\tau, & j\le t<j+1,\\ 0, & otherwise.\end{cases} : i_0=1,...,N\right] \\ &=: \left[w_{i_0}(t,i,j) : i_0=1,...,N\right]; \end{aligned}$$

  Set $W(i,j)=\left[\int_{\alpha_{i,i_0}}^{\alpha_{i,i_0}+\frac{1}{2^{k-1}}} w_{j_0}(\tau,i,j)\psi_{i_0}(\tau-j)d\tau : i_0=1,...,N, j_0=1,...,N\right]$;

  Set $v(t,i)=\left[v_{i_0}(t,i) : i_0=1,...,N\right]$, where $v_{i_0}(t,i)$ is defined as follows:

$$v_{i_0}(t,i)=\begin{cases}\int_{\alpha_{i,i_0}}^{t} p(t,\tau)\psi_{i_0}(\tau-i)d\tau, & \alpha_{i,i_0}\le t<\alpha_{i,i_0}+\frac{1}{2^{k-1}},\\ \int_{\alpha_{i,i_0}}^{\alpha_{i,i_0}+\frac{1}{2^{k-1}}} p(t,\tau)\psi_{i_0}(\tau-i)d\tau, & \alpha_{i,i_0}+\frac{1}{2^{k-1}}\le t\le i+1,\\ 0, & otherwise.\end{cases}$$

  Set

$$V(i)=\left[\int_{\alpha_{i,i_0}}^{\alpha_{i,i_0}+\frac{1}{2^{k-1}}} v_{j_0}(\tau,i)\psi_{i_0}(\tau-i)d\tau : i_0=1,...,N, j_0=1,...,N\right];$$

- Step2 For $l=0,1,...,n_0-1$ do
  Set $rhsv=\left[\int_{\alpha_{l,i}}^{\alpha_{l,i}+\frac{1}{2^{k-1}}} r(\tau)\psi_i(\tau-l)d\tau : i=1,...,N\right]$;
  Solve the system $(I-V(l))X=rhsv+lagv$, and obtain X, then join X to ϕ. Set $lagv=\sum_{s=0}^{n_0} W(s,l+1)\left[\phi_{i+s\times N} : i=1,...,N\right]$, and if $l<n_0-1$, put $l=l+1$, and repeat Step2.

- Step3 Set $\widetilde{\phi}(t)=\sum_{l=0}^{n_0-1}\sum_{i=1}^{N}\psi_i(t-l)\phi_{i+l\times N}$, as the approximated solution of (11).

5 Convergence Analysis

We give some definitions, lemmas and theorems associated with this section.

Definition 1. *Let X and Y be normed spaces, and let $A : X \to Y$ be linear operator. Then A is compact if the set $\{A\phi : \|\phi\| \leq 1\}$ has compact closure in Y. This is equivalent to saying that for every bounded sequence $\{\phi_n\} \subseteq X$, the sequence $\{A\phi_n\}$ has a subsequence that is convergent to some point in Y. Compact operators are also called completely continuous operators (see also [2] section 2.8, [24] chapter 4 and [17] chapter12.)*

Definition 2. *A sequence $\mathcal{A}_n : X \to Y$ of linear operators from a normed space X into a normed space Y is called collectively compact if each sequence from the set $\{\mathcal{A}_n\phi : \phi \in X, \|\phi\| \leq 1, n \in \mathbb{N}\}$ contains a convergent subsequence.*

Theorem 3. *Let $I = [0,T], D = \{(t,s) : 0 \leq s \leq t \leq T\}$, for some positive $0 < T$, $p_\alpha(t-s) = \begin{cases} (t-s)^{-\alpha} & 0<\alpha<1, \\ Log(t-s) & \alpha = 1. \end{cases}$ and $(\mathcal{V}_\alpha\phi)(t) = \int_0^t p_\alpha(t-s)K(t,s)\phi(s)ds$, where $K \in C(D)$. Then for $r \in C(I)$ the linear, weakly singular Volterra integral equation $\phi(t) - (\mathcal{V}_\alpha\phi)(t) = r(t), t \in I$ possesses a unique solution $\phi \in C(I)$.*

Proof. See Theorems 6.1.2 and 6.1.7 of [10]. □

Theorem 4. *Let $A : X \to X$,be a compact operator in a normed linear space X. Then $I - A$ is surjective if and only if it is injective. If the inverse operator $(I-A)^{-1} : X \to X$ exists, it is bounded.*

Proof. See Theorem 12.2 of [17]. □

Theorem 5. *Let X be a Banach space, let $A_n : X \to X$ be a collectively compact sequence, and let $B_n : X \to X$ be a pointwise convergent sequence with limit operator $B : X \to X$. Then* $\lim_{n\to\infty} \|(B_n - B)A_n\| = 0$.

Proof. See Theorem 12.9 of [17]. □

Theorem 6. *Let $A : X \to X$ be a compact linear operator on a Banach space X such that $I - A$ is injective, and assume that the sequence $A_n : X \to X$ of linear operators is collectively compact and pointwise convergent; i.e.,* $\lim_{n\to\infty} A_n\phi = A\phi$ *for all $\phi \in X$. Then for sufficiently large n the invers operators $(I - A_n)^{-1} : X \to X$ exist and are uniformly bounded. For the solutions of the equations*

$$\phi - A\phi = f, \tag{18}$$

and

$$\phi_n - A_n\phi_n = f_n, \tag{19}$$

we have an error estimate $\|\phi_n - \phi\| \leq C\{\|(A_n - A)\phi\| + \|f_n - f\|\}$ for some constant C.

Proof. See Theorem 12.10 of [17]. □

We rewrite equations (14) and (16) in the form of (18) and (19) respectively. For this aim, define

$$A\phi(t) := \int_l^t \phi(\tau)p(t-\tau)d\tau,\ \phi \in X := L^2(l, l+1),$$

$$f(t) := r(t) + \sum_{s=0}^{l-1} \int_s^{s+1} \phi(\tau)p(t-\tau)d\tau,$$

$$B_n\phi(t) := \left(\Phi^{(l)}\right)^T \Psi(t-l),\ \Phi^{(l)} = \Phi_n^{(l)} = (\phi_1^{(l)}, \phi_2^{(l)}, ..., \phi_{2^{n-1}\times n^2}^{(l)})^T,$$

$$\phi_n(t) := B_n\phi(t), \qquad A_n := B_n A.$$

Obviously (14) is in the form of (18). By defining

$$f_n(t) := \left\{ \left(R^{(l)}\right)^T + \sum_{s=0}^{l-1} \left(\Phi^{(s)}\right)^T \left(W^{(s)}\right)^T \right\} \Psi(t-l) = B_n f(t),$$

we see that

$$A\phi_n(t) = AB_n\phi(t) = \left(\Phi^{(l)}\right)^T \int_l^t \Psi(\tau - l)p(t-\tau)d\tau,$$

and hence

$$A_n\phi_n(t) = B_n A\phi_n(t) = \left(\Phi^{(l)}\right)^T \left(V^{(l)}\right)^T \Psi(t-l),$$

these means (16) is in the form of (19).

To terminate this section, it is sufficient to show that all hypotheses of theorem 6 are true for the produced operators. The operator A is compact (see Section 2.8 of [2]). Theorem 3 shows that the operator $I - A$ is surjective, and theorem 4 says that $I - A$ is injective. Compactness of A shows that the fixed sequence $\mathcal{A}_n := A$ be a collectively compact sequence. Equation (8) says that $\lim_{n\to\infty} B_n\phi = B\phi = I\phi = \phi$, and hence Theorem 5 forces that $\lim_{n\to\infty}\|A_n - A\| = \lim_{n\to\infty}\|B_nA - IA\| = \lim_{n\to\infty}\|(B_n - B)\mathcal{A}_n\| = 0$. A_n converges to A in norm and this forces either $A_n : X \to X$ be collectively compact and pointwise convergent. These analyses show that all hypotheses of Theorem 6 are true.

For $l = 0,\ f_n(t) := \left(R^{(0)}\right)^T \Psi(t-l)$ and equation (10) shows that $\lim_{n\to\infty}\|f_n - f\|_{L^2(0,1)} = 0$, then the error estimate of Theorem 6 forces $\|\phi_n - \phi\|_{L^2(0,1)} \le C\left\{\left\|(A_n - A)\|_{L^2(0,1)}\|\phi\right\|_{L^2(0,1)} + \|f_n - f\|_{L^2(0,1)}\right\} \to 0\ as\ n \to \infty$, and hence $\lim_{n\to\infty}\|f_n - f\|_{L^2(1,2)} = 0$. The error estimate of Theorem 6 forces $\|\phi_n - \phi\|_{L^2(1,2)} \le C\left\{\left\|(A_n - A)\|_{L^2(1,2)}\|\phi\right\|_{L^2(1,2)} + \|f_n - f\|_{L^2(1,2)}\right\} \to 0\ as\ n \to \infty$, and so on. By induction, we see that for finite $l \in \mathbb{N}$, $\lim_{n\to\infty}\|\phi_n - \phi\|_{L^2(l,l+1)} = 0$.

6 Application in heat conduction

We give some definitions, lemmas and theorems associated with this section

Definition 3. *The fundamental solution of heat equation is denoted by $K(x,t)$, and the Neumann's function is denoted by $N(x,\xi,t)$,*

$$K(x,t) := \frac{1}{\sqrt{4\pi t}} \exp\left\{-\frac{x^2}{4t}\right\}, \quad N(x,\xi,t) := K(x-\xi,t) + K(x+\xi,t).$$

Lemma 2. *For any integrable function f that satisfies $|f(x)| \le C_1\exp\{C_2x^2\}$, where C_1 and C_2 are positive constants, $\lim_{t\downarrow 0}\int_{-\infty}^{\infty} K(x-\xi,t)f(\xi)d\xi = f(x),\ 0 < t$, at the point x of continuity of f.*

Proof. See Lemma 3.4.3 of [12]. □

Lemma 3. *At a point of continuity of g, $\lim_{x\downarrow 0} -2\int_0^t \frac{\partial K}{\partial x}(x,t-\tau)g(\tau)d\tau = g(t)$.*

Proof. See Lemma 4.2.1 of [12]. □

Theorem 7. *The problem of determining the unique bounded solution u that satisfies (2)- (5), where $C_i, i = 1,2$, are positive constants, and where f, α, and g are piecewise-continuous functions, and F is an integrable function on $\{(t,\tau)|0 \le \tau \le t\}$, is equivalent to the problem of determining the unique piecewise-continuous solution ϕ to the integral equation*

$$\phi(t)-\frac{1}{\sqrt{\pi}}\int_0^t\left\{\alpha(t)+\int_\tau^t F(t,s)ds\right\}\frac{\phi(\tau)}{\sqrt{t-\tau}}d\tau=r(t)\quad 0<t, \tag{20}$$

where

$$r(t)=g(t)+\frac{\alpha(t)}{\sqrt{\pi t}}\int_0^\infty \exp\left\{-\frac{\xi^2}{4t}\right\}f(\xi)d\xi-\int_0^t\frac{F(t,\tau)}{\sqrt{\pi\tau}}\int_0^\infty\exp\left\{-\frac{\xi^2}{4\tau}\right\}f(\xi)d\xi d\tau. \tag{21}$$

And the solution u has the representation

$$u(x,t)=-2\int_0^t K(x,t-\tau)\phi(\tau)d\tau+\int_0^\infty N(x,\xi,t)f(\xi)d\xi. \tag{22}$$

Proof. We are going to search $u(x,t)=u_1(x,t)+u_2(x,t)$, such that u_1,u_2 satisfy heat equation and each of them establish one of the equations (3), (4). For this aim let $u_1(x,t)=-2\int_0^t K(x,t-\tau)\phi(\tau)d\tau,\ u_2(x,t)=\int_0^\infty N(x,\xi,t)f(\xi)d\xi$. From [12], chapter one, both of u_1 and u_2 are solutions of equation (2). Lemma 2 leads $u(x,0)=u_2(x,0)=\lim_{t\downarrow 0}\int_0^\infty N(x,\xi,t)f(\xi)d\xi=\lim_{t\downarrow 0}\int_{-\infty}^\infty K(x-\xi,t)f_e(\xi)d\xi=f(x)$, where f_e is the even extension of f to $-\infty<x<\infty$. Elementary evaluations lead $N_x(0,\xi,t)=0$, and $u_x(0,t)=\lim_{x\downarrow 0}\int_0^\infty N_x(x,\xi,t)f(\xi)d\xi+\lim_{x\downarrow 0}-2\int_0^t\frac{\partial K}{\partial x}(x,t-\tau)\phi(\tau)d\tau=\int_0^\infty N_x(0,\xi,t)f(\xi)d\xi+\phi(t)=\phi(t)$, where we apply Lemma 3 for the function ϕ. On the other hand

$$\begin{aligned}u(0,t)=u_1(0,t)+u_2(0,t)&=-2\int_0^t K(0,t-\tau)\phi(\tau)d\tau+\int_0^\infty N(0,\xi,t)f(\xi)d\xi\\&=-\frac{2}{\sqrt{4\pi}}\int_0^t\frac{\phi(\tau)}{\sqrt{t-\tau}}d\tau+\frac{1}{\sqrt{\pi t}}\int_0^\infty\exp\left\{-\frac{\xi^2}{4t}\right\}f(\xi)d\xi.\end{aligned}$$

Substitution of $u_x(0,t),u(0,t)$ in Eq. (4) and using Fubini's theorem reduces to Eq. (20), where the known right hand side function $r(t)$ is given by (21). By consideration of chapter3 of [12] the solution u in the class (5) is unique, and hence the proof is completed. □

7 Numerical examples

Example 1. *In the problem (2)-(5), for $f(x)=1,F(t,\tau)=\frac{\sqrt{\pi}}{\sqrt{t-\tau}},\alpha(t)=\sqrt{\pi}$, the integral equation associated with this problem is, $\phi(t)-\int_0^t\phi(\tau)p(t,\tau)d\tau=r(t)$, where $p(t,\tau)=2+\frac{1}{\sqrt{t-\tau}},r(t)=-2+3e^{-t}-2DawsonF(\sqrt{t})$, which has the exact solution $\phi(t)=e^{-t}$. The Dawson integral is defined by $DawsonF(t)=e^{-t^2}\int_0^t e^{y^2}dy$, is a special function, defined in many programming languages such as Mathematica.The exact solution of the problem (2)-(5) is*

$$u(x,t)=1-\frac{1}{2}e^{-t-ix}\left(-i+e^{2ix}\left(i+erfi\left[\frac{2t-ix}{2\sqrt{t}}\right]\right)+erfi\left[\frac{2t+ix}{2\sqrt{t}}\right]\right),$$

where i is the imaginary unit. In Table 1, column2 shows absolute errors of $\widetilde{\phi}$ at $t=0.5i,i=1,2,3,4,5,6$, ϕ is exact solution and $\widetilde{\phi}$ is evaluated by Legendre wavelets technique with $M=8,k=3,n_0=3$. Figure 1 shows variations of these solutions as functions of t for Example 1.
In Table 1, columns $3,4,5,6,7,8$, show absolute errors of $\widetilde{u}$ at $(x,t)=(0.5i,0.5j),i,j=1,2,3,4,5,6$, u is exact solution and $\widetilde{u}$ is the approximated solution evaluated numerically by substitution of $\widetilde{\phi}$, instead of ϕ in u representation Eq.(22). Here $e_{ij},i,j=1,2,3,4,5,6$ is the absolute error of $\widetilde{u}$ at $(0.5i,0.5j)$, and for example $1.84D-14$ means 1.84×10^{-14}, which shows convergence of the method. Figures 3 and 4 show variations of these solutions as functions of (x,t) for Example 1. As these Figures show, $\widetilde{u}$ is a good approximation of u.

Table 1: Absolute errors of $\widetilde{\phi}$ and $\widetilde{u}$ for Example 1.

i	$\lvert\phi-\widetilde{\phi}\rvert_i$	e_{i1}	e_{i2}	e_{i3}	e_{i4}	e_{i5}	e_{i6}
1	$1.85D-14$	$5.72D-15$	$1.67D-15$	$5.57D-14$	$1.53D-12$	$4.02D-11$	$1.05D-9$
2	$3.15D-14$	$3.19D-15$	$2.33D-15$	$1.40D-14$	$4.25D-13$	$1.12D-11$	$2.93D-10$
3	$5.45D-13$	$1.15D-15$	$1.53D-15$	$2.55D-15$	$1.16D-13$	$3.12D-12$	$8.17D-11$
4	$1.41D-11$	$1.53D-16$	$9.58D-16$	$7.77D-16$	$2.89D-14$	$8.70D-13$	$2.28D-11$
5	$3.69D-10$	$2.64D-16$	$6.38D-16$	$2.35D-15$	$5.43D-15$	$2.45D-13$	$6.35D-12$
6	$9.65D-9$	$2.49D-16$	$1.06D-15$	$2.18D-15$	$9.02D-16$	$6.96D-14$	$1.77D-12$

Example 2. *In problems* (2)-(5)*, for* $f(x)=1, F(t,\tau)=1, \alpha(t)=1$*, the integral equation associated with this problem is,* $\phi(t)-\int_0^t \phi(\tau)p(t,\tau)d\tau = r(t)$*, where* $p(t,\tau)=\frac{1}{\sqrt{\pi(t-\tau)}}+\frac{\sqrt{t-\tau}}{\sqrt{\pi}}, r(t)=\sqrt{t}-\frac{1}{8}\sqrt{\pi}t(4+t)$*, which has the exact solution* $\phi(t)=\sqrt{t}$*.The exact solution of the problem* (2)-(5) *is*

$$u(x,t)=1+\frac{1}{2}e^{-\frac{x^2}{4t}}\sqrt{t}x-\frac{\sqrt{\pi}}{4}\left(2t+x^2\right)erfc\left[\frac{x}{2\sqrt{t}}\right].$$

In Table 2, column2 shows absolute errors of $\widetilde{\phi}$ *at* $t=0.5i, i=1,2,3,4,5,6$*,* ϕ *is exact solution and* $\widetilde{\phi}$ *is evaluated by Legendre wavelets technique with* $M=8, k=3, n_0=3$*. Figure 2 shows variation of these solutions as functions of t for Example 2.*
In Table 2, columns $3,4,5,6,7,8$*, show absolute errors of* $\widetilde{u}$ *at* $(x,t)=(0.5i,0.5j), i,j=1,2,3,4,5,6$*,* u *is exact solution and* $\widetilde{u}$ *is the approximated solution evaluated numerically by substitution of* $\widetilde{\phi}$*, instead of* ϕ *in* u *representation Eq.*(22)*. Here* $e_{ij}, i,j=1,2,3,4,5,6$ *is the absolute error of* $\widetilde{u}$ *at* $(0.5i,0.5j)$*. Figures 5 and 6 show variations of these solutions as functions of* (x,t) *for Example 2. As these Figures show,* $\widetilde{u}$ *is a good approximation of* u*.*

Table 2: Absolute errors of $\widetilde{\phi}$ and $\widetilde{u}$ for Example 2.

i	$\lvert\phi-\widetilde{\phi}\rvert_i$	e_{i1}	e_{i2}	e_{i3}	e_{i4}	e_{i5}	e_{i6}
1	$1.05D-8$	$9.37D-10$	$1.54D-11$	$7.83D-6$	$1.68D-5$	$3.80D-5$	$8.77D-5$
2	$9.72D-4$	$4.03D-11$	$3.96D-11$	$3.86D-6$	$8.80D-6$	$1.99D-5$	$4.58D-5$
3	$1.78D-5$	$6.91D-13$	$1.62D-10$	$1.66D-6$	$4.50D-6$	$1.05D-5$	$2.37D-5$
4	$4.04D-5$	$4.82D-11$	$1.35D-9$	$6.01D-7$	$2.23D-6$	$5.05D-6$	$1.26D-5$
5	$9.34D-5$	$3.13D-10$	$2.43D-9$	$1.84D-7$	$1.16D-6$	$2.73D-6$	$8.51D-6$
6	$2.17D-4$	$3.51D-10$	$3.47D-9$	$8.70D-8$	$2.86D-7$	$1.58D-6$	$3.01D-7$

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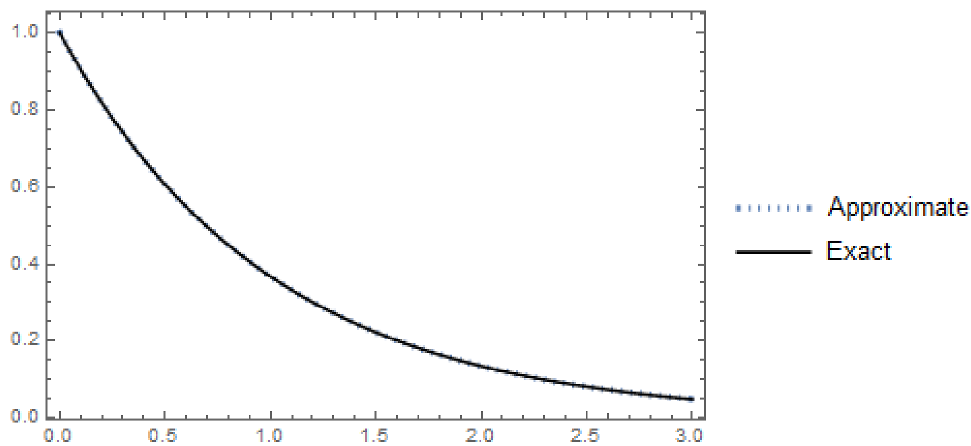

Figure 1: Variations of $\phi(t)$ and $\widetilde{\phi}(t)$ as functions of t for Example 1.

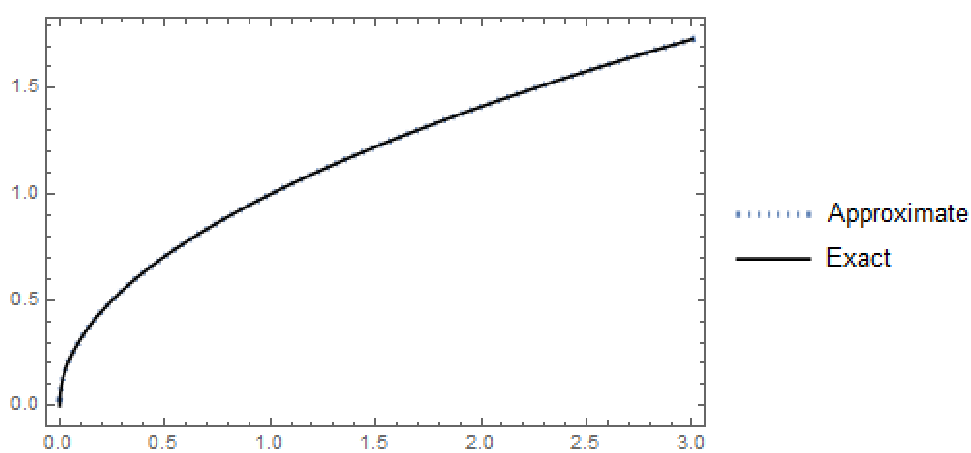

Figure 2: Variations of $\phi(t)$ and $\widetilde{\phi}(t)$ as functions of t for Example 2.

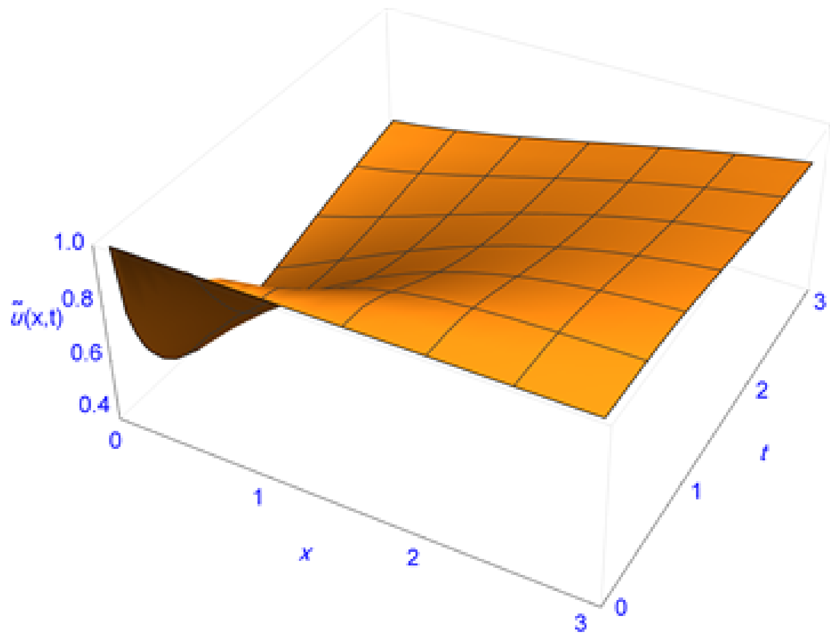

Figure 3: Variation of the $\widetilde{u}(x,t)$ as a function of (x,t) for Example 1.

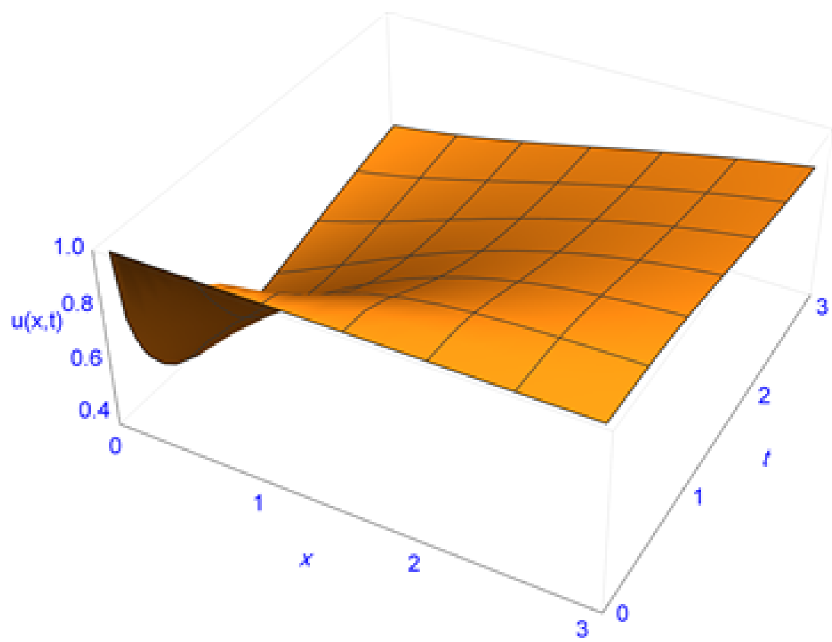

Figure 4: Variation of the $u(x,t)$ as a function of (x,t) for Example 1.

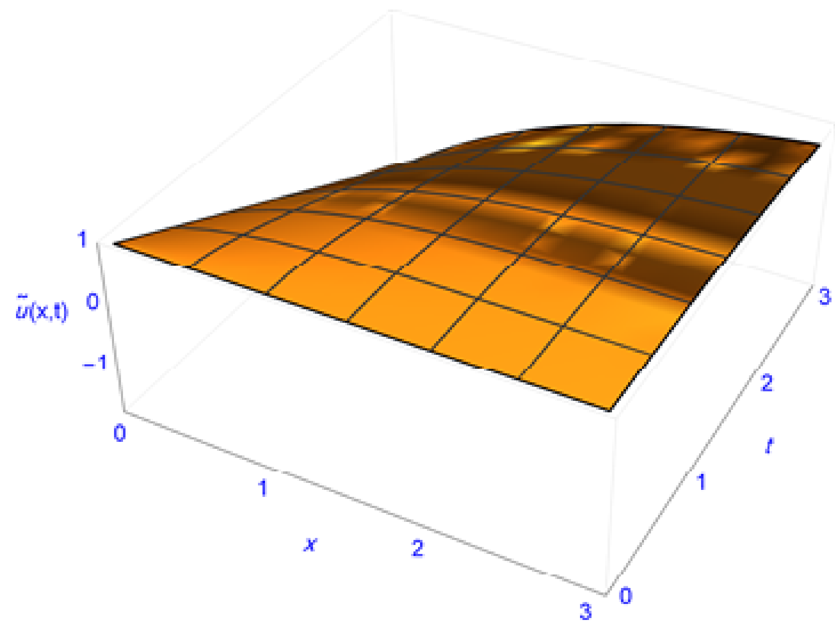

Figure 5: Variation of the $\widetilde{u}(x,t)$ as a function of (x,t) for Example 2.

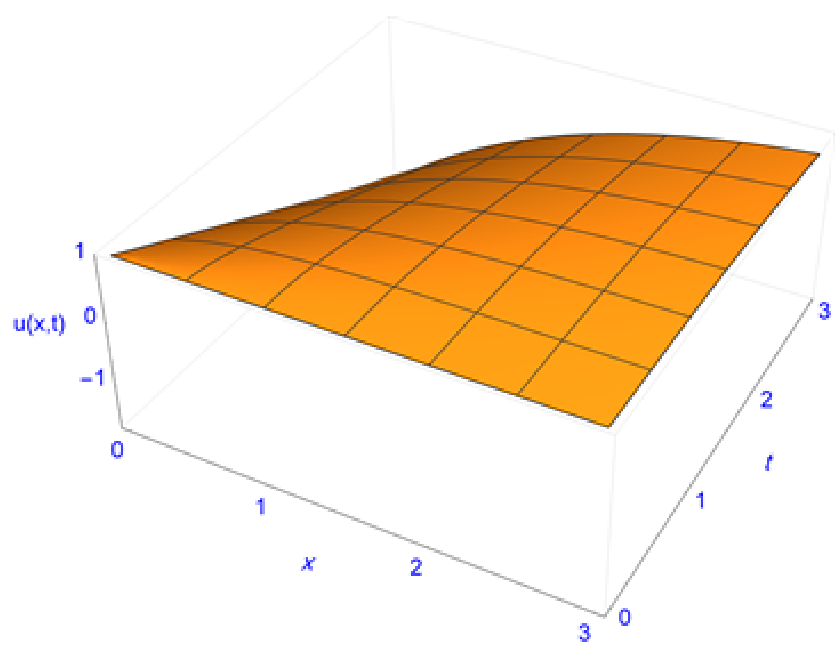

Figure 6: Variation of the $u(x,t)$ as a function of (x,t) for Example 2.