Existence and controllability of higher-order nonlinear fractional integrodifferential systems via resolvent operator

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Abstract: This work analyzes the existence of solution and approximate controllability for higher order non-linear fractional integro-differential systems with Riemann-Liouville derivatives in Banach spaces. Firstly, the definition of mild solution for the system is derived. Then a set of sufficient conditions for the existence of mild solution and approximate controllability of the system is obtained. The discussions are based on fixed point approach, and the theory of convolution and fractional resolvent. To illustrate the feasibility of developed theory, an example is given.

Keywords: Fractional Systems, Mild Solutions, Fixed Point, Approximate Controllability

1 Introduction

Let V and V' be Banach spaces, and $Z = L_p([0,\lambda]; V)$ and $U = L_p([0,\lambda]; V')$ be function spaces. Consider the non-linear fractional system

$$\begin{cases} D_t^\kappa z(t) = Az(t) + Bu(t) + f\left(t, z(t), \displaystyle\int_0^t \psi(t,s,z(s))\,ds\right), \quad t \in (0,\lambda], \\ \left(I_t^{2-\kappa} z(t)\right)_{t=0} = y_0 \in V, \\ \left(D_t^{\kappa-1} z(t)\right)_{t=0} = y_1 \in V, \end{cases} \tag{1.1}$$

where $1 < \kappa \leq 2$, $p < \frac{1}{2-\kappa}$ (when $\kappa \neq 2$) and D_t^κ stands for κ-order Riemann-Liouville derivative. The control $u \in U$, the state $z \in Z$; $A : D(A) \subseteq V \to V$ is the generator of a Riemann-Liouville fractional κ-order resolvent $\wp_\kappa(t)$, where $D(A)$ is dense in V. $B : U \to Z$ is a linear map. The operators $f : [0,\lambda] \times V \times V \to V$ and $\psi : \Delta \times V \to V$ are non-linear, here $\Delta = \{(t,s) : 0 \leq s \leq t \leq \lambda\}$.

In many physical, biological and engineering problems, differential systems of fractional order are found to be suitable models. Therefore, in last twenty years, they attracted more attention from researchers. In fact, for the illustration of memory and hereditary properties, fractional derivatives give a better instrument. For this reason, they have been broadly applied in the areas of physics, electrodynamics, economics, aerodynamics, control theory, viscoelasticity and heat conduction. In recent years, noteworthy achievements of fractional systems have been made in the theory as well as applications [2–14].

In many areas such as nuclear reactor dynamics and thermoelasticity, it is required to reflect the systems' memory effect in the model. In the modeling of such systems, if differential equations are used, which involve functions at any specific time and space, the impact of history result is omitted. Therefore, to incorporate the memory effect in these systems, a term of integration is added in the differential system, which turns to integro-differential system. The integro-differential systems have been broadly applied in viscoelastic mechanics, fluid dynamics, thermoelastic contact, control theory, heat conduction, industrial

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mathematics, financial mathematics, biological models, chemical kinetics and aerospace systems, etc. (see [16–22]).

The existence and controllability results for various types of systems are proved by many researchers [?, 15, 21–42, 44–54, 56, 57]. In recent years, the controllability properties of Caputo fractional systems have been extensively studied. However, the work on the controllability of Riemann-Liouville fractional systems is still in the initial phase. Liu and Li [38] studied the approximate controllability of Riemann-Liouville fractional systems of the form $D_t^\kappa z(t) = Az(t) + f(t, z(t)) + (Bu)(t)$ $(0 < \kappa \leq 1)$ with the integral initial condition $\left(I_t^{1-\kappa} z(t)\right)_{t=0} = y_0$ in Banach spaces using the theory of C_0-semigroup together with the probability density function. Ibrahim et al. [34] determined the existence and controllability results for the same system with the initial condition $\lim_{t\to 0^+} \Gamma(\kappa) t^{1-\kappa} z(t) = y_0$ using the concept of κ-order resolvent rather than C_0-semigroup. Mahmudov and McKibben [32] determined the approximate controllability of fractional systems with generalized Riemann-Liouville derivatives. Yang and Wang [31] investigated the approximate controllability of fractional differential inclusions with Riemann-Liouville derivatives.

However, as for as we know, the controllability for fractional integro-differential systems of order $\kappa \in (1, 2]$ using Riemann-Liouville fractional resolvent have not been discussed in the literature. In order to fill this gap, we determine the existence and uniqueness of solution and approximate controllability of the Riemann-Liouville fractional integro-differential system (1.1).

Main contributions of the paper are:

- The definition of mild solution in terms of resolvent operator is derived using the theory of convolution.
- Existence and uniqueness of mild solutions is proven using generalized fixed point theorem.
- Approximate controllability of the non-linear system is proven using sequence method. For this, some additional results are derived in Lemma 5.4 for proposed system under some hypotheses.
- The concept of Riemann-Liouville fractional resolvent is used rather than fractional cosine family.

2 Preliminaries

In this section, some definitions and preliminaries are presented which are used throughout the paper. Denote by $\mathcal{L}(V)$ the space of linear and bounded operators from V to V.

Definition 2.1. *The Riemann-Liouville integral of order κ is given by*

$$I_t^\kappa \varphi(t) = \frac{1}{\Gamma(\kappa)} \int_0^t (t-s)^{\kappa-1} \varphi(s)\, ds, \quad \kappa > 0,$$

where Γ is the gamma function.

Definition 2.2. *The Riemann-Liouville derivative of order κ is given by*

$$D_t^\kappa \varphi(t) = \frac{1}{\Gamma(m-\kappa)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-\kappa-1} \varphi(s)\, ds, \quad \kappa > 0,$$

where $m = [\kappa] + 1$.

Definition 2.3. *The Mittag-Leffler function $E_{\kappa,\widehat{\kappa}}$ is given by*

$$E_{\kappa,\widehat{\kappa}}(\xi) = \sum_{j=0}^{\infty} \frac{\xi^j}{\Gamma(\kappa j + \widehat{\kappa})}.$$

For $\widehat{\kappa}=1$, it is denoted by E_κ.

Theorem 2.4. *[43]. Suppose $\kappa>0$, $\varphi_1(t)$ is non-negative and locally integrable over $0\le t\le\lambda$ and $\varphi_2(t)$ is a non-negative, non-decreasing continuous bounded function over $0\le t\le\lambda$. If $\xi(t)$ be locally integrable and non-negative over $0\le t\le\lambda$ satisfying*

$$\xi(t)\le\varphi_1(t)+\varphi_2(t)\int_0^t(t-s)^{\kappa-1}\xi(s)\,ds.$$

Then

$$\xi(t)\le\varphi_1(t)+\int_0^t\left(\sum_{\alpha=1}^{\infty}\frac{(\varphi_2(t)\Gamma(\kappa))^{\alpha}}{\Gamma(\alpha\kappa)}(t-s)^{\alpha\kappa-1}\varphi_1(s)\right)ds.$$

Corollary 2.5. *[43]. Under the assumption of Theorem 2.4, if $\varphi_1(t)$ is non-decreasing, then*

$$\xi(t)\le\varphi_1(t)E_\kappa\big(\varphi_2(t)t^{\kappa}\Gamma(\kappa)\big).$$

Definition 2.6. *[1]. A family $\{\wp_\kappa(t)\mid t>0\}\subset\mathcal{L}(V)$ is said to be Riemann-Liouville κ-order fractional resolvent, if*

(i) for $y\in V$, $\wp_\kappa(\cdot)y\in C((0,\infty),V)$ and

$$\lim_{t\to0^+}\Gamma(\kappa-1)t^{2-\kappa}\wp_\kappa(t)y=y,$$

(ii) $\wp_\kappa(s)$ and $\wp_\kappa(t)$ commute for $s,t>0$,

(iii) for $s,t>0$

$$\wp_\kappa(s)I_t^{\kappa}\wp_\kappa(t)-I_s^{\kappa}\wp_\kappa(s)\wp_\kappa(t)=\frac{s^{\kappa-2}}{\Gamma(\kappa-1)}I_t^{\kappa}\wp_\kappa(t)-\frac{t^{\kappa-2}}{\Gamma(\kappa-1)}I_s^{\kappa}\wp_\kappa(s).$$

Definition 2.7. *The operator A given by*

$$Ay=\Gamma(2\kappa-1)\lim_{t\to0^+}\frac{t^{2-\kappa}\wp_\kappa(t)y-\frac{1}{\Gamma(\kappa-1)}y}{t^{\kappa}}\quad\text{for } y\in D(A),$$

where

$$D(A)=\left\{y\in V:\lim_{t\to0^+}\frac{t^{2-\kappa}\wp_\kappa(t)y-\frac{1}{\Gamma(\kappa-1)}y}{t^{\kappa}}\ \text{exists}\right\},$$

is known as the generator of Riemann-Liouville κ-order fractional resolvent $\wp_\kappa(t)$.

Remark 2.8. *Since $\wp_\kappa(\cdot)$ has singularity at $t=0$ (when $\kappa\neq2$). Therefore we can not assume the uniform boundedness of $\wp_\kappa(t)$ on any interval $(0,r]$ $(r>0)$. However, $t^{2-\kappa}\wp_\kappa(t)$ is uniformly bounded on every bounded interval contained in $\mathbb{R}^+$.*

Lemma 2.9. *[1]. Let A be the generator of a Riemann-Liouville κ-order fractional resolvent $\wp_\kappa(t)$, then*

(i) $\wp_\kappa(t)y\in D(A)$ and $A\wp_\kappa(t)y=\wp_\kappa(t)Ay$ for $y\in D(A)$,

(ii) for $y\in V$, $t>0$,

$$\wp_\kappa(t)y=\frac{t^{\kappa-2}}{\Gamma(\kappa-1)}y+AI_t^{\kappa}\wp_\kappa(t)y,$$

(iii) for $y\in D(A)$, $t>0$,

$$\wp_\kappa(t)y=\frac{t^{\kappa-2}}{\Gamma(\kappa-1)}y+I_t^{\kappa}\wp_\kappa(t)Ay.$$

Throughout this article we assume that there is a constant $l_\wp>0$ such that $\|t^{2-\kappa}\wp_\kappa(t)\|\le l_\wp$.

3 Definition of mild solutions

In this section, we derive the definition of mild solution of (1.1). For this, we consider the system

$$\begin{cases} D_t^\kappa z(t) = Az(t) + \xi(t), \qquad t \in (0,\lambda], \\ \left(I_t^{2-\kappa} z(t)\right)_{t=0} = y_0 \in V, \\ \left(D_t^{\kappa-1} z(t)\right)_{t=0} = y_1 \in V, \end{cases} \tag{3.1}$$

where $\xi \in L_p([0,\lambda]; V)$.

By a mild solution of (3.1) we mean a function $z \in C((0,\lambda]; V)$ satisfying

$$z(t) = \frac{t^{\kappa-2}}{\Gamma(\kappa-1)} y_0 + \frac{t^{\kappa-1}}{\Gamma(\kappa)} y_1 + AI_t^\kappa z(t) + I_t^\kappa \xi(t). \tag{3.2}$$

From the definition of convolution, it is not difficult to prove the next lemma

Lemma 3.1. *Let ς_1 and ς_2 be any two functions such that $\varsigma_1 * \varsigma_2$ exists. Then for $t > 0$, the integral $\int_0^t (\varsigma_1 * \varsigma_2)(t)\, dt$ makes sense and*

$$\int_0^t (\varsigma_1 * \varsigma_2)(t)\, dt = \left(\int_0^{(\cdot)} \varsigma_1(t)\, dt * \varsigma_2 \right)(t) = \left(\varsigma_1 * \int_0^{(\cdot)} \varsigma_2(t)\, dt \right)(t).$$

Using above lemma, we prove that (3.2) is equivalent to the integral equation

$$z(t) = \wp_\kappa(t) y_0 + \int_0^t \wp_\kappa(t) y_1\, dt + \int_0^t \int_0^{t-s} \wp_\kappa(\sigma)\xi(s)\, d\sigma\, ds. \tag{3.3}$$

Theorem 3.2. *A function $z \in C((0,\lambda]; V)$ is a mild solution of (3.1) iff it satisfies (3.3).*

Proof. Let $\varsigma_\kappa(t) = \frac{t^{\kappa-2}}{\Gamma(\kappa-1)}$ and $\eta_\kappa(t) = \frac{t^{\kappa-1}}{\Gamma(\kappa)}$. By (ii) of Lemma 2.9, one has

$$\varsigma_\kappa(t) = \wp_\kappa(t) - \left(A\eta_\kappa * \wp_\kappa\right)(t).$$

Now

$$\begin{aligned} \varsigma_\kappa * z &= \left(\wp_\kappa - A\eta_\kappa * \wp_\kappa\right) * z \\ &= \left(\wp_\kappa - A\left(\int_0^{(\cdot)} \varsigma_\kappa(t)\, dt \right) * \wp_\kappa \right) * z \\ &= \wp_\kappa * z - \wp_\kappa * \left(A\left(\int_0^{(\cdot)} \varsigma_\kappa(t)\, dt \right) * z \right) \\ &= \wp_\kappa * \left(z - A\left(\int_0^{(\cdot)} \varsigma_\kappa(t)\, dt \right) * z \right) \\ &= \wp_\kappa * \left(\varsigma_\kappa y_0 + \int_0^{(\cdot)} \varsigma_\kappa(t) y_1\, dt + \int_0^{(\cdot)} \varsigma_\kappa(t)\, dt * \xi \right) \\ &= \varsigma_\kappa * \left(\wp_\kappa y_0 + \int_0^{(\cdot)} \wp_\kappa(t) y_1\, dt + \int_0^{(\cdot)} \wp_\kappa(t)\, dt * \xi \right), \\ \implies z(t) &= \left(\wp_\kappa y_0 + \int_0^{(\cdot)} \wp_\kappa(t) y_1\, dt + \int_0^{(\cdot)} \wp_\kappa(t)\, dt * \xi \right)(t) \end{aligned}$$

$$= \wp_\kappa(t)y_0 + \int_0^t \wp_\kappa(t)y_1\,dt + \int_0^t \int_0^{t-s} \wp_\kappa(\sigma)\xi(s)\,d\sigma\,ds.$$

Conversely, we assume that (3.3) is satisfied. Then one has

$$\begin{aligned}
&\left(s^{2-\kappa}\wp_\kappa(s) - \frac{1}{\Gamma(\kappa-1)}\right) I_t^\kappa z(t)\\
&= \left(s^{2-\kappa}\wp_\kappa(s) - \frac{1}{\Gamma(\kappa-1)}\right) I_t^\kappa \left(\wp_\kappa(t)y_0 + \int_0^t \wp_\kappa(t)y_1 dt + \int_0^t \int_0^{t-s} \wp_\kappa(\sigma)\xi(s)\,d\sigma\,ds\right)\\
&= \left(s^{2-\kappa}\wp_\kappa(s) - \frac{1}{\Gamma(\kappa-1)}\right) \left(I_t^\kappa \wp_\kappa(t)y_0 + I_t^\kappa \int_0^t \wp_\kappa(t)y_1\,dt + I_t^\kappa \left(\int_0^{(\cdot)} \wp_\kappa(\sigma)\,d\sigma * \xi\right)(t)\right)\\
&= \left(s^{2-\kappa}\wp_\kappa(s) I_t^\kappa \wp_\kappa(t)y_0 - \frac{1}{\Gamma(\kappa-1)} I_t^\kappa \wp_\kappa(t)y_0\right)\\
&\quad + \left(s^{2-\kappa}\wp_\kappa(s) I_t^\kappa \int_0^t \wp_\kappa(t)y_1\,dt - \frac{1}{\Gamma(\kappa-1)} I_t^\kappa \int_0^t \wp_\kappa(t)y_1\,dt\right)\\
&\quad + \left(s^{2-\kappa}\wp_\kappa(s) I_t^\kappa \left(\int_0^{(\cdot)} \wp_\kappa(\sigma)\,d\sigma * \xi\right)(t) - \frac{1}{\Gamma(\kappa-1)} I_t^\kappa \left(\int_0^{(\cdot)} \wp_\kappa(\sigma)\,d\sigma * \xi\right)(t)\right)\\
&= s^{2-\kappa}\left(\wp_\kappa(s) I_t^\kappa \wp_\kappa(t)y_0 - \frac{s^{\kappa-2}}{\Gamma(\kappa-1)} I_t^\kappa \wp_\kappa(t)y_0\right)\\
&\quad + s^{2-\kappa}\left(\wp_\kappa(s) I_t^\kappa \int_0^t \wp_\kappa(t)y_1\,dt - \frac{s^{\kappa-2}}{\Gamma(\kappa-1)} I_t^\kappa \int_0^t \wp_\kappa(t)y_1\,dt\right)\\
&\quad + s^{2-\kappa}\left(\wp_\kappa(s) I_t^\kappa \left(\int_0^{(\cdot)} \wp_\kappa(\sigma)\,d\sigma * \xi\right)(t) - \frac{s^{\kappa-2}}{\Gamma(\kappa-1)} I_t^\kappa \left(\int_0^{(\cdot)} \wp_\kappa(\sigma)\,d\sigma * \xi\right)(t)\right)\\
&= s^{2-\kappa}\left(I_s^\kappa \wp_\kappa(s)\wp_\kappa(t)y_0 - \frac{t^{\kappa-2}}{\Gamma(\kappa-1)} I_s^\kappa \wp_\kappa(s)y_0\right)\\
&\quad + s^{2-\kappa}\left(I_s^\kappa \wp_\kappa(s) \int_0^t \wp_\kappa(t)y_1\,dt - \frac{t^{\kappa-1}}{\Gamma(\kappa)} I_s^\kappa \wp_\kappa(s)y_1\right)\\
&\quad + s^{2-\kappa}\left(I_s^\kappa \wp_\kappa(s) \left(\int_0^{(\cdot)} \wp_\kappa(\sigma)\,d\sigma * \xi\right)(t) - I_s^\kappa \wp_\kappa(s) \left(\int_0^{(\cdot)} \frac{t^{\kappa-2}}{\Gamma(\kappa-1)}\,dt * \xi\right)(t)\right)\\
&= s^{2-\kappa} I_s^\kappa \wp(s)\Bigg(\left(\wp_\kappa(t)y_0 + \int_0^t \wp_\kappa(t)y_1\,dt + \int_0^t \int_0^{t-s} \wp_\kappa(\sigma)\xi(s)\,d\sigma\,ds\right)\\
&\quad - \left(\frac{t^{\kappa-2}}{\Gamma(\kappa-1)}y_0 + \frac{t^{\kappa-1}}{\Gamma(\kappa)}y_1 + I_t^\kappa \xi(t)\right)\Bigg)\\
&= s^{2-\kappa} I_s^\kappa \wp(s)\left(z(t) - \frac{t^{\kappa-2}}{\Gamma(\kappa-1)}y_0 - \frac{t^{\kappa-1}}{\Gamma(\kappa)}y_1 - I_t^\kappa \xi(t)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
A I_t^\kappa z(t) &= \Gamma(2\kappa-1) \lim_{s\to 0^+} \frac{\left(s^{2-\kappa}\wp_\kappa(s) - \frac{1}{\Gamma(\kappa-1)}\right) I_t^\kappa z(t)}{s^\kappa}\\
&= \Gamma(2\kappa-1) \lim_{s\to 0^+} s^{2-2\kappa} I_s^\kappa \wp_\kappa(s)\left(z(t) - \frac{t^{\kappa-2}}{\Gamma(\kappa-1)}y_0 - \frac{t^{\kappa-1}}{\Gamma(\kappa)}y_1 - I_t^\kappa \xi(t)\right). \qquad (3.4)
\end{aligned}$$

Now, for any $y \in V$

$$\left\|\Gamma(2\kappa-1)s^{2-2\kappa} I_s^\kappa \wp_\kappa(s)y - y\right\| = \left\|\frac{\Gamma(2\kappa-1)}{\Gamma(\kappa)} \int_0^s s^{2-2\kappa}(s-\vartheta)^{\kappa-1}\wp_\kappa(\vartheta)y\,d\vartheta - y\right\|$$

$$
\begin{aligned}
&= \left\| \frac{\Gamma(2\kappa-1)}{\Gamma(\kappa)} \int_0^1 s^{2-\kappa}(1-\vartheta)^{\kappa-1}\wp_\kappa(s\vartheta)y\,d\vartheta - y \right\| \\
&= \left\| \frac{\Gamma(2\kappa-1)}{\Gamma(\kappa)} \int_0^1 \vartheta^{\kappa-2}(1-\vartheta)^{\kappa-1}(s\vartheta)^{2-\kappa}\wp_\kappa(s\vartheta)y\,d\vartheta \right. \\
&\quad \left. - \frac{\Gamma(2\kappa-1)}{\Gamma(\kappa)\Gamma(\kappa-1)} \int_0^1 \vartheta^{\kappa-2}(1-\vartheta)^{\kappa-1}y\,d\vartheta \right\| \\
&\le \sup_{\vartheta\in(0,1]} \left\| \Gamma(\kappa-1)(s\vartheta)^{2-\kappa}\wp_\kappa(s\vartheta)y - y \right\|
\end{aligned}
$$

$$
\implies \lim_{s\to 0^+} \Gamma(2\kappa-1)s^{2-2\kappa}I_s^\kappa \wp_\kappa(s)y = y.
$$

Hence from (3.4)

$$
\begin{aligned}
AI_t^\kappa z(t) &= z(t) - \frac{t^{\kappa-2}}{\Gamma(\kappa-1)}y_0 - \frac{t^{\kappa-1}}{\Gamma(\kappa)}y_1 - I_t^\kappa \xi(t) \\
\implies z(t) &= \frac{t^{\kappa-2}}{\Gamma(\kappa-1)}y_0 + \frac{t^{\kappa-1}}{\Gamma(\kappa)}y_1 + AI_t^\kappa z(t) + I_t^\kappa \xi(t).
\end{aligned}
$$

□

Now we are in position to give the definition of mild solution of the original system in terms of $\wp_\kappa$. For this, we consider the Banach space $C_{2-\kappa}([0,\lambda];V) = \{z : t^{2-\kappa}z(t) \in C([0,\lambda];V)\}$ with the norm $\|z\|_{C_{2-\kappa}} = \sup_{t\in[0,\lambda]}\{t^{2-\kappa}\|z(t)\|_V\}$.

Definition 3.3. *A function $z \in C_{2-\kappa}([0,\lambda];V)$ is said to be a mild solution of* (1.1) *if it satisfies*

$$
\begin{aligned}
z(t) &= \wp_\kappa(t)y_0 + \int_0^t \wp_\kappa(t)y_1\,dt \\
&\quad + \int_0^t \int_0^{t-s} \wp_\kappa(\sigma)\left(Bu(s) + f\left(s, z(s), \int_0^s \psi(s,\vartheta,z(\vartheta))\,d\vartheta\right)\right) d\sigma\,ds. \qquad (3.5)
\end{aligned}
$$

If we denote by $z(t,u)$ the mild solution of (1.1) corresponding to a given $u \in U$, then the set $\mathfrak{R}_\lambda(f) = \{z(\lambda,u) \in V : u \in U\}$ is known as the **reachable set** of (1.1). Further, the system (1.1) is called **approximately controllable** on $[0,\lambda]$ if $\overline{\mathfrak{R}_\lambda(f)} = V$.

Remark 3.4. *It should be noted that, for $\kappa \neq 2$, $C_{2-\kappa}([0,\lambda];V)$ is dense subset of Z if $p < \frac{1}{2-\kappa}$.*

4 Existence and uniqueness of mild solutions

To study the mild solution, we assume the following conditions:

(A_1) there exist positive constants l_f and l_ψ satisfying

$$
\begin{aligned}
&(i)\ \|f(t,y_1,\widetilde{y}_1) - f(t,y_2,\widetilde{y}_2)\| \le l_f\left(\|y_1-y_2\| + \|\widetilde{y}_1-\widetilde{y}_2\|\right)\ \forall\, y_\alpha, \widetilde{y}_\alpha \in V,\ \alpha = 1,2, \\
&(ii)\ \|\psi(t,s,y_1) - \psi(t,s,y_2)\| \le l_\psi\|y_1-y_2\|\ \forall\, y_\alpha \in V,\ \alpha = 1,2,
\end{aligned}
$$

(A_2) there exist $\varphi_1, \varphi_2 \in L_p[0,\lambda]$ and $l'_f > 0$ such that

$$
\begin{aligned}
&(i)\ \|f(t,y,\widetilde{y})\| \le \varphi_1(t) + l'_f t^{2-\kappa}\left(\|y\| + \|\widetilde{y}\|\right) \text{ for } a.e.\ t \in [0,\lambda] \text{ and all } y, \widetilde{y} \in V, \\
&(ii)\ \|\psi(t,s,y)\| \le \varphi_2(s) \text{ for } (t,s) \in \Delta \text{ and } y \in V.
\end{aligned}
$$

Theorem 4.1. *Under assumptions (A_1) and $(A2)$, the non-linear system* (1.1) *admits exactly one mild solution in $C_{2-\kappa}([0,\lambda];V)$ for each given $u \in U$.*

Proof. Define the operator $\mathcal{Q} : C_{2-\kappa}([0,\lambda];V) \to C_{2-\kappa}([0,\lambda];V)$ by

$$\begin{aligned}(\mathcal{Q}z)(t) &= \wp_\kappa(t)y_0 + \int_0^t \wp_\kappa(t)y_1\,dt \\ &\quad + \int_0^t\int_0^{t-s}\wp_\kappa(\sigma)\left(Bu(s) + f\left(s, z(s), \int_0^s \psi(s,\vartheta,z(\vartheta))\,d\vartheta\right)\right)d\sigma\,ds.\end{aligned}$$

For $z, \widetilde{z} \in C_{2-\kappa}([0,\lambda];V)$, one has

$$\begin{aligned}&t^{2-\kappa}\|(\mathcal{Q}z)(t) - (\mathcal{Q}\widetilde{z})(t)\| \\ &\leq t^{2-\kappa}\int_0^t\left\|\int_0^{t-s}\wp_\kappa(\sigma)\bigg(f\left(s,z(s),\int_0^s\psi(s,\vartheta,z(\vartheta))\,d\vartheta\right)\right. \\ &\quad \left. - f\left(s,\widetilde{z}(s),\int_0^s\psi(s,\vartheta,\widetilde{z}(\vartheta))\,d\vartheta\right)\bigg)d\sigma\right\|ds \\ &\leq \frac{l_\wp l_f}{\kappa-1}t^{2-\kappa}\int_0^t(t-s)^{\kappa-1}\bigg(\|z(s)-\widetilde{z}(s)\| \\ &\quad + \int_0^s\|\psi(s,\vartheta,z(\vartheta)) - \psi(s,\vartheta,\widetilde{z}(\vartheta))\|\,d\vartheta\bigg)ds \\ &\leq \frac{l_\wp l_f}{\kappa-1}t^{2-\kappa}\int_0^t(t-s)^{\kappa-1}\bigg(s^{\kappa-2}s^{2-\kappa}\|z(s)-\widetilde{z}(s)\| \\ &\quad + l_\psi\int_0^s\vartheta^{\kappa-2}\vartheta^{2-\kappa}\|z(\vartheta)-\widetilde{z}(\vartheta)\|\,d\vartheta\bigg)ds \\ &\leq \frac{l_\wp l_f}{\kappa-1}t^{2-\kappa}\int_0^t(t-s)^{\kappa-1}\left(s^{\kappa-2} + l_\psi\frac{s^{\kappa-1}}{\kappa-1}\right)ds\,\|z-\widetilde{z}\|_{C_{2-\kappa}} \\ &= \frac{l_\wp l_f}{\kappa-1}t^{2-\kappa}\left(\frac{\Gamma(\kappa)\Gamma(\kappa-1)}{\Gamma(2\kappa-1)}t^{2\kappa-2} + \frac{l_\psi\Gamma(\kappa)\Gamma(\kappa-1)}{\Gamma(2\kappa)}t^{2\kappa-1}\right)\|z-\widetilde{z}\|_{C_{2-\kappa}} \\ &\leq l_\wp l_f t^\kappa\frac{(\Gamma(\kappa-1))^2}{\Gamma(2\kappa-1)}\left(1 + \frac{l_\psi\lambda}{2\kappa-1}\right)\|z-\widetilde{z}\|_{C_{2-\kappa}}.\end{aligned}$$

By inductions, one can obtain

$$\begin{aligned}&t^{2-\kappa}\|(\mathcal{Q}^n z)(t) - (\mathcal{Q}^n\widetilde{z})(t)\| \\ &\quad\leq (l_\wp l_f t^\kappa)^n\frac{(\Gamma(\kappa-1))^{n+1}}{\Gamma((n+1)\kappa-1)}\left(\prod_{i=1}^n\left(1+\frac{l_\psi\lambda}{(i+1)\kappa-1}\right)\right)\|z-\widetilde{z}\|_{C_{2-\kappa}} \\ &\quad\leq \frac{\Gamma(\kappa-1)\left(l_\wp l_f\lambda^\kappa\Gamma(\kappa-1)\left(1+l_\psi\lambda\right)\right)^n}{\Gamma((n+1)\kappa-1)}\|z-\widetilde{z}\|_{C_{2-\kappa}} \\ &\Longrightarrow \|\mathcal{Q}^n z - \mathcal{Q}^n\widetilde{z}\|_{C_{2-\kappa}} \leq \frac{\Gamma(\kappa-1)\left(l_\wp l_f\lambda^\kappa\Gamma(\kappa-1)\left(1+l_\psi\lambda\right)\right)^n}{\Gamma((n+1)\kappa-1)}\|z-\widetilde{z}\|_{C_{2-\kappa}}.\end{aligned}$$

It is known that the Mittag-Leffler series

$$E_{\kappa,\kappa-1}\left(l_\wp l_f\lambda^\kappa\Gamma(\kappa-1)\left(1+l_\psi\lambda\right)\right) = \sum_{\alpha=0}^\infty\frac{\left(l_\wp l_f\lambda^\kappa\Gamma(\kappa-1)\left(1+l_\psi\lambda\right)\right)^\alpha}{\Gamma(\alpha\kappa+(\kappa-1))}$$

is convergent. Therefore $\frac{(l_\wp l_f\lambda^\kappa\Gamma(\kappa-1)(1+l_\psi\lambda))^n}{\Gamma((n+1)\kappa-1)} < \frac{1}{\Gamma(\kappa-1)}$ for some integer n. Hence from Banach contraction principle, $\mathcal{Q}$ has exactly one fixed point in $C_{2-\kappa}([0,\lambda];V)$, which is the mild solution of the original system. □

5 Main results

Let us define the Nemytskii operator $\Theta_f : C_{2-\kappa}([0,\lambda];V) \to Z$ given by

$$(\Theta_f(z))(t) = f\left(t, z(t), \int_0^t \psi(t,s,z(s))\,ds\right),\ z \in C_{2-\kappa}([0,\lambda];V)$$

and the continuous linear map $\Phi : Z \to V$ given by

$$\Phi(z) = \int_0^\lambda \int_0^{\lambda-s} \wp_\kappa(\sigma) z(s)\,d\sigma\,ds,\ z \in Z.$$

Remark 5.1. *The reachable set $\mathfrak{R}_\lambda(f)$ is dense in V iff for given $\varepsilon > 0$ and a $\widehat{y} \in V$, there is a $u_\varepsilon \in U$ such that the mild solution z_ε corresponding to u_ε satisfies*

$$\left\| \widehat{y} - \wp_\kappa(\lambda)y_0 - \int_0^\lambda \wp(t)y_1\,dt - \Phi(\Theta_f(z_\varepsilon)) - \Phi(Bu_\varepsilon) \right\| \le \varepsilon.$$

For further development, we assume the following conditions:

(A_3) there exist positive constants $\widehat{l}_f$ and $\widehat{l}_\psi$ satisfying

$$(i)\ \|f(t,y_1,\widetilde{y}_1) - f(t,y_2,\widetilde{y}_2)\| \le \widehat{l}_f t^{2-\kappa}\left(\|y_1 - y_2\| + \|\widetilde{y}_1 - \widetilde{y}_2\|\right)\ \forall y_\alpha, \widetilde{y}_\alpha \in V,\ \alpha = 1,2,$$
$$(ii)\ \|\psi(t,s,y_1) - \psi(t,s,y_2)\| \le \widehat{l}_\psi s^{2-\kappa}\|y_1 - y_2\|\ \forall y_\alpha \in V,\ \alpha = 1,2,$$

(A_4) for given $\varepsilon > 0$ and $z \in Z$, there exists a $u \in U$ satisfying

$$\|\Phi z - \Phi(Bu)\|_V \le \varepsilon \text{ and } \|Bu\|_Z \le b\|z\|_Z,$$

where $b > 0$ is constant and it doesn't dependent on z.

Remark 5.2. *Note that (A_1) is a weaker assumption than (A_3) hence by Theorem 4.1, for a fixed $u \in U$, the system (1.1) has exactly one mild solution in $C_{2-\kappa}([0,\lambda];V)$ if assumptions (A_2) and (A_3) are true.*

Remark 5.3. *It is easy to verify that the assumption (A_4) is satisfied if $B(U)$ is dense in Z.*

First we derive the next lemma:

Lemma 5.4. *Under assumptions (A_2) and (A_3), any mild solutions of (1.1) satisfy the following*

(i) $\|z(\cdot,u)\|_{C_{2-\kappa}} \le \mathcal{C}_1 E_\kappa\left(\frac{l_\wp l'_f \lambda^2 \Gamma(\kappa)}{\kappa - 1}\right)$ *for $u \in U$,*

(ii) $\|z_1(\cdot,u_1) - z_2(\cdot,u_2)\|_{C_{2-\kappa}} \le \mathcal{C}_2 E_\kappa\left(\frac{l_\wp l'_f \lambda^2 \Gamma(\kappa)}{\kappa - 1}\right)\|Bu_1 - Bu_2\|_Z$ *for $u_1, u_2 \in U$ provided that*

$$\kappa(\kappa - 1) > l_\wp \widehat{l}_f \widehat{l}_\psi \lambda^{5-\kappa} E_\kappa\left(\frac{l_\wp \widehat{l}_f \lambda^2 \Gamma(\kappa)}{\kappa - 1}\right);$$

where

$$\mathcal{C}_1 = l_\wp\left(\|y_0\| + \frac{\lambda}{\kappa - 1}\|y_1\| + \frac{\lambda^{2-\frac{1}{p}}}{\kappa - 1}\left(\frac{p-1}{p\kappa - 1}\right)^{1-\frac{1}{p}}\left(\|Bu\|_Z + \|\varphi_1\|_{L_p}\right) + \frac{l'_f \lambda^{5-\kappa-\frac{1}{p}}}{\kappa(\kappa-1)}\|\varphi_2\|_{L_p}\right)$$

and

$$\mathcal{C}_2 = \frac{\kappa l_\wp \lambda^{2-\frac{1}{p}}\left(\frac{p-1}{p\kappa-1}\right)^{1-\frac{1}{p}}}{\kappa(\kappa-1) - l_\wp \widehat{l}_f \widehat{l}_\psi \lambda^{5-\kappa} E_\kappa\left(\frac{l_\wp \widehat{l}_f \lambda^2 \Gamma(\kappa)}{\kappa-1}\right)}.$$

Proof. (i) If $z \in C_{2-\kappa}([0,\lambda];V)$ is the mild solution of (1.1) corresponding to $u \in U$, then

$$\begin{aligned} z(t) &= \wp_\kappa(t)y_0 + \int_0^t \wp_\kappa(t)y_1\,dt \\ &\quad + \int_0^t \int_0^{t-s} \wp_\kappa(\sigma)\Big(Bu(s) + f\Big(s, z(s), \int_0^s \psi(s,\vartheta,z(\vartheta))\,d\vartheta\Big)\Big)\,d\sigma\,ds. \end{aligned}$$

Therefore

$$\begin{aligned} t^{2-\kappa}\|z(t)\|_V &\le t^{2-\kappa}\|\wp_\kappa(t)y_0\| + t^{2-\kappa}\int_0^t \|\wp_\kappa(t)y_1\|\,dt \\ &\quad + t^{2-\kappa}\int_0^t \Big\|\int_0^{t-s} \wp_\kappa(\sigma)\Big(Bu(s) + f\Big(s,z(s),\int_0^s \psi(s,\vartheta,z(\vartheta))\,d\vartheta\Big)\Big)\,d\sigma\Big\|\,ds \\ &\le l_\wp\Big(\|y_0\| + \frac{t}{\kappa-1}\|y_1\| + \frac{t^{2-\kappa}}{\kappa-1}\int_0^t (t-s)^{\kappa-1}\|Bu(s)\|\,ds \\ &\quad + \frac{t^{2-\kappa}}{\kappa-1}\int_0^t (t-s)^{\kappa-1}\Big(\varphi_1(s) + l_f' s^{2-\kappa}\|z(s)\|_V + l_f' s^{2-\kappa}\int_0^s \varphi_2(\vartheta)\,d\vartheta\Big)\,ds\Big) \\ &\le l_\wp\Bigg(\|y_0\| + \frac{\lambda}{\kappa-1}\|y_1\| + \frac{\lambda^{2-\frac{1}{p}}}{\kappa-1}\left(\frac{p-1}{p\kappa-1}\right)^{1-\frac{1}{p}}\left(\|Bu\|_Z + \|\varphi_1\|_{L_p}\right) \\ &\quad + \frac{l_f'\lambda^{5-2\kappa-\frac{1}{p}}}{\kappa-1}\int_0^t (t-s)^{\kappa-1}\,ds\|\varphi_2\|_{L_p} + \frac{l_f'\lambda^{2-\kappa}}{\kappa-1}\int_0^t (t-s)^{\kappa-1}s^{2-\kappa}\|z(s)\|_V\,ds\Bigg) \\ &\le \mathcal{C}_1 + \frac{l_\wp l_f'\lambda^{2-\kappa}}{\kappa-1}\int_0^t (t-s)^{\kappa-1}s^{2-\kappa}\|z(s)\|_V\,ds. \end{aligned}$$

From Corollary 2.5, we obtain

$$\begin{aligned} t^{2-\kappa}\|z(t)\|_V &\le \mathcal{C}_1 E_\kappa\left(\frac{l_\wp l_f'\lambda^2\Gamma(\kappa)}{\kappa-1}\right) \\ \implies \|z\|_{C_{2-\kappa}} &\le \mathcal{C}_1 E_\kappa\left(\frac{l_\wp l_f'\lambda^2\Gamma(\kappa)}{\kappa-1}\right). \end{aligned}$$

(ii) Let $z_\alpha \in C_{2-\kappa}([0,\lambda];V)$ be the mild solution of (1.1) corresponding to $u_\alpha \in U,\ \ \alpha = 1,2$. Then

$$\begin{aligned} z_\alpha(t) &= \wp_\kappa(t)y_0 + \int_0^t \wp_\kappa(t)y_1\,dt \\ &\quad + \int_0^t\int_0^{t-s} \wp_\kappa(\sigma)\Big(Bu_\alpha(s) + f\Big(s, z_\alpha(s), \int_0^s \psi(s,\vartheta,z_\alpha(\vartheta))\,d\vartheta\Big)\Big)\,d\sigma\,ds. \end{aligned}$$

Therefore

$$\begin{aligned} & t^{2-\kappa}\|z_1(t) - z_2(t)\|_V \\ &\le \frac{l_\wp t^{2-\kappa}}{\kappa-1}\Big(\int_0^t (t-s)^{\kappa-1}\|Bu_1(s) - Bu_2(s)\|\,ds + \int_0^t (t-s)^{\kappa-1}\cdot \\ &\quad \Big\|f\Big(s, z_1(s), \int_0^s \psi(s,\vartheta,z_1(\vartheta))\,d\vartheta\Big) - f\Big(s, z_2(s), \int_0^s \psi(s,\vartheta,z_2(\vartheta))\,d\vartheta\Big)\Big\|\,ds\Big) \\ &\le \frac{l_\wp\lambda^{2-\frac{1}{p}}}{\kappa-1}\left(\frac{p-1}{p\kappa-1}\right)^{1-\frac{1}{p}}\|Bu_1 - Bu_2\|_Z + \frac{l_\wp\widehat{l}_f\lambda^{2-\kappa}}{\kappa-1}\int_0^t (t-s)^{\kappa-1}s^{2-\kappa}. \end{aligned}$$

$$
\begin{aligned}
&\left(\|z_1(s) - z_2(s)\| + \widehat{l}_\psi \int_0^s \vartheta^{2-\kappa}\|z_1(\vartheta) - z_2(\vartheta)\|\, d\vartheta\right) ds \\
&\le \frac{l_\wp \lambda^{2-\frac{1}{p}}}{\kappa - 1}\left(\frac{p-1}{p\kappa - 1}\right)^{1-\frac{1}{p}} \|Bu_1 - Bu_2\|_Z + \frac{l_\wp \widehat{l}_f \lambda^{2-\kappa}}{\kappa - 1}\Bigg(\int_0^t (t-s)^{\kappa-1} s^{2-\kappa}\cdot \\
&\quad \|z_1(s) - z_2(s)\|\, ds + \widehat{l}_\psi \int_0^t (t-s)^{\kappa-1}\lambda^{3-\kappa}\, ds \|z_1 - z_2\|_{C_{2-\kappa}}\Bigg) \\
&\le \frac{l_\wp \lambda^{2-\frac{1}{p}}}{\kappa - 1}\left(\frac{p-1}{p\kappa - 1}\right)^{1-\frac{1}{p}} \|Bu_1 - Bu_2\|_Z + \frac{l_\wp \widehat{l}_f \widehat{l}_\psi \lambda^{5-\kappa}}{\kappa(\kappa - 1)}\|z_1 - z_2\|_{C_{2-\kappa}} \\
&\quad + \frac{l_\wp \widehat{l}_f \lambda^{2-\kappa}}{\kappa - 1}\int_0^t (t-s)^{\kappa-1} s^{2-\kappa}\|z_1(s) - z_2(s)\|\, ds.
\end{aligned}
$$

From Corollary 2.5, we obtain

$$
\begin{aligned}
t^{2-\kappa}\|z_1(t) - z_2(t)\|_V &\le \Bigg(\frac{l_\wp \lambda^{2-\frac{1}{p}}}{\kappa - 1}\left(\frac{p-1}{p\kappa - 1}\right)^{1-\frac{1}{p}} \|Bu_1 - Bu_2\|_Z \\
&\quad + \frac{l_\wp \widehat{l}_f \widehat{l}_\psi \lambda^{5-\kappa}}{\kappa(\kappa - 1)}\|z_1 - z_2\|_{C_{2-\kappa}}\Bigg) E_\kappa\left(\frac{l_\wp \widehat{l}_f \lambda^2 \Gamma(\kappa)}{\kappa - 1}\right) \\
\implies \|z_1 - z_2\|_{C_{2-\kappa}} &\le \Bigg(\frac{l_\wp \lambda^{2-\frac{1}{p}}}{\kappa - 1}\left(\frac{p-1}{p\kappa - 1}\right)^{1-\frac{1}{p}} \|Bu_1 - Bu_2\|_Z \\
&\quad + \frac{l_\wp \widehat{l}_f \widehat{l}_\psi \lambda^{5-\kappa}}{\kappa(\kappa - 1)}\|z_1 - z_2\|_{C_{2-\kappa}}\Bigg) E_\kappa\left(\frac{l_\wp \widehat{l}_f \lambda^2 \Gamma(\kappa)}{\kappa - 1}\right) \\
\implies \|z_1 - z_2\|_{C_{2-\kappa}} &\le \frac{\kappa l_\wp \lambda^{2-\frac{1}{p}}\left(\frac{p-1}{p\kappa-1}\right)^{1-\frac{1}{p}} E_\kappa\left(\frac{l_\wp \widehat{l}_f \lambda^2\Gamma(\kappa)}{\kappa-1}\right)}{\kappa(\kappa - 1) - l_\wp \widehat{l}_f \widehat{l}_\psi \lambda^{5-\kappa} E_\kappa\left(\frac{l_\wp \widehat{l}_f \lambda^2\Gamma(\kappa)}{\kappa-1}\right)}\|Bu_1 - Bu_2\|_Z \\
&= \mathcal{C}_2 E_\kappa\left(\frac{l_\wp \widehat{l}_f \lambda^2\Gamma(\kappa)}{\kappa - 1}\right)\|Bu_1 - Bu_2\|_Z.
\end{aligned}
$$

□

Theorem 5.5. *Under assumptions* (A_2)*-*(A_4)*, the non-linear system* (1.1) *is approximately controllable if*

$$
0 < \frac{\kappa l_\wp \widehat{l}_f b \lambda^2\left(1 + \widehat{l}_\psi \lambda^{3-\kappa}\right)\left(\frac{p-1}{p\kappa-1}\right)^{1-\frac{1}{p}} E_\kappa\left(\frac{l_\wp \widehat{l}_f \lambda^2\Gamma(\kappa)}{\kappa-1}\right)}{\kappa(\kappa - 1) - l_\wp \widehat{l}_f \widehat{l}_\psi \lambda^{5-\kappa} E_\kappa\left(\frac{l_\wp \widehat{l}_f \lambda^2\Gamma(\kappa)}{\kappa-1}\right)} < 1. \tag{5.1}
$$

Proof. We need to prove that $D(A) \subseteq \overline{\mathfrak{R}_\lambda(f)}$, that is, for given $\varepsilon > 0$ and a $\widehat{y} \in D(A)$, there is a $u_\varepsilon \in U$ such that

$$
\left\|\widehat{y} - \wp_\kappa(\lambda)y_0 - \int_0^\lambda \wp(t)y_1\, dt - \Phi(\Theta_f(z_\varepsilon)) - \Phi(Bu_\varepsilon)\right\|_V \le \varepsilon,
$$

where $z_\varepsilon(t) = z(t, u_\varepsilon)$. One can see that there exists a $\widetilde{z} \in Z$ such that $\Phi\widetilde{z} = \widetilde{y}$, where $\widetilde{y} = \widehat{y} - \wp_\kappa(\lambda)y_0 - \int_0^\lambda \wp(t)y_1\, dt$, for example

$$
\widetilde{z}(t) = \frac{[\Gamma(\kappa - 1)]^2}{\lambda}\left((\lambda - t)^{2(2-\kappa)}\wp_\kappa(\lambda - t)\widetilde{y} + 2t(\lambda - t)^{(2-\kappa)}\frac{d}{dt}\left((\lambda - t)^{2-\kappa}\wp_\kappa(\lambda - t)\widetilde{y}\right)\right).
$$

Let $u_1 \in U$. By assumption (A_4), there is a $u_2 \in U$ such that

$$
\left\|\widehat{y} - \wp_\kappa(\lambda)y_0 - \int_0^\lambda \wp(t)y_1\, dt - \Phi(\Theta_f(z_1)) - \Phi(Bu_2)\right\|_V \le \frac{\varepsilon}{3^2},
$$

where $z_1(t) = z(t, u_1)$. Denote $z_2(t) = z(t, u_2)$, again by assumption (A_4) there is a $\omega_2 \in U$ such that

$$\big\|\Phi(\Theta_f(z_2) - \Theta_f(z_1)) - \Phi(B\omega_2)\big\|_V \le \frac{\varepsilon}{3^3}$$

and

$$\begin{aligned}
\|B\omega_2\|_Z &\le b\big\|\Theta_f(z_2) - \Theta_f(z_1)\big\|_Z \\
&= b\left(\int_0^\lambda \left\|f\left(t, z_2(t), \int_0^t \psi(t,\vartheta,z_2(\vartheta))\,d\vartheta\right) - f\left(t, z_1(t), \int_0^t \psi(t,\vartheta,z_1(\vartheta))\,d\vartheta\right)\right\|_V^p dt\right)^{\frac{1}{p}} \\
&\le b\widehat{l}_f\left(\int_0^\lambda \left(t^{2-\kappa}\|z_2(t) - z_1(t)\| + \widehat{l}_\psi t^{2-\kappa}\int_0^t \vartheta^{2-\kappa}\|z_2(\vartheta) - z_1(\vartheta)\|\,d\vartheta\right)^p dt\right)^{\frac{1}{p}} \\
&\le b\widehat{l}_f\left(\int_0^\lambda \left(1 + \widehat{l}_\psi\lambda^{3-\kappa}\right)^p dt\right)^{\frac{1}{p}} \|z_2 - z_1\|_{C_{2-\kappa}} \\
&\le b\widehat{l}_f\lambda^{\frac{1}{p}}\left(1 + \widehat{l}_\psi\lambda^{3-\kappa}\right)\|z_2 - z_1\|_{C_{2-\kappa}}. \qquad (5.2)
\end{aligned}$$

Since (5.1) implies that $\kappa(\kappa-1) > l_\wp\widehat{l}_f\widehat{l}_\psi\lambda^{5-\kappa}E_\kappa\left(\frac{l_\wp\widehat{l}_f\lambda^2\Gamma(\kappa)}{\kappa-1}\right)$. Therefore by previous lemma and (5.2), we get

$$\begin{aligned}
\|B\omega_2\|_Z &\le b\widehat{l}_f\lambda^{\frac{1}{p}}\left(1 + \widehat{l}_\psi\lambda^{3-\kappa}\right)\mathcal{C}_2E_\kappa\left(\frac{l_\wp\widehat{l}_f\lambda^2\Gamma(\kappa)}{\kappa-1}\right)\|Bu_1 - Bu_2\|_Z \\
&= \frac{\kappa l_\wp\widehat{l}_f b\lambda^2\left(1 + \widehat{l}_\psi\lambda^{3-\kappa}\right)\left(\frac{p-1}{p\kappa-1}\right)^{1-\frac{1}{p}}E_\kappa\left(\frac{l_\wp\widehat{l}_f\lambda^2\Gamma(\kappa)}{\kappa-1}\right)}{\kappa(\kappa-1) - l_\wp\widehat{l}_f\widehat{l}_\psi\lambda^{5-\kappa}E_\kappa\left(\frac{l_\wp\widehat{l}_f\lambda^2\Gamma(\kappa)}{\kappa-1}\right)}\|Bu_1 - Bu_2\|_Z.
\end{aligned}$$

Now, if we define

$$u_3(t) = u_2(t) - \omega_2(t), \quad u_3 \in U,$$

then

$$\begin{aligned}
&\left\|\widehat{y} - \wp_\kappa(\lambda)y_0 - \int_0^\lambda \wp(t)y_1\,dt - \Phi(\Theta_f(z_2)) - \Phi(Bu_3)\right\|_V \\
&\quad\le \left\|\widehat{y} - \wp_\kappa(\lambda)y_0 - \int_0^\lambda \wp(t)y_1\,dt - \Phi(\Theta_f(z_1)) - \Phi(Bu_2)\right\|_V \\
&\qquad + \big\|\Phi(\Theta_f(z_2) - \Theta_f(z_1)) - \Phi(B\omega_2)\big\|_V \\
&\quad\le \left(\frac{1}{3^2} + \frac{1}{3^3}\right)\varepsilon.
\end{aligned}$$

By inductions, we have a sequence $\{u_n\}$ in U so that

$$\left\|\widehat{y} - \wp_\kappa(\lambda)y_0 - \int_0^\lambda \wp(t)y_1\,dt - \Phi(\Theta_f(z_n)) - \Phi(Bu_{n+1})\right\|_V \le \left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n+1}}\right)\varepsilon,$$

where $z_n(t) = z(t, u_n)$, and

$$\|Bu_{n+1} - Bu_n\|_Z \le \frac{\kappa l_\wp\widehat{l}_f b\lambda^2\left(1 + \widehat{l}_\psi\lambda^{3-\kappa}\right)\left(\frac{p-1}{p\kappa-1}\right)^{1-\frac{1}{p}}E_\kappa\left(\frac{l_\wp\widehat{l}_f\lambda^2\Gamma(\kappa)}{\kappa-1}\right)}{\kappa(\kappa-1) - l_\wp\widehat{l}_f\widehat{l}_\psi\lambda^{5-\kappa}E_\kappa\left(\frac{l_\wp\widehat{l}_f\lambda^2\Gamma(\kappa)}{\kappa-1}\right)}\|Bu_n - Bu_{n-1}\|_Z.$$

Clearly, the sequence $\{Bu_n\}$ is Cauchy in Z. Completeness of Z and continuity of Φ make it clear that $\{\Phi(Bu_n)\}$ is a Cauchy sequence in V and hence for some positive integer n_0, one has

$$\left\|\Phi(Bu_{n_0+1}) - \Phi(Bu_{n_0})\right\|_V \leq \frac{\varepsilon}{3}.$$

Now

$$\begin{aligned}
&\left\|\widehat{y} - \wp_\kappa(\lambda)y_0 - \int_0^\lambda \wp(t)y_1\,dt - \Phi(\Theta_f(z_{n_0})) - \Phi(Bu_{n_0})\right\|_V \\
&\quad\leq \left\|\widehat{y} - \wp_\kappa(\lambda)y_0 - \int_0^\lambda \wp(t)y_1\,dt - \Phi(\Theta_f(z_{n_0})) - \Phi(Bu_{n_0+1})\right\|_V \\
&\qquad + \left\|\Phi(Bu_{n_0+1}) - \Phi(Bu_{n_0})\right\|_V \\
&\quad\leq \left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n_0+1}}\right)\varepsilon + \frac{\varepsilon}{3} \\
&\quad< \varepsilon.
\end{aligned}$$

Hence $\overline{\mathfrak{R}_\lambda(f)} = V$. □

6 Example

Consider the following boundary value problem with $\kappa \in (1,2]$ and $0 \leq \sigma \leq 1$

$$\begin{cases}
D_t^\kappa \zeta(t,\sigma) = \frac{\partial^2}{\partial\sigma^2}\zeta(t,\sigma) + \upsilon(t,\sigma) + f\left(t, \zeta(t,\sigma), \int_0^t \psi(t,s,\zeta(s,\sigma))ds\right), & 0 < t \leq 1, \\
\zeta(t,0) = \zeta(t,1) = 0, & 0 < t \leq 1, \\
\left(I_t^{2-\kappa}\zeta(t,\sigma)\right)_{t=0} = \varsigma_0(\sigma), & 0 \leq \sigma \leq 1, \\
\left(D_t^{\kappa-1}\zeta(t,\sigma)\right)_{t=0} = \varsigma_1(\sigma), & 0 \leq \sigma \leq 1.
\end{cases} \tag{6.1}$$

Let $V = V' = L_2[0,1]$ and $A : D(A) \subset V \to V$ is defined as

$$Ay = \frac{d^2y}{d\sigma^2}$$

where

$$D(A) = \{y \in W^{2,2}[0,1] \mid y(0) = y(1) = 0\}.$$

Then the κ-order fractional resolvent $\wp_\kappa(t)$ generated by A is given by

$$(\wp_\kappa(t)y)(\sigma) = \sum_{\alpha=1}^{\infty} t^{\kappa-2}E_{\kappa,\kappa-1}\left(-\alpha^2\pi^2t^\kappa\right)y_\alpha \sin(\alpha\pi\sigma),$$

where $\sin(\pi\sigma), \sin(2\pi\sigma), \ldots$ are eigenfunctions of A associated with the eigenvalues $-\pi^2, -2^2\pi^2, \ldots$; respectively and $y(\sigma) = \sum_{\alpha=1}^{\infty} y_\alpha \sin(\alpha\pi\sigma)$ (see [1]).

The abstract form of (6.1) is

$$\begin{cases}
D_t^\kappa z(t) = Az(t) + Bu(t) + f\left(t, z(t), \int_0^t \psi(t,s,z(s))\,ds\right), & t \in (0,1], \\
\left(I_t^{2-\kappa}z(t)\right)_{t=0} = y_0, \\
\left(D_t^{\kappa-1}z(t)\right)_{t=0} = y_1,
\end{cases}$$

where $z(t) = \zeta(t, \cdot),\ u(t) = \upsilon(t, \cdot),\ y_0 = \varsigma_0(\cdot),\ y_1 = \varsigma_1(\cdot)$ and B is the identity operator.

If we take

$$\widetilde{\zeta}(t, \sigma) = \int_0^t \psi(t, s, \zeta(s, \sigma))\, ds$$

and

$$\begin{aligned} f\left(t, \zeta(t, \sigma), \widetilde{\zeta}(t, \sigma)\right) &= f\left(t, \zeta(t, \sigma), \int_0^t \psi(t, s, \zeta(s, \sigma)) ds\right) \\ &= (1 + t^2) + l_0 t \left(\zeta(t, \sigma) + \int_0^t l_1 \left(t^2 + s^2\right) s^3 \cos(ts) \sin(\zeta(s, \sigma))\, ds\right), \end{aligned}$$

where

$$\psi(t, s, \zeta(s, \sigma)) = l_1 \left(t^2 + s^2\right) s^3 \cos(ts) \sin(\zeta(s, \sigma)).$$

Then (i) of (A_2), and (A_3) are satisfied with $l'_f = \widehat{l}_f = |l_0|$ and $\widehat{l}_\psi = 2|l_1|$.

Also,

$$\|\psi(t, s, \zeta(s, \sigma))\| \le |l_1| \left(1 + s^2\right) s^3 = \varphi_2(s) \in L_p[0, 1].$$

Hence (ii) of (A_2) is satisfied. If we select l_0 and l_1 sufficiently close to zero so that (5.1) is satisfied, then by Theorem 5.5 the approximately controllability of (6.1) follows if $\kappa > \frac{3}{2}$.

7 Concluding remarks

In this paper, we have investigated the approximate controllability of higher order Riemann-Liouville fractional integro-differential systems with integral initial conditions in Banach spaces. The results for existence and uniqueness have been derived by using the theory of fixed point together with Lipschitz continuity of non-linear functions. For this, the definition of mild solution in terms of fractional resolvent $\wp_\kappa(\cdot)$ has been derived. Making use of these techniques, one can study the approximate controllability of higher order Riemann-Liouville fractional integro-differential systems with non-instantaneous impulses. Also, these results can be extended for non-autonomous systems with Riemann-Liouville derivatives.

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