Blending type by Bezier-Summation-Integral type operators

Article in Communications · January 2018

2 authors:

Tuncer Acar
Kirikkale University

Arun Kajla
Central University of Haryana

BLENDING TYPE APPROXIMATION BY BÉZIER-SUMMATION-INTEGRAL TYPE OPERATORS

Tuncer Acar[1] and Arun Kajla[2]

[1]Department of Mathematics, Faculty of Science and Arts,
Kirikkale University, TR-71450, Yahsihan, Kirikkale, Turkey
e-mail: tunceracar@ymail.com

[2]Department of Mathematics
Central University of Haryana
Haryana-123031, India
e-mail: rachitkajla47@gmail.com

Abstract. In this note we construct the Bézier variant of summation integral type operators based on a non-negative real parameter. We present a direct approximation theorem by means of the first order modulus of smoothness and the rate of convergence for absolutely continuous functions having a derivative equivalent to a function of bounded variation. In the last section, we study the quantitative Voronovskaja type theorem.
Keywords: Bézier operators, summation integral type operators, rate of convergence, bounded variation.
Mathematics Subject Classification(2010): 41A25, 26A15.

1. Introduction

In 1912 Bernstein introduced the most famous algebraic polynomials $B_n(f;x)$ in approximation theory in order to give a constructive proof of Weierstrass's theorem which is given by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1],$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and he proved that if $f \in C[0,1]$ then $B_n(f;x)$ converges uniformly to $f(x)$ in $[0,1]$.
The Bernstein operators have been used in many branches of mathematics and computer science. Since their useful structure, Bernstein polynomials and their modifications have been intensively studied. Among the other we refer the readers to (cf. [2, 3, 8, 12, 23, 27]).

For $f \in C[0,1]$, Chen et al. [10] introduced a generalization of the Bernstein operators based on a non-negative parameter α $(0 \leq \alpha \leq 1)$ as follows:

$$T_n^{(\alpha)}(f;x) = \sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1] \tag{1.1}$$

where $p_{n,k}^{(\alpha)}(x) = \left[\binom{n-2}{k}(1-\alpha)x + \binom{n-2}{k-2}(1-\alpha)(1-x) + \binom{n}{k}\alpha x(1-x)\right] x^{k-1}(1-x)^{n-k-1}$ and $n \geq 2$. They proved the rate of convergence, Voronovskaja type asymptotic formula and Shape preserving properties for these operators. For the special case, $\alpha = 1$, these operators

reduce the well-known Bernstein operators.
In [19], Kajla and Acar introduced a sequence of summation-integral type operators as follows:

$$D_n^{(\alpha)}(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x)\int_0^1 p_{n,k}(t)\, f(t)dt, \tag{1.2}$$

where $f \in L_1[0,1]$ (the space of all Lebesgue integrable functions on $[0,1]$),
$p_{n,k}(t) = \binom{n}{k} t^k(1-t)^{n-k}$ and $p_{n,k}^{(\alpha)}(x)$ is defined as above. In [19], Voronoskaja type asymptotic formula, rate of convergence, local and global convergence results were established for these operators (1.2).

The aim of this paper is to introduce Bézier variant of the operators (1.2) and obtain the direct approximation results. Furthermore we study the rate of convergence for an absolutely continuous function f having a derivative f' equivalent with a function of bounded variation on $[0,1]$ and quantitative Voronovskaja type theorem.

A Bézier curve is a parametric curve frequently used in computer graphics and image processing. These are mainly used in approximation, interpolation, curve fitting etc. Bézier-Bernstein type operators were established by many mathematicians. The pioneer works in this direction are due to [3, 5, 9, 13, 24, 26, 28–30]. In these works, the direct approximation results were obtained and the rate of convergence for functions of bounded variation were established. The order of approximation of the summation-integral type operators for functions with derivatives of bounded variation is estimated in [1, 4, 6, 7, 14–18, 20–22, 25].

For $f \in L_1[0,1]$, we define the Bézier variant of the operators $D_n^{(\alpha)}(f;x)$ as

$$\mathcal{S}_{n,\alpha}^{(\rho)}(f;x) = (n+1)\sum_{k=0}^{n} Q_{n,k,\alpha}^{(\rho)}(x)\int_0^1 p_{n,k}(t)f(t)dt, \quad x \in [0,1], \tag{1.3}$$

where $\rho \geq 1$, $Q_{n,k,\alpha}^{(\rho)}(x) = [J_{n,k,\alpha}(x)]^\rho - [J_{n,k+1,\alpha}(x)]^\rho$ and $J_{n,k,\alpha}(x) = \sum_{j=k}^{n} p_{n,j}^{(\alpha)}(x)$, when $k \leq n$ and 0 otherwise.

Alternatively we may rewrite the operators (1.3) as

$$\mathcal{S}_{n,\alpha}^{(\rho)}(f;x) = \int_0^1 \mathcal{M}_{n,\alpha,\rho}(x,t)f(t)dt, \;\; x \in [0,1], \tag{1.4}$$

where

$$\mathcal{M}_{n,\alpha,\rho}(x,t) = (n+1)\sum_{k=0}^{n} Q_{n,k,\alpha}^{(\rho)}(x)p_{n,k}(t).$$

If $\rho = 1$ then the operators $\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)$ reduce to the operators $D_n^{(\alpha)}(f;x)$.
Throughout this article, C denotes a positive constant independent of n and x, not necessarily the same at each occurrence.

To express our results we give the following auxiliary results.

Lemma 1. *[19] Let* $e_i(t) = t^i, i = \overline{0,4}$, *then we have*

(i) $D_n^{(\alpha)}(e_0;x) = 1;$

(ii) $D_n^{(\alpha)}(e_1;x) = x + \dfrac{1-2x}{(n+2)}$;

(iii) $D_n^{(\alpha)}(e_2;x) = x^2 + \dfrac{2x^2(\alpha-3n-4)}{(n+2)(n+3)} + \dfrac{2x(2n-\alpha+1)}{(n+2)(n+3)} + \dfrac{2}{(n+2)(n+3)}$;

(iv) $D_n^{(\alpha)}(e_3;x) = x^3 + \dfrac{6x^3(-n(5+2n-\alpha)-2(1+\alpha))}{(n+2)(n+3)(n+4)} + \dfrac{3x^2(n(3n-2\alpha-1)+10(\alpha-1))}{(n+2)(n+3)(n+4)}$
$+ \dfrac{18x(n-\alpha+1)}{(n+2)(n+3)(n+4)} + \dfrac{6}{(n+2)(n+3)(n+4)}$;

(v) $D_n^{(\alpha)}(e_4;x) = x^4 + \dfrac{x^4(-4(n+3)(16+n(3+5n))+12\alpha(n-3)(n-2))}{(n+2)(n+3)(n+4)(n+5)}$
$+ \dfrac{4x^3(n-2)(n(4n-3\alpha-1)+33(\alpha-1))}{(n+2)(n+3)(n+4)(n+5)} + \dfrac{24x^2\left(n+3n^2+14(\alpha-1)-4n\alpha\right)}{(n+2)(n+3)(n+4)(n+5)}$
$+ \dfrac{48x(2n-3\alpha+3)}{(n+2)(n+3)(n+4)(n+5)} + \dfrac{24}{(n+2)(n+3)(n+4)(n+5)}$.

Hence we get

$$D_n^{(\alpha)}((t-x);x) = \frac{1-2x}{n+2} < \frac{\lambda_1(1-x)}{n+1}, \quad \forall x \in [0,1] \text{ and } \forall n \in \mathbb{N} \tag{1.5}$$

with $\lambda_1 \geq 2$ and

$$D_n^{(\alpha)}((t-x)^2;x) = \frac{2x(1-x)(n-\alpha-2)}{(n+2)(n+3)} + \frac{2}{(n+2)(n+3)}.$$

From [19], we have

$$D_n^{(\alpha)}((t-x)^2;x) < \frac{2}{(n+2)}\gamma_n^2(x), \quad \forall x \in [0,1] \text{ and } \forall n \in \mathbb{N}$$

where $\gamma_n^2(x) = \varphi^2(x) + \frac{1}{(n+2)}$ and $\varphi^2(x) = x(1-x)$. Then we can write

$$D_n^{(\alpha)}((t-x)^2;x) < \frac{\lambda_2\gamma_n^2(x)}{n+2}, \ \lambda_2 \geq 2. \tag{1.6}$$

$$\begin{aligned} D_n^{(\alpha)}((t-x)^4;x) = {} & \frac{12x^3(x-2)(n(n-2\alpha-19)+46\alpha-36)}{(n+2)(n+3)(n+4)(n+5)} + \frac{12x^2(n(n-2\alpha-25)+58\alpha-38)}{(n+2)(n+3)(n+4)(n+5)} \\ & + \frac{24x(3n-6\alpha+1)}{(n+2)(n+3)(n+4)(n+5)} + \frac{24}{(n+2)(n+3)(n+4)(n+5)} \end{aligned} \tag{1.7}$$

Remark 1. *We have*

$$\begin{aligned} \mathcal{S}_{n,\alpha}^{(\rho)}(e_0;x) &= \sum_{k=0}^{n} Q_{n,k,\alpha}^{(\rho)}(x) = [J_{n,0,\alpha}(x)]^\rho \\ &= \left[\sum_{j=0}^{n} p_{n,j}^{(\alpha)}(x)\right]^\rho = 1, \ \textit{since} \ \sum_{j=0}^{n} p_{n,j}^{(\alpha)}(x) = 1. \end{aligned}$$

Lemma 2. *[19] Let $f \in C[0,1]$. Then for $x \in [0,1]$ we have*

$$\| D_n^{(\alpha)}(f) \| \leq \| f \| .$$

2. Direct Estimates

To describe our results, we recall the definitions of the first order modulus of smoothness and the K-functional [11]. Let $\varphi(x) = \sqrt{x(1-x)}$, $f \in C[0,1]$. The first order modulus of smoothness is given by

$$\omega_{\varphi}(f;t) = \sup_{0<h\leq t}\left\{ \left| f\left(x+\frac{h\varphi(x)}{2}\right) - f\left(x-\frac{h\varphi(x)}{2}\right)\right|, x \pm \frac{h\varphi(x)}{2} \in [0,1]\right\},$$

and the appropriate Peetre's K-functional is defined by

$$\overline{K}_{\varphi}(f;t) = \inf_{g\in W_{\varphi}}\{||f-g|| + t||\varphi g'|| + t^2||g'||\} \ (t>0),$$

where $W_{\varphi} = \{g : g \in AC_{loc}, ||\varphi g^{'}|| < \infty, ||g^{'}|| < \infty\}$ and $||.||$ is the uniform norm on $C[0,1]$. It is well known that ([11], Thm. 3.1.2) $\overline{K}_{\varphi}(f;t) \sim \omega_{\varphi}(f;t)$ which means that there exists a constant $M > 0$ such that

$$M^{-1}\omega_{\varphi}(f;t) \leq \overline{K}_{\varphi}(f;t) \leq M\omega_{\varphi}(f;t). \tag{2.1}$$

Lemma 3. *Let $f \in C[0,1]$. Then, for $x \in [0,1]$, we have*

$$\| \mathcal{S}_{n,\alpha}^{(\rho)}(f) \| \leq \rho \| f \| .$$

Proof. Applying the inequality $| a^{\rho} - b^{\rho} | \leq \rho \mid a - b \mid$ with $0 \leq a, b \leq 1, \rho \geq 1$ and from definition of $Q_{n,k,\alpha}^{(\rho)}(x)$, we may write

$$\begin{aligned} 0 &< [J_{n,k,\alpha}(x)]^{\rho} - [J_{n,k+1,\alpha}(x)]^{\rho} \leq \rho\,(J_{n,k,\alpha}(x) - J_{n,k+1,\alpha}(x)) \\ &= \rho\; p_{n,k}^{(\alpha)}(x). \end{aligned}$$

Hence from the definition $\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)$ and Lemma 2, we obtain

$$\| \mathcal{S}_{n,\alpha}^{(\rho)}(f) \| \leq \rho \| D_n^{(\alpha)}(f) \| \leq \rho \| f \| .$$

□

Now we study a direct approximation theorem for the operators $\mathcal{S}_{n,\alpha}^{(\rho)}$.

Theorem 1. *Suppose that f be in $C[0,1]$ and $\varphi(x) = \sqrt{x(1-x)}$ then for every $x \in [0,1)$, we have*

$$| \mathcal{S}_{n,\alpha}^{(\rho)}(f;x) - f(x) | < C\omega_{\varphi}\left(f;\frac{1}{\sqrt{n+2}}\right), \tag{2.2}$$

where C is a constant independent of n and x.

Proof. By the definition of $\overline{K}_{\varphi}(f;t)$, for fixed n, x, we can choose $g = g_{n,x} \in W_{\varphi}$ such that

$$||f-g|| + \frac{1}{\sqrt{n+2}}||\varphi g'|| + \frac{1}{n+2}||g'|| \leq 2\overline{K}_{\varphi}\left(f;\frac{1}{\sqrt{n+2}}\right). \tag{2.3}$$

Using Remark 1, we can write

$$\begin{aligned} | \mathcal{S}_{n,\alpha}^{(\rho)}(f;x) - f(x) | &\leq | \mathcal{S}_{n,\alpha}^{(\rho)}(f-g;x) | + |f-g| + | \mathcal{S}_{n,\alpha}^{(\rho)}(g;x) - g(x) | \\ &\leq 2||f-g|| + | \mathcal{S}_{n,\alpha}^{(\rho)}(g;x) - g(x) | . \end{aligned} \tag{2.4}$$

We only need to compute the second term in the above equation. We will have to split the estimate into two domains, i.e. $x \in F_n^c = [0, 1/n]$ and $x \in F_n = (1/n, 1)$.

Using the representation $g(t) = g(x) + \int_x^t g'(u)du$, we get

$$\left|\mathcal{S}_{n,\alpha}^{(\rho)}(g;x) - g(x)\right| = \left|\mathcal{S}_{n,\alpha}^{(\rho)}\left(\int_x^t g'(u)du; x\right)\right|. \tag{2.5}$$

If $x \in F_n = (1/n, 1)$ then $\gamma_n(x) \sim \varphi(x)$. We have

$$\left|\int_x^t g'(u)du\right| \leq ||\varphi g'|| \left|\int_x^t \frac{1}{\varphi(u)}du\right|. \tag{2.6}$$

For any $x, t \in (0, 1)$, we find that

$$\begin{aligned}
\left|\int_x^t \frac{1}{\varphi(u)}du\right| &= \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}du\right| \\
&\leq \left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\right)du\right| \\
&\leq 2\left(\ |\ \sqrt{t} - \sqrt{x}\ | + |\ \sqrt{1-t} - \sqrt{1-x}\ |\ \right) \\
&= 2|t-x|\left(\frac{1}{\sqrt{t} + \sqrt{x}} + \frac{1}{\sqrt{1-t} + \sqrt{1-x}}\right) \\
&< 2|t-x|\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right) \\
&\leq \frac{2\sqrt{2}\ |t-x|}{\varphi(x)}.
\end{aligned} \tag{2.7}$$

Combining (2.5)-(2.7) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|\mathcal{S}_{n,\alpha}^{(\rho)}(g;x) - g(x)| &< 2\sqrt{2}||\varphi g'||\varphi^{-1}(x)\mathcal{S}_{n,\alpha}^{(\rho)}(|t-x|;x) \\
&\leq 2\sqrt{2}||\varphi g'||\varphi^{-1}(x)\left(\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)\right)^{1/2} \\
&\leq 2\sqrt{2}||\varphi g'||\varphi^{-1}(x)\left(\rho\ D_n^{(\alpha)}((t-x)^2;x)\right)^{1/2}.
\end{aligned}$$

From (1.6), we get

$$|\mathcal{S}_{n,\alpha}^{(\rho)}(g;x) - g(x)| < \frac{C||\varphi g'||}{\sqrt{n+2}}. \tag{2.8}$$

For $x \in F_n^c = [0, 1/n]$, $\gamma_n(x) \sim \dfrac{1}{\sqrt{n+2}}$ and

$$\left|\int_x^t g'(u)du\right| \leq ||g'||\ |t-x|.$$

Therefore, using Cauchy-Schwarz inequality we have

$$\begin{aligned}
|\mathcal{S}_{n,\alpha}^{(\rho)}(g;x) - g(x)| &\leq ||g'||\mathcal{S}_{n,\alpha}^{(\rho)}(|t-x|;x) \\
&\leq C||g'||\frac{\gamma_n(x)}{\sqrt{n+2}} < \frac{C}{n+2}||g'||.
\end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we have

$$|\mathcal{S}_{n,\alpha}^{(\rho)}(g;x)-g(x)|<C\left(\frac{||\varphi g'||}{\sqrt{n+2}}+\frac{1}{n+2}||g'||\right). \tag{2.10}$$

Using $\overline{K}_{\varphi}(f;t)\sim\omega_{\varphi}(f;t)$ and (2.3), (2.4), (2.10), we get the desired relation (2.2). This completes the proof of the theorem.

□

3. Rate of Convergence

In this section we would like to obtain the rate of convergence of the operators $\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)$ for an absolutely continuous function f having a derivative f' equivalent to a function of bounded variation on $[0,1]$.

Throughout this section $DBV[0,1]$ will denote the class of all absolutely continuous functions f defined on $[0,1]$ and having on $(0,1)$, a derivative $f^{'}$ equivalent with a function of bounded variation on $[0,1]$. We notice that the functions $f\in DBV[0,1]$ possess a representation

$$f(x)=\int_0^x g(t)dt+f(0),$$

where $g\in BV[0,1]$, i.e., g is a function of bounded variation on $[0,1]$.

Lemma 4. *Let $x\in(0,1]$, then for $\rho\geq 1,\lambda_2\geq 2$ and sufficiently large n, we have*

(i) $\vartheta_{n,\alpha,\rho}(x,y)=\displaystyle\int_0^y\mathcal{M}_{n,\alpha,\rho}(x,t)dt<\frac{\rho\lambda_2}{(n+2)}\frac{\gamma_n^2(x)}{(x-y)^2}$, $0\leq y<x$,

(ii) $1-\vartheta_{n,\alpha,\rho}(x,z)=\displaystyle\int_z^1\mathcal{M}_{n,\alpha,\rho}(x,t)dt<\frac{\rho\lambda_2}{(n+2)}\frac{\gamma_n^2(x)}{(z-x)^2}$, $x<z<1$.

Proof. (i) From Lemmas 1 and 2, we get

$$\begin{aligned}\vartheta_{n,\alpha,\rho}(x,y) &= \int_0^y\mathcal{M}_{n,\alpha,\rho}(x,t)dt\leq\int_0^y\left(\frac{x-t}{x-y}\right)^2\mathcal{M}_{n,\alpha,\rho}(x,t)dt\\ &= \mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)\ (x-y)^{-2}\leq\rho D_n^{(\alpha)}((t-x)^2;x)\ (x-y)^{-2}\\ &< \frac{\rho\lambda_2}{(n+2)}\frac{\gamma_n^2(x)}{(x-y)^2}.\end{aligned}$$

The proof of (ii) is similar to the proof of (i). Hence it is omitted. □

Theorem 2. *Let $f\in DBV(0,1),\rho\geq 1$ and let $\upsilon_a^b(f_x')$ be the total variation of f_x' on $[a,b]\subset[0,1]$. Then for every $x\in(0,1)$ and for sufficiently large n, we have*

$$\begin{aligned}\left|\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)\right| &< \frac{1}{\rho+1}\left|f'(x+)+\rho f'(x-)\right|\sqrt{\frac{\rho\lambda_2}{(n+2)}}\ \gamma_n(x)\\ &\quad+\sqrt{\frac{\rho\lambda_2}{(n+2)}}\ \gamma_n(x)\frac{\rho}{\rho+1}\left|f'(x+)-f'(x-)\right|\\ &\quad+\rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}x^{-1}\sum_{k=1}^{[\sqrt{n}\ ]}\upsilon_{x-(x/k)}^{x}(f_x')+\frac{x}{\sqrt{n}}\upsilon_{x-(x/\sqrt{n})}^{x}(f_x')\\ &\quad+\rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}(1-x)^{-1}\sum_{k=1}^{[\sqrt{n}\ ]}\upsilon_x^{x+((1-x)/k)}(f_x')+\frac{1-x}{\sqrt{n}}\upsilon_x^{x+((1-x)/\sqrt{n})}(f_x'),\end{aligned}$$

where $\lambda_2 \geq 2$ *and the auxiliary function and* f'_x *is defined by*

$$f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x \\ 0, & t = x \\ f'(t) - f'(x+) & x < t \leq 1. \end{cases}$$

Proof. Using the fact that $\int_0^1 \mathcal{M}_{n,\alpha,\rho}(x,t)dt = \mathcal{S}^{(\rho)}_{n,\alpha}(e_0;x) = 1$, we have

$$\begin{aligned} \mathcal{S}^{(\rho)}_{n,\alpha}(f;x) - f(x) &= \int_0^1 [f(t) - f(x)]\mathcal{M}_{n,\alpha,\rho}(x,t)dt \\ &= \int_0^1 \Big(\int_x^t f'(u)du \Big) \mathcal{M}_{n,\alpha,\rho}(x,t)dt. \end{aligned} \tag{3.1}$$

From definition of the function f'_x, for any $f \in DBV(0,1)$, we can write

$$\begin{aligned} f'(t) &= \frac{1}{\rho+1}\Big(f'(x+) + \rho f'(x-) \Big) + f'_x(t) \\ &\quad + \frac{1}{2}\Big(f'(x+) - f'(x-) \Big)\Big(sgn(t-x) + \frac{\rho-1}{\rho+1} \Big) \\ &\quad + \delta_x(t)\Big(f'(x) - \frac{1}{2}\Big(f'(x+) + f'(x-) \Big)\Big), \end{aligned} \tag{3.2}$$

where

$$\delta_x(t) = \begin{cases} 1\ , & x = t \\ 0\ , & x \neq t. \end{cases}$$

It is clear that

$$\int_0^1 \mathcal{M}_{n,\alpha,\rho}(x,t) \int_x^t \Big[f'(x) - \frac{1}{2}\Big(f'(x+) + f'(x-) \Big)\Big] \delta_x(t)dudt = 0.$$

By (1.4) and simple computations, we have

$$\begin{aligned} P_1 &= \int_0^1 \Big(\int_x^t \frac{1}{\rho+1}\Big(f'(x+) + \rho f'(x-) \Big) du \Big) \mathcal{M}_{n,\alpha,\rho}(x,t)dt \\ &= \frac{1}{\rho+1}\Big| f'(x+) + \rho f'(x-) \Big| \int_0^1 |t-x| \mathcal{M}_{n,\alpha,\rho}(x,t)dt \\ &\leq \frac{1}{\rho+1}\Big| f'(x+) + \rho f'(x-) \Big| \Big(\mathcal{S}^{(\rho)}_{n,\alpha}((t-x)^2;x) \Big)^{1/2} \end{aligned}$$

and

$$
\begin{aligned}
P_2 &= \int_0^1 \left(\int_x^t \frac{1}{2}\Big(f'(x+) - f'(x-) \Big) \left(sgn(u-x) + \frac{\rho-1}{\rho+1} \right) du \right) \mathcal{M}_{n,\alpha,\rho}(x,t)dt \\
&= \frac{1}{2}\left(f'(x+) - f'(x-) \right) \Bigg[-\int_0^x \left(\int_t^x \left(sgn(u-x) + \frac{\rho-1}{\rho+1} \right) du \right) \mathcal{M}_{n,\alpha,\rho}(x,t)dt \\
&\quad + \int_x^1 \left(\int_x^t \left(sgn(u-x) + \frac{\rho-1}{\rho+1} \right) du \right) \mathcal{M}_{n,\alpha,\rho}(x,t)dt \Bigg] \\
&\leq \frac{\rho}{\rho+1}\left(f'(x+) - f'(x-) \right) \int_0^1 |t-x| \, \mathcal{M}_{n,\alpha,\rho}(x,t)dt \\
&= \frac{\rho}{\rho+1}\left(f'(x+) - f'(x-) \right) \mathcal{S}_{n,\alpha}^{(\rho)} \left(|t-x| ; x \right) \\
&\leq \frac{\rho}{\rho+1}\left(f'(x+) - f'(x-) \right) \left(\mathcal{S}_{n,\alpha}^{(\rho)} ((t-x)^2; x) \right)^{1/2}.
\end{aligned}
$$

By using (1.6) and considering (3.1), (3.2) we obtain the following estimate

$$
\begin{aligned}
\left| \mathcal{S}_{n,\alpha}^{(\rho)}(f;x) - f(x) \right| &< \left| E_{n,\alpha,\rho}(f_x', x) + F_{n,\alpha,\rho}(f_x', x) \right| \\
&\quad + \frac{1}{\rho+1} \left| f'(x+) + \rho f'(x-) \right| \sqrt{\frac{\rho\lambda_2}{(n+2)}} \, \gamma_n(x) \\
&\quad + \frac{\rho}{\rho+1} \left| f'(x+) - f'(x-) \right| \sqrt{\frac{\rho\lambda_2}{(n+2)}} \, \gamma_n(x),
\end{aligned}
\tag{3.3}
$$

where

$$
E_{n,\alpha,\rho}(f_x', x) = \int_0^x \left(\int_x^t f_x'(u)du \right) \mathcal{M}_{n,\alpha,\rho}(x,t)dt
$$

and

$$
F_{n,\alpha,\rho}(f_x', x) = \int_x^1 \left(\int_x^t f_x'(u)du \right) \mathcal{M}_{n,\alpha,\rho}(x,t)dt.
$$

To complete the proof, it is sufficient to estimate the terms $E_{n,\alpha,\rho}(f_x', x), F_{n,\alpha,\rho}(f_x', x)$. Since $\int_a^b d_t \vartheta_{n,\alpha,\rho}(x,t) \leq 1$ for all $[a,b] \subseteq [0,1]$, using integration by parts and applying Lemma 4 with $y = x - (x/\sqrt{n})$, we have

$$
\begin{aligned}
\left| E_{n,\alpha,\rho}(f_x', x) \right| &= \left| \int_0^x \left(\int_x^t f_x'(u)du \right) d_t \vartheta_{n,\alpha,\rho}(x,t) \right| \\
&= \left| \int_0^x \vartheta_{n,\alpha,\rho}(x,t) f_x'(t) dt \right| \\
&\leq \left(\int_0^y + \int_y^x \right) \left| f_x'(t) \right| \left| \vartheta_{n,\alpha,\rho}(x,t) \right| dt \\
&< \rho \frac{\lambda_2 \gamma_n^2(x)}{(n+2)} \int_0^y \upsilon_t^x (f_x')(x-t)^{-2} dt + \int_y^x \upsilon_t^x (f_x') dt \\
&\leq \rho \frac{\lambda_2 \gamma_n^2(x)}{(n+2)} \int_0^y \upsilon_t^x (f_x')(x-t)^{-2} dt + \frac{x}{\sqrt{n}} \upsilon_{x-(x/\sqrt{n})}^x (f_x').
\end{aligned}
$$

By the substitution of $u = x/(x-t)$, we get

$$\begin{aligned}
\rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}\int_0^{x-(x/\sqrt{n})}(x-t)^{-2}\upsilon_t^x(f_x')dt &= \rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}x^{-1}\int_1^{\sqrt{n}}\upsilon_{x-(x/u)}^x(f_x')du \\
&\leq \rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}x^{-1}\sum_{k=1}^{[\sqrt{n}\ ]}\int_k^{k+1}\upsilon_{x-(x/u)}^x(f_x')du \\
&< \rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}x^{-1}\sum_{k=1}^{[\sqrt{n}\ ]}\upsilon_{x-(x/k)}^x(f_x').
\end{aligned}$$

Hence we reach the following result

$$\left|E_{n,\alpha,\rho}(f_x',x)\right| < \rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}x^{-1}\sum_{k=1}^{[\sqrt{n}\ ]}\upsilon_{x-(x/k)}^x(f_x') + \frac{x}{\sqrt{n}}\upsilon_{x-(x/\sqrt{n})}^x(f_x'). \tag{3.4}$$

Using integration by parts and applying Lemma 4 with $z = x + ((1-x)/\sqrt{n})$, we have

$$\begin{aligned}
|F_{n,\alpha,\rho}(f_x',x)| &= \left|\int_x^1\left(\int_x^t f_x'(u)du\right)\mathcal{M}_{n,\alpha,\rho}(x,t)dt\right| \\
&= \left|\int_x^z\left(\int_x^t f_x'(u)du\right)d_t(1-\vartheta_{n,\alpha,\rho}(x,t)) + \int_z^1\left(\int_x^t f_x'(u)du\right)d_t(1-\vartheta_{n,\alpha,\rho}(x,t))\right| \\
&= \left|\left[\left(\int_x^t f_x'(u)du\right)(1-\vartheta_{n,\alpha,\rho}(x,t))\right]_x^z - \int_x^z f_x'(t)(1-\vartheta_{n,\alpha,\rho}(x,t))dt\right. \\
&\quad \left. + \int_z^1\left(\int_x^t f_x'(u)du\right)d_t(1-\vartheta_{n,\alpha,\rho}(x,t))\right| \\
&= \left|\int_x^z f_x'(u)du(1-\vartheta_{n,\alpha,\rho}(x,z)) - \int_x^z f_x'(t)(1-\vartheta_{n,\alpha,\rho}(x,t))dt\right. \\
&\quad \left. + \left[\int_x^t f_x'(u)du(1-\vartheta_{n,\alpha,\rho}(x,t))\right]_z^1 - \int_z^1 f_x'(t)(1-\vartheta_{n,\alpha,\rho}(x,t))dt\right| \\
&= \left|\int_x^z f_x'(t)(1-\vartheta_{n,\alpha,\rho}(x,t))dt + \int_z^1 f_x'(t)(1-\vartheta_{n,\alpha,\rho}(x,t))dt\right| \\
&< \rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}\int_z^1\upsilon_x^t(f_x')(t-x)^{-2}dt + \int_x^z\upsilon_x^t(f_x')dt \\
&\leq \rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}\int_{x+((1-x)/\sqrt{n})}^1\upsilon_x^t(f_x')(t-x)^{-2}dt + \frac{(1-x)}{\sqrt{n}}\upsilon_x^{x+((1-x)/\sqrt{n})}(f_x').
\end{aligned}$$

By the substitution of $u = (1-x)/(t-x)$, we get

$$\begin{aligned}
\rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}\int_{x+((1-x)/\sqrt{n})}^1\upsilon_x^t(f_x')(t-x)^{-2}dt &= \rho\frac{\lambda_2\gamma_n^2(x)}{(1-x)(n+2)}\int_1^{\sqrt{n}}\upsilon_x^{x+((1-x)/u)}(f_x')du \\
&< \rho\frac{\lambda_2\gamma_n^2(x)}{(1-x)(n+2)}\sum_{k=1}^{[\sqrt{n}\ ]}\int_k^{k+1}\upsilon_x^{x+((1-x)/u)}(f_x')du \\
&\leq \rho\frac{\lambda_2\gamma_n^2(x)}{(1-x)(n+2)}\sum_{k=1}^{[\sqrt{n}\ ]}\upsilon_x^{x+((1-x)/k)}(f_x').
\end{aligned}$$

Thus, we get

$$\begin{aligned}|F_{n,\alpha,\rho}(f'_x,x)| &< \rho\frac{\lambda_2\gamma_n^2(x)}{(1-x)(n+2)}\sum_{k=1}^{[\sqrt{n}\ ]}\upsilon_x^{x+((1-x)/k)}(f'_x)\\ &\quad+\frac{1-x}{\sqrt{n}}\upsilon_x^{x+((1-x)/\sqrt{n})}(f'_x). \qquad (3.5)\end{aligned}$$

Collecting the estimates (3.3)-(3.5), we get the required result. This completes the proof of theorem. □

4. quantitative Voronovskaja-type theorem

Now we are going to study a quantitative Voronovskaja-type result for the operators $\mathcal{S}_{n,\alpha}^{(\rho)}$. This result is given using the first order Ditzian-Totik modulus of smoothness.

Theorem 3. *Let $f\in C^2[0,1]$. Then there hold*

$$\begin{aligned}\left|\sqrt{n}\left(\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)\right)\right| &\leq \sqrt{2\rho\left\{\varphi^2(x)+\frac{1}{n+2}\right\}}||f''||+||f''||\frac{\rho}{\sqrt{n}}\varphi^2(x)\\ &\quad+\frac{C}{\sqrt{n}}\omega_\varphi(x)\left(f'';\frac{2\sqrt{3}}{\sqrt{n}}\varphi(x)\right)+\circ(n^{-1});\\ \left|\sqrt{n}\left(\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)\right)\right| &\leq \sqrt{2\rho\left\{\varphi^2(x)+\frac{1}{n+2}\right\}}||f''||+||f''||\frac{\rho}{\sqrt{n}}\varphi^2(x)\\ &\quad+\frac{C}{\sqrt{n}}\omega_\varphi(x)\varphi(x)\left(f'';\frac{2\sqrt{3}}{\sqrt{n}}\right)+\circ(n^{-1})\end{aligned}$$

Proof. Let $f\in C^2[0,1]$ and $x,t\in[0,1]$. Then Taylor's expansion, we may write

$$f(t)-f(x)=(t-x)f'(x)+\int_x^t(t-u)f''(u)du.$$

Thus,

$$f(t)-f(x)=f'(x)(t-x)-\frac{1}{2}(t-x)^2f''(x)+\int_x^t(t-u)f''(u)du-\int_x^t(t-u)f''(u)du.$$

Operating $\mathcal{S}_{n,\alpha}^{(\rho)}(\cdot;x)$ to both sides of the above relation, we get

$$\begin{aligned}|\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)| &= |f'(x)|\mathcal{S}_{n,\alpha}^{(\rho)}(|t-x|;x)+\frac{1}{2}|f''(x)|\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)\\ &\quad+\mathcal{S}_{n,\alpha}^{(\rho)}\left(\left|\int_x^t|t-u|\ |f''(u)-f''(x)|du\right|;x\right). \qquad (4.1)\end{aligned}$$

Therefore, $g\in W_\varphi$ we have

$$\left|\int_x^t|t-u|\ |f''(u)-f''(x)|du\right|\leq||f''-g||(t-x)^2+2||\varphi g'||\varphi^{-1}(x)|t-x|^3. \qquad (4.2)$$

Thus, in view of (4.1), (4.2), (1.7) and using Cauchy-Schwarz inequality, we may write

$$\begin{aligned}
\left|\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)\right| &\leq |f'(x)|\mathcal{S}_{n,\alpha}^{(\rho)}(|t-x|;x)+\frac{1}{2}|f''(x)|\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)\\
&\quad +||f''-g||\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)+2||\varphi g'||\varphi^{-1}(x)\mathcal{S}_{n,\alpha}^{(\rho)}(|t-x|^3;x)\\
&\leq ||f'||\left(\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)\right)^{1/2}+\frac{1}{2}||f''||\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)+||f''-g||\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)\\
&\quad +2||\varphi g'||\varphi^{-1}(x)\left(\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)\right)^{1/2}\left(\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^4;x)\right)^{1/2}\\
&= \sqrt{\frac{2\rho}{(n+2)}\left\{\varphi^2(x)+\frac{1}{n+2}\right\}}||f''||+||f''||\frac{\rho}{n+2}\left\{\varphi^2(x)+\frac{1}{n+2}\right\}\\
&\quad +\frac{2\rho}{n+2}\left\{\varphi^2(x)+\frac{1}{n+2}\right\}||f''-g||+2||\varphi g'||\varphi^{-1}(x)\left\{\varphi^2(x)+\frac{1}{n+2}\right\}\\
&\quad \times\Bigg(\rho\Bigg(\frac{12x^3(x-2)\left(n(n-2\alpha-19)+46\alpha-36\right)}{(n+2)(n+3)(n+4)(n+5)}\\
&\quad +\frac{12x^2\left(n(n-2\alpha-25)+58\alpha-38\right)}{(n+2)(n+3)(n+4)(n+5)}+\frac{24x(3n-6\alpha+1)}{(n+2)(n+3)(n+4)(n+5)}\\
&\quad +\frac{24}{(n+2)(n+3)(n+4)(n+5)}\Bigg)\Bigg)^{1/2}\\
&\leq \sqrt{\frac{2\rho}{(n+2)}\left\{\varphi^2(x)+\frac{1}{n+2}\right\}}||f''||+||f''||\frac{\rho}{n+2}\varphi^2(x)\\
&\quad +\frac{2\rho}{n+2}\left\{\varphi^2(x)||f''-g||+||\varphi g'||\varphi(x)\frac{2\sqrt{3}}{\sqrt{n}}\right\}+\circ(n^{-3/2}).
\end{aligned}$$

Since $\varphi^2(x)\leq\varphi(x)\leq 1, x\in[0,1]$, we have

$$\begin{aligned}
\left|\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)\right| &\leq \sqrt{\frac{2\rho}{(n+2)}\left\{\varphi^2(x)+\frac{1}{n+2}\right\}}||f''||+||f''||\frac{\rho}{n+2}\varphi^2(x)\\
&\quad +\frac{2\rho}{n+2}\left\{||f''-g||+||\varphi g'||\varphi(x)\frac{2\sqrt{3}}{\sqrt{n}}\right\}+\circ(n^{-3/2}). \qquad (4.3)\\
\left|\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)\right| &\leq \sqrt{\frac{2\rho}{(n+2)}\left\{\varphi^2(x)+\frac{1}{n+2}\right\}}||f''||+||f''||\frac{\rho}{n+2}\varphi^2(x)\\
&\quad +\frac{2\rho}{n+2}\varphi(x)\left\{||f''-g||+||\varphi g'||\frac{2\sqrt{3}}{\sqrt{n}}\right\}+\circ(n^{-3/2}). \qquad (4.4)
\end{aligned}$$

By taking the infimum on the right hand side of the above relations over $g \in W_\phi$, we get

$$\begin{aligned}\left|\sqrt{n}\left(\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)\right)\right| &\leq \sqrt{2\rho\left\{\varphi^2(x)+\frac{1}{n+2}\right\}}\|f''\|+\|f''\|\frac{\rho}{\sqrt{n}}\varphi^2(x)\\ &\quad+\frac{C}{\sqrt{n}}\overline{K_\varphi}\left(f'';\frac{2\sqrt{3}}{\sqrt{n}}\varphi(x)\right)+\circ(n^{-1});\\ \left|\sqrt{n}\left(\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)\right)\right| &\leq \sqrt{2\rho\left\{\varphi^2(x)+\frac{1}{n+2}\right\}}\|f''\|+\|f''\|\frac{\rho}{\sqrt{n}}\varphi^2(x)\\ &\quad+\frac{C}{\sqrt{n}}\varphi(x)\overline{K_\varphi}\left(f'';\frac{2\sqrt{3}}{\sqrt{n}}\right)+\circ(n^{-1}).\end{aligned}$$

Applying $\overline{K_\varphi}(f,t)\sim\omega_\varphi(f,t)$, the theorem is proved. $\square$

References

[1] U. Abel, V. Gupta and M. Ivan, Asymptotic approximation of functions and their derivatives by generalized Baskakov-Szász-Durrmeyer operators, Analysis Theory Appl. 21 (2005), 15-26.
[2] T. Acar and A. Aral, On pointwise convergence of q-Bernstein operators and their q-derivatives, Numer. Funct. Anal. Optim., 36 (3), (2015), 287-304.
[3] T. Acar, P. N. Agrawal and T. Neer, Bézier variant of the Bernstein-Durrmeyer type operators, Results. Math., DOI: 10.1007/s00025-016-0639-3.
[4] P. N. Agrawal, V. Gupta, A. S. Kumar and A. Kajla, Generalized Baskakov-Szász type operators, Appl. Math. Comput. 236 (2014), 311-324.
[5] P. N. Agrawal, N. Ispir and A. Kajla, Approximation properties of Bézier-summation-integral type operators based on Polya-Bernstein functions, Appl. Math. Comput. 259 (2015) 533-539.
[6] R. Bojanic and F. H. Cheng, Rate of convergence of Bernstein polynomials for functions with derivatives of bounded variation, J. Math. Anal. Appl. 141 (1) (1989), 136-151.
[7] R. Bojanic and F. Cheng, Rate of convergence of Hermite-Fejer polynomials for functions with derivatives of bounded variation, Acta Math. Hungar. 59 (1992), 91-102.
[8] D. Cárdenas-Morales, P. Garrancho and I. Raşa, Asymptotic formulae via a Korovkin type result, Abstr. Appl. Anal. (2012). Art.ID 217464, 12pp.
[9] G. Chang, Generalized Bernstein-Bezier polynomials, J. Comput. Math. 1 (4) (1983), 322-327.
[10] X. Chen, J. Tan, Z. Liu and J. Xie, Approximation of functions by a new family of generalized Bernstein operators, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2016.12.075
[11] Z. Ditzian and V. Totik, Moduli of Smoothness, Springer, New York, (1987).
[12] A. D. Gadjiev and A. M. Ghorbanalizaeh, Approximation properties of a new type Bernstein-Stancu polynomials of one and two variables, Appl. Math. Comput. 216 (2010) 890-901.
[13] S. S. Guo, G. F. Liu and Z. J. Song, Approximation by Bernstein-Durrmeyer-Bezier operators in L_p spaces, Acta Math. Sci. Ser. A Chin. Ed. 30 (6) (2010), 1424-1434.
[14] V. Gupta and R. P. Agarwal, Convergence Estimates in Approximation Theory, Springer, Berlin (2014).
[15] V. Gupta and H. Karsli, Rate of convergence for the Bézier variant of the MKZD operators, Georgian Math. J. 14 (2007), 651-659.
[16] V. Gupta, V. Vasishtha and M. K. Gupta, Rate of convergence of summation-integral type operators with derivatives of bounded variation, J. Inequal. Pure Appl. Math. 4 (2) (2003), Article 34.
[17] N. Ispir, Rate of convergence of generalized rational type Baskakov operators, Math. Comput. Modelling 46 (2007), 625-631.
[18] S. Jain and R. K. Gangwar, Approximation degree for generalized integral operators, Rev. Un. Mat. Argentina 50 (1) (2009), 61-68.
[19] A. Kajla and T. Acar, Blending type approximation by generalized Bernstein-Durrmeyer type operators, Miskolc Mathematical Notes (To Appear).
[20] A. Kajla, A. M. Acu and P. N. Agrawal, Baskakov-Szász type operators based on inverse Pólya-Eggenberger distribution, Ann. Funct. Anal. 8 (2017) 106-123.

[21] H. Karsli, Rate of convergence of new Gamma type operators for functions with derivatives of bounded variation, Math. Comput. Modelling 45 (2007), 617-624.

[22] T. Neer, A. M. Acu and P. N. Agrawal, Bézier variant of genuine-Durrmeyer type operators based on Pólya distribution. Carpathian J. Math. 1 (2017).

[23] M. Mursaleen, K. J. Ansari and A. Khan, On $(p, q)-$analogue of Bernstein operators, Appl. Math. Comput. 266 (2015) 874-882.

[24] M.-Y. Ren and X. M. Zeng, Approximation of a kind of new type Bézier operators, J. Inequal. Appl. (2015) 2015: 412 DOI 10.1186/s13660-015-0940-9

[25] H. M. Srivastava and V. Gupta, Rate of convergence for the Bézier variant of the Bleimann-Butzer-Hahn operators, Appl. Math. Letters, 18 (2005) 849-857.

[26] P. Wang and Y. Zhou, A new estimate on the rate of convergence of Durrmeyer-Bezier Operators, J. Inequal. Appl. 2009, Article ID 702680.

[27] M. Yang, D. Yu and P. Zhou, On the approximation by operators of Bernstein-Stancu types, Appl. Math. Comput. 246 (2014) 79-87.

[28] X. M. Zeng, On the rate of convergence of two Bernstein-Bézier type operators for bounded variation functions, J. Approx. Theory 95 (1998), 369-387.

[29] X. M. Zeng, On the rate of convergence of two Bernstein-Bézier type operators for bounded variation functions II, J. Approx. Theory 104 (2000), 330-344.

[30] X. M. Zeng and A. Piriou, On the rate of convergence of two Bernstein-Bezier type operators for bounded variation functions. J. Approx. Theory 95 (1998), 369-387.