New exact soliton solutions, bifurcation and multistability behaviors of traveling waves for the (3+1)-dimensional modified Zakharov-Kuznetsov equation with higher order dispersion

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Abstract

The goal of the present paper is to obtain and analyze new exact travelling wave solutions and bifurcation behavior of modified Zakharov-Kuznetsov (mZK) equation with higher order dispersion term. For this purpose, first and second simple methods are used to build soliton solutions of travelling wave solutions. Furthermore, bifurcation behavior of traveling waves including new type of quasiperiodic and multi-periodic traveling wave motions have been examined depending on the physical parameters. Multistability for the nonlinear mZK equation has been investigated depending on fixed values of physical parameters with various initial conditions. The suggested methods for the analytical solutions are powerful and benefical tools to obtain the exact travelling wave solutions of nonlinear evolution equations (NLEEs). Two and three-dimensional plots are also provided to illustrate the new solutions. Bifurcation and multistability behaviors of traveling wave solution of the nonlinear mZK equation with higher order dispersion will add some value in the literature of mathematical and plasma physics.

Keywords: Modified Zakharov-Kuznetsov equation; Quasiperiodic motion; First simple method; Second simple method; Bifurcation.

1 Introduction

NLEEs involve nonlinear complex physical phenomena and play an outstanding role while characterizing complicated phenomena rooting in different branches of science for example fluid flow, wave propagations, fluid mechanics, nonlinear optics, optical fibres, chemical kinematics, chemical physics, plasma physics, solid-state physics, hydrodynamic, nonlinear transmission lines, plasma physics, geochemistry, biology and soil consolidations. Therefore, obtaining exact solutions of such nonlinear equations are a rich area of research for the scientists because the resulting solutions can describe physical behaviour of concerned problems in the best way [1, 2, 3]. These solutions define various phenomena in nature, such as vibrations, solitons and propagation with a finite speed [4]. In the recent past, many researcher developed a wide range of methods and still trying to construct new methods to establish analytical and solitary traveling wave solutions of the NLEEs. Some of these methods are: inverse scattering method [5, 6], Backlund transformation method [7], modified simple equation method [8], homogeneous balance method [9], direct algebraic method [10], Hirota bilinear transformation method [11], tanh–sech method [12, 13], extended tanh method [14, 15, 16], Jacobi elliptic function expansion method [17, 18, 19] , generalized Riccati equation method [20], sine–cosine method [21], F-expansion method [22, 23], homogeneous balance method [24], Exp function method [25, 26, 27], Cole-Hopf transformation method [28], Adomian decomposition method [29], homotopy analysis method [30], homotopy perturbation method [31], first and second simple metod [32, 33], bifurcation method [34, 35] and first integral method [36].

The nonlinear Zakharov-Kuznetsov (NZK) equation is an another alternative version of nonlinear model describing (2+1)-dimensional modulation of a KdV soliton equation in fluid mechanics [37, 38]. In two-and three-dimensional spaces, the NZK equation is given by

$$u_t + auu_x + bu_{xxx} + cu_{xyy} = 0, \tag{1}$$

and

$$u_t + auu_x + bu_{xxx} + c(u_{xyy} + u_{xzz}) = 0, \tag{2}$$

respectively, where a is known as the coefficient of nonlinear term and $b,\ c$ are called the coefficients of dispersion terms. Here x, y, z are space variables, t is time and u is acoustic wave potential. This equation was first derived to model the propagation of weakly nonlinear ion-acoustic waves in plasma, which involves cold ions and hot-isothermal electrons in a medium with a uniform magnetic field. The equation is also used to define different types of acoustic waves in magnetized plasmas [39]. It has been shown the equation is not integrable by means of the inverse scattering transform method. It was found that the solitary-wave solutions of the ZK equation are inelastic. Hesam et al. [40] developed differential transform method for Zakharov equation. The nonlinear ZK equation with higher order dispersion term is given by

$$u_t + auu_x + bu_{xxx} + c(u_{xyy} + u_{xzz}) + du_{xxxxx} = 0, \tag{3}$$

where a, b, c are same as Eq. (2) and d is the coefficient of fifth order dispersion. With an appropriately modified form of the electron number density given in [41], Munro and Parkes [37] demonstrated that reductive perturbation can induce following modified Zakharov–Kuznetsov (mZK) equation

$$16(u_t - ku_x) + 30u^{1/2}u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0, \tag{4}$$

here k is a positive constant. The mZK equation have solutions that symbolize plane-periodic and solitary traveling waves propagating. It is noted that the mZK equation is a high dimensional nonlinear evolution equation and, thus, the study of its reduction problem is of theoretical interest [42]. Park et al. [43] applied modified Khater method to equation. The extended mapping method is developed to study the traveling wave solution for a mZK equation by Peng [44]. In our manuscript, we study the following nonlinear modified ZK equation with higher order dispersion term as

$$u_t + au^2u_x + bu_{xxx} + c(u_{xyy} + u_{xzz}) + du_{xxxxx} = 0. \tag{5}$$

Multistability alludes to an interesting phenomenon where a dynamical system provides more than one numerical solution for a fixed values of the parameters at various initial conditions [45, 46]. Arecchi et al. [47] performed experimental observation of multistability behavior in a Q-switch laser system. Natiq et al. [48] experienced coexisting features involving chaotic and quasi-periodic phenomena and the coexistence of symmetric Hopf bifurcations. Morfu et al. [49] reported multistability in Cellular Nonlinear Network in image processing. Rahim et al. [50] investigated multistability behavior in a hyperchaotic system. Li and Sprott [51] studied multistability phenomenon in the famous Lorenz system in a special parametric range space. In various fields of plasmas, multistability behavior also known as coexisting features were extensively investigated in discharge plasmas [52], plasma diodes [53], solar wind plasma [54], electron-ion plasma [55], and in various quantum plasmas [56, 57].

The aims of this study are twofold and will take place for the first time in the literature. Firstly, we introduce the soliton solutions of the mZK equation with higher order dispersion term using different typies of two simple methods:

· First simple method was suggested by Nikolay A. Kudryashov [58], its applications have also been shown in [32, 59] and

· Second simple method was suggested by Khalid K. Ali [33].

Secondly, we examined bifurcation behavior of traveling waves including quasiperiodic, multi-periodic and multistability motion for mZK equation with higher order dispersion depending on the physical parameters. Thus, we construct new exact and travelling wave solutions in soliton.

The remnant of this paper is systematized as follows: an introduction is given in Section 1. The main steps of the First and Second simple methods are specified in Section 2. At the next section, in Section 3, we apply these methods in detail with finding exact travelling wave solutions of the mZK equation. In Section 4, some figures are presented in the two and three-dimensional to display the solutions given in Section 3. Bifurcation behavior of travelling wave solution containing: Quasiperiodic, multi-periodic and multistability wave motion of the mZK equation is investigated in Section 5. Finally the paper end with a conclusion in Section 6.

2 Overview of the methods

2.1 First simple method [58]

Let's consider the

$$F(u, u_t, u_x, u_y, u_z, u_{tt}, u_{xx}, u_{yy}, u_{zz}, ...) = 0, \tag{6}$$

nonlinear partial differential equation where $u = u(x, y, z, t)$ is the unknown function.

Step 1: Use the following wave transformation:

$$u(x, y, z, t) = u(\xi), \quad \xi = lx + my + nz - vt, \tag{7}$$

where l, m, n are constants and v is velocity of the traveling wave.

$$P(u', u'', u''', ...) = 0, u' = \frac{du}{d\xi}. \tag{8}$$

By using above terms, Eq.(6) is reduced to a non-linear ordinary differential equation.

Step 2: Assume solution of (8) takes form of a finite series

$$u(\xi) = \sum_{i=0}^{N} (A_i(Q(\xi))^i, \tag{9}$$

$A_i(i = 0, 1, 2, ..., N), A_N \neq 0$, are unknowns with $(A_i \neq 0)$ to be calculated. N is a positive integer and will be computed by homogeneous balance algorithm.

Step 3: The function $Q(\xi)$ satisfies auxiliary differential equation:

$$(Q'(\xi))^2 = \alpha^2 Q(\xi)^2 (1 - \Omega Q(\xi)^2), \tag{10}$$

(10) gives the following solution:

$$Q(\xi) = \frac{4\sigma \exp(-\alpha\xi)}{4\sigma^2 + \Omega \exp(-2\alpha\xi)}. \tag{11}$$

Step 4: By substituting (9) and (10) into (8) and collecting all terms with the same power of $Q(\xi)$ together, (8) turn into a polynomial, taking each coefficient equal to zero, a system of algebraic equations are obtained.

Step 5: By using the Mathematica program, we can obtain the exact solution of (8).

2.2 Second simple method [33]

We illustrate modified Kudryashov method in this section as follows:

Step 1: Assume a solution of (8) given in a series form:

$$u(\xi) = \sum_{i=0}^{N} (A_i(Q(\xi))^i, \tag{12}$$

where A_i is the same as in First simple method.

Step 2: Function $Q(\xi)$ fulfills the differential equation:

$$(Q'(\xi))^2 = \alpha^2 (\log(C))^2 Q(\xi)^2 (1 - \Omega Q(\xi)^2), \tag{13}$$

the solution of (13) is introduced by:

$$Q(\xi) = \frac{4\sigma C^{(-\alpha\xi)}}{4\sigma^2 + \Omega C^{(-2\alpha\xi)}}. \tag{14}$$

Step 3: Putting (12) and (13) into (8), we procure a polynomial of $Q(\xi)$. Setting all the coefficients of the like powers of $Q(\xi)$ to zero, a system of algebraic equations are obtained.

Step 4: System of equations are solved by Mathematica program. Consequently, we can obtain exact solution of (8).

3 Implementations of the methods

We employ the transformation (7) with $l^2 + m^2 + n^2 = 1$. Then, the Eq. (5) becomes

$$-(v - alu^2)u_\xi + (bl^3 + clm^2 + cln^2)u_{\xi\xi\xi} + dl^5 u_{\xi\xi\xi\xi\xi} = 0. \tag{15}$$

Integrating equation (15) according to ξ,

$$-(v - \frac{alu^2}{3})u + (bl^3 + clm^2 + cln^2)u_{\xi\xi} + dl^5 u_{\xi\xi\xi\xi} = c_1, \tag{16}$$

is obtained and here c_1 is an integrating constant. Applying the boundary conditions $u \to 0,\ u_\xi \to 0,\ u_{\xi\xi} \to 0,\ \ u_{\xi\xi\xi} \to 0,\ \ u_{\xi\xi\xi\xi} \to 0$ as $\xi \to \pm\infty$ in equation (16), one can obtain $c_1 = 0$. Then Eq. (16) becomes

$$-(v - \frac{alu^2}{3})u + (bl^3 + cl(1 - l^2))u_{\xi\xi} + dl^5 u_{\xi\xi\xi\xi} = 0. \tag{17}$$

Balancing u^3 with $u_{\xi\xi\xi\xi}$ in (17), following relation is obtained:

$$3N = N + 4 \Rightarrow N = 2. \tag{18}$$

3.1 First simple method

From (9) and (18), the solution of (17) is written in the form:

$$u(\xi) = A_0 + A_1 Q(\xi) + A_2 Q^2(\xi), \tag{19}$$

By setting above solution in Eq. (17) and equating factors of each power of $Q(\xi)$ in resulting equation to zero, we reach following nonlinear algebraic system:

$$\frac{1}{3}aA_0^3 l - A_0 v = 0,$$

$$aA_0^2 A_1 l + \alpha^2 A_1 l \left(bl^2 - cl^2 + c + \alpha^2 dl^4\right) - A_1 v = 0,$$

$$aA_0A_1^2l + aA_0^2A_2l + 4\alpha^2A_2l\left(bl^2 - cl^2 + c + 4\alpha^2dl^4\right) - A_2v = 0,$$
$$\frac{1}{3}A_1l\left(aA_1^2 - 6\alpha^2\Omega\left(bl^2 - cl^2 + c + 10\alpha^2dl^4\right)\right) + 2aA_0A_1A_2l = 0,$$
$$aA_0A_2^2l + aA_1^2A_2l - 6\alpha^2A_2bl^3\Omega + 6\alpha^2A_2cl^3\Omega - 6\alpha^2A_2cl\Omega - 120\alpha^4A_2dl^5\Omega = 0,$$
$$aA_1A_2^2l + 24\alpha^4A_1dl^5\Omega^2 = 0,$$
$$\frac{1}{3}aA_2^3l + 120\alpha^4A_2dl^5\Omega^2 = 0.$$

Solving the previous system, we obtain the following solutions:

$$\begin{aligned} &A_0 = 0, \quad A_1 = 0, \quad A_2 = \mp\frac{3\sqrt{\frac{5}{2}}\sqrt{v}\Omega}{2\sqrt{a}\sqrt{l}}, \\ &d = -\frac{v}{64\alpha^4l^5}, \quad b = -\frac{c}{l^2} + c + \frac{5v}{16\alpha^2l^3}. \end{aligned} \tag{20}$$

Substituting (20) in (19) with (11) and (7), we get the following solutions of (5):

$$u_{1,2}(x,y,z,t) = \mp\frac{3\sqrt{\frac{5}{2}}\sqrt{v}\Omega}{2\sqrt{a}\sqrt{l}}\left(\frac{4\sigma\exp(-\alpha(lx+my+nz-vt))}{4\sigma^2 + \Omega\exp(-2\alpha(lx+my+nz-vt))}\right)^2. \tag{21}$$

3.2 Second simple method

From (12) and (18), the solution of (17) is written in the form:

$$u(\xi) = A_0 + A_1Q(\xi) + A_2Q^2(\xi). \tag{22}$$

By setting above solution (22) in (17) and equating coefficients of like powers of $Q(\xi)$, we obtain following set of non-linear algebraic equations:

$$\frac{1}{3}aA_0^3l - A_0v = 0,$$
$$aA_0^2A_1l + \alpha^2A_1l\log^2(C)\left(bl^2 - cl^2 + c + \alpha^2dl^4\log^2(C)\right) - A_1v = 0,$$
$$aA_0A_1^2l + aA_0^2A_2l + 4\alpha^2A_2l\log^2(C)\left(bl^2 - cl^2 + c + 4\alpha^2dl^4\log^2(C)\right) - A_2v = 0,$$
$$\frac{1}{3}aA_1^3l + 2aA_0A_1A_2l + 2\alpha^2A_1l^3\Omega(c-b)\log^2(C)$$
$$-2\alpha^2A_1cl\Omega\log^2(C) - 20\alpha^4A_1dl^5\Omega\log^4(C) = 0,$$
$$aA_0A_2^2l + aA_1^2A_2l - 6\alpha^2A_2bl^3\Omega\log^2(C) + 6\alpha^2A_2cl^3\Omega\log^2(C)$$
$$-6\alpha^2A_2cl\Omega\log^2(C) - 120\alpha^4A_2dl^5\Omega\log^4(C) = 0,$$

$$aA_1A_2^2l + 24\alpha^4A_1dl^5\Omega^2\log^4(C) = 0,$$
$$\frac{1}{3}aA_2^3l + 120\alpha^4A_2dl^5\Omega^2\log^4(C) = 0.$$

Now, the following new exact solutions for (5) will be produced:

$$\begin{aligned} &A_0 = 0, \quad A_1 = 0, \quad A_2 = -\frac{3\sqrt{\frac{5}{2}}\sqrt{v}\Omega}{2\sqrt{a}\sqrt{l}}, \\ &d = -\frac{v}{64\alpha^4l^5\log^4(C)}, \quad b = \frac{16\alpha^2cl^3\log^2(C) - 16\alpha^2cl\log^2(C) + 5v}{16\alpha^2l^3\log^2(C)}. \end{aligned} \tag{23}$$

Substituting (23) in (22) with (14) and (7), we get the following solutions of (5):

$$u_{1,2}(x, y, z, t) = -\frac{3\sqrt{\frac{5}{2}}\sqrt{v}\Omega}{2\sqrt{a}\sqrt{l}}\left(\frac{4\sigma C^{(-\alpha(lx+my+nz-vt))}}{4\sigma^2 + \Omega C^{(-2\alpha(lx+my+nz-vt))}}\right)^2. \tag{24}$$

4 Graphical illustrations

Now, some figures in two and three dimensional have been drawn to exemplify solutions given above. The graph of (21) using the First simple method at $c = 0.2, \sigma = 5, a = 4, \Omega = 6, v = 0.5, l = 0.55, m = 0.35, n = 0.1, y = z = 2$ is introduced in Fig. (1). Finally, we shown the graph of (24) using the Second simple method at $c = 0.2, a = 4, \sigma = 5, k = 0.001, \Omega = 6, v = 0.5, l = 0.55, m = 0.35, n = 0.1, C = 0.4, y = z = 2$ in Fig. (2).

5 Bifurcation analysis

We investigate bifurcation behavior of traveling wave solution of the nonlinear modified ZK equation with higher order dispersion (5) for the first time in the literature. To discover all possible traveling wave solutions of nonlinear modified ZK equation (5), we form the following dynamical system [60-65] (with parameters a, b, c, d, l and v) from equation (17):

$$\begin{cases} u_\xi = X, \\ X_\xi = Y, \\ Y_\xi = Z, \\ Z_\xi = (v - \frac{al}{3}u^2)\frac{u}{dl^5} - \frac{(bl^2+c(1-l^2))Y}{dl^4}. \end{cases} \tag{25}$$

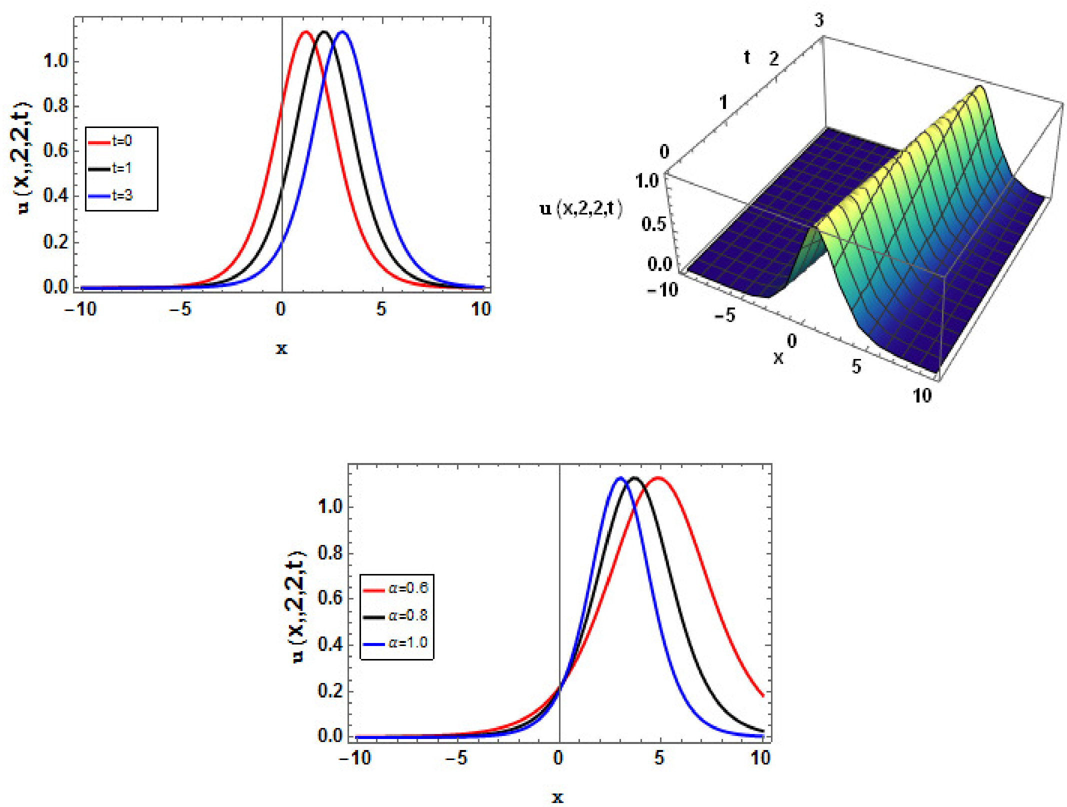

Figure 1: Profile of (21) using the first simple method at $c = 0.2, \sigma = 5, a = 4, \Omega = 6, v = 0.5, l = 0.55, m = 0.35, n = 0.1$.

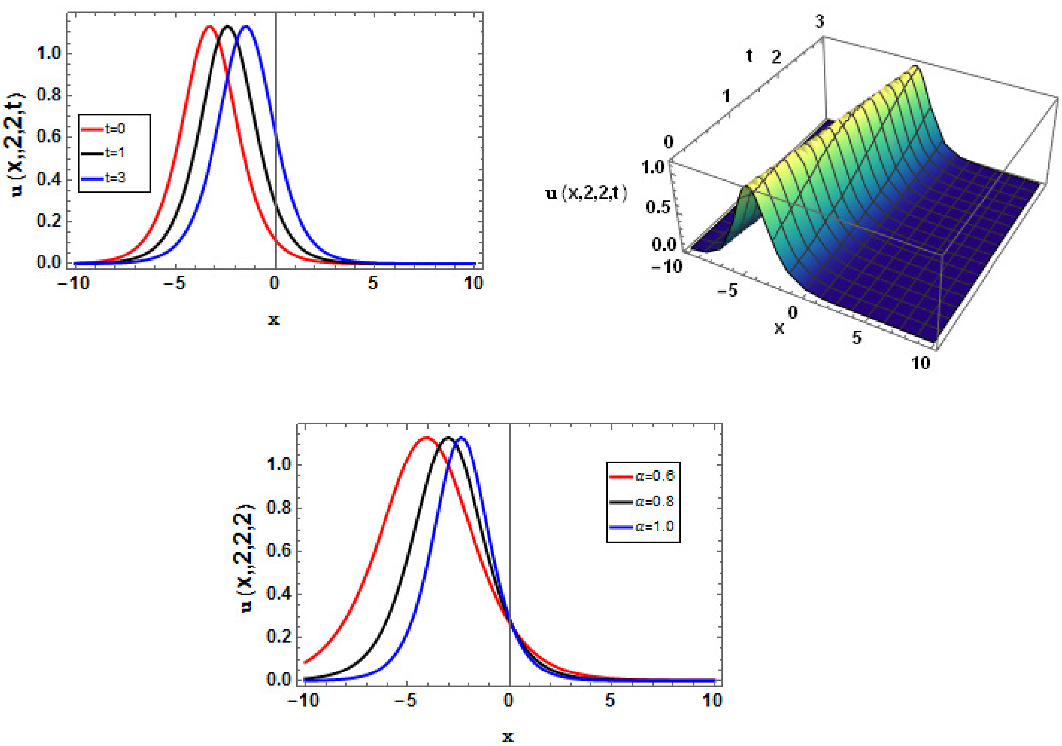

Figure 2: Profile of (24) using the second simple method at $c = 0.2, a = 4, \sigma = 5, k = 0.001, \Omega = 6, v = 0.5, l = 0.55, m = 0.35, n = 0.1, C = 0.4$.

Let $\overrightarrow{F} = \big(X,\ Y,\ Z,\ (v - \frac{al}{3}u^2)\frac{u}{dl^5} - \frac{(bl^2+c(1-l^2))Y}{dl^4}\big)$. Then divergence of $\overrightarrow{F}$ is:

$$\overrightarrow{\nabla} \cdot \overrightarrow{F} = \frac{\partial u_\xi}{\partial u} + \frac{\partial X_\xi}{\partial X} + \frac{\partial Y_\xi}{\partial Y} + \frac{\partial Z_\xi}{\partial Z} = 0. \tag{26}$$

Thus one can make a conclusion on the conservativeness of the dynamical system (25). The singular points of the system (25) are given by solutions of the following set of equations:

$$\begin{cases} X = 0, \\ Y = 0, \\ Z = 0, \\ (v - \frac{al}{3}u^2)\frac{u}{dl^5} - \frac{(bl^2+c(1-l^2))Y}{dl^4} = 0. \end{cases} \tag{27}$$

The dynamical system (25) has three equilibrium points at $P_1(u_1, 0, 0, 0)$, $P_2(u_2, 0, 0, 0)$ and $P_3(u_3, 0, 0, 0)$, where $u_1 = 0$, $u_2 = \sqrt{\frac{3v}{al}}$, and $u_3 = -\sqrt{\frac{3v}{al}}$.

The stability of the singular point based on the character of eigenvalues of the Jacobian matrix J_P. After making linearisation of the dynamical system (25) at the singular point $P(u,\ X,\ Y,\ Z)$, the Jacobian matrix J_P can be written as:

$$J_P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{dl^5}(v - alu^2) & 0 & -\frac{1}{dl^4}(bl^2 + c(1-l^2)) & 0 \end{pmatrix}. \tag{28}$$

One can acquire eigenvalues of the system (25) at $P(u,\ X,\ Y,\ Z)$ by making solution of the following equation:

$$|\lambda I - J_P| = 0. \tag{29}$$

Then one can obtain the following characteristic equation as:

$$\lambda^4 + M_1\lambda^3 + M_2\lambda^2 + M_3\lambda + M_4 = 0, \tag{30}$$

where

$$\begin{cases} M_1 = 0, \\ M_2 = -\frac{1}{dl^4}(bl^2 + c(1-l^2)), \\ M_3 = 0, \\ M_4 = \frac{1}{dl^5}(alu^2 - v). \end{cases}$$

The singular point $P(u,\ X,\ Y,\ Z)$ is considered as stable if all possible solutions of equation (30) having real parts less than zero for the singular point or it will be considered as unstable.

5.1 Quasiperiodic and multi-periodic traveling wave motions

Some new types of quasiperiodic and multi-periodic motions for the travelling wave solutions of the modified ZK equation (5) are investigated through the conservative dynamical system (25) based on suitable values of the parameters a, b, c, d, l, and v in Figures (3-7).

In Figure (3), we present phase space and variation of wave profile u for a quasiperiodic motion of the modified ZK equation (5) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.3$ and $v = 6$ with initial condition $(3.1, 1.1, -0.1, -0.2)$. In this case, the phase space forms a torus connected with two leafs faced to each other. In Figure (4), we present phase space and variation of wave profile u for a quasiperiodic motion of the modified ZK equation (5) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.4$ and $v = 6$ with initial condition $(3.1, 1.1, -0.1, -0.2)$. In this case, the phase space looks like a heart-shape. In Figure (5), we present phase space and variation of wave profile u for a quasiperiodic motion of the modified ZK equation (5) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.48$ and $v = 6$ with initial condition $(3.1, 1.1, -0.1, -0.2)$. In this case, the phase space forms a torus connected with two leafs faced to each other with multi-bends. In Figure (6), we present phase space and variation of wave profile u for a quasiperiodic motion of the modified ZK equation (5) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.494$ and $v = 6$. with initial condition $(3.1, 1.1, -0.1, -0.2)$. In this case, the phase space forms a torus connected with two sets of multi-torus structures faced to each other.

There exists a period-9 motion of the dynamical system (25) and corresponding phase portrait is shown in Figure (7) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.5$ and $v = 6$. with initial condition $(3.1, 1.1, -0.1, -0.2)$. It is important to note that all phase spaces, presented in Figures (3-7), are symmetric with respect to Y-axis. Such phase spaces, presented in Figures (3-7), are observed for the first time in the literature of nonlinear modified ZK equation (5) with higher order dispersion term.

5.2 Multistability of traveling wave motion

Multistability behaviors for the travelling wave solutions of modified ZK equation (5) are examined through conservative dynamical system (25) based on fixed values of the parameters a, b, c, d, l, and v in Figure (8) with different initial conditions: (a) $(0.1, 1.1, -0.1, -0.2)$, (b) $(0.5, 1.1, -0.1, -0.2)$, (c) $(0.1, 0.1, -0.1, -0.2)$, (d) $(0.1, 0.1, 0.1, -0.2)$, (e) $(0.1, 0.1, 0.1, 0.2)$,

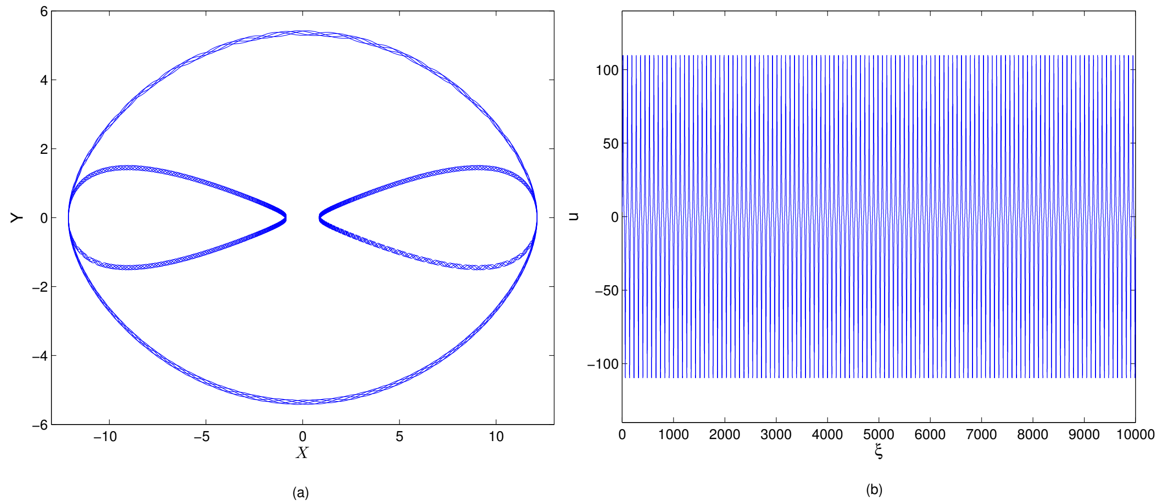

Figure 3: Phase space of the system (25) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.3$ and $v = 6$.

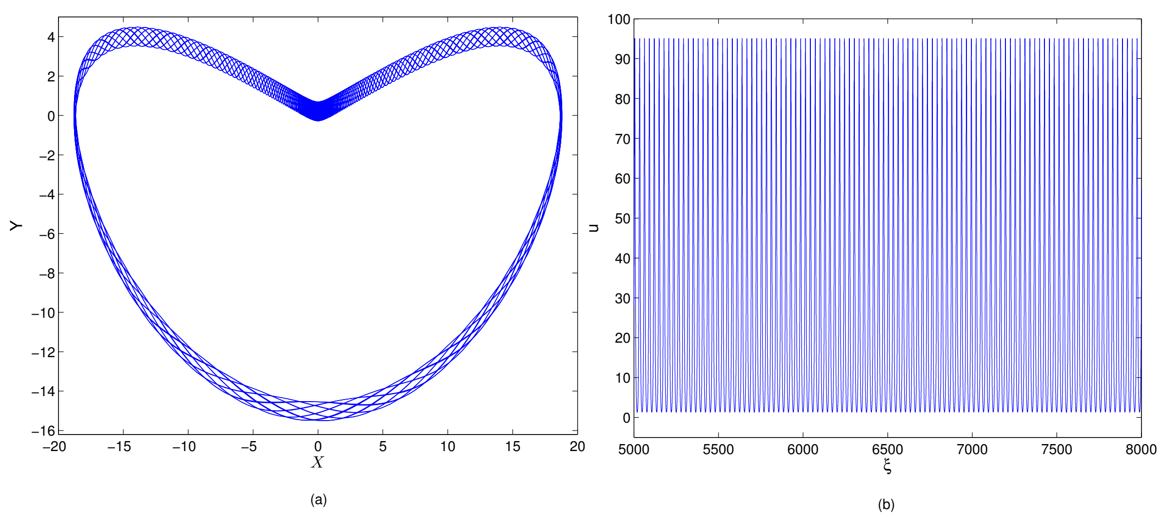

Figure 4: Phase space of the system (25) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.4$ and $v = 6$.

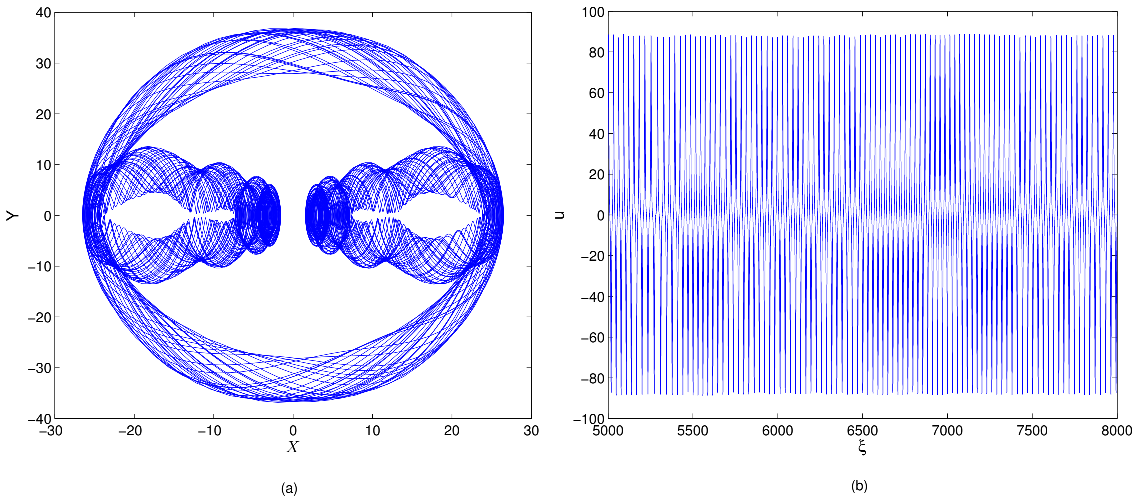

Figure 5: Phase space of the system (25) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.48$ and $v = 6$.

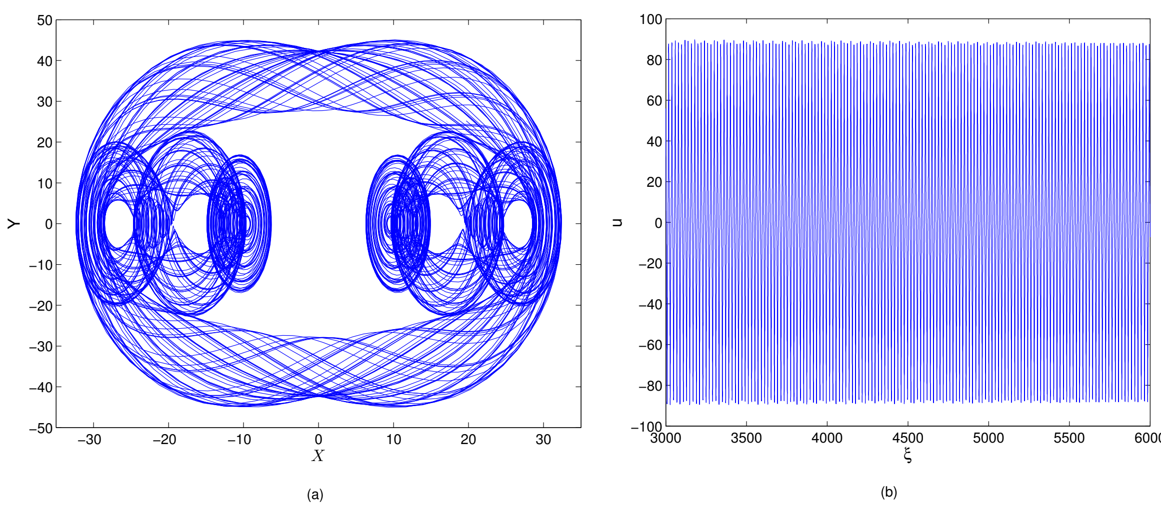

Figure 6: Phase space of the system (25) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.494$ and $v = 6$.

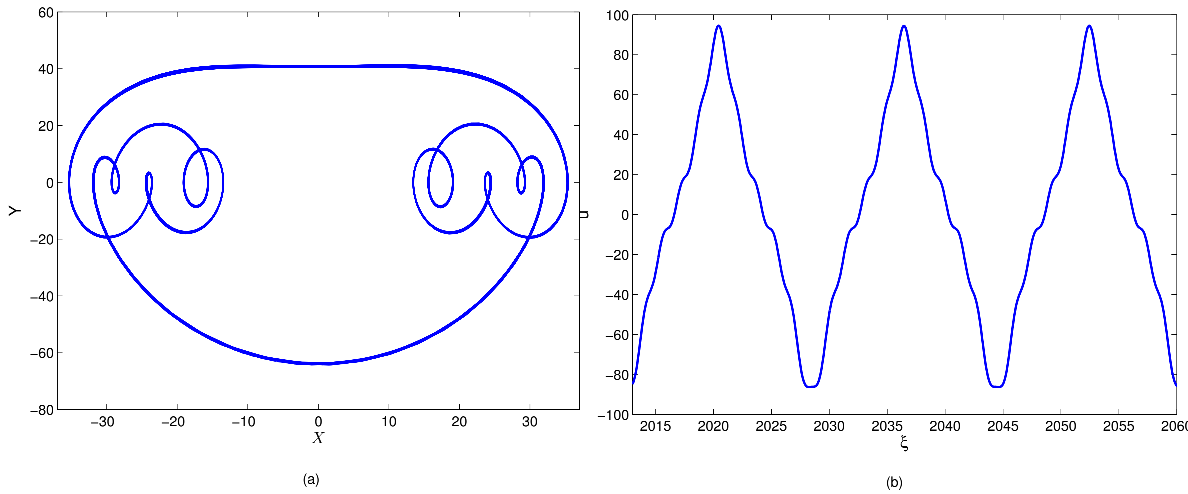

Figure 7: Phase space of the system (25) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.5$ and $v = 6$.

and (f) $(0.1, 1.1, 0.9, 0.2)$. All these phase spaces are qualitatively different from each other. It is important to note that all phase spaces of Figure (8) are symmetric in nature with respect to Y-axis. This kind of multistability behaviors for the travelling wave solutions of the modified ZK equation (5) with higher order dispersion term are reported for the first time in the literature.

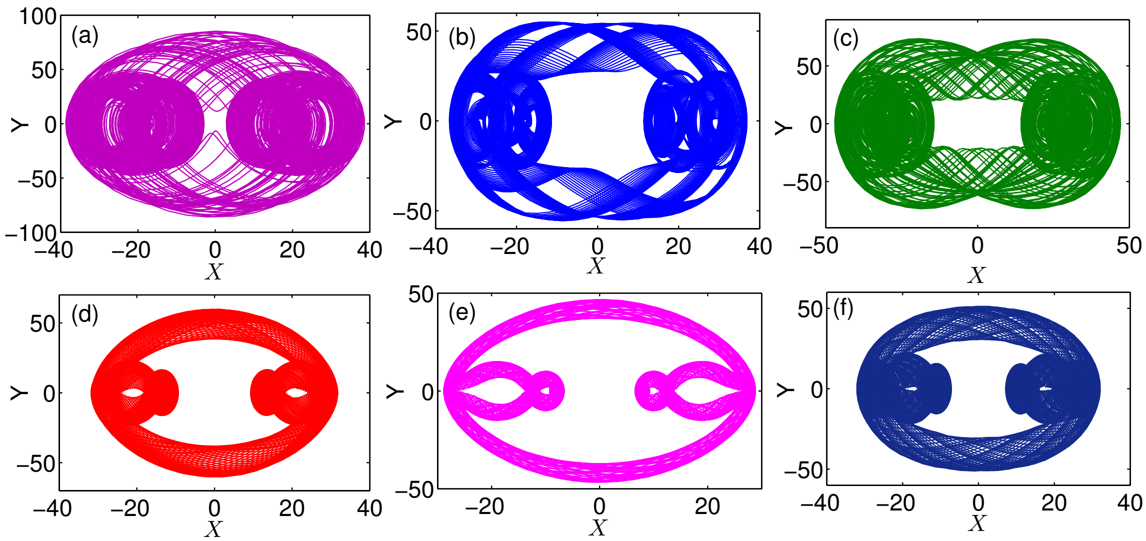

Figure 8: Phase spaces of the system (25) for $a = 0.01,\ b = 1,\ c = 1,\ d = 1,\ l = 0.494$ and $v = 6$. with different initial conditions: (a) $(0.1, 1.1, -0.1, -0.2)$, (b) $(0.5, 1.1, -0.1, -0.2)$, (c) $(0.1, 0.1, -0.1, -0.2)$, (d) $(0.1, 0.1, 0.1, -0.2)$, (e) $(0.1, 0.1, 0.1, 0.2)$, and (f) $(0.1, 1.1, 0.9, 0.2)$.

6 Conclusions

In this paper, by successfully implementing the First and Second simple method, traveling wave solutions for the nonlinear (3+1) dimensional mZK equation have been obtained. New soliton solutions are derived. For a clear understanding, solutions are illustrated with details in 2D and 3D. These solutions have many applications and can supply a beneficial contribution for researchers to examine and discover the waves features in several areas of physics and applied sciences. Bifurcation behavior of travelling wave solutions of the mZK equation was also analysed. A collections of new types of quasiperiodic motions was reported for the first time in the literature of the mZK equation with higher order dispersion term. Considering fixed values of the parameters, multistability behavior of the mZK equation was shown at different initial conditions. As a conclusion, it can be easily seen that the methods used in this paper may further be improved to solve and analyse qualitative behaviour of nonlinear traveling wave solutions for other NLEEs in mathematical and plasma physics.

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