Bézier variant of Păltănea operators based on Gould Hopper polynomials

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Keywords and phrases: Bézier operators, Gould Hopper polynomials, Modulus of continuity, Rate of convergence, Bounded variation, Weighted space.

AMS Subject Classifications (2010): 26A15, 41A10, 41A25, 41A36.

1. Introduction

In the theory of approximation Szász operators play a significant role. Several authors have investigated many amusing properties of these operators in [11, 13, 15, 16]. Jakimovski and Leviatan generalized these operators in [7] using Appell polynomials. Later Ismail [6] generalized these operators by using Sheffer polynomials. In order to approximate Lebesgue integrable function Kantorovich and Durrmeyer type amendment have been explored for the positive linear operators. For $\alpha > 0$, $\rho > 0$ and $x \in \mathbb{R}_0^+ = [0, \infty)$, Păltănea [10] recognized the two parameter integral modification of Szász operators as

$$L_\alpha^\rho(f; x) = \sum_{k=1}^{\infty} s_{\alpha,k}(x) \int_0^\infty \Theta_{\alpha,k}^\rho(t) f(t) dt + e^{-\alpha x} f(0), \tag{1.1}$$

where

$$s_{\alpha,k}(x) = e^{-\alpha x} \frac{(\alpha x)^k}{k!}$$

and

$$\Theta_{\alpha,k}^\rho(t) = \frac{\alpha\rho}{\Gamma(k\rho)} \cdot e^{-\alpha\rho t} (-\alpha\rho t)^{k\rho-1}.$$

A connection between orthogonal polynomial and positive linear operators (see [14]) were introduced by Varma *et al.* in 2012. They formulated positive linear operators concerning Brenke polynomials. Later on many adaptation of Szász operators have been considered by involving different types of orthogonal polynomials.

A Bézier curve is a parametric curve successively used in computer aided geometric design, image processing and curve fitting. Zeng and Piriou [19] pioneered the Bézier variant of Bernstein operators. Subsequently, many researchers established the Bézier variant of various operators [4, 5, 18].

Recently, In [3] the authors acknowledged the Bézier form of the Jakimovski-Leviatan-Păltănea operators based on Appell polynomials and conferred some direct approximation theorem and rate of convergence for the functions having a derivative of bounded variation. Motivated by this, we study farther in this direction and introduce the Bézier variant of Păltănea operators based on Gould Hopper polynomials. Gould Hopper polynomials have generating function of the type

$$e^{ht^{d+1}}e^{xt}=\sum_{k=0}^{\infty}g_k^{d+1}(x,h)\frac{t^k}{k!}, \tag{1.2}$$

and explicit description as

$$g_k^{d+1}(x,h)=\sum_{s=0}^{[\frac{k}{d+1}]}\frac{k!}{s!(k-(d+1)s)!}h^s x^{k-(d+1)s},$$

where [.] denotes the integer part.
For $n\in\mathbb{N},\ \rho>0,\ \theta\geq 1$ and all real valued continuous and bounded function f on $\mathbb{R}_0^+$, we define Bézier-Păltănea operators based on Gould Hopper polynomials as

$$G_{n,h,\rho}^{d,\theta}(f;x)=\sum_{k=1}^{\infty}X_{n,h,k}^{d,\theta}(x)\int_0^{\infty}\Theta_{n,k}^{\rho}(t)f(t)dt+X_{n,h,0}^{d,\theta}f(0), \tag{1.3}$$

where

$$X_{n,h,k}^{d,\theta}(x)=[\zeta_{n,h,k}^{d}(x)]^{\theta}-[\zeta_{n,h,k+1}^{d}(x)]^{\theta},$$

$$\zeta_{n,h,k}^{d}(x)=\begin{cases}\sum\limits_{j=k}^{\infty}l_{n,h,j}^{d}(x) & when\ k\leq n,\\ 0 & otherwise.\end{cases}$$

$$l_{n,h,k}^{d}(x)=e^{-nx-h}\cdot\frac{g_k^{d+1}(nx,h)}{k!}$$

and

$$\Theta_{n,k}^{\rho}(t)=\frac{n\rho}{\Gamma(k\rho)}\cdot e^{-n\rho t}(-n\rho t)^{k\rho-1}.$$

Some important properties of $\zeta_{n,h,k}^{d}(x)$ are as follows:

(1) $\zeta_{n,h,k}^{d}(x)-\zeta_{n,h,k+1}^{d}(x)=l_{n,h,k}^{d}(x)$,
(2) $\zeta_{n,h,0}^{d}(x)>\zeta_{n,h,1}^{d}(x)>\zeta_{n,h,2}^{d}(x)>\cdots>\zeta_{n,h,n}^{d}(x)>0,\ \forall x\in\mathbb{R}_0^+$.

Also, the operators $G_{n,h,\rho}^{d,\theta}$ have the integral representation

$$G_{n,h,\rho}^{d,\theta}(f;x)=\int_0^{\infty}K_{n,h,\rho}^{d,\theta}(x,t)f(t)dt, \tag{1.4}$$

where $K_{n,h,\rho}^{d,\theta}(x,t)$ is the kernal defined by

$$K_{n,h,\rho}^{d,\theta}(x,t)=\sum_{k=1}^{\infty}X_{n,h,k}^{d,\theta}(x)\Theta_{n,k}^{\rho}(t)+X_{n,h,0}^{d,\theta}(x)\delta(t), \tag{1.5}$$

$\delta(t)$ is the Dirac-delta function.

If we take $\theta = 1$ then operator (1.3) reduces to operators $G^{d,1}_{n,h,\rho}(f,x) = G^{d}_{n,h,\rho}(f,x)$ given by

$$G^{d}_{n,h,\rho}(f;x) = \sum_{k=1}^{\infty} l^{d}_{n,h,k}(x) \int_0^{\infty} \Theta^{\rho}_{n,k}(t) f(t) dt + l^{d}_{n,h,0} f(0), \tag{1.6}$$

where

$$l^{d}_{n,h,k}(x) = e^{-nx-h} \cdot \frac{g_k^{d+1}(nx,h)}{k!}$$

and

$$\Theta^{\rho}_{n,k}(t) = \frac{n\rho}{\Gamma(k\rho)} \cdot e^{-n\rho t}(-n\rho t)^{k\rho-1}.$$

The main objective of this paper is to introduce the Bézier variant of linear positive operators that have been derived from the d-orthogonal polynomials e.g. Gould Hopper polynomials and study some direct theorem by means of Ditzian-Totik modulus of smoothness and rate of convergence for function having a derivative of bounded variation.

2. Auxiliary result

Lemma 2.1. *Let $G^{d}_{n,h,\rho}$ be the operators defined by (1.6). Then, we have*

(1) $G^{d}_{n,h,\rho}((t-x);x) = \frac{h(d+1)}{n}$,
(2) $G^{d}_{n,h,\rho}((t-x)^2;x) = \frac{x}{n\rho}(1+\rho) + \frac{h(d+1)}{n^2\rho}(\rho(d+1)(h+1)+1)$,
(3) $G^{d}_{n,h,\rho}((t-x)^4;x) = \frac{3x^2}{n^2}\left(1+\frac{2}{\rho}+\frac{1}{\rho^2}\right) + \frac{x}{n^3}\Big(6h^2(d+1)^2 - 4h(d+1)^3 + 14h(d+1)^2 + 4hd^2(d+1) + 1 + \frac{1}{\rho}(6h(h+1)(d+1)^2 + 18h(d+1) + 6) + \frac{1}{\rho^2}(14h(d+1)+11) + \frac{1}{\rho^3}\Big) + \frac{1}{n^4}\Big(h(d+1)^4(h^3+6h^2+7h+1) + \frac{6h}{\rho}(d+1)^3(h^2+3h+1) + \frac{11h}{\rho^2}(h+1)(d+1)^2 + \frac{6}{\rho^3}h(d+1)\Big)$.

Let $C_B(\mathbb{R}_0^+)$ denote the space of all bounded and continuous functions on $\mathbb{R}_0^+$ endowed with the norm

$$\|f\| = \sup_{x\in\mathbb{R}_0^+} |f(x)|$$

Lemma 2.2. *For $f \in C_B(\mathbb{R}_0^+)$, we have*

$$\|G^{d}_{n,h,\rho} f\| \leq \|f\|.$$

Remark 2.3. We have

$$\begin{aligned} G_{n,h,\rho}^{d,\theta}(1;x) &= \sum_{k=1}^{\infty} X_{n,h,k}^{d,\theta}(x)\int_0^{\infty}\Theta_{n,k}^{\rho}(t)dt + X_{n,h,0}^{d,\theta} \\ &= \sum_{k=0}^{\infty} X_{n,h,k}^{d,\theta}(x) \\ &= \left(\sum_{k=0}^{\infty} l_{n,h,k}^{d}(x)\right)^{\theta},\ \textit{since} \sum_{k=0}^{\infty} l_{n,h,k}^{d}(x) = 1 \\ &= 1. \end{aligned}$$

Lemma 2.4. *Let $G_{n,h,\rho}^{d,\theta}$ be the operator defined by (1.3), then*

(1) $G_{n,h,\rho}^{d,\theta}((t-x);x) \leq \theta\frac{h(d+1)}{n}$,

(2) $G_{n,h,\rho}^{d,\theta}((t-x)^2;x) \leq \theta\Big\{\frac{x}{n\rho}(1+\rho) + \frac{h(d+1)}{n^2\rho}(\rho(d+1)(h+1)+1)\Big\}$.

(3) $G_{n,h,\rho}^{d}((t-x)^4;x) \leq \theta\Big\{\frac{3x^2}{n^2}\Big(1+\frac{2}{\rho}+\frac{1}{\rho^2}\Big) + \frac{x}{n^3}\Big(6h^2(d+1)^2 - 4h(d+1)^3 + 14h(d+1)^2 + 4hd^2(d+1) + 1 + \frac{1}{\rho}(6h(h+1)(d+1)^2 + 18h(d+1) + 6) + \frac{1}{\rho^2}(14h(d+1)+11) + \frac{1}{\rho^3}\Big) + \frac{1}{n^4}\Big(h(d+1)^4(h^3+6h^2+7h+1) + \frac{6h}{\rho}(d+1)^3(h^2+3h+1) + \frac{11h}{\rho^2}(h+1)(d+1)^2 + \frac{6}{\rho^3}h(d+1)\Big)\Big\}$.

Proof. From equation (1.3), we have

$$\begin{aligned} G_{n,h,\rho}^{d,\theta}(f;x) &= \sum_{k=1}^{\infty} X_{n,h,k}^{d,\theta}(x)\int_0^{\infty}\Theta_{n,k}^{\rho}(t)f(t)dt + X_{n,h,0}^{d,\theta}f(0), \\ &= \sum_{k=1}^{\infty}([\zeta_{n,h,k}^{d}(x)]^{\theta} - [\zeta_{n,h,k+1}^{d}(x)]^{\theta})\int_0^{\infty}\Theta_{n,k}^{\rho}(t)f(t)dt \\ &\quad +([\zeta_{n,h,0}^{d}(x)]^{\theta} - [\zeta_{n,h,1}^{d}(x)]^{\theta})f(0). \end{aligned}$$

Using the well known inequality $|a^{\beta} - b^{\beta}| \leq \beta|a-b|$ with $0 \leq a,b \leq 1,\ \beta \geq 1$ and (1), we have

$$\begin{aligned} G_{n,h,\rho}^{d,\theta}(f;x) &\leq \theta\Big\{\sum_{k=1}^{\infty} l_{n,h,k}^{d}(x)\int_0^{\infty}\Theta_{n,k}^{\rho}(t)f(t)dt + l_{n,h,0}^{d}f(0)\Big\}, \\ &\leq \theta G_{n,h,\rho}^{d}(f;x). \end{aligned}$$

In view of Lemma 2.1, we find

$$\begin{aligned} G_{n,h,\rho}^{d,\theta}((t-x);x) &\leq \theta G_{n,h,\rho}^{d}((t-x);x) \\ &\leq \theta\frac{h(d+1)}{n}, \end{aligned}$$

and

$$\begin{aligned} G^{d,\theta}_{n,h,\rho}((t-x)^2;x) &\leq \theta G^{d}_{n,h,\rho}((t-x)^2;x) \\ &\leq \theta\left\{\frac{x}{n\rho}(1+\rho)+\frac{h(d+1)}{n^2\rho}(\rho(d+1)(h+1)+1)\right\}. \end{aligned}$$

□

Remark 2.5. (1) For $C_1>1,\ \rho>0,\ \theta\geq 1,\ x\in(0,\infty)$ and sufficiently large n

$$G^{d,\theta}_{n,h,\rho}((t-x)^2;x)\leq C_1\theta\frac{x(1+\rho)}{n\rho}.$$

(2) For $C_2>1,\ \rho>0,\ \theta\geq 1,\ x\in(0,\infty)$ and sufficiently large n

$$G^{d,\theta}_{n,h,\rho}((t-x)^4;x)\leq C_2\theta\left(\frac{x(1+\rho)}{n\rho}\right)^2.$$

Lemma 2.6. *Let $f\in C_B(\mathbb{R}^+_0)$, then*

$$\|G^{d,\theta}_{n,h,\rho}f\|\leq\|f\|.$$

Proof. In view of equation (1.3) and Remark 2.3, we have

$$\begin{aligned} |G^{d,\theta}_{n,h,\rho}(f;x)| &= \left|\sum_{k=1}^{\infty}X^{d,\theta}_{n,h,k}(x)\int_0^\infty\Theta^{\rho}_{n,k}(t)f(t)dt+X^{d,\theta}_{n,h,0}f(0)\right| \\ &\leq \left(\sum_{k=1}^{\infty}X^{d,\theta}_{n,h,k}(x)\int_0^\infty\Theta^{\rho}_{n,k}(t)dt+X^{d,\theta}_{n,h,0}\right)\|f\| \\ &\leq G^{d,\theta}_{n,h,\rho}(1;x)\|f\|=\|f\|, \end{aligned}$$

which completes the proof. □

3. Direct Approximation

In this section first we recall the definition of well known Ditizian-Totik modulus of smoothness $\omega_{\phi^\tau}(f;t)$ and Peetre's $K-$functional [2].
Let $\phi(x)=\sqrt{x}$ and $f\in C_B(\mathbb{R}^+_0)$. For $0\leq\tau\leq 1$, define

$$\omega_{\phi^\tau}(f;t)=\sup_{0\leq h\leq t}\ \sup_{x\pm\frac{h\phi^\tau(x)}{2}\in\mathbb{R}^+_0}\left|f\left(x+\frac{h\phi^\tau(x)}{2}\right)-f\left(x-\frac{h\phi^\tau(x)}{2}\right)\right|, \tag{3.1}$$

and the $K-$functional

$$K_{\phi^\tau}(f;t)=\inf_{g\in W_\tau}\left\{\|f-g\|+t\|\phi^\tau g'\|\right\}$$

where

$$W_\tau=\{g:\ g\in AC_{loc};\ \|\phi^\tau g'\|<\infty\}$$

AC_{loc} denotes the space of locally absolutely continuous function on $\mathbb{R}^+_0$ and $\|.\|$ is the uniform norm on $C_B(\mathbb{R}^+_0)$. Also,

$$\omega_{\phi^\tau}(f;t)\sim K_{\phi^\tau}(f;t),$$

which means that there exists a constant $C > 0$ such that

$$C^{-1}\omega_{\phi^\tau}(f;t) \leq K_{\phi^\tau}(f;t) \leq C\omega_{\phi^\tau}(f;t). \tag{3.2}$$

Lemma 3.1. *For $f \in W_\tau$, $\phi(x) = \sqrt{x}$, $0 \leq \tau \leq 1$ and $t, x > 0$, we have*

$$\left|\int_x^t f'(u)du\right| \leq 2^\tau x^{-\frac{\tau}{2}}|t-x|\|\phi^\tau f'\|.$$

Proof. Since

$$\left|\int_x^t f'(u)du\right| \leq \|\phi^\tau f'\|\left|\int_x^t \frac{du}{\phi^\tau(u)}\right|, \tag{3.3}$$

applying Hölder inequality with $p = \frac{1}{\tau}$ and $q = \frac{1}{1-\tau}$, we get

$$\begin{aligned}
\left|\int_x^t \frac{du}{\phi^\tau(u)}\right| &\leq \left|\int_x^t \frac{1}{\sqrt{u}}du\right|^\tau |t-x|^{1-\tau} \\
&\leq 2^\tau|\sqrt{t}-\sqrt{x}|^\tau|t-x|^{1-\tau} \\
&\leq 2^\tau \frac{|t-x|^\tau}{x^{\frac{\tau}{2}}}|t-x|^{1-\tau}.
\end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we have

$$\left|\int_x^t f'(u)du\right| \leq 2^\tau x^{-\frac{\tau}{2}}|t-x|\|\phi^\tau f'\|.$$

□

Theorem 3.2. *For $f \in C_B(\mathbb{R}_0^+)$, we have*

$$|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| \leq C\omega_{\phi^\tau}\left(f; \frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right),$$

where ω_{ϕ^τ} is given by (3.1) and C is a constant independent of n and x.

Proof. Let $g \in W_\tau$, using Lemma 2.6, we have

$$\begin{aligned}
|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| &\leq |G_{n,h,\rho}^{d,\theta}(f-g;x)| + |f(x)-g(x)| + |G_{n,h,\rho}^{d,\theta}(g;x) - g(x)| \\
&\leq 2\|f-g\| + |G_{n,h,\rho}^{d,\theta}(g;x) - g(x)|.
\end{aligned} \tag{3.5}$$

Since $g(t) = g(x) + \int_x^t g'(u)du$ and $G_{n,h,\rho}^{d,\theta}(1,x) = 1$, we obtain

$$|G_{n,h,\rho}^{d,\theta}(g;x) - g(x)| \leq \left|G_{n,h,\rho}^{d,\theta}\left(\int_x^t g'(u)du\right)\right|.$$

By Lemma 3.1, we get

$$|G^{d,\theta}_{n,h,\rho}(g;x)-g(x)| \leq 2^{\tau}x^{-\frac{\tau}{2}}\|\phi^{\tau}g'\|G^{d,\theta}_{n,h,\rho}(|t-x|;x). \tag{3.6}$$

Using Cauchy-Schwarz inequality and Remark 2.5, we have

$$\begin{aligned} G^{d,\theta}_{n,h,\rho}(|t-x|;x) &\leq \sqrt{G^{d,\theta}_{n,h,\rho}(1;x)}\sqrt{G^{d,\theta}_{n,h,\rho}((t-x)^2;x)} \\ &\leq \sqrt{\frac{C_1\theta x(1+\rho)}{n\rho}} \end{aligned} \tag{3.7}$$

Combining (3.5)-(3.7), we get

$$|G^{d,\theta}_{n,h,\rho}(f;x)-f(x)| \leq 2\|f-g\| + C_2\|\phi^{\tau}g'\|\frac{\phi^{1-\tau}(x)}{\sqrt{n}}.$$

Taking infimum over all $g \in W_{\tau}$, we have

$$|G^{d,\theta}_{n,h,\rho}(f;x)-f(x)| \leq C_3K_{\phi^{\tau}}\left(f;\frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right).$$

Using (3.2), we get

$$|G^{d,\theta}_{n,h,\rho}(f;x)-f(x)| \leq C\omega_{\phi^{\tau}}\left(f;\frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right),$$

which is the required result. □

Remark 3.3. Taking $\tau = 0$, we get error estimation in terms of the classical modulus of continuity. i.e.,

$$|G^{d,\theta}_{n,h,\rho}(f;x)-f(x)| \leq C\omega\left(f;\sqrt{\frac{x}{n}}\right).$$

Now, we give the following local approximation result for the function belonging to two parameter family of Lipschitz-type space:
For $a \geq 0,\ b > 0$ be fixed, consider the Lipschitz-type space([9]):

$$Lip^{a,b}_M(\beta) = \left\{f \in C_B(\mathbb{R}^+_0) : |f(y)-f(x)| \leq M\frac{|y-x|^{\beta}}{(y+ax^2+bx)^{\frac{\beta}{2}}}\ and\ x,y \in (0,\infty)\right\}, \tag{3.8}$$

where M is any positive constant and $0 < \beta \leq 1$. For $a = 0$ and $b = 1$, space $Lip^{0,1}_M(\beta)$ coincides with the space $Lip^{*}_M(\beta)$ given by O. Szász [12].

Theorem 3.4. *Let $f \in Lip^{a,b}_M(\beta)$. Then for every $n \in \mathbb{N},\ \rho > 0,\ \theta \geq 1$ and $x \in (0,\infty)$, we have*

$$|G^{d,\theta}_{n,h,\rho}(f;x)-f(x)| \leq M\left(\frac{\theta G^{d}_{n,h,\rho}((t-x)^2;x)}{ax^2+bx}\right)^{\frac{\beta}{2}},$$

where

$$G^{d}_{n,h,\rho}((t-x)^2;x) = \frac{x}{n\rho}(1+\rho) + \frac{h(d+1)}{n^2\rho}(\rho(d+1)(h+1)+1).$$

Proof. First of all we prove the theorem for $\beta = 1$. Consider $f \in Lip_M^{a,b}(\beta)$ and $x \in (0, \infty)$, then

$$\begin{aligned}
|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| &\leq |G_{n,h,\rho}^{d,\theta}(f(t) - f(x);x)| \\
&\leq G_{n,h,\rho}^{d,\theta}(|f(t) - f(x)|;x) \\
&\leq G_{n,h,\rho}^{d,\theta}\left(M\frac{|t-x|}{\sqrt{t + ax^2 + bx}};x\right) \\
&\leq \frac{M}{\sqrt{ax^2+bx}} G_{n,h,\rho}^{d,\theta}(|t-x|;x).
\end{aligned}$$

Applying Cauchy-Schwarz inequality and using the fact $G_{n,h,\rho}^{d,\theta}(1;x) = 1$, we have

$$\begin{aligned}
|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| &\leq \frac{M}{\sqrt{ax^2+bx}}(G_{n,h,\rho}^{d,\theta}((t-x)^2;x))^{\frac{1}{2}} \\
&\leq \frac{M}{\sqrt{ax^2+bx}}(\theta G_{n,h,\rho}^{d}((t-x)^2;x))^{\frac{1}{2}} \\
&\leq M\left(\frac{\theta G_{n,h,\rho}^{d}((t-x)^2;x)}{ax^2+bx}\right)^{\frac{1}{2}},
\end{aligned}$$

where $G_{n,h,\rho}^{d}((t-x)^2;x)$ is given in Lemma 2.1. Hence the result holds for $\beta = 1$. Now assume that $0 < \beta < 1$ and $f \in Lip_M^{a,b}(\beta)$, then we have

$$\begin{aligned}
|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| &\leq G_{n,h,\rho}^{d,\theta}(|f(t) - f(x)|;x) \\
&\leq G_{n,h,\rho}^{d,\theta}\left(M\frac{|t-x|^\beta}{(t + ax^2 + bx)^{\frac{\beta}{2}}};x\right) \\
&\leq G_{n,h,\rho}^{d,\theta}\left(M\frac{|t-x|^\beta}{(ax^2 + bx)^{\frac{\beta}{2}}};x\right) \\
&\leq \frac{M}{(ax^2+bx)^{\frac{\beta}{2}}} G_{n,h,\rho}^{d,\theta}(|t-x|^\beta;x)
\end{aligned}$$

Now applying Hölder inequality with $p = \frac{1}{\beta}$ and $q = \frac{1}{1-\beta}$, we have

$$|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| \leq \frac{M}{(ax^2+bx)^{\frac{\beta}{2}}}(G_{n,h,\rho}^{d,\theta}(|t-x|;x))^{\beta}$$

Finally, by Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|G^{d,\theta}_{n,h,\rho}(f;x)-f(x)| &\leq \frac{M}{(ax^2+bx)^{\frac{\beta}{2}}}(G^{d,\theta}_{n,h,\rho}((t-x)^2;x))^{\frac{\beta}{2}} \\
&\leq \frac{M}{(ax^2+bx)^{\frac{\beta}{2}}}(\theta G^{d,\theta}_{n,h,\rho}((t-x)^2;x))^{\frac{\beta}{2}} \\
&\leq M\left(\frac{\theta G^{d}_{n,h,\rho}((t-x)^2;x)}{ax^2+bx}\right)^{\frac{\beta}{2}},
\end{aligned}$$

which is the desired result. □

Now, we give the rate of convergence of our constructed operators in the weighted space: Weighted space of function which are defined on semi-axis $\mathbb{R}_0^+$ and satisfy the inequality $|f(x)|\leq M_f(1+x^2)$, where M_f is a constant depending on f, is denoted by $B_{x^2}(\mathbb{R}_0^+)$. i.e.,

$$B_{x^2}(\mathbb{R}_0^+)=\left\{f:|f(x)|\leq M_f(1+x^2), M_f \textit{ is a constant connected with } f\right\}.$$

Introduce

$$C_{x^2}(\mathbb{R}_0^+)=\left\{f\in B_2(\mathbb{R}_0^+):\ f \textit{ is continuous}\right\}.$$

$$C^*_{x^2}(\mathbb{R}_0^+)=\left\{f\in C_2(\mathbb{R}_0^+): \exists \lim_{x\to\infty}\frac{|f(x)|}{1+x^2}<\infty\right\}.$$

These spaces are endowed with the norm

$$\|f\|_{x^2}=\sup_{x\in\mathbb{R}_0^+}\frac{|f(x)|}{1+x^2}.$$

In general the first and second order modulus of continuity do not tend to zero with $\delta\to 0$ on $\mathbb{R}_0^+$, so we use the following weighted modulus of continuity [17]:

$$\Omega(f;\delta)=\sup_{x\geq 0}\sup_{0<|t|\leq\delta}\frac{|f(x+t)-f(x)|}{1+(x+t)^2} \tag{3.9}$$

Theorem 3.5. *Let $f\in C^*_{x^2}(\mathbb{R}_0^+)$ and $\Omega(f;.)$ be its modulus of continuity. Then for $x\in\mathbb{R}_0^+$, $\rho,\delta>0$, $\theta\geq 1$ and sufficiently large n, we have*

$$\begin{aligned}
|G^{d,\theta}_{n,h,\rho}(f;x)-f(x)| \leq\ & 2(1+x^2)\Omega\left(f;\frac{1}{\sqrt{n}}\right)\left(1+C_1\theta\frac{x(1+\rho)}{n\rho}+\sqrt{\theta C_1}\left(\frac{x(1+\rho)}{\rho}\right)^{\frac{1}{2}}\right. \\
&\left.\times\left(1+\sqrt{\theta C_2}\left(\frac{x(1+\rho)}{n\rho}\right)\right)\right),
\end{aligned}$$

where $C_1,C_2>1$ are constants independent of x and n.

Proof. For $t,x\in\mathbb{R}_0^+$, $\delta>0$ and by the definition of weighted modulus of continuity, we have

$$|f(t)-f(x)|\leq 2(1+x^2)(1+(t-x)^2)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f;\delta). \tag{3.10}$$

Applying $G^{d,\theta}_{n,h,\rho}$ on the inequality (3.10) and then Cauchy-Schwarz inequality in the last term, we obtain

$$\begin{aligned} G^{d,\theta}_{n,h,\rho}(|f(t)-f(x)|;x) &\leq 2(1+x^2)\Omega(f;\delta)\bigg(G^{d,\theta}_{n,h,\rho}(1+(t-x)^2;x) \\ &\quad +G^{d,\theta}_{n,h,\rho}\bigg(\frac{(1+(t-x)^2)|t-x|}{\delta};x\bigg)\bigg) \\ &\leq 2(1+x^2)\Omega(f;\delta)\bigg(G^{d,\theta}_{n,h,\rho}(1;x)+G^{d,\theta}_{n,h,\rho}((t-x)^2;x) \\ &\quad +G^{d,\theta}_{n,h,\rho}\bigg(\frac{(1+(t-x)^2)|t-x|}{\delta};x\bigg)\bigg) \\ &\leq 2(1+x^2)\Omega(f;\delta)\bigg(G^{d,\theta}_{n,h,\rho}(1;x)+G^{d,\theta}_{n,h,\rho}((t-x)^2;x) \\ &\quad +\frac{1}{\delta}(G^{d}_{n,h,\rho}((t-x)^2;x))^{\frac{1}{2}}+\frac{1}{\delta}(G^{d,\theta}_{n,h,\rho}((t-x)^4;x))^{\frac{1}{2}} \\ &\quad \times(G^{d,\theta}_{n,h,\rho}((t-x)^2;x))^{\frac{1}{2}}\bigg). \end{aligned} \tag{3.11}$$

From remark 2.5, we have

$$G^{d,\theta}_{n,h,\rho}((t-x)^2;x) \leq C_1\theta\frac{x(1+\rho)}{n\rho} \tag{3.12}$$

$$G^{d,\theta}_{n,h,\rho}((t-x)^4;x) \leq C_2\theta\bigg(\frac{x(1+\rho)}{n\rho}\bigg)^2, \tag{3.13}$$

for some positive constants C_1 and C_2.
Combining the estimates (3.11)-(3.13) and choosing $\delta=\frac{1}{\sqrt{n}}$, we get the required result. $\square$

4. Quantitative Voronovskaja-type theorem

In this section, we give Voronovskaja-type theorem for $G^{d,\theta}_{n,h,\rho}$. By using Ditzian-Totik modulus of smoothness of first order we will prove this theorem:

Theorem 4.1. *Let $f\in C_B(\mathbb{R}^+_0)$ such that $f',f''\in C_B(\mathbb{R}^+_0)$. Then there hold*

$$\begin{aligned} |\sqrt{n}(G^{d,\theta}_{n,h,\rho}(f;x)-f(x))| &\leq \|f'\|\bigg(C_1\theta\frac{x(1+\rho)}{\rho}\bigg)^{\frac{1}{2}}+\frac{1}{2}\|f''\|C_1\theta\frac{x(1+\rho)}{\sqrt{n}\rho} \\ &\quad +C_3\theta\frac{x(1+\rho)}{\sqrt{n}\rho}\omega_{\phi^\tau}\bigg(f;\frac{\phi^{1-\tau}(x)}{\sqrt{n}}\bigg), \end{aligned}$$

where $C_1>1$ and C_3 are constants independent of n and x.

Proof. By Taylor formula, we write

$$f(t)=f(x)+(t-x)f'(x)+\int_x^t(t-u)f''(u)du.$$

Thus,

$$f(t)-f(x)=(t-x)f'(x)-\frac{1}{2}(t-x)^2f''(u)+\int\limits_x^t(t-u)(f''(u)-f''(x))du, \quad (4.1)$$

operating $G_{n,h,\rho}^{d,\theta}(.;x)$ to both side of above relation, we get

$$\begin{aligned}|G_{n,h,\rho}^{d,\theta}(f;x)-f(x)| &\leq |f'(x)||G_{n,h,\rho}^{d,\theta}((t-x);x)|+\frac{1}{2}|f''(x)||G_{n,h,\rho}^{d,\theta}((t-x)^2;x)| \\ &\quad +\left|G_{n,h,\rho}^{d,\theta}\left(\int\limits_x^t(t-u)(f''(u)-f''(x))du;x\right)\right| \\ &\leq |f'(x)|G_{n,h,\rho}^{d,\theta}(|t-x|;x)+\frac{1}{2}|f''(x)|G_{n,h,\rho}^{d,\theta}((t-x)^2;x) \\ &\quad +G_{n,h,\rho}^{d,\theta}\left(\left|\int\limits_x^t(t-u)(f''(u)-f''(x))du\right|;x\right) \end{aligned} \quad (4.2)$$

Therefore $g\in W_\tau$, we have

$$\left|\int\limits_x^t(t-u)(f''(u)-f''(x))du\right|\leq\|f''-g\|(t-x)^2+2^\tau\phi^{-\tau}(x)\|\phi^\tau f'\||t-x|^3 \quad (4.3)$$

From (4.2), we have

$$\begin{aligned}|G_{n,h,\rho}^{d,\theta}(f;x)-f(x)| &\leq \|f'\|G_{n,h,\rho}^{d,\theta}(|t-x|;x)+\frac{1}{2}\|f''\|G_{n,h,\rho}^{d,\theta}((t-x)^2;x) \\ &\quad +\|f''-g\|G_{n,h,\rho}^{d,\theta}((t-x)^2;x)+2^\tau\phi^{-\tau}(x)\|\phi^\tau f'\|G_{n,h,\rho}^{d,\theta}(|t-x|^3;x)\end{aligned}$$

In view of Remark 2.5 and using Cauchy-Schwarz inequality in the first and last term, we obtain

$$\begin{aligned}
|G^{d,\theta}_{n,h,\rho}(f;x)-f(x)| &\leq \|f'\|(G^{d,\theta}_{n,h,\rho}((t-x)^2;x))^{\frac{1}{2}}+\frac{1}{2}\|f''\||G^{d,\theta}_{n,h,\rho}((t-x)^2;x)| \\
&\quad +\|f''-g\|G^{d,\theta}_{n,h,\rho}((t-x)^2;x)+2^{\tau}\phi^{-\tau}(x)\|\phi^{\tau}g'\|(G^{d,\theta}_{n,h,\rho}((t-x)^2;x))^{\frac{1}{2}} \\
&\quad \times(G^{d,\theta}_{n,h,\rho}((t-x)^4;x))^{\frac{1}{2}} \\
&\leq \|f'\|\left(C_1\theta\frac{x(1+\rho)}{n\rho}\right)^{\frac{1}{2}}+\frac{1}{2}\|f''\|C_1\theta\frac{x(1+\rho)}{n\rho} \\
&\quad +\|f''-g\|C_1\theta\frac{x(1+\rho)}{n\rho}+2^{\tau}\phi^{-\tau}(x)\|\phi^{\tau}g'\|\left(C_1\theta\frac{x(1+\rho)}{n\rho}\right)^{\frac{1}{2}} \\
&\quad \times\left(C_2\theta\left(\frac{x(1+\rho)}{n\rho}\right)^2\right)^{\frac{1}{2}} \\
&\leq \|f'\|\left(C_1\theta\frac{x(1+\rho)}{n\rho}\right)^{\frac{1}{2}}+\frac{1}{2}\|f''\|C_1\theta\frac{x(1+\rho)}{n\rho} \\
&\quad +C_1\theta\frac{x(1+\rho)}{n\rho}\left\{\|f''-g\|+M^*\frac{\phi^{1-\tau}(x)}{\sqrt{n}}\|\phi^{\tau}g'\|\right\}.
\end{aligned}$$

Taking taking the infimum on the right hand side of the above relations over $g\in W_{\tau}$, we get

$$\begin{aligned}
|\sqrt{n}(G^{d,\theta}_{n,h,\rho}(f;x)-f(x))| &\leq \|f'\|\left(C_1\theta\frac{x(1+\rho)}{\rho}\right)^{\frac{1}{2}}+\frac{1}{2}\|f''\|C_1\theta\frac{x(1+\rho)}{\sqrt{n}\rho} \\
&\quad +C_1\theta\frac{x(1+\rho)}{\sqrt{n}\rho}K_{\phi^{\tau}}\left(f;M^*\frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right).
\end{aligned}$$

Now using the inequality (3.2), theorem is proved. □

5. Rate of convergence for functions of Bounded Variation

The rate of convergence for functions having derivative of bounded variation is a fascinating topic, several researchers have studied in this direction [1, 8]. In this section, we shall obtain the rate of convergence of $G^{d,\theta}_{n,h,\rho}(.;.)$ for functions having derivative of bounded variation.
Let $BV(\mathbb{R}^+_0)$ be the space of functions on $\mathbb{R}^+_0$ having derivative of bounded variation on every finite subinterval of $\mathbb{R}^+_0$. Now, we consider the space

$$DBV(\mathbb{R}^+_0)=\{f:\mathbb{R}^+_0\to\mathbb{R}:f\in BV(\mathbb{R}^+_0)\ and\ |f(x)|\leq M_f(1+x^2)\ for\ some\ M_f>0\}$$

Let $f\in DBV(\mathbb{R}^+_0)$ then f can be interpreted as

$$f(x)=\int\limits_0^x g(t)dt+f(0),$$

where g is a function of bounded variation on each finite subinterval of $\mathbb{R}^+_0$.

Lemma 5.1. *Let $x \in \mathbb{R}_0^+$ and $K_{n,h,\rho}^{d,\theta}(x,t)$ be the kernel defined by (1.5). Then for $C_1 > 1$ and sufficiently large n, we have*

(1) $\xi_{n,\rho}^{\theta}(x,y) = \int\limits_0^y K_{n,h,\rho}^{d,\theta}(x,t)dt \leq \frac{\theta C_1 x(1+\rho)}{n\rho}\frac{1}{(x-y)^2},\ 0 \leq y < x.$

(2) $1 - \xi_{n,\rho}^{\theta}(x,z) = \int\limits_z^{\infty} K_{n,h,\rho}^{d,\theta}(x,t)dt \leq \frac{\theta C_1 x(1+\rho)}{n\rho}\frac{1}{(z-x)^2},\ x < z < \infty$

Proof. Using Lemma 2.2, we get

$$\begin{aligned}
\xi_{n,\rho}^{\theta}(x,y) &= \int_0^y K_{n,h,\rho}^{d,\theta}(x,t)dt \\
&\leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 K_{n,h,\rho}^{d,\theta}(x,t)dt \\
&\leq \frac{1}{(x-y)^2} G_{n,h,\rho}^{d,\theta}((t-x)^2;x) \\
&\leq \frac{\theta C_1 x(1+\rho)}{n\rho}\frac{1}{(x-y)^2}.
\end{aligned}$$

Similarly, we can show (2) hence proof is omitted. ☐

Theorem 5.2. *Let $f \in DBV(\mathbb{R}_0^+)$. Then for every $x \in (0,\infty)$, $\rho > 0$, $\theta \geq 1$ and sufficiently large n, we have*

$$\begin{aligned}
|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| &\leq \frac{\sqrt{\theta}}{\theta+1}|f'(x+) + \theta f'(x-)|\left(\frac{C_1 x(1+\rho)}{n\rho}\right)^{\frac{1}{2}} \\
&+\frac{\theta^{\frac{3}{2}}}{\theta+1}|f'(x+) - f'(x-)|\left(\frac{C_1 x(1+\rho)}{n\rho}\right)^{\frac{1}{2}} \\
&+\theta\frac{C_1(1+\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right) + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x\right) \\
&+\frac{x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f'_x\right) + \theta\frac{C_1(1+\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{u}} f'_x
\end{aligned}$$

where $C_1 > 1$ is a positive constant and $\bigvee_c^d f$ denotes the total variation of f on $[c,d]$ and f'_x is defined by

$$f'_x(t) = \begin{cases} f'(t) - f'(x-) & \text{if } 0 \leq t < x, \\ 0 & \text{if } t = x, \\ f'(t) - f'(x+) & \text{if } x < t < \infty. \end{cases} \tag{5.1}$$

Proof. For any $f \in DBV(\mathbb{R}_0^+)$, from the definition of $f'_x(t)$, we can write

$$\begin{aligned}
f'(t) &= \frac{1}{\theta+1}(f'(x+) + \theta f'(x-)) + f'_x(t) + \frac{1}{2}(f'(x+) - f'(x-))\left(sgn(t-x) + \frac{\theta-1}{\theta+1}\right) \\
&+\delta_x(t)\left(f'(x) - \frac{1}{2}(f'(x+) + f'(x-))\right),
\end{aligned} \tag{5.2}$$

where

$$\delta_x(t)=\begin{cases}1 & \text{if } x=t,\\ 0 & \text{if } x\neq t.\end{cases}$$

Now since $G_{n,h,\rho}^{d,\theta}(1;x)=1$, we have

$$\begin{aligned}G_{n,h,\rho}^{d,\theta}(f;x)-f(x) &= G_{n,h,\rho}^{d,\theta}(f(t)-f(x);x)\\ &= \int_0^\infty K_{n,h,\rho}^{d,\theta}(x,t)(f(t)-f(x))dt\\ &= \int_0^\infty K_{n,h,\rho}^{d,\theta}(x,t)\left(\int_x^t f'(u)du\right)dt.\end{aligned}$$

From (5.2), we obtain

$$\begin{aligned}G_{n,h,\rho}^{d,\theta}(f;x)-f(x) &= \int_0^\infty K_{n,h,\rho}^{d,\theta}(x,t)\left(\int_x^t\left\{\frac{1}{\theta+1}(f'(x+)+\theta f'(x-))+f'_x(u)\right.\right.\\ &\quad+\frac{1}{2}(f'(x+)-f'(x-))\left(sgn(u-x)+\frac{\theta-1}{\theta+1}\right)\\ &\quad\left.\left.+\delta_x(u)\left(f'(x)-\frac{1}{2}(f'(x+)+f'(x-))\right)\right\}du\right)dt\end{aligned} \tag{5.3}$$

From the definition of $\delta_x(t)$, it is clear that

$$\int_0^\infty K_{n,h,\rho}^{d,\theta}(x,t)\left(\int_x^t\left(f'(x)-\frac{1}{2}(f'(x+)+f'(x-))\right)\delta_x(u)du\right)dt=0 \tag{5.4}$$

Now, consider

$$\begin{aligned}&\left|\int_0^\infty K_{n,h,\rho}^{d,\theta}(x,t)\left(\int_x^t\frac{1}{\theta+1}(f'(x+)+\theta f'(x-))du\right)dt\right|\\ =&\left|\frac{1}{\theta+1}(f'(x+)+\theta f'(x-))\int_0^\infty K_{n,h,\rho}^{d,\theta}(x,t)(t-x)dt\right|\\ \leq&\frac{1}{\theta+1}|f'(x+)+\theta f'(x-)|\int_0^\infty K_{n,h,\rho}^{d,\theta}(x,t)|t-x|dt\end{aligned}$$

Applying Cauchy Schwarz inequality and Remark 2.5 for sufficiently large n, we have

$$\begin{aligned}
&\Big|\int_0^\infty K_{n,h,\rho}^{d,\theta}(x,t)\Big(\int_x^t \frac{1}{\theta+1}(f'(x+)+\theta f'(x-))du\Big)dt\Big| \\
\leq\ & \frac{1}{\theta+1}|f'(x+)+\theta f'(x-)|\sqrt{G_{n,h,\rho}^{d,\theta}((t-x)^2;x)} \\
\leq\ & \frac{1}{\theta+1}|f'(x+)+\theta f'(x-)|\Big(\frac{\theta C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}} \\
\leq\ & \frac{\sqrt{\theta}}{\theta+1}|f'(x+)+\theta f'(x-)|\Big(\frac{C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}}.
\end{aligned} \tag{5.5}$$

Similarly, we obtain

$$\begin{aligned}
&\Big|\int_0^\infty K_{n,h,\rho}^{d,\theta}(x,t)\Big(\int_x^t \frac{1}{2}(f'(x+)-f'(x-))\Big(sgn(t-x)+\frac{\theta-1}{\theta+1}\Big)du\Big)dt\Big| \\
\leq\ & \frac{\theta}{\theta+1}|f'(x+)-f'(x-)|\Big(\frac{\theta C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}} \\
\leq\ & \frac{\theta^{\frac{3}{2}}}{\theta+1}|f'(x+)-f'(x-)|\Big(\frac{C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}}
\end{aligned} \tag{5.6}$$

Considering (5.3)-(5.6), we obtain the following estimates

$$\begin{aligned}
|G_{n,h,\rho}^{d,\theta}(f;x)-f(x)| \ \leq\ & \Big|\int_0^\infty K_{n,h,\rho}^{d,\theta}(x,t)\Big(\int_x^t f_x'(u)du\Big)dt\Big| \\
&+\frac{\sqrt{\theta}}{\theta+1}|f'(x+)+\theta f'(x-)|\Big(\frac{C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}} \\
&+\frac{\theta^{\frac{3}{2}}}{\theta+1}|f'(x+)-f'(x-)|\Big(\frac{C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}} \\
\leq\ & \Big|A_{n,h,\rho}^{d,\theta}(f_x';x)+B_{n,h,\rho}^{d,\theta}(f_x';x)\Big| \\
&+\frac{\sqrt{\theta}}{\theta+1}|f'(x+)+\theta f'(x-)|\Big(\frac{C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}} \\
&+\frac{\theta^{\frac{3}{2}}}{\theta+1}|f'(x+)-f'(x-)|\Big(\frac{C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}},
\end{aligned} \tag{5.7}$$

where

$$A^{d,\theta}_{n,h,\rho}(f'_x;x)=\int_0^x\left(\int_x^t f'_x(u)du\right)K^{d,\theta}_{n,h,\rho}(x,t)dt$$

and

$$B^{d,\theta}_{n,h,\rho}(f'_x;x)=\int_x^\infty\left(\int_x^t f'_x(u)du\right)K^{d,\theta}_{n,h,\rho}(x,t)dt.$$

Now, we estimate the terms $A^{d,\theta}_{n,h,\rho}(f'_x;x)$ and $B^{d,\theta}_{n,h,\rho}(f'_x;x)$:
Using the definition of $\xi^{\theta}_{n,\rho}(.,.)$ given is Lemma 5.1 and applying the definition of integral by parts, we can write

$$\begin{aligned}A^{d,\theta}_{n,h,\rho}(f'_x;x) &= \int_0^x\left(\int_x^t f'_x(u)du\right)\frac{\partial\xi^{\theta}_{n,\rho}(x,t)}{\partial t}dt\\ &= \int_0^x f'_x(t)\xi^{\theta}_{n,\rho}(x,t)dt.\end{aligned}$$

Thus,

$$\begin{aligned}|A^{d,\theta}_{n,h,\rho}(f'_x;x)| &= \left|\int_0^x f'_x(t)\xi^{\theta}_{n,\rho}(x,t)dt\right|\\ &\leq \int_0^{x-\frac{x}{\sqrt{n}}}|f'_x(t)|\xi^{\theta}_{n,\rho}(x,t)dt+\int_{x-\frac{x}{\sqrt{n}}}^x|f'_x(t)|\xi^{\theta}_{n,\rho}(x,t)dt.\end{aligned}$$

Since $f'_x(x)=0$ and $\xi^{\theta}_{n,\rho}(x,t)\leq 1$, we get

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^x|f'_x(t)|\xi^{\theta}_{n,\rho}(x,t)dt &= \int_{x-\frac{x}{\sqrt{n}}}^x|f'_x(t)-f'_x(x)|\xi^{\theta}_{n,\rho}(x,t)dt\\ &\leq \int_{x-\frac{x}{\sqrt{n}}}^x|f'_x(t)-f'_x(x)|dt\\ &\leq \int_{x-\frac{x}{\sqrt{n}}}^x\left(\bigvee_t^x f'_x\right)dt\\ &\leq \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right).\end{aligned}$$

Now consider $\int\limits_0^{x-\frac{x}{\sqrt{n}}}|f'_x(t)|\xi^{\theta}_{n,\rho}(x,t)dt$ and using Lemma 5.1, we have

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}}|f'_x(t)|\xi^{\theta}_{n,\rho}(x,t)dt &\leq \theta\frac{C_1x(1+\rho)}{n\rho}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{|f'_x(t)|}{(x-t)^2}dt\\ &= \theta\frac{C_1x(1+\rho)}{n\rho}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{|f'_x(t)-f'_x(x)|}{(x-t)^2}dt\\ &\leq \theta\frac{C_1x(1+\rho)}{n\rho}\int_0^{x-\frac{x}{\sqrt{n}}}\left(\bigvee_t^x f'_x\right)\frac{dt}{(x-t)^2}.\end{aligned}$$

Assuming $t = x - \frac{x}{u}$, we have

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}} |f_x'(t)|\xi_{n,\rho}^{\theta}(x,t)dt &\leq \theta\frac{C_1(1+\rho)}{n\rho}\int_1^{\sqrt{n}}\Big(\bigvee_{x-\frac{x}{u}}^{x} f_x'\Big)du \\ &\leq \theta\frac{C_1(1+\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\Big(\bigvee_{x-\frac{x}{k}}^{x} f_x'\Big).\end{aligned}$$

Therefore,

$$|A_{n,h,\rho}^{d,\theta}(f_x';x)| \leq \theta\frac{C_1(1+\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\Big(\bigvee_{x-\frac{x}{k}}^{x} f_x'\Big) + \frac{x}{\sqrt{n}}\Big(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x'\Big). \tag{5.8}$$

Consider

$$|B_{n,h,\rho}^{d,\theta}(f_x';x)| = \Big|\int_x^{\infty}\Big(\int_x^t f_x'(u)du\Big)K_{n,h,\rho}^{d,\theta}(x,t)dt\Big|,$$

using integration by parts and applying Lemma 5.1 with $z = x + \frac{x}{\sqrt{n}}$, we have

$$\begin{aligned}|B_{n,h,\rho}^{d,\theta}(f_x';x)| &\leq \Big|\int_x^z f_x'(t)(1-\xi_{n,\rho}^{\theta}(x,t))dt\Big| + \Big|\int_z^{\infty} f_x'(t)(1-\xi_{n,\rho}^{\theta}(x,t))dt\Big| \\ &\leq \int_x^z \bigvee_x^t f_x' dt + \theta\frac{C_1 x(1+\rho)}{n\rho}\int_z^{\infty}\bigvee_x^t f_x'\frac{1}{(t-x)^2}dt \\ &\leq \frac{x}{\sqrt{n}}\Big(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'\Big) + \theta\frac{C_1 x(1+\rho)}{n\rho}\int_{x+\frac{x}{\sqrt{n}}}^{\infty}\bigvee_x^t f_x'(t-x)^{-2}dt,\end{aligned} \tag{5.9}$$

Put $t = x + \frac{x}{u}$, we get

$$\begin{aligned}\theta\frac{C_1 x(1+\rho)}{n\rho}\int_{x+\frac{x}{\sqrt{n}}}^{\infty}\bigvee_x^t f_x'(t-x)^{-2}dt &= \theta\frac{C_1(1+\rho)}{n\rho}\int_0^{\sqrt{n}}\bigvee_x^{x+\frac{x}{u}} f_x' du, \\ &\leq \theta\frac{C_1(1+\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{u}} f_x',\end{aligned} \tag{5.10}$$

Combining 5.9 and 5.10, we have

$$|B_{n,h,\rho}^{d,\theta}(f_x';x)| \leq \frac{x}{\sqrt{n}}\Big(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'\Big) + \theta\frac{C_1(1+\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{u}} f_x'. \tag{5.11}$$

Now collecting the estimates 5.7, 5.8 and 5.11, we get the required result. □

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