# Several Turán-type inequalities for the hypergeometric superhyperbolic functions

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**Abstract**: In this paper, we investigate some Turán-type inequalities for the hypergeometric superhyperbolic sine and cosine functions associated with Kummer confluent hypergeometric series of first type.

**Key words:** Cauchy-Bunyakovsky-Schwarz inequality; Turán-type inequality; gamma function; beta function; Kummer confluent hypergeometric series of first type.

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## 1. Introduction

It is well-known that Turán-type inequality plays an important role in mathematics and physics [1, 2]. Many experts have done considerable research on Turán-type inequality [3–10].

Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$ be the sets of the complex, real, and natural numbers, respectively.

The generalized Cauchy-Bunyakovsky-Schwarz ($CBS$) inequality is defined as [11]

$$\{\int_a^b [\alpha(t)]^\eta[\beta(t)]^\eta[\gamma(t)]^\eta dt\}^2 \le \{\int_a^b [\alpha(t)]^{\eta-r}[\beta(t)]^{\eta-s}[\gamma(t)]^{\eta-o}dt\}\{\int_a^b [\alpha(t)]^{\eta+r}[\beta(t)]^{\eta+s}[\gamma(t)]^{\eta+o}dt\}, \tag{1}$$

where $\eta$, $r$, $s$, $o \in \mathbb{R}$, $\alpha$, $\beta$, and $\gamma$ are real integrable functions.

For $p, q, z \in \mathbb{C}$, $k \in \mathbb{N}$, $\{\Re(q), \Re(p)\} > 0$, and $\Re(q) > \Re(p)$, Kummer confluent hypergeometric series of first type is given by [12, 13]

$${}_1F_1(p;q;z) = \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)}\int_0^1 t^{p-1}(1-t)^{q-p-1}e^{zt}dt, \tag{2}$$

where

$$(p)_k = \frac{\Gamma(p+k)}{\Gamma(p)}, \tag{3}$$

is Pochhammer's symbol (see [14], p.8-9).

In this paper, it is important that we prove the following theorems:

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**Theorem 1.** *If $p, q, z \in \mathbb{C}$, $\lambda \in \mathbb{R}$, $\{\Re(q), \Re(p)\} > 0$, $\Re(q) > \Re(p)$, $p > |\chi|$, $q - p > |\varpi|$, and $|z| \leq 1$, then*

$$\begin{aligned}[{}_1Supersinh_1(p;q;\lambda z)]^2 \leq & \frac{B(p-\chi, q-p-\varpi)B(p+\chi, q-p+\varpi)}{[B(p,q-p)]^2} \\ & \times {}_1Supersinh_1[p-\delta; q-(\chi+\varpi); \lambda z] \\ & \times {}_1Supersinh_1[p+\delta; q+(\chi+\varpi); \lambda z],\end{aligned} \tag{4}$$

*where the hypergeometric superhyperbolic sine based on Kummer confluent hypergeometric series of first type is defined as (see [15], p.257)*

$${}_1Supersinh_1(p;q;\lambda z) = \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \int_0^1 t^{p-1}(1-t)^{q-p-1}\frac{1}{2}(e^{\lambda zt} - e^{-\lambda zt})dt. \tag{5}$$

**Theorem 2.** *Let $p, q, z \in \mathbb{C}$, $\lambda \in \mathbb{R}$, $\{\Re(q), \Re(p)\} > 0$, $\Re(q) > \Re(p)$, $p > |\psi|$, $q - p > |\beta|$, and $|z| \leq 1$. Then*

$$\begin{aligned}[{}_1Supercosh_1(p;q;\lambda z)]^2 \leq & \frac{B(p-\psi, q-p-\beta)B(p+\psi, q-p+\beta)}{[B(p,q-p)]^2} \\ & \times {}_1Supercosh_1[p-\vartheta; q-(\psi+\beta); \lambda z] \\ & \times {}_1Supercosh_1[p+\vartheta; q+(\psi+\beta); \lambda z],\end{aligned} \tag{6}$$

*where the hypergeometric superhyperbolic cosine via Kummer confluent hypergeometric series of first type can be expressed as (also see [15], p.258)*

$${}_1Supercosh_1(p;q;\lambda z) = \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \int_0^1 t^{p-1}(1-t)^{q-p-1}\frac{1}{2}(e^{\lambda zt} + e^{-\lambda zt})dt. \tag{7}$$

The aim of this paper is to study of Turán-type inequalities for the hypergeometric superhyperbolic sin and cosine functions via Kummer confluent hypergeometric series of first type.

The structure of the paper is as follows. In Section 2, we prove Theorem 1. In Section 3, we give the proof of Theorem 2.

# 2. Proof of Theorem 1

Let us prove Theorem 1.

Let $\alpha(t) = t^{p-1}, \beta(t) = (1-t)^{q-p-1}$, and $\gamma(t) = \frac{1}{2}(e^{\lambda zt} - e^{-\lambda zt})$. Then (5) can be shown as

$$\begin{aligned}\frac{\Gamma(p)\Gamma(q-p)}{\Gamma(q)}\, {}_1Supersinh_1(p;q;\lambda z) &= \frac{1}{2}\int_0^1 t^{p-1}(1-t)^{q-p-1}(e^{\lambda zt} - e^{-\lambda zt})dt \\ &= \int_0^1 \alpha(t)\beta(t)\gamma(t)dt.\end{aligned} \tag{8}$$

According to (1), we have

$$\begin{aligned}\{\int_0^1 t^{\sigma(p-1)}(1-t)^{\sigma(q-p-1)}(e^{\lambda zt} - e^{-\lambda zt})^{\sigma}dt\}^2 \leq & \int_0^1 t^{(\sigma-r)(p-1)}(1-t)^{(\sigma-s)(q-p-1)}(e^{\lambda zt} - e^{-\lambda zt})^{\sigma-o}dt \\ & \times \int_0^1 t^{(\sigma+r)(p-1)}(1-t)^{(\sigma+s)(q-p-1)}(e^{\lambda zt} - e^{-\lambda zt})^{\sigma+o}dt.\end{aligned} \tag{9}$$

By (8) and (9), we obtain

$$\begin{aligned}[\,_1Supersinh_1(\sigma p;\sigma(q-2)+2;\lambda z)]^2 \leq & \frac{\{\Gamma[\sigma(q-2)+2]\}^2}{\{\Gamma[\sigma(p-1)+1]\}^2\{\Gamma[\sigma(p-q-1)+1]\}^2} \\ & \times \frac{\Gamma[(\sigma-r)(p-1)+1]\Gamma[(\sigma-s)(q-p-1)+1]}{\Gamma[(\sigma-r)(p-1)+(\sigma-s)(q-p-1)+2]} \\ & \times \frac{\Gamma[(\sigma+r)(q-1)+1]\Gamma[(\sigma+s)(q-p-1)+1]}{\Gamma[(\sigma+r)(p-1)+(\sigma+s)(q-p-1)+2]} \\ & \times \,_1Supersinh_1[(\sigma-o)p;(\sigma-s)(q-p-1)+2;\lambda z] \\ & \times \,_1Supersinh_1[(\sigma+o)p;(\sigma+s)(q-p-1)+2;\lambda z]. \end{aligned} \tag{10}$$

Furthermore, if $g_1=\sigma(p-1)+1$, $g_2=\sigma(q-p-1)+1$, $g_3=\sigma p$, $h_1=r(p-1)$, $h_2=s(q-p-1)$, and $h_3=op$, then (10) can be expressed as

$$\begin{aligned}[\,_2Supersinh_1(g_3;g_1+g_2;\lambda z)]^2 \leq & \frac{[\Gamma(g_1+g_2)]^2}{[\Gamma(g_1)]^2[\Gamma(g_2)]^2}\frac{\Gamma(g_1-h_1)\Gamma(g_2-h_2)}{\Gamma[(g_1-h_1)+(g_2-h_2)]}\frac{\Gamma(g_1+h_1)\Gamma(g_2+h_2)}{\Gamma[(g_1+h_1)+(g_2+h_2)]} \\ & \times \,_2Supersinh_1[g_3-h_3;(g_1-h_1)+(g_2-h_2);\lambda z] \\ & \times \,_2Supersinh_1[g_3+h_3;(g_1+h_1)+(g_2+h_2);\lambda z]. \end{aligned} \tag{11}$$

Thus, by gamma and beta functions [13]

$$B(s,o)=\frac{\Gamma(s)\Gamma(o)}{\Gamma(s+o)} \quad s,o>0, \tag{12}$$

Turán-type inequality is given by

$$\begin{aligned}[\,_1Supersinh_1(g_3,g_1;g_1+g_2;\lambda z)]^2 \leq & \frac{B(g_1-h_1,g_2-h_2)B(g_1+h_1,g_2+h_2)}{[B(g_1,g_2)]^2} \\ & \times \,_1Supersinh_1[g_3-h_3,g_1-h_1;(g_1-h_1)+(g_2-h_2);\lambda z] \\ & \times \,_1Supersinh_1[g_3+h_3,g_1+h_1;(g_1+h_1)+(g_2+h_2);\lambda z], \end{aligned} \tag{13}$$

for $g_1>|h_1|$, $g_2>|h_2|$, and $|z|\leq 1$.

Meanwhile, if $\sigma=1$, then (10) can be shown as

$$\begin{aligned}[\,_1Supersinh_1(p;q;\lambda z)]^2 \leq & \frac{[\Gamma(q)]^2}{[\Gamma(p)]^2[\Gamma(q-p)]^2}\frac{\Gamma[p-r(p-1)]\Gamma[(q-p)-s(q-p-1)]}{\Gamma[q-r(p-1)-s(q-p-1)]} \\ & \times \frac{\Gamma[p+r(p-1)]\Gamma[(q-p)+s(q-p-1)]}{\Gamma[q+r(p-1)+s(q-p-1)]} \\ & \times \,_1Supersinh_1[(1-o)p;q-r(p-1)-s(q-p-1);\lambda z] \\ & \times \,_1Supersinh_1[(1+o)p;q+r(p-1)+s(q-p-1);\lambda z]. \end{aligned} \tag{14}$$

Finally, suppose $r(p-1)=\chi$, $s(q-p-1)=\varpi$, $op=\delta$, $q-p>|\varpi|$, and $|z|\leq 1$. Then

$$\begin{aligned}[\,_1Supersinh_1(p;q;\lambda z)]^2 \leq & \frac{B(p-\chi,q-p-\varpi)B(p+\chi,q-p+\varpi)}{[B(p,q-p)]^2} \\ & \times \,_1Supersinh_1[p-\delta;q-(\chi+\varpi);\lambda z] \\ & \times \,_1Supersinh_1[p+\delta;q+(\chi+\varpi);\lambda z]. \end{aligned} \tag{15}$$

Therefore, (4) can be proved.

## 3. Proof of Theorem 2

Now, we concentrate on the proof of Theorem 2.

Let $\alpha(t) = t^{p-1}$, $\beta(t) = (1-t)^{q-p-1}$, and $\gamma(t) = \frac{1}{2}(e^{\lambda zt} + e^{-\lambda zt})$. Then

$$\begin{aligned}\frac{\Gamma(p)\Gamma(q-p)}{\Gamma(q)}\, {}_1Supercosh_1(p;q;\lambda z) &= \frac{1}{2}\int_0^1 t^{p-1}(1-t)^{q-p-1}(e^{\lambda zt}+e^{-\lambda zt})dt \\ &= \int_0^1 \alpha(t)\beta(t)\gamma(t)dt.\end{aligned} \tag{16}$$

Based on (1), we have

$$\begin{aligned}[\int_0^1 t^{\eta(p-1)}(1-t)^{\eta(q-p-1)}(e^{\lambda zt}+e^{-\lambda zt})^{\eta}dt]^2 \le &\int_0^1 t^{(\eta-r)(p-1)}(1-t)^{(\eta-s)(q-p-1)}(e^{\lambda zt}+e^{-\lambda zt})^{\eta-o}dt \\ &\times \int_0^1 t^{(\eta+r)(p-1)}(1-t)^{(\eta+s)(q-p-1)}(e^{\lambda zt}+e^{-\lambda zt})^{\eta+o}dt.\end{aligned} \tag{17}$$

By (16) and (17), we get

$$\begin{aligned}[{}_1Supercosh_1(\eta p;\eta(q-2)+2;\lambda z)]^2 \le &\frac{\{\Gamma[\eta(q-2)+2]\}^2}{\{\Gamma[\eta(p-1)+1]\}^2\{\Gamma[\eta(q-p-1)+1]\}^2} \\ &\times \frac{\Gamma[(\eta-r)(p-1)+1]\Gamma[(\eta-s)(q-p-1)+1]}{\Gamma[(\eta-r)(p-1)+(\eta-s)(q-p-1)+2]} \\ &\times \frac{\Gamma[(\eta+r)(p-1)+1]\Gamma[(\eta+s)(q-p-1)+1]}{\Gamma[(\eta+r)(p-1)+(\eta+s)(q-p-1)+2]} \\ &\times {}_1Supercosh_1[(\eta-o)p;(\eta-r)(p-1)+(\eta-s)(q-p-1)+2;\lambda z] \\ &\times {}_1Supercosh_1[(\eta+o)p;(\eta+r)(p-1)+(\eta+s)(q-p-1)+2;\lambda z].\end{aligned} \tag{18}$$

After that, let $j_1 = \eta(p-1)+1$, $j_2 = \eta(q-p-1)+1$, $j_3 = \eta p$, $k_1 = r(p-1)$, $k_2 = s(q-p-1)$, and $k_3 = op$. Then we obtain

$$\begin{aligned}[{}_1Supercosh_1(j_3;j_1+j_2;\lambda z)]^2 \le &\frac{[\Gamma(j_1+j_2)]^2}{[\Gamma(j_1)]^2[\Gamma(j_2)]^2}\frac{\Gamma(j_1-k_1)\Gamma(j_2-k_2)}{\Gamma[(j_1-k_1)+(j_2-k_2)]}\frac{\Gamma(j_1+k_1)\Gamma(j_2+k_2)}{\Gamma[(j_1+k_1)+(j_2+k_2)]} \\ &\times {}_1Supercosh_1[j_3-k_3;(j_1-k_1)+(j_2-k_2);\lambda z] \\ &\times {}_1Supercosh_1[j_3+k_3;(j_1+k_1)+(j_2+k_2);\lambda z].\end{aligned} \tag{19}$$

What's more, for $j_1 > |k_1|$, $j_2 > |k_2|$, and $|z| \le 1$, by (12), we have

$$\begin{aligned}[{}_1Supercosh_1(j_3;j_1+j_2;\lambda z)]^2 \le &\frac{B(j_1-k_1,j_2-k_2)B(j_1+k_1,j_2+k_2)}{[B(j_1,j_2)]^2} \\ &\times {}_1Supercosh_1[j_3-k_3;(j_1-k_1)+(j_2-k_2);\lambda z] \\ &\times {}_1Supercosh_1[j_3+k_3;(j_1+k_1)+(j_2+k_2);\lambda z].\end{aligned} \tag{20}$$

Afterwards, if $\eta = 1$, then (18) can be shown as

$$\begin{aligned}
[{}_1Supercosh_1(p;q;\lambda z)]^2 \leq & \frac{[\Gamma(q)]^2}{[\Gamma(p)]^2[\Gamma(q-p)]^2} \frac{\Gamma[p-r(p-1)]\Gamma[(q-p)-s(q-p-1)]}{\Gamma[q-r(p-1)-s(q-p-1)]} \\
& \times \frac{\Gamma[p+r(p-1)]\Gamma[(q-p)+s(q-p-1)]}{\Gamma[q+r(p-1)+s(q-p-1)]} \\
& \times {}_1Supercosh_1[(1-o)p; q-r(p-1)-s(q-p-1); \lambda z] \\
& \times {}_1Supercosh_1[(1+o)p; q+r(p-1)+s(q-p-1); \lambda z].
\end{aligned} \tag{21}$$

Lastly, for $r(p-1) = \psi$, $s(q-p-1) = \beta$, $op = \vartheta$, $q-p > |\beta|$, and $|z| \leq 1$, by (12), (21) can be written as

$$\begin{aligned}
[{}_1Supercosh_1(p;q;\lambda z)]^2 \leq & \frac{B(p-\psi, q-p-\beta)B(p+\psi, q-p+\beta)}{[B(p,q-p)]^2} \\
& \times {}_1Supercosh_1[p-\vartheta; q-(\psi+\beta); \lambda z] \\
& \times {}_1Supercosh_1[p+\vartheta; q+(\psi+\beta); \lambda z].
\end{aligned} \tag{22}$$

Thus, (6) is true.

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## Conflict of Interest

There is no conflict of interest in this work.

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