The well-posedness and long-time behavior of the nonlocal diffusion porous medium equations with nonlinear term

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Abstract

In this paper, we mainly consider the well-posedness and long-time behavior of solutions for the nonlocal diffusion porous medium equations with nonlinear term. Firstly, we obtain the well-posedness of the solutions in $L^1(\Omega)$ for the equations. Secondly, we prove the existence of a global attractor by proving there exists a compact absorbing set. Finally, we apply index theory to consider the dimension of the attractor and prove that there exists an infinite dimensional attractor of the equations under proper conditions.

Keywords: nonlocal diffusion; attractors; porous medium equation.
MSC: 35B40; 35B41; 76S99.

1. Introduction

In this paper, we consider the long-time behavior of solutions for following equation:

$$\begin{cases} u_t + (-\Delta)^{\sigma/2}(|u|^{m-1}u) + g(u) = h, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $(-\Delta)^{\sigma/2}$ is the spectral fractional Laplacian operator, $\sigma \in (0,2)$, $m > 1$, $\Omega \subset\subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, $u_0 \in L^1(\Omega)$ is the initial data, $h = h(x) \in L^\infty(\Omega)$ is a given external force, the nonlinear term $g \in C^1(\mathbb{R})$ satisfies the dissipativity condition:

$$-C_1 + k_1|s|^{q-1} \leq g'(s), \tag{1.2}$$

where $q > 1$, k_1 and C_1 are some positive constants.

Recently, a great attention in the literature has been devoted to the study of nonlocal operators, both for their mathematical interest and for their applications in concrete models. The fractional Laplacian operator is a kind of nonlocal operator, which arises in several areas and it can be understood as the infinitesimal generator of a stable Lévy process (see [3, 6, 33]). The fractional diffusion partial differential equations is nowadays intensively studied both from theoretical and experimental point of views, since it conveniently explains a large number of phenomena in physics, finance, biology, ecology, geophysics, and others. Some authors have investigated important properties of fractional partial differential equations. For example, a lot of the fractional linear or nonlinear elliptic partial differential equations were studied in [4, 5, 26, 32]; the linear reaction-diffusion equations with the fractional diffusion,

$$u_t + (-\Delta)^{\sigma/2}u + f(u) = h,$$

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appear in [1, 24, 25, 40, 41]. Since the work of A. de Pablo et al. [29], the nonlocal (fractional) porous medium equation (which is one of the nonlinear nonlocal diffusion equations)

$$\begin{cases} u_t + (-\Delta)^{\sigma/2}(u^m) = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) = u_0, & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

have been paid more and more attention. The existence of weak solutions and the continuity with respect to exponent σ and m of the fractional porous medium equation in the whole space $\mathbb{R}^n$ was established in [30]. Infinite speed of propagation was presented in [38], which is different from the classical porous medium equation. The existence of self-similar solutions with conserved finite mass was discussed by [38]. In [7] the fractional porous medium equation in bounded domain was investigated. They proved the existence of H^*-solution by subdifferential operator theory and considered asymptotic behaviour of the equation.

The fractional porous medium equation with nonlinear term,

$$\begin{cases} u_t + (-\Delta)^{\sigma/2}(u^m) = g(u), & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) = u_0, & x \in \mathbb{R}^n, \end{cases}$$

was considered in [35] for $0 < \sigma < 2$, $m > m_c = (n-\sigma)_+/n$ and $g(u) = u - u^2$. The author investigated the propagation properties of nonnegative and bounded solutions, and proved that level sets propagate exponentially fast in time which is different from standard KPP problem (see [20]).

As for $\sigma = 2$ ($(-\Delta)^{\sigma/2} = -\Delta$), that is classical porous medium equation with nonlinear term, there are a number of papers consider the existence of attractors (see [2, 15, 16, 17, 19] and references therein). In [18], the authors have proved that the existence of global attractor and considered the dimension of the attractor by estimating the Kolmogorov entropy. The infinite dimensionality of the global attractor has been considered in [15], which is obtained by showing that their ε-Kolmogorov entropy behaves as a polynomial of the variable $1/\varepsilon$ as ε tends to zero.

Compared to above papers, our work encounters some difficulties when dealing with problem (1.1). Because the nonlinear term g is without an upper growth restriction, which provoke some mathematical difficulties. There make the study of (1.1) particularly interesting. In the recent paper [42], we have consider the equation for initial date $u_0 \in L^{m+1}$. However, in [42], the uniqueness and continuity of the solution was not studied. Therefore, it seems difficult to consider the long-time behavior. Furthermore, we apply $\mathbb{Z}_2$-index theory to consider the dimension of the attractors, which is very different with Kolmogorov entropy estimation as in [15, 18].

The main aim of the present paper is to consider the well-posedness and the long-time behavior of solutions for the equation (1.1) in L^1. In the first part, we will prove that there exists a solution which generated a continuous semigroup in $L^1(\Omega)$. Firstly, in light of the fractional porous medium equation is a degenerate equation, it seems to be difficult to directly multiply a function to the equation. Hence, we consider a non-degenerate approximate equation analogue of (1.1) and prove that there exists a solution with sufficiently smooth initial data for the approximate equation by a discretization process in time as [44]. Secondly, in consideration of the nonlinear term without an upper growth restriction, we get a $L^1 - L^\infty$-estimates for the solution. Furthermore, in order to relax the smoothness assumption of initial date, we use an estimate (3.10) to get a weak limit of the solution for the non-degenerate equation. Lastly, we show that the weak limit is a unique weak solution of equation (1.1), which generated a continuous semigroup. In addition, since the fractional Laplacian operator is a kind of nonlocal operator, it is difficult to get proper estimate for the main term $((-\Delta)^{\sigma/2}(|u|^{m-1}u))$. We use the method of σ-harmonic extension to overcome the problem. In the second part, we will consider long-time behavior of the solution for fractional porous medium equation. We get that the existence of a global attractor by proving there exists a compact absorbing set. Moreover, we apply index theory to consider the dimension of the attractor and prove that there exists an infinite dimensional attractor of the equation on some conditions.

The paper is organized as follows. The preliminary things are discussed in Section 2. The well-posedness of the approximate equation is verified in Section 3. The global well-posedness of equation (1.1) is given in Section 4. The existence of a global attractor is established in Section 5. Moreover, under some proper condition, we show that the dimension of the attractor is infinity.

2. Some preliminaries

In this section, we first consider the work space and recall the results of de Pablo et al. ([7, 29, 30]) related to the method of σ-harmonic extension.

The fractional laplacian operator $(-\Delta)^{\sigma/2}$ in a bounded domain is defined by a spectral decomposition [8, 14, 29, 30, 36]. Let $\{\varphi_k, \lambda_k\}_{k=1}^{\infty}$ be the eigenfunctions and the corresponding eigenvalue of $-\Delta$ in Ω with Dirichlet boundary condition. The operator $(-\Delta)^{\sigma/2}$ is defined by, for any $u \in C_0^{\infty}(\Omega)$, $u = \Sigma_{k=1}^{\infty} u_k\varphi_k$,

$$(-\Delta)^{\sigma/2}u = \Sigma_{k=1}^{\infty}\lambda_k^{\sigma/2}u_k\varphi_k.$$

This operator can be extended by density for u in the Hilbert space

$$H_0^{\sigma/2}(\Omega) = \{u \in L^2(\Omega) : \|u\|^2_{H_0^{\sigma/2}} = \Sigma_{k=1}^{\infty}\lambda_k^{\sigma/2}u_k^2 < \infty\}.$$

The fractional laplacian can be also defined by $\sigma-$harmonic extension which was introduced by Caffarelli and Silvestre for the case of the whole space in [11], and extended to bounded domains in [8, 13], see also [9, 36].

If $u(x)$ is a smooth bounded function defined on Ω, its extension to the upper half-cylinder $C_\Omega = \Omega \times (0, \infty)$, $U = E(u)$, is unique smooth bounded solution of the equation

$$\begin{cases} \nabla \cdot (y^{1-\sigma}\nabla U) = 0, & \text{in } C_\Omega, \\ U = 0, & \text{on } \partial\Omega \times [0, \infty), \\ U(x, 0) = u(x), & \text{on } \Omega. \end{cases} \tag{2.1}$$

Then, for $\mu_\sigma = 2^{\sigma-1}\Gamma(\sigma/2)\Gamma(1 - \sigma/2) > 0$,

$$-\mu_\sigma \lim_{y\to 0^+} y^{1-\sigma}\frac{\partial U}{\partial y} = (-\Delta)^{\sigma/2}u(x).$$

The operator E can be extended to $H_0^{\sigma/2}(\Omega)$. We need to consider the energy space $X_0^{\sigma}(C_\Omega)$, the closure of $C_c^{\infty}(C_\Omega)$ with respect the norm

$$\|U\|_{X_0^\sigma} = \left(\mu_\sigma \int_{C_\Omega} y^{1-\sigma}|\nabla U|^2 dxdy\right)^{1/2}.$$

Then, we have following Lemmas.

Lemma 2.1. *([30]) The operator $E : H_0^{\sigma/2}(\Omega) \to X_0^{\sigma}(C_\Omega)$ is an isometry, that is,*

$$\int_\Omega (-\Delta)^{\sigma/4}\psi(-\Delta)^{\sigma/4}\varphi dx = \mu_\sigma \int_{C_\Omega} y^{1-\sigma}\langle\nabla E(\psi)\nabla E(\varphi)\rangle dxdy.$$

Lemma 2.2. *([30]) Let $\Phi_1, \Phi_2 \in X_0^{\sigma}(C_\Omega)$ and $Tr(\Phi_1) = Tr(\Phi_2)$. Then, for some function $\varphi \in H_0^{\sigma/2}(\Omega)$, we have*

$$\mu_\sigma \int_{C_\Omega} y^{1-\sigma}\langle\nabla E(\varphi)\nabla\Phi_1\rangle dxdy = \mu_\sigma \int_{C_\Omega} y^{1-\sigma}\langle\nabla E(\varphi)\nabla\Phi_2\rangle dxdy.$$

Lemma 2.3. *([13])*

$$H_0^s(\Omega) = \begin{cases} W^{s,2}(\Omega), & 0 < s < 1/2, \\ H_{00}^{1/2}(\Omega) & s = 1/2, \\ W_0^{s,2}(\Omega) & 1/2 < s < 1. \end{cases}$$

where $W^{s,2}(\Omega)$ and $W_0^{s,2}(\Omega)$ are classical Sobolev space with the norm $\|u\|^2_{W^{s,2}(\Omega)} = \|u\|^2_{L^2(\Omega)} + [u]^2_{W^{s,2}(\Omega)}$. $H_{00}^{1/2}(\Omega)$ is Lions-Magenes space, that is, the set comprised by the function in $L^2(\Omega)$ and $[u]_{W^{1/2,2}(\Omega)} < \infty$ and $\int_\Omega \frac{u(x)^2}{dist(x,\partial\Omega)}dx < \infty$.

Next, we recall the definitions and results of the attractor [21, 22, 27, 31, 44] and the fractal dimension [21, 22, 27, 31].

Definition 2.4. *([31, 44]) Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a Banach space X. A set $\mathcal{A} \subset X$ is said to be a global attractor if the following conditions hold: (i) $\mathcal{A}$ is compact in X. (ii) $\mathcal{A}$ is strictly invariant, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for any $t \geq 0$. (iii) For any bounded subset $B \subset X$ and for any neighborhood $O = O(\mathcal{A})$ of $\mathcal{A}$ in X, there exists a time $\tau_0 = \tau_0(B)$ such that $S(t)B \subset O(\mathcal{A})$ for any $t \geq \tau_0$.*

Lemma 2.5. *([31, 44]) Let X be a (subset of) Banach space and $(S(t), X)$ be a dynamical system which possesses a compact absorbing set B, that is, for any bounded set X there exists a $\tau_0(X)$ such that $S(t)X \subset B$ for all $t \geq \tau_0(X)$. Then, there exists a global attractor $\mathcal{A} = \omega(B)$.*

Definition 2.6. *([31]) Let A be a compact set in a metric space X. Then, for every $\varepsilon > 0$, A can be covered by the finite number of ε-balls in X. Let $N(X, \varepsilon)$ be the minimal number of such balls. The fractal dimension of the set A can be expressed by*

$$d_F(A) = \limsup_{\varepsilon\to 0^+} \frac{\log \mathcal{N}(A, \varepsilon)}{\log(1/\varepsilon)}.$$

Remark 2.7. *([23]) Let fractal dimension of a set less than n, then the set homeomorphic to a subset of $\mathbb{R}^m$, where $m \leq 2n + 1$.*

Finally, we review the theory of Z_2-index [37].

Definition 2.8. *([37]) For $A \in \Sigma$, let*

(i) *$\gamma(A) = 0$, if $A = \emptyset$;*

(ii) *$\gamma(A) = \inf\{m : \exists h \in C^0(A; \mathbb{R}^m \setminus \{0\}), h(-u) = -h(u)\}$.*

(iii) *$\gamma(A) = +\infty$, if $\{m : \exists h \in C^0(A; \mathbb{R}^m \setminus \{0\}), h(-u) = -h(u)\} = \varnothing$, in particular, if $0 \in A$.*

Then, the function $\gamma : \Sigma \to \mathbb{Z}^+ \cup +\infty$ is called the Z_2-index on Σ.

Lemma 2.9. *([37]) A Z_2-index defined on Σ satisfies*

(i) *$\gamma(A) = 0 \Leftrightarrow A = \emptyset$;*

(ii) *For every $A, B \in \Sigma$, if $A \subset B$, then $\gamma(A) \leq \gamma(B)$;*

(iii) *Let $\phi : X \to X$ be a continuous and odd function. Then, $\gamma(A) \leq \gamma(\overline{\phi(A)})$ for every $A \in \Sigma$;*

(iv) *Let $A \in \Sigma$ be a compact set, then $\exists\delta > 0$ such that $\gamma(\overline{N_\delta(A)}) = \gamma(A)$, $N_\delta(A)$ is a symmetric δ-neighborhood of A;*

3. The well-posedness of the approximate equation

In [42], we have obtained the solutions with initial data $u_0 \in L^{m+1}(\Omega)$ for equations (1.1). We will prove that there exists a solutions semigroup, which is also a C^0 semigroup, in $L^1(\Omega)$ for equations (1.1). In order to show that there exists a solution in $L^1(\Omega)$ with $u_0 \in L^1(\Omega)$, we prove that there exists a solution with sufficiently smooth initial data satisfies (3.10), and then we get a solution in $L^1(\Omega)$ by (3.10).

We now consider the approximate equation with sufficiently smooth initial data $u_0 \in C_c^\infty(\Omega)$, which is a non-degenerate equation.

$$\begin{cases} u_t + (-\Delta)^{\sigma/2}(|u|^{m-1}u) + \mu(-\Delta)^{\sigma/2}u + g(u) = h, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0, & x \in \Omega, \end{cases} \tag{3.1}$$

where $\mu > 0$ is a parameter. The solution $u(x,t)$ is relate to μ, hence, we also denote that as $u_\mu(x,t)$. Analogously to [42], the corresponding difference equation of (3.1) is as follows

$$\begin{cases} \partial_t^{-\varepsilon} u_\varepsilon + (-\Delta)^{\sigma/2}(|u_\varepsilon|^{m-1}u_\varepsilon) + \mu(-\Delta)^{\sigma/2}u_\varepsilon + g(u_\varepsilon) = h(x), & x \in \Omega \times [0,T], \\ u_\varepsilon = 0, & x \in \partial\Omega \times [0,T], \end{cases} \tag{3.2}$$

where $\partial_t^{-\varepsilon} u_\varepsilon = \dfrac{u(x,t) - u(x,t-\varepsilon)}{\varepsilon}$. There exists a solution of the difference equation (3.2) which satisfies the following energy estimate,

$$\begin{aligned} &\int_0^T \int_\Omega [(-\Delta)^{\sigma/4}(u_\varepsilon^m + \mu u_\varepsilon)]^2 dxdt + \frac{1}{m+1}\int_\Omega |u_\varepsilon(T)|^{m+1} + \frac{1}{2}\int_\Omega |u_\varepsilon(T)|^2 dx \\ &+ C\int_0^T \int_\Omega |u_\varepsilon|^{m+q} dxdt + C\int_0^T \int_\Omega |u_\varepsilon|^{1+q} dxdt \\ &\le C\|h\|_{L^\infty(\Omega)} + \frac{1}{m+1}\int_\Omega |u_0|^{m+1}dx + \frac{1}{2}\int_\Omega |u_0|^2 dx + CT|\Omega|. \end{aligned} \tag{3.3}$$

Let $\varepsilon \to 0$, we get a solution $u_\mu(x,t)$ of the non-degenerate equation (3.1) and satisfies the following inequality,

$$\begin{aligned} &\int_0^T \int_\Omega [(-\Delta)^{\sigma/4}(u_\mu^m + \mu u_\mu)]^2 dxdt + \frac{1}{m+1}\int_\Omega |u_\mu(T)|^{m+1} + \frac{1}{2}\int_\Omega |u_\mu(T)|^2 dx \\ &+ C\int_0^T \int_\Omega |u_\mu|^{m+q} dxdt + C\int_0^T \int_\Omega |u_\mu|^{1+q} dxdt \\ &\le C\|h\|_{L^\infty(\Omega)} + \frac{1}{m+1}\int_\Omega |u_0|^{m+1}dx + \frac{1}{2}\int_\Omega |u_0|^2 dx + CT|\Omega|, \end{aligned} \tag{3.4}$$

where C is independent of μ.

1. $L^\infty - L^\infty$-estimate

In view of the absence of an upper growth restriction for g, we now investigate the standard $L^1 - L^\infty$-estimate. In order to get the $L^1 - L^\infty$-estimate, we first show the $L^\infty - L^\infty$-estimate and $L^1 - L^1$-estimate.

Lemma 3.1. *Let condition* (1.2) *hold and let u be a weak solution of* (3.1) *with initial data $u_0 \in L^\infty(\Omega)$, then $u(t) \in L^\infty(\Omega)$.*

Proof. As before, we multiply (3.2) by $u_\varepsilon^k = |u_\varepsilon|^{k-1}u_\varepsilon$ and integrate over $\Omega \times [0,T]$, we have

$$\begin{aligned} \int_0^T \int_\Omega [(-\Delta)^{\sigma/2}u_\varepsilon^m]u_\varepsilon^k dxdt + \mu\int_0^T \int_\Omega [(-\Delta)^{\sigma/2}u_\varepsilon]u_\varepsilon^k dxdt + \int_0^T \int_\Omega (\partial_t^{-\varepsilon}u_\varepsilon)u_\varepsilon^k dxdt \\ + \int_0^T \int_\Omega g(u_\varepsilon)u_\varepsilon^k dxdt = \int_0^T \int_\Omega hu_\varepsilon^k dxdt. \end{aligned}$$

Applying Lemma 2.2, we get

$$\begin{aligned} \int_0^T \int_\Omega [(-\Delta)^{\sigma/2}u_\varepsilon^m]u_\varepsilon^k dxdt &= \int_0^T \int_{C_\Omega} y^{1-\sigma}(\nabla E(u_\varepsilon^m))(\nabla E(u_\varepsilon^k))dxdt \\ &= \int_0^T \int_{C_\Omega} y^{1-\sigma}(\nabla(E(u_\varepsilon))^m)(\nabla(E(u_\varepsilon))^k)dxdt \\ &\ge 0, \end{aligned}$$

and

$$\int_0^T \int_\Omega [(-\Delta)^{\sigma/2}u_\varepsilon]u_\varepsilon^k dxdt \ge 0$$

In view of

$$\int_0^T \int_\Omega (\partial_t^{-\varepsilon} u_\varepsilon) u_\varepsilon^k dxdt \geq \frac{1}{k+1} \int_\Omega (|u_\varepsilon(T)|^{k+1} - |u_\varepsilon(0)|^{k+1}) dx$$

and (1.2), we have

$$\frac{1}{k+1} \int_\Omega |u_\varepsilon(T)|^{k+1} dx + \int_0^T \int_\Omega |u_\varepsilon|^{k+q} dxdt \leq C^k + \frac{1}{k+1} |u_\varepsilon(0)|^{k+1} dx,$$

where C is independent of k. Hence

$$\left(\int_\Omega |u_\varepsilon(T)|^{k+1} dx \right)^{\frac{1}{k+1}} \leq (k+1)^{\frac{1}{k+1}} C^{\frac{k}{k+1}} + \left(\int_\Omega |u_\varepsilon(0)|^{k+1} dx \right)^{\frac{1}{k+1}}.$$

Let $k \to \infty$, we deduce that $\|u_\varepsilon(T)\|_{L^\infty(\Omega)} \leq C + \|u_0\|_{L^\infty(\Omega)}$. Therefore, $\|u(T)\|_{L^\infty(\Omega)} \leq C + \|u_0\|_{L^\infty(\Omega)}$. □

2. A priori estimates

It is easy to check that u_t of (3.1) is a functional as [42]. In order to show that u_t is actually a function, we need the following lemmas.

Lemma 3.2. *Let $u(x,t) \in L^1(\Omega \times [0,T])$, $u_\varepsilon \to u$ a.e. $(x,t) \in \Omega \times [0,T]$. Assume that u_ε and $\dfrac{u_\varepsilon(t) - u_\varepsilon(t-\varepsilon)}{\varepsilon}$ is uniformly bounded in $L^p(\Omega \times [0,T])$, where $p > 1$. Then the weak derivative $u'(x,t) \in L^p(\Omega \times [0,T])$.*

Proof. Let $\phi \in C_c^\infty(\Omega \times (0,T))$. Because $\dfrac{u_\varepsilon(t) - u_\varepsilon(t-\varepsilon)}{\varepsilon}$ is uniformly bounded in $L^p(\Omega \times [0,T])$ with respect to ε, we have a weakly convergent subsequence which is convergent to $g(x,t) \in L^p(\Omega \times [0,T])$, that is,

$$\int_{\Omega \times [0,T]} \frac{u_\varepsilon(t) - u_\varepsilon(t-\varepsilon)}{\varepsilon} \phi(x,t) dxdt \to \int_{\Omega \times [0,T]} g(x,t) \phi(x,t) dxdt, \quad \text{as } \varepsilon \to 0.$$

In addition,

$$\begin{aligned} \int_{\Omega \times [0,T]} u(x,t) \phi'(x,t) dxdt &= \lim_{\varepsilon \to 0} \int_{\Omega \times [0,T]} u_\varepsilon(x,t) \frac{\phi(x,t+\varepsilon) - \phi(x,t)}{\varepsilon} dxdt \\ &= -\lim_{\varepsilon \to 0} \int_{\Omega \times [0,T]} \frac{u_\varepsilon(t) - u_\varepsilon(t-\varepsilon)}{\varepsilon} \phi(x,t) dxdt. \end{aligned}$$

Therefore,

$$\int_{\Omega \times [0,T]} u(x,t) \phi'(x,t) dxdt = -\int_{\Omega \times [0,T]} g(x,t) \phi(x,t) dxdt.$$

Hence, we get $u'(x,t) = g(x,t) \in L^p(\Omega \times [0,T])$ □

Lemma 3.3. *([29]) Let $m > 0$, then there exists a positive constant c (depend on m) such that*

$$(x^m - 1)(x - 1) \geq c(x^{\frac{m+1}{2}} - 1)^2, \quad \forall x \geq 1.$$

and

$$(x^m + 1)(x + 1) \geq c(x^{\frac{m+1}{2}} + 1)^2, \quad \forall x \geq 1.$$

We now show that u_t is actually a function.

Theorem 3.4. *Let the condition (1.2) hold, $u_0 \in C_c^\infty(\Omega)$ and let u be a the weak solution of (3.1), then for all $\delta > 0$, $u_t \in L^2(\Omega \times [\delta, T])$.*

Proof. Indeed, let

$$\partial_t^{-\varepsilon}(u_\varepsilon^m) = \frac{u_\varepsilon^m(x,t) - u_\varepsilon^m(x,t-\varepsilon)}{\varepsilon}$$
$$= m|\theta u_\varepsilon(x,t) + (1-\theta)u_\varepsilon(x,t-\varepsilon)|^{m-1}\partial_t^{-\varepsilon}u_\varepsilon,$$

where $\theta \in [0,1]$. Then, multiplying equation (3.2) by $\partial_t^{-\varepsilon}(u_\varepsilon^m + \mu u_\varepsilon)$ and integrating over $\Omega \times [s,T]$, where $0 \le s \le T$. We divide the time interval [0,T] in n subinterval and denote t_s as right extreme point of the subinterval which include s. The subinterval include s denote by s_1th subinterval.

We now consider the inequality term by term. Because of the Lemma 3.3, we get

$$C\int_s^T \int_\Omega (\partial_t^{-\varepsilon} u_\varepsilon^{\frac{m+1}{2}})^2 dxdt \le \int_s^T \int_\Omega \partial_t^{-\varepsilon}(u_\varepsilon)\partial_t^{-\varepsilon}(u_\varepsilon^m)dxdt.$$

For the second term, we have

$$\begin{aligned}
&\int_s^T \int_\Omega (-\Delta)^{\sigma/2}(u_\varepsilon^m + \mu u_\varepsilon)\cdot \partial_t^{-\varepsilon}(u_\varepsilon^m + \mu u_\varepsilon)dxdt \\
&= \int_s^T \int_\Omega (-\Delta)^{\sigma/4}(u_\varepsilon^m + \mu u_\varepsilon)\cdot \partial_t^{-\varepsilon}[(-\Delta)^{\sigma/4}(u_\varepsilon^m + \mu u_\varepsilon)]dxdt \\
&\ge \frac{1}{2}\{\int_\Omega [(-\Delta)^{\sigma/4}(u_\varepsilon^m(x,T) + \mu u_\varepsilon(x,T))]^2 dx \\
&- (1-\zeta)\int_\Omega [(-\Delta)^{\sigma/4}(u_\varepsilon^m(x,s) + \mu u_\varepsilon(x,s))]^2 dx \\
&- \zeta \int_\Omega [(-\Delta)^{\sigma/4}(u_\varepsilon^m(x,s-\varepsilon) + \mu u_\varepsilon(x,s-\varepsilon))]^2 dx\},
\end{aligned}$$

where $\zeta = \frac{t_s - s}{\varepsilon}$. Defined a function $\widehat{g}(s) = g(s) + C_1 s$, then $\widehat{g}'(s) \ge 0$. We get the following estimate,

$$\begin{aligned}
&\int_s^T \int_\Omega \widehat{g}(u_\varepsilon)\partial_t^{-\varepsilon}(u_\varepsilon^m)dxdt \\
&= \sum_{l=s_1}^n \int_\Omega \widehat{g}(u_\varepsilon(x,t_l))(u_\varepsilon^m(x,t_l) - u_\varepsilon^m(x,t_{l-1}))dx + \int_s^{t_s}\int_\Omega \widehat{g}(u_\varepsilon)\partial_t^{-\varepsilon}(u_\varepsilon^m)dxdt \\
&\ge \sum_{l=s_1}^n \int_\Omega (\widehat{G}(u_\varepsilon(x,t_l)) - \widehat{G}(u_\varepsilon(x,t_{l-1})))dx + m\zeta\int_\Omega (\widehat{G}(u_\varepsilon(x,s)) - \widehat{G}(u_\varepsilon(x,s-\varepsilon)))dx \\
&= m\{\int_\Omega \widehat{G}(u_\varepsilon(x,T))dx - (1-\zeta)\int_\Omega \widehat{G}(u_\varepsilon(x,s))dx - \zeta\int_\Omega \widehat{G}(u_\varepsilon(x,s-\varepsilon))dx\},
\end{aligned}$$

where $\widehat{G}(u) = \int_0^u \widehat{g}(s)|s|^{m-1}ds$.

Accordingly, we deduce that

$$\begin{aligned}
&\int_s^T \int_\Omega \widehat{g}(u_\varepsilon)\partial_t^{-\varepsilon}(u_\varepsilon)dxdt \\
&= \sum_{l=s_1}^n \int_\Omega \widehat{g}(u_\varepsilon(x,t_l))(u_\varepsilon(x,t_l) - u_\varepsilon(x,t_{l-1}))dx + \int_s^{t_s}\int_\Omega \widehat{g}(u_\varepsilon)\partial_t^{-\varepsilon}(u_\varepsilon)dxdt \\
&\ge \sum_{l=s_1}^n \int_\Omega (\widetilde{G}(u_\varepsilon(x,t_l)) - \widetilde{G}(u_\varepsilon(x,t_{l-1})))dx + \zeta\int_\Omega (\widetilde{G}(u_\varepsilon(x,s)) - \widetilde{G}(u_\varepsilon(x,s-\varepsilon)))dx \\
&= \int_\Omega \widetilde{G}(u_\varepsilon(x,T))dx - (1-\zeta)\int_\Omega \widetilde{G}(u_\varepsilon(x,s))dx - \zeta\int_\Omega \widetilde{G}(u_\varepsilon(x,s-\varepsilon))dx,
\end{aligned}$$

where $\widetilde{G}(u)=\int_0^u \widehat{g}(s)ds$.

In addition, we get

$$
\begin{aligned}
&\int_s^T \int_\Omega u_\varepsilon \cdot \partial_t^{-\varepsilon}(u_\varepsilon^m)dxdt\\
&=\frac{2m}{m+1}\int_s^T \int_\Omega u_\varepsilon \cdot |\theta' u_\varepsilon^{\frac{m+1}{2}}(x,t)+(1-\theta')u_\varepsilon^{\frac{m+1}{2}}(x,t-\varepsilon)|^{\frac{m-1}{m+1}} \cdot \partial_t^{-\varepsilon} u_\varepsilon^{\frac{m+1}{2}} dxdt\\
&\le \tau \int_s^T \int_\Omega (\partial_t^{-\varepsilon} u_\varepsilon^{\frac{m+1}{2}})^2 dxdt\\
&+C\int_s^T \int_\Omega u_\varepsilon^2 \cdot |\theta' u_\varepsilon^{\frac{m+1}{2}}(x,t)+(1-\theta')u_\varepsilon^{\frac{m+1}{2}}(x,t-\varepsilon)|^{\frac{2(m-1)}{m+1}} dxdt,
\end{aligned}
$$

where τ is small enough.

For the last term, we have the following equation,

$$
\begin{aligned}
&\int_s^T \int_\Omega h(x)\partial_t^{-\varepsilon}(u_\varepsilon^m)dxdt\\
&=\sum_{l=s_1}^{n}\int_\Omega h(x)(u_\varepsilon^m(x,t_l)-u_\varepsilon^m(x,t_{l-1}))dx+\int_s^{t_s}\int_\Omega h(x)\partial_t^{-\varepsilon}(u_\varepsilon^m)dxdt\\
&=\int_\Omega h(x)u_\varepsilon^m(T)dx-(1-\zeta)\int_\Omega h(x)u_\varepsilon^m(s)dx-\zeta\int_\Omega h(x)u_\varepsilon^m(s-\varepsilon)dx.
\end{aligned}
$$

Accordingly,

$$
\begin{aligned}
&\int_s^T \int_\Omega h(x)\partial_t^{-\varepsilon}(u_\varepsilon)dxdt\\
&=\int_\Omega h(x)u_\varepsilon(T)dx-(1-\zeta)\int_\Omega h(x)u_\varepsilon(s)dx-\zeta\int_\Omega h(x)u_\varepsilon(s-\varepsilon)dx.
\end{aligned}
$$

Combining with Lemma 3.1 and the above estimate, we conclude that

$$
\begin{aligned}
&\int_s^T \int_\Omega (\partial_t^{-\varepsilon} u_\varepsilon^{\frac{m+1}{2}})^2 dxdt+\mu\int_s^T \int_\Omega (\partial_t^{-\varepsilon} u_\varepsilon)^2 dxdt\\
&+\frac{1}{2}\int_\Omega [(-\Delta)^{\sigma/4}(u_\varepsilon^m(x,T)+\mu u_\varepsilon(x,T))]^2 dx+m\int_\Omega \widehat{G}(u_\varepsilon(x,T))dx+\mu\int_\Omega \widetilde{G}(u_\varepsilon(x,T))dx\\
&\le \int_\Omega [(-\Delta)^{\sigma/4}(u_\varepsilon^m(x,s)+\mu u_\varepsilon(x,s))]^2 dx\\
&+\int_\Omega [(-\Delta)^{\sigma/4}(u_\varepsilon^m(x,s-\varepsilon)+\mu u_\varepsilon(x,s-\varepsilon))]^2 dx+C.
\end{aligned}
$$

where C depend on $\|u_0\|_{L^\infty(\Omega)}, g, T$ and $\|h\|_{L^\infty(\Omega)}$, but is independent of ε.
Integrating the above inequality over $[0,T]$ with respect to s, we infer that

$$
\begin{aligned}
&\int_0^T t\int_\Omega (\partial_t^{-\varepsilon} u_\varepsilon^{\frac{m+1}{2}})^2 dxdt+\mu\int_0^T t\int_\Omega (\partial_t^{-\varepsilon} u_\varepsilon)^2 dxdt\\
&+\frac{T}{2}\int_\Omega [(-\Delta)^{\sigma/4}(u_\varepsilon^m(x,T)+\mu u_\varepsilon(x,T))]^2 dx+mT\int_\Omega \widehat{G}(u_\varepsilon(x,T))dx+\mu T\int_\Omega \widetilde{G}(u_\varepsilon(x,T))dx\\
&\le \int_0^T \int_\Omega [(-\Delta)^{\sigma/4}(u_\varepsilon^m(x,s)+\mu u_\varepsilon(x,s))]^2 dx\\
&+\int_0^T \int_\Omega [(-\Delta)^{\sigma/4}(u_\varepsilon^m(x,s-\varepsilon)+\mu u_\varepsilon(x,s-\varepsilon))]^2 dx+TC.
\end{aligned}
$$

For all δ, T and $0 < \delta < T$, the inequality (3.3) imply that

$$\delta \int_\delta^T \int_\Omega (\partial_t^{-\varepsilon} u_\varepsilon)^2 dxdt \leq \int_0^T t \int_\Omega (\partial_t^{-\varepsilon} u_\varepsilon)^2 dxdt \leq C.$$

Applying Lemma 3.2, we finish the prove of Lemma 3.4. □

Remark 3.5. *Due to*

$$u \in L^\infty(0,T;L^{m+1}(\Omega)) \hookrightarrow L^1(0,T;L^1(\Omega) + H^{-\sigma/2}(\Omega)),$$

and

$$u_t \in L^1(0,T;L^1(\Omega) + H^{-\sigma/2}(\Omega)),$$

thanks to Proposition 7.1 in [31], we get

$$u \in C(0,T;L^1(\Omega) + H^{-\sigma/2}(\Omega)).$$

Hence,

$$u \in C(0,T;L^1(\Omega)). \tag{3.5}$$

3. $L^1 - L^1$-estimate

Lemma 3.6. *Let condition* (1.2) *hold, let initial data $u_0 \in L^\infty(\Omega)$ and let $u(t)$ be the weak solution of* (3.1). *Then,*

$$\|u(T)\|_{L^1(\Omega)} \leq C + \|u_0\|_{L^1(\Omega)}, \tag{3.6}$$

where C is independent of T.

Proof. We consider a function $p_j(s)$, $p_j(s) \to \text{sgn}s$ as $j \to \infty$ and exists $M_j > 0$, such that $-1 < p_j(s) < 1$, $p_j(0) = 0$ and $0 < p_j^{'}(s) < M_j$.
Applying Lemma 2.3 and Lemma 3.1, we get that $p_j(u^m) \in H_0^{\sigma/2}(\Omega)$. Hence, thanks to Lemma 2.2, we deduce that

$$\begin{aligned}
&\int_\Omega (-\Delta)^{\sigma/2} u^m \cdot p_j(u^m) dxdt \\
&= \int_{C_\Omega} y^{1-\sigma} \nabla E(u^m) \cdot \nabla E(p_j(u^m)) dxdydt \\
&= \int_{C_\Omega} y^{1-\sigma} \nabla E(u^m) \cdot \nabla p_j(E(u^m)) dxdydt \\
&= \int_{C_\Omega} y^{1-\sigma} (\nabla E(u^m))^2 \cdot p_j^{'}(E(u^m)) dxdydt \\
&\geq 0.
\end{aligned}$$

Similarly, we have

$$\int_\Omega (-\Delta)^{\sigma/2} u \cdot p_j(u^m) dxdt \geq 0.$$

In light of

$$
\begin{aligned}
&\int_\Omega g(u)p_j(u^m)dx \\
&\geq \int_\Omega (-C + |u|^q)|p_j(u^m)|dx \\
&\geq -C|\Omega| + \int_\Omega |u|^q|p_j(u^m)|dx. \\
&\geq -C|\Omega| + k\int_\Omega |u||p_j(u^m)|dx - C\int_\Omega |p_j(u^m)|dx \\
&\geq -C|\Omega| + k\int_\Omega |u||p_j(u^m)|dx,
\end{aligned}
$$

Multiplying equation (3.1) by $e^{kt}p_j(u^m)$ and integrating (x,t) over $\Omega\times[\delta,T]$, we obtain

$$
\begin{aligned}
\int_\delta^T e^{kt}p_j(u^m)u_t dxdt &+ k\int_\delta^T\int_\Omega e^{kt}|u||p_j(u^m)|dxdt \\
&\leq \int_\delta^T\int_\Omega Ce^{kt}dxdt + \int_\delta^T\int_\Omega |h|e^{kt}dxdt.
\end{aligned}
$$

Passing to the limit $j\to\infty$, we get the following inequality,

$$
\begin{aligned}
&\int_\delta^T e^{kt}\int_\Omega \frac{d}{dt}|u|dxdt + k\int_\delta^T e^{kt}\int_\Omega |u|dxdt \\
&\leq \int_\delta^T e^{kt}Cdt + \int_\delta^T e^{kt}\int_\Omega |h|dxdt \\
&\leq Ce^{kT}.
\end{aligned}
$$

Therefore, we conclude that

$$
\|u(T)\|_{L^1(\Omega)} \leq C + e^{\delta-T}\|u(\delta)\|_{L^1(\Omega)}.
$$

Thanks to the remark (3.5), passing to the limit $\delta\to 0$, we get (3.6). □

4. $L^1 - L^\infty$-estimate

Theorem 3.7. *Let condition* (1.2) *hold. Suppose $u_0\in L^\infty(\Omega)$. Assume that $u(t)$ is a weak solution of equation* (3.1). *Then there exists a constant M such that*

$$
\|u(t)\|_{L^\infty(\Omega)} \leq C(1+\frac{1}{t^{\frac{1}{q-1}}}) := M,\ \forall t>0. \tag{3.7}
$$

where M is independent of $\|u_0\|_{L^\infty(\Omega)}$, but depend on $\frac{1}{t}$.

Proof. We multiply the equation (3.1) by $u^{k(q-1)} = |u|^{k(q-1)-1}u$ and integrate over $\Omega\times[0,T]$. This implies that

$$
\frac{1}{1+k(q-1)}\frac{d}{dt}\|u(t)\|^{1+k(q-1)}_{L^{1+k(q-1)}} + \frac{k_1}{4q}\|u(t)\|^{1+(k+1)(q-1)}_{L^{1+(k+1)(q-1)}} \leq C_{k+1}, \tag{3.8}
$$

where

$$
\begin{aligned}
C_{k+1} = \|h\|_{L^\infty}^{\frac{(k+1)(q-1)+1}{q}} &\cdot \left[\frac{(k+1)(q-1)+1}{k(q-1)}\cdot\frac{k_1}{4q}\right]^{-\frac{k(q-1)}{q}}. \\
&+\left(\frac{2q}{k_1}\right)^{\frac{k(q-1)}{q}}\cdot|\Omega|.
\end{aligned}
$$

Then, integrating the equation (3.8) over $[s,t]$ with respect to t, we have

$$\begin{aligned}&\frac{1}{1+k(q-1)}\|u(t)\|_{L^{1+k(q-1)}}^{1+k(q-1)}+\frac{k_1}{4q}\int_s^t\|u(\xi)\|_{L^{1+(k+1)(q-1)}}^{1+(k+1)(q-1)}d\xi\\&\le C_{k+1}(t-s)+\frac{1}{1+k(q-1)}\|u(s)\|_{L^{1+k(q-1)}}^{1+k(q-1)}.\end{aligned}$$

Let us now multiply above inequality by s^k and integrate over $[0,t]$ with respect to s. Then, we conclude that

$$\begin{aligned}&\frac{t^{k+1}}{1+k(q-1)}\|u(t)\|_{L^{1+k(q-1)}}^{1+k(q-1)}+\frac{k_1}{4q(k+1)}\int_0^t s^{k+1}\|u(s)\|_{L^{1+(k+1)(q-1)}}^{1+(k+1)(q-1)}ds\\&\le C_{k+1}\cdot\frac{t^{k+2}}{(k+1)(k+2)}+\int_0^t\frac{s^k}{1+k(q-1)}\|u(s)\|_{L^{1+k(q-1)}}^{1+k(q-1)}ds.\end{aligned}\tag{3.9}$$

Iterating the inequality (3.9) and combining with Lemma 3.6, we arrive at

$$\begin{aligned}&\frac{t^{k+1}}{1+k(q-1)}\|u(t)\|_{L^{1+k(q-1)}}^{1+k(q-1)}\\&\le C_{k+1}\cdot\frac{t^{k+2}}{(k+1)(k+2)}+\cdots+C_1\frac{t^2}{2}+Ct+t\|u(0)\|_{L^1(\Omega)}\\&\le C\|u(0)\|_{L^1(\Omega)}\cdot C_{k+1}(t^{k+2}+\cdots+t).\end{aligned}$$

We divided the above inequality by t^{k+1} and raised to the power of $\frac{1}{1+k(q-1)}$. This implies that

$$\begin{aligned}&\left(\frac{1}{1+k(q-1)}\right)^{\frac{1}{1+k(q-1)}}\|u(t)\|_{L^{1+k(q-1)}}\\&\le(C\|u(0)\|_{L^1(\Omega)}\cdot C_{k+1})^{\frac{1}{1+k(q-1)}}\cdot\left(\frac{1-t^{k+2}}{(1-t)t^k}\right)^{\frac{1}{1+k(q-1)}}.\end{aligned}$$

Letting $k\to\infty$ for $t>0$, we have

$$\|u(t)\|_{L^\infty(\Omega)}\le C\left(1+\frac{1}{t^{\frac{1}{q-1}}}\right)\ \forall t>0.$$

□

Remark 3.8. *In fact, the M is independent of $\|u_0\|_{L^1(\Omega)}$ by the Theorem 3.7.*

5. Continuous dependence of the solution

The following lemma show the uniqueness of the solution for equation (3.1) and which also can be used to obtain a weak solution with initial data $u_0\in L^1(\Omega)$.

Lemma 3.9. *Let the condition* (1.2) *hold and let $u_1(t)$ and $u_2(t)$ be two solution of equation* (3.1) *with initial data $u_0^1\in C_c^\infty(\Omega)$ and $u_0^2\in C_c^\infty(\Omega)$, respectively. Then the following estimate hold:*

$$\|u_1(t)-u_2(t)\|_{L^1(\Omega)}\le e^{Kt}\|u_0^1-u_0^2\|_{L^1(\Omega)},\tag{3.10}$$

where $K=\max_{s\in\mathbb{R}}\{-g'(s)\}$.

Proof. Indeed, let $v(t)=u_1(t)-u_2(t)$. Then, the function satisfies the following equation

$$\begin{cases}v_t+(-\Delta)^{\sigma/2}(l_1(t)+\mu)v+l_2(t)v=0,\\ v|_{\partial\Omega}=0,\\ v|_{t=0}=u_0^1-u_0^2.\end{cases}$$

where

$$l_1(t) = \int_0^1 |su_1(t) + (1-s)u_2(t)|^{m-1} ds \geq 0$$
$$l_2(t) = \int_0^1 g'(su_1(t) + (1-s)u_2(t)) ds \geq -K.$$

Multiplying the above equation by $p_j((l_1(t)+\mu)v)$ and integrating over Ω, we get

$$\int_\Omega v_t p_j((l_1(t)+\mu)v)dx \leq K \int_\Omega v p_j((l_1(t)+\mu)v)dx.$$

Let us now multiply the inequality by e^{-Kt} and integrate over $[\delta, T]$ with respect to t. This implies that

$$\int_\delta^T e^{-Kt} \int_\Omega v_t p_j((l_1(t)+\mu)v)dxdt \leq K \int_\delta^T e^{-Kt} \int_\Omega v p_j((l_1(t)+\mu)v)dxdt.$$

Letting $j \to \infty$, we have

$$\int_\delta^T e^{-Kt} \frac{d}{dt} \int_\Omega |v| dxdt \leq K \int_\delta^T e^{-Kt} \int_\Omega |v| dxdt,$$

that is,

$$\int_\delta^T \frac{d}{dt}(e^{-Kt} \int_\Omega |v| dx) dt \leq 0.$$

Thus,

$$\int_\Omega |v(T)| dx \leq e^{K(T-\delta)} \int_\Omega |v(\delta)| dx.$$

Letting $\delta \to 0$ and combining with remark 3.5, we get

$$||u_1(t) - u_2(t)||_{L^1(\Omega)} \leq e^{Kt} ||u_0^1 - u_0^2||_{L^1(\Omega)}.$$

□

4. Solutions in $L^1(\Omega)$

We now formulate the definition of a weak solution with initial data in $L^1(\Omega)$ of that problem.

Definition 4.1. *We say that a function u is a weak solution of* (1.1) *for every $\delta > 0$ and $u_0 \in L^1(\Omega)$, if*

$$u \in C([0,T], L^1(\Omega)), \quad u \in L^\infty(\Omega \times [\delta, T]), \quad u^m \in L^2([\delta, T], H_0^{\sigma/2}(\Omega))$$

and it satisfies (1.1) *in the sense of distributions.*

Remark 4.2. *In fact, thanks to* (3.7) *and remark 3.5, we know that the weak solution of the approximate equation* (3.1) *satisfy the condition of the Definition 4.1.*

Lemma 4.3. *Assume that the condition* (1.2) *hold and initial data $u_0 \in C_c^\infty(\Omega)$. Then, there exists a weak solution of the equation* (1.1) *for the Definition 4.1.*

Proof. Indeed, let u_{μ_n} be a weak solution of the approximate equation (3.1) with initial data u_0, where μ_n is the parameter and $\mu_n \to 0$ as $n \to 0$. Then, u_{μ_n} satisfy the estimate (3.4). We conclude that $u^m_{\mu_n}$ is uniformly bounded in $L^2(0,T;H_0^{\sigma/2}(\Omega))$. Hence, we can extract a weakly convergent subsequence with $u^m_{\mu_n} \rightharpoonup u^m$ in $L^2(0,T;H_0^{\sigma/2}(\Omega))$. Applying the Aubin-Lions Lemma, we get $u_{\mu_n} \to u$ a.e. in $\Omega\times[0,T]$ (analogously to the proof of Theorem 2.3 in [42]). Thus, $\|u_{\mu_n}(t)-u(t)\|_{L^1(\Omega)} \to 0$ a.e. in $[0,T]$. Thanks to Lemma 3.9, we infer that the limit $u(x,t)$ satisfy (3.10). We now show that the $u(x,t)$ is what we need. In addition, thanks to (3.7), we get $\|u_{\mu_n}\|_{L^\infty(\Omega\times[\delta,T])} \leq M$. Hence, there exists a weakly star convergent subsequence of u_{μ_n} such that $u_{\mu_n} \overset{*}{\rightharpoonup} u$. Therefore, $\|u\|_{L^\infty(\Omega\times[\delta,T])} \leq \liminf_{n\to\infty} \|u_{\mu_n}\|_{L^\infty(\Omega\times[\delta,T])} \leq M$. We now show that $u \in C([0,T],L^1(\Omega))$. Let $0 \leq s < t \leq T$ and $\varepsilon > 0$, because of (3.10), we have

$$
\begin{aligned}
&\|u(s) - u(t)\|_{L^1(\Omega)}\\
\leq& e^{K(t-s)}\|u_0 - u(t-s)\|_{L^1(\Omega)}\\
\leq& e^{K(t-s)}\|u_0 - u_{u_{\mu_n}}(t-s) + u_{u_{\mu_n}}(t-s) - u(t-s)\|_{L^1(\Omega)}\\
\leq& e^{K(t-s)}\|u_0 - u_{u_{\mu_n}}(t-s)\|_{L^1(\Omega)} + e^{K(t-s)}\|u_{u_{\mu_n}}(t-s) - u(t-s)\|_{L^1(\Omega)}\\
\leq& \varepsilon,
\end{aligned}
$$

where μ_n and $t-s$ is small enough. Hence, we arrive at $u \in C([0,T],L^1(\Omega))$. It is easy show that for every test function $\varphi \in C_c^\infty(\Omega\times(0,T))$, we have $\mu_n \int_0^T \int_\Omega (-\Delta)^{\sigma/4} u_{\mu_n} (-\Delta)^{\sigma/4}\varphi dxdt \to 0$ as $\mu_n \to 0$. Actually ,

$$
\begin{aligned}
&\mu_n \int_0^T \int_\Omega (-\Delta)^{\sigma/4} u_{\mu_n} (-\Delta)^{\sigma/4}\varphi dxdt\\
=&\mu_n \int_0^T \int_\Omega u_{\mu_n} (-\Delta)^{\sigma/2}\varphi dxdt\\
\leq&\mu_n \left(\int_0^T \int_\Omega |u_{\mu_n}|^2 dxdt\right)^{1/2} \cdot \left(\int_0^T \int_\Omega ((-\Delta)^{\sigma/2}\varphi)^2 dxdt\right)^{1/2}\\
\leq&\mu_n \cdot C \to 0 \text{ as } \mu_n \to 0.
\end{aligned}
$$

Using the analogous method of Theorem 2.3 in [42] for nonlinear term, we get that $u(x,t)$ satisfy the equation (1.1) in sense of distributions. □

In order to consider the uniqueness of the weak solution for (1.1), we need to study the following equation

$$
\begin{cases}
-\partial_t w + l_1(t)(-\Delta)^{\sigma/2} w + \varepsilon(-\Delta)^{\sigma/2} w = 0 & x \in \Omega\times[0,T],\\
w(x,t) = 0 & x \in \partial\Omega\times[0,T],\\
w(x,T) = w_T & x \in \Omega,
\end{cases}
\tag{4.1}
$$

where the $l_1(t)$ is the same as that in Lemma 3.9.

According to the standard Fatou-Galerkin method and the maximum principle, it is easy to obtain the solvability result for the equation (4.1) as Lemma 1.4 in [18]. Here we only state the result as follows.

Lemma 4.4. *Assume that $l_1 \in L^\infty(\Omega\times[0,T])$. Then for every $w_T \in H_0^{\sigma/2}(\Omega)$ and every $\varepsilon > 0$, there exists a unique solution $w \in W^{(1,\sigma),2}(\Omega\times[0,T])$ of the equation* (4.1) *and the following estimate holds:*

$$
\|(-\Delta)^{\sigma/4} w(t)\|^2_{L^2(\Omega)} + 2\varepsilon \int_0^T \|(-\Delta)^{\sigma/2} w(t)\|^2_{L^2(\Omega)} dt \leq \|(-\Delta)^{\sigma/4} w(T)\|^2_{L^2(\Omega)}.
\tag{4.2}
$$

Moreover, if in addition, $C_1 \leq w_T(x) \leq C_2$, then

$$
C_1 \leq w(x,t) \leq C_2, \textit{ for } t \in [0,T].
\tag{4.3}
$$

Now we are ready to state our main result in this section.

Theorem 4.5. *Assume that the condition* (1.2) *hold and initial data* $u_0 \in L^1(\Omega)$. *Then, there exists a unique weak solution of* (1.1) *for the Definition 4.1.*

Proof. Let $u_0^n \in C_0^\infty(\Omega)$ and $u_0^n \to u_0$ in $L^1(\Omega)$. Then, thanks to the Lemma 4.3, we get that there exists a weak solution $u^n(x,t)$ for equation (1.1) with initial data u_0^n. Because of (3.10), we get that $u^n(t)$ is the Cauchy sequence in $L^1(\Omega)$. Hence, there exists $u(t) \in L^1(\Omega)$ such that $u^n(t) \to u(t)$ in $L^1(\Omega)$ for every $t \geq 0$. Let us show that $u(x,t)$ is the weak solution of equation (1.1). Taking into account u^n is the weak solution of equation (1.1) with initial data u_0^n, we conclude that $(u^n)^m$ is uniformly bounded in $L^2(0,T;H_0^{\sigma/2}(\Omega))$. Thus, we can extract a weakly convergent subsequence with $(u^n)^m \rightharpoonup u^m$ in $L^2(0,T;H_0^{\sigma/2}(\Omega))$. In addition, thanks to (3.7), we get $\|u^n\|_{L^\infty(\Omega\times[\delta,T])} \leq M$. Then, we can extract a weakly star convergent subsequence with $u^n \overset{*}{\rightharpoonup} u$. Hence, $\|u\|_{L^\infty(\Omega\times[\delta,T])} \leq \liminf_{n\to\infty} \|u^n\|_{L^\infty(\Omega\times[\delta,T])} \leq M$. Besides, let $u_{1,0}, u_{2,0} \in L^1(\Omega)$. Then, there exists $u_{1,0}^n, u_{2,0}^n \in C_c^\infty(\Omega)$ such that $u_{1,0}^n \to u_{1,0}$ in $L^1(\Omega)$ and $u_{2,0}^n \to u_{2,0}$ in $L^1(\Omega)$. Moreover, there exists weak solution $u_1^n(x,t)$ and $u_2^n(x,t)$ with initial data $u_{1,0}^n$ and $u_{2,0}^n$, respectively. Applying (3.10), we get that there exists $u_1(x,t)$ and $u_2(x,t)$ such that $u_1^n(t) \to u_1(t)$ in $L^1(\Omega)$ and $u_2^n(t) \to u_2(t)$ in $L^1(\Omega)$ for every $t \geq 0$. Thanks to Lemma 4.3, we have

$$
\begin{aligned}
&\|u_1(t) - u_2(t)\|_{L^1(\Omega)} \\
=&\|u_1(t) - u_1^n(t) + u_1^n(t) - u_2^n(t) + u_2^n(t) - u_2(t)\|_{L^1(\Omega)} \\
\leq&\|u_1(t) - u_1^n(t)\|_{L^1(\Omega)} + \|u_1^n(t) - u_2^n(t)\|_{L^1(\Omega)} + u_2^n(t) - u_2(t)\|_{L^1(\Omega)} \\
\leq&\|u_1(t) - u_1^n(t)\|_{L^1(\Omega)} + e^{Kt}\|u_{1,0}^n - u_{2,0}^n\|_{L^1(\Omega)} + \|u_2^n(t) - u_2(t)\|_{L^1(\Omega)} \\
\to& e^{Kt}\|u_{1,0} - u_{2,0}\|_{L^1(\Omega)} \text{ as } n \to \infty.
\end{aligned}
$$

Hence, $u_1(x,t)$ and $u_2(x,t)$ satisfy (3.10). We now show $u \in C([0,T], L^1(\Omega))$. Actually, let $0 \leq s < t \leq T$, applying (3.10), we have following inequality hold for every $\varepsilon > 0$:

$$
\begin{aligned}
&\|u(s) - u(t)\|_{L^1(\Omega)} \\
\leq& e^{K(t-s)}\|u_0 - u(t-s)\|_{L^1(\Omega)} \\
\leq& e^{K(t-s)}\|u_0 - u_0^n + u_0^n - u^n(t-s) + u^n(t-s) - u(t-s)\|_{L^1(\Omega)} \\
\leq& e^{K(t-s)}\|u_0 - u_0^n\|_{L^1(\Omega)} + \|u_0^n - u^n(t-s)\|_{L^1(\Omega)} + \|u^n(t-s) - u(t-s)\|_{L^1(\Omega)} \\
\leq& \varepsilon.
\end{aligned}
$$

where $t-s$ small enough and n large enough. Using the analogous method of Theorem 2.3 in [42] for nonlinear term, we have $u(x,t)$ satisfy the equation (1.1) in sense of distributions. Actually, $u(x,t)$ is a approximate solution of (1.1).

To show the uniqueness and continuous dependence. We consider the following equation

$$v_t + (-\Delta)^{\sigma/2}(l_1(t)v) + l_2(t)v = 0 \tag{4.4}$$

where $v(t) = u_1(t) - u_2(t)$ is the difference between two solutions of the equation (1.1) and the l_i are the same as that in Lemma 3.9. We assume, in addition, that $u_i \in L^\infty(\Omega \times [0,T])$. Because of the absence of enough regularity for the equation, we can not multiply the equation by $sgn(v)$. Hence, in order to overcome the difficulty, we multiply the equation by the solution $w(t)$ of equation (4.1) and integrate over $\Omega \times [\delta, T]$. We have

$$(v(T), w(T)) - (v(0), w(0)) - \varepsilon \int_0^T ((-\Delta)^{\sigma/2} w(t), v(t))dt + \int_0^T (l_2(t)v(t), w(t))dt = 0. \tag{4.5}$$

Approximating the function $w_T^0 = sgn(v(T))$ in $L^2(\Omega)$ by $w_T^n \in H_0^{\sigma/2}(\Omega)$ and $-1 \leq w_T^n \leq 1$ and constructing the solution $w^n(t)$ of the equation (4.1), we infer that $-1 \leq w_T^n(x,t) \leq 1$ and

$$(v(T), w_T^n) - \varepsilon \int_0^T ((-\Delta)^{\sigma/2} w^n(t), v(t))dt \leq \|v(0)\|_{L^1(\Omega)} + L_2 \int_0^T \|v(t)\|_{L^1(\Omega)}dt \tag{4.6}$$

where $L_2 = \|l_2(x,t)\|_{L^\infty(\Omega\times[0,T])}$. Passing to the limit $\varepsilon \to 0$ and using the inequality (4.6) and

$$-\varepsilon\int_0^T((-\Delta)^{\sigma/2}w^n(t),v(t))dt \leq \varepsilon^{1/4}(\varepsilon^{1/2}\|(-\Delta)^{\sigma/2}w^n\|^2_{L^2(\Omega\times[0,T])}+\|v\|^2_{L^2(\Omega\times[0,T])}),$$

we have

$$(v(T),w_T^n) \leq \|v(0)\|_{L^1(\Omega)} + L_2\int_0^T\|v(t)\|_{L^1(\Omega)}dt. \tag{4.7}$$

We now pass to the limit $\varepsilon \to \infty$ in (4.7). Then, we conclude

$$\|v(T)\|_{L^1(\Omega)} \leq \|v(0)\|_{L^1(\Omega)} + L_2\int_0^T\|v(t)\|_{L^1(\Omega)}dt.$$

Thanks to the Gronwall inequality, we have

$$\|v(t)\|_{L^1(\Omega)} \leq e^{L_2t}\|v(0)\|_{L^1(\Omega)}. \tag{4.8}$$

Hence,we have prove the uniqueness of weak solution for the equation (1.1) under the additional assumption $u \in L^\infty(\Omega\times[0,T])$. Moreover, a weak solution is a approximate solution. Therefore, all weak solution satisfy (3.10). We now consider the general case, that is, u_i belong only to $L^\infty(\Omega\times[\delta,T])$ for every $\delta > 0$. Hence, we have

$$\|u_1(t)-u_2(t)\|_{L^1(\Omega)} \leq e^{K(t-\delta)}\|u_1(\delta)-u_2\delta\|_{L^1(\Omega)}.$$

Passing to the limit $\delta \to 0$, we conclude that all the weak solutions satisfy the inequality (3.10), that is, the weak solution is unique. In addition, it is imply that $u \in C([0,T],L^1(\Omega))$. □

We have obtained existence and uniqueness of weak solutions and their continuous dependence on initial conditions. Hence, we can define the operator semigroup $\{S(t)\}_{t\geq 0}$, which is continuous in $L^1(\Omega)$.

5. The infinite-dimensional global attractor

In this section, we consider the existence and dimension of the attractor for the semigroup $\{S(t)\}_{t\geq 0}$ in $L^1(\Omega)$. It is easy to show that there exists a bounded absorbing set of the semigroup in $L^1(\Omega)$ by the Theorem 4.5. We now prove that the semigroup has a compact absorbing set.

Theorem 5.1. *Let the condition* (1.2) *hold. Then the semigroup* $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set in* $W^{\frac{\sigma}{2m},2m}(\Omega)$*, that is, there exists a bounded set* $\mathfrak{B}$ *in* $W^{\frac{\sigma}{2m},2m}(\Omega)$*, such that for every bounded subset* B *in* $L^1(\Omega)$*, there exists a positive constant* $T_0 = T_0(\|B\|_{L^1})$*, such that*

$$S(t)B \subset \mathfrak{B}, \quad \textit{for every} \ \ t \geq T_0.$$

Proof. Let $T > 2$ and $T - \frac{1}{2} \geq \delta \geq T-1$. Multiplying the equation (3.1) by $u_\mu^m + \mu u_\mu$, integrating over $\Omega\times[\delta,T]$ and combining with (3.7), we get

$$\int_\delta^T\int_\Omega[(-\Delta)^{\sigma/4}(u_\mu^m+\mu u_\mu)]^2dxdt \leq C,$$

where C is independent of T and δ.
We multiply the equation (3.1) by $(t-\delta)(u_\mu^m+\mu u_\mu)_t$ and integrate over $\Omega\times[\delta,T]$. It implies that

$$\begin{aligned}&\int_\delta^T\int_\Omega(t-\delta)(m|u_\mu|^{m-1}+\mu)u_{\mu t}^2dxdt + \frac{1}{2}(T-\delta)\int_\Omega[(-\Delta)^{\sigma/4}(u_\mu^m(T)+\mu u_\mu(T))]^2dx\\&+(T-\delta)\int_\Omega G(u_\mu(T))dx\\&=\frac{1}{2}\int_\delta^T\int_\Omega[(-\Delta)^{\sigma/4}(u_\mu^m+\mu u_\mu)]^2dxdt + \int_\delta^T\int_\Omega G(u_\mu(t))dxdt\\&+(T-\delta)\int_\Omega h(u_\mu^m+\mu u_\mu)dx - \int_\delta^T\int_\Omega h(u_\mu^m+\mu u_\mu)dxdt,\end{aligned}$$

where $G(u_\mu) = \int_0^{u_\mu} g(s)(|s|^{m-1}+1)ds$. Applying (3.7) again, we have

$$\int_\Omega [(-\Delta)^{\sigma/4} u_\mu^m(T)]^2 dx \le C,$$

where C is independent of T and δ. Hence, let $u(t)$ be weak solution of equation (1.1). Then,

$$\int_\Omega [(-\Delta)^{\sigma/4} u^m(T)]^2 dx \le C.$$

Therefore,

$$\|u(t)\|_{W^{\frac{\sigma}{2m},2m}(\Omega)} \le C, \quad t > 2.$$

□

Based on the above theorem, we get the existence of the global attractor of the semigroup $\{S(t)\}_{t\ge 0}$ by Lemma 2.5.

Theorem 5.2. *Let the condition* (1.2) *hold. Then the semigroup* $\{S(t)\}_{t\ge0}$ *has a global attractor in* $L^1(\Omega)$.

We now investigate the dimension of the attractor by the Z_2 index theory. Let us consider the following equation.

$$\begin{cases} u_t + (-\Delta)^{\sigma/2}(|u|^{m-1}u) - |u|^{s-1}u + g(u) = 0, & (x,t) \in \Omega\times\mathbb{R}^+, \\ u(x,t) = 0, & (x,t)\in\partial\Omega\times\mathbb{R}^+, \\ u(x,0) = u_0, & x\in\Omega, \end{cases} \tag{5.1}$$

where $g(u)$ satisfies the condition (1.2), and g is an odd function. Assume that $1 \le s < min\{m,q\}$ and there is a constant $\alpha > s$ such that

$$\lim_{|\tau|\to 0} \frac{g(\tau)\tau}{|\tau|^{\alpha+1}} = 0. \tag{5.2}$$

Scince g is an odd function, applying (5.2) we get that $g(\tau)\tau \le C|\tau|^{\alpha+1}$, where τ small enough; and

$$G(s) := \int_0^s g(\tau)|\tau|^{m-1} d\tau \le \frac{C}{m+\alpha}|s|^{m+\alpha}. \tag{5.3}$$

Theorem 5.3. *Let g is an odd function and satisfies the condition* (1.2) *and* (5.2)*. Then, the semigroup* $\{S(t)\}_{t\ge0}$ *generated by problem* (5.1) *is odd and admits a global attractor* $\mathcal{A}$ *in* $L^1(\Omega)$*. Moreover, the global attractor* $\mathcal{A}$ *is symmetric, that is,* $-\mathcal{A} = \mathcal{A}$.

Proof. For every $u_0 \in L^1(\Omega)$, we know that $-u_0 \in L^1(\Omega)$. Let $u(t)$ be the weak solution of problem (5.1) with initial data u_0. Then, $-u(t)$ is the weak solution of problem (5.1) with initial data $-u_0 \in L^1(\Omega)$. Therefore, $S(t)(-u_0) = -u(t) = -S(t)u_0$, that is, $\{S(t)\}_{t\ge0}$ is odd.

It is easy to show that $-|u|^{s-1}u + g(u) \ge -C + k|u|^q$, where C, k are positive constants. Hence, the existence of a global attractor $\mathcal{A}$ is an immediate consequence of Theorem 5.2.

We now show that the attractor $\mathcal{A}$ is symmetric.

$$\mathcal{A} = \omega(B) = \cap_{\tau\ge0}\overline{\cup_{t\ge\tau}S(t)B},$$

where B is an absorbing set and $\omega(B)$ is its ω limit set. Indeed, let $B_0 = B(0,R) = \{u \in L^1(\Omega) : \|u\|_1 \le R\}$ be the symmetric absorbing set of semigroup $S(t)$. Assume $y \in \mathcal{A}$, then there exists a sequence $\{x_n\}_{n=1}^\infty \subset B_0$ and $t_n \to \infty$, such that

$$S(t_n)x_n \to y, \text{ in } L^1(\Omega).$$

Because $\{-x_n\}_{n=1}^\infty \subset B_0$ and $S(t)$ is odd, we have

$$S(t_n)(-x_n) = -S(t_n)x_n \to -y, \text{ in } L^1(\Omega).$$

Hence, $-y \in \mathcal{A}$ and $\mathcal{A}$ is symmetric. □

Lemma 5.4. *[28, 43] Let $\{S(t)\}_{t\geq 0}$ be an odd semigroup on a Banach space X, which possesses a symmetric global attractor $\mathcal{A}$. For any positive integer k, if there exists a symmetric set $B \subset X$ such that $\gamma(B) \geq k,\ 0 \notin \omega(B)$, then there exists a neighborhood $O(0)$ of 0 such that $\gamma(\mathcal{A}\backslash O(0)) \geq k$*

Theorem 5.5. *Let above assumption hold and let $u_0 \in L^1(\Omega)$. Then, problem (5.1) possesses a symmetric global attractor $\mathcal{A}$ in $L^1(\Omega)$, of which the fractal dimension is infinite.*

Proof. Applying Remark 2.7, we only need show that for every integer $k > 0$, there exists a neighborhood $O(0)$ of 0, such that

$$\gamma(\mathcal{A}\backslash O(0)) \geq k. \tag{5.4}$$

Actually, the inequality (5.4) implies that for every m, the odd continuous mapping from $\mathcal{A}\backslash O(0)$ to $\mathbb{R}^m \setminus \{0\}$ does not exists. Moreover, the homeomorphic mapping from $\mathcal{A}$ to a subset of $\mathbb{R}^m$ does not exists. In light of Remark 2.7, we can conclude that the dimension of $\mathcal{A}$ is infinite. To this end, thanks to Lemma 5.4, we need to find a symmetric set B_k such that $\gamma(B_k) \geq k$ and $\omega(B_k) \subset \mathcal{A}\backslash\{0\}$.

Consider the functional

$$\Phi(u) = \int_\Omega \left\{\frac{1}{2}[(-\Delta)^{\sigma/4}(|u|^{m-1}u)]^2 - \frac{1}{s+m}|u|^{s+m} + G(u)\right\} dx,$$

where $0 < s < min\{m, q\}$. We conclude that

$$\Phi(u(t_1)) - \Phi(u(t_2)) = -\int_{t_2}^{t_1}\int_\Omega (u_t^{\frac{m+1}{2}})^2 dxdt \leq 0.$$

Hence, for every $u_0 \in H_0^{\sigma/2}(\Omega) \cap L^\infty(\Omega)$, the function $\Phi(u(t))$ is nonincreasing with respect to t.

For every integer $k > 0$, let V_k be an k-dimensional subspace of $H_0^{\sigma/2}(\Omega) \cap L^\infty(\Omega)$. Setting $A_k = \{u \in V_k : \|(-\Delta)^{\sigma/4}u\|_{L^2(\Omega)} + \|u\|_{L^\infty(\Omega)} = 1\}$, then A_k is compact in $H_0^{\sigma/2}(\Omega) \cap L^\infty(\Omega)$, and there exists $\delta > 0$ such that

$$\inf_{u\in A_m} \|u\|^{m+s}_{L^{m+s}(\Omega)} = \delta.$$

Setting $\varepsilon A_k = \{\varepsilon u : u \in A_k\}$, then $\gamma(\varepsilon A_k) = \gamma(A_k) = k$. let $v = \varepsilon u \in \varepsilon A_k$, combine with (5.3), for every sufficiently small ε, we have

$$\begin{aligned}
\Phi(v) &= \int_\Omega \left\{\frac{1}{2}[(-\Delta)^{\sigma/4}(|v|^{m-1}v)]^2 - \frac{1}{s+m}|v|^{s+m} + G(v)\right\} dx \\
&\leq \frac{1}{2}\int_{C_\Omega} y^{1-\sigma}|E(v)|^{2(m-1)}(\nabla v)^2 dxdy - \frac{1}{s+m}\int_\Omega |v|^{s+m}dx + \frac{C}{m+\alpha}\int_\Omega |v|^{m+\alpha}dx \\
&\leq \frac{\varepsilon^{2(m-1)}}{2}\|v\|^2_{H_0^{\sigma/2}(\Omega)} - \frac{\delta}{s+m}\varepsilon^{s+m} + \frac{C}{m+\alpha}\varepsilon^{m+\alpha} \\
&\leq \frac{1}{2}\varepsilon^{2m} - \frac{\delta}{s+m}\varepsilon^{s+m} + \frac{C}{m+\alpha}\varepsilon^{m+\alpha}.
\end{aligned}$$

Since $s < min\{m, \alpha\}$, for every $v = \varepsilon A_k$, we have $\Phi(v) < 0$. Owing to $\Phi(0) = 0$ and the function $\Phi(u(t))$ is nonincreasing with respect to t, we infer that $\omega(\varepsilon A_k) \subset \mathcal{A}\backslash\{0\}$. Applying Lemma 5.4 and Z_2 index theory, we have

$$\gamma(\mathcal{A}\backslash O(0)) \geq k.$$

□

Remark 5.6. *In the case of fractional diffusion non-degenerate equations ($m = 1$) in bounded domains, it is easy to get the finite dimensionality of the attractor by estimating the Kolmogorov entropy. However, for $m > 1$, the dimension of attractor is infinite in Theorem 5.5. It is because the power of the first term (that is equivalent to $2m$) of the functional Φ is greater than the second term ($s + m$). The coefficient of the second term is negative. Hence, the functional is decreasing and then increasing in the domain of the original point. Moreover, for every eigenvalue of $(-\Delta)^{\sigma/2}$, there exists a complete bounded orbit which lie in attractor. The complete bounded orbits are different from each other. Accordingly, the dimension of attractor is infinite.*

Acknowledgements

C. Zhang was supproted by NSFC(11801228) and the dual creative (innovative and entrepreneurial) talents project in Jiangsu Province. F. Meng was supported by NSFC(11701230) and QingLan Project, Jiangsu Overseas visiting scholar Program for University Prominent Young Middle aged Teachers. C. Liu was supported by the Natural Science Foundation of Jiangsu Province (Grants No BK20170308) and NSFC(11801227).

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