

Fractional Schrödinger–Poisson systems with indefinite potentials*

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Abstract

This paper is devoted to the following fractional Schrödinger–Poisson systems:

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi(x)u = f(x, u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi(x) = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $(-\Delta)^s$ is the fractional Laplacian, $s, t \in (0, 1)$, $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous. In contrast to most studies, we consider that the potentials V is indefinite. With the help of Morse theory, the existence of nontrivial solutions for the above problem is obtained.

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1 Introduction

In this paper, we are concerned with the existence solutions for the fractional Schrödinger–Poisson systems:

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi(x)u = f(x, u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi(x) = u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.1)$$

where $s, t \in (0, 1)$ satisfy $2t + 4s > 3$, the potential function $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous. $(-\Delta)^s$ is the fractional Laplace operator which, up to a normalization constant, is defined as

$$(-\Delta)^s \varphi(x) = 2 \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B_\varepsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x - y|^{3+2s}} dy, \quad x \in \mathbb{R}^3,$$

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along functions $\varphi \in C_0^\infty(\mathbb{R}^3)$, $B_\varepsilon(x)$ denotes the ball of \mathbb{R}^3 centered at $x \in \mathbb{R}^3$ and radius $\varepsilon > 0$. For details on the introduction to fractional Laplace operator we can refer the readers to [13, 18].

(1.1) can be seen as the fractional form of the following systems:

$$\begin{cases} -\Delta u + V(x)u + \phi(x)u = f(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi(x) = u^2 & \text{in } \mathbb{R}^3. \end{cases} \quad (1.2)$$

Equations (1.2) was introduced by Benci and Fortunato [3] as a physical model describing solitary waves for nonlinear Schrödinger type equations interacting with an unknown electrostatic field. The first equation of (1.2) is coupled with a Poisson equation, which means that the potential is determined by the charge of the wave function. The term ϕu is nonlocal and concerns the interaction with the electric field.

It is well known that many researchers have devoted to the existence and multiplicity of solutions for the system like (1.1) via critical point theory under various assumptions on the potential V and the nonlinearity; for example, see [11]. In [19], Zhang et al. considered the following fractional Schrödinger–Poisson systems:

$$\begin{cases} (-\Delta)^s u + \lambda \phi u = g(u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = \lambda u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ and g satisfies subcritical or critical growth conditions. Moreover, by using a perturbation approach, the authors in [19] obtained the existence of positive solutions for small λ and studied the asymptotic of solutions for $\lambda \rightarrow 0^+$.

Recently, many researchers have devoted themselves to the systems with many potentials (see [2, 5, 14, 15]). In [14], Pucci et al. established Bartsch–Wang type compact embedding theorem for the fractional Sobolev spaces, then they obtained multiple solutions for nonhomogeneous Schrödinger–Kirchhoff type equations involving the fractional p -Laplacian in \mathbb{R}^N by using the Ekeland variational principle and the Mountain Pass theorem. Tang and Lan in [15] established the existence of solutions to a class of p -Kirchhoff equations via morse theory. It should be noted that the authors in all these papers only considered the case where the Schrödinger operator $-\Delta + V$ is positive definite. In this point, the zero function $u = 0$ is a local minimizer of I and one may apply the Mountain Pass theorem [1] to obtain the results. So, a natural question is how about the existence result if V may be negative somewhere. Here, we give a positive answer to this problem in Theorem 2.1.

In order to overcome the lack of compactness of the Sobolev embedding

$$H^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3),$$

we will apply the method in [7], roughly speaking, V is coercive so that the work space is compactly embedding into $L^2(\mathbb{R}^3)$, then the local linking theory of Li and Willem [10] can be applied to obtain critical points of I .

2 Preliminaries and main results

In this section, we outline the variational framework and summarize some properties of the nonlocal term ϕ_u appearing in (1.1).

The fractional Sobolev space $H^s(\mathbb{R}^3)$ can be described by means of the Fourier transform as follows

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2s} + 1) |\hat{u}|^2 d\xi < \infty \right\},$$

which is endowed with the standard scalar product and norm

$$(u, v) = \int_{\mathbb{R}^3} (|\xi|^{2s} + 1) \hat{u} \bar{\hat{v}} d\xi, \quad \|u\|_{H^s}^2 = \int_{\mathbb{R}^3} (|\xi|^{2s} + 1) |\hat{u}|^2 d\xi.$$

In view of Plancherel's theorem, we have

$$(u, v) = \int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + uv) dx, \quad \|u\|_{H^s}^2 = \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + |u|^2) dx.$$

The homogeneous fractional Sobolev space $D^s(\mathbb{R}^3) = \{u \in L^{2^*_s}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}|^2 d\xi < \infty\}$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $[u]_s^2$, where

$$[u]_s^2 = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dx dy.$$

It is well known that $H^s(\mathbb{R}^3)$ is continuously embedded into $L^p(\mathbb{R}^3)$ for $2 \leq p \leq 2^*_s$, and for any $s \in (0, 1)$, there exists a best constant $S_s > 0$ of such that

$$S_s = \inf_{u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^{2^*_s} dx \right)^{2/2^*_s}}. \quad (2.1)$$

We know that problem (1.1) can be reduced to a single equation with a nonlocal term. Since $s, t \in (0, 1)$ satisfy $2t + 4s > 3$, there holds $12/(3 + 2t) < 6/(3 - 2s)$ and thus $H^s(\mathbb{R}^3) \hookrightarrow L^{12/(3+2t)}(\mathbb{R}^3)$. For all $u \in H^s(\mathbb{R}^3)$, let us define the linear functional L_u by $D^t(\mathbb{R}^3)$

$$L_u(v) = \int_{\mathbb{R}^3} u^2 v dx, \quad \forall v \in D^t(\mathbb{R}^3).$$

Then, by the Hölder inequality and (2.1), there exist $C_1, C_2 > 0$ such that

$$\begin{aligned} |L_u(v)| &\leq \left(\int_{\mathbb{R}^3} |u(x)|^2 \frac{6}{3+2t} dx \right)^{\frac{3+2t}{6}} \left(\int_{\mathbb{R}^3} |v(x)|^2 \frac{6}{3-2t} dx \right)^{\frac{3-2t}{6}} \\ &\leq C_1 S_t^{-\frac{1}{2}} \|u\|_{H^t}^2 [v]_t = C_2 \|u\|_{H^t}^2 [v]_t. \end{aligned} \quad (2.2)$$

Therefore, according the Lax-Milgram theorem, for every $u \in H^t(\mathbb{R}^3)$, there exists a unique $\phi_u^t \in D^t(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} u^2 v dx = \int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi_u^t \cdot (-\Delta)^{\frac{t}{2}} v dx \quad (2.3)$$

for any $v \in D^t(\mathbb{R}^3)$, that is, ϕ_u^t is a weak solution of

$$(-\Delta)^t \phi_u^t = u^2 \quad \text{in } \mathbb{R}^3,$$

and the representation formula

$$\phi_u^t = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} dy = c_t \frac{1}{|x|^{3-2t}} * u^2, \quad \forall x \in \mathbb{R}^3 \quad (2.4)$$

holds, which is called t -Riesz potential, where

$$c_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(3-2t)}{\Gamma(t)}.$$

We can get $\phi_u^t > 0$ for all $u \neq 0$. Moreover, by (2.2), (2.3) and the Sobolev inequality, there exist $C_3, C_4, C_5 > 0$ such that

$$\begin{aligned} [\phi_u^t]_s &\leq C_3 \|u\|_{H^s}^2, \\ \|\phi_u^t\|_{L^{2^*_s}(\mathbb{R}^3)} &\leq C_4 [\phi_u^t]_s, \end{aligned} \quad (2.5)$$

and

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x-y|^{3-2t}} u^2(x) u^2(y) dx dy = \int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq C_5 \|u\|_{H^t}^4. \quad (2.6)$$

Substituting ϕ_u^t in problem (1.1), we obtain the following single fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u + \phi_u^t(x)u = f(x, u) \quad \text{in } \mathbb{R}^3. \quad (2.7)$$

Obviously, solutions of problem (2.7) can be obtained by looking for critical points of the functional $I : H^s(\mathbb{R}^3) \rightarrow \mathbb{R}$ defined by

$$I(u) = \frac{1}{2} \left([u]_s^2 + \int_{\mathbb{R}^3} V(x)|u|^2 dx \right) + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t(x) u^2 dx - \int_{\mathbb{R}^3} F(x, u) dx,$$

where $F(\xi) := \int_0^\xi f(\tau) d\tau$. In addition, (2.5) and (2.6) imply that I is a well-defined \mathcal{C}^1 functional, and for all $v \in H^s(\mathbb{R}^3)$, we get

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^3} [(-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x)uv] dx + \int_{\mathbb{R}^3} \phi_u^t uv dx - \int_{\mathbb{R}^3} f(x, u)v dx.$$

Hence, if $u \in H^s(\mathbb{R}^3)$ is a critical point of I , then the pair (u, ϕ_u^t) , with ϕ_u^t as in (2.4), is a solution of problem (1.1).

Let us define the operator $\Phi : H^s(\mathbb{R}^3) \rightarrow D^t(\mathbb{R}^3)$ as follows: $\Phi[u] = \phi_u^t$. In the next lemma we summarize some properties of Φ , which is useful for the study of our problem.

Lemma 2.1. (see [16, 19]) *For any $u \in H^s(\mathbb{R}^3)$, we have*

- (1) Φ is continuous;

- (2) Φ maps bounded sets into bounded sets;
- (3) If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$ then $\Phi[u_n] \rightharpoonup \Phi[u]$ in $D^s(\mathbb{R}^3)$;
- (4) $\Phi[\varsigma u] = \varsigma^2 \Phi[u]$ for all $\varsigma \in \mathbb{R}$;
- (5) If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$ and $u_n \rightarrow u$ in $L^r(\mathbb{R}^3)$ for $2 \leq r < 2_s^*$, then

$$\int_{\mathbb{R}^3} \phi_{u_n}^t(x) u_n v dx \rightarrow \int_{\mathbb{R}^3} \phi_u^t(x) u v dx \text{ for all } v \in H^s(\mathbb{R}^3),$$

and

$$\int_{\mathbb{R}^3} \phi_{u_n}^t(x) u_n^2 dx \rightarrow \int_{\mathbb{R}^3} \phi_u^t(x) u^2 dx.$$

In this paper, motivated by [7, 9, 12], we consider the case where the potential V is indefinite when the compact embedding may not be true. We make the following assumption on the potential V .

(V) $V \in \mathcal{C}(\mathbb{R}^3)$ is bounded such that the quadratic form

$$Q(u) = \left([u]_s^2 + \int_{\mathbb{R}^3} V(x) |u|^2 dx \right)$$

is nondegenerate and the negative space of Q is finite-dimensional.

We may choose an equivalent norm $\|\cdot\|$ on $E = H^s(\mathbb{R}^3)$ such that

$$Q(u) = \|u^+\|^2 - \|u^-\|^2, u = u^+ + u^-, u^\pm \in E^\pm,$$

where E^+ and E^- are the positive and negative spaces of Q respectively, $E = E^+ \oplus E^-$. For example, $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a continuous function such that

$$\lim_{|x| \rightarrow \infty} V(x) = \nu, \quad 0 < \nu < +\infty$$

and 0 does not happen to be a spectrum point of the Schrödinger operator, then V satisfies our assumption.

We assume that f satisfies:

(f₁) $f \in \mathcal{C}(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and there exist C_6 and $p \in (4, 2_s^*)$ such that

$$|f(x, \tau)| \leq C_6(1 + |\tau|^{p-1}).$$

(f₂) $f(x, \tau) = o(|\tau|)$ as $\tau \rightarrow 0$ uniformly in $x \in \mathbb{R}^3$ and

$$\lim_{|\tau| \rightarrow \infty} \frac{F(x, \tau)}{\tau^4} = +\infty, \text{ for all } x \in \mathbb{R}^3. \quad (2.8)$$

(f₃) $0 < 4F(x, \tau) \leq \tau f(x, \tau)$ for $x \in \mathbb{R}^3$ and $\tau \neq 0$.

(f_4) for any $r > 0$, we have

$$\lim_{|x| \rightarrow \infty} \sup_{0 < |\tau| \leq r} \left| \frac{f(x, \tau)}{\tau} \right| = 0.$$

Remark 2.1. If $a : \mathbb{R}^3 \rightarrow (0, \infty)$ be continuous, $\lim_{|x| \rightarrow \infty} a(x) = 0$, $p \in (4, \frac{6}{3-2s})$. We consider $f(x, \tau) = a(x)|\tau|^{p-2}\tau$, it is clear that $f(x, \tau)$ satisfies $(f_1) - (f_4)$.

We are now ready to recall some concepts and results from infinite-dimensional Morse theory [4].

Let Y be a real Banach space and $\mathcal{J} \in C^1(Y, \mathbb{R})$, $\mathcal{K} = \{u \in Y : \mathcal{J}'(u) = 0\}$. Then the q th critical group of \mathcal{J} at an isolated critical point $u \in \mathcal{K}$ with $\mathcal{J}(u) = c$ is defined by

$$C_q(\mathcal{J}, u) := H_q(\mathcal{J}^c \cap \tilde{U}, \mathcal{J}^c \cap \tilde{U} \setminus \{u\}), \quad q \in \mathbb{N} := \{0, 1, 2, \dots\},$$

where $\mathcal{J}^c = \{u \in Y : \mathcal{J}(u) \leq c\}$, \tilde{U} is any neighborhood of u , containing the unique critical point, H_q is the singular relative homology with coefficients in an Abelian group G .

If \mathcal{J} satisfies the $(PS)_c$ -condition and the critical values of \mathcal{J} are bounded from below by some $a < \inf \mathcal{J}(\mathcal{K})$, then the critical groups of \mathcal{J} at infinity were introduced by Bartsch and Li [2] as

$$C_q(\mathcal{J}, \infty) := H_q(X, \mathcal{J}^a), \quad q \in \mathbb{N}. \quad (2.9)$$

If \mathcal{J} satisfies the $(PS)_c$ -condition, then \mathcal{J} satisfies the deformation condition. By the deformation lemma, the right-hand side of (2.9) does not depend on the choice of a .

Proposition 2.1. (see [2]) If $\mathcal{J} \in C^1(X, \mathbb{R})$ satisfies the $(PS)_c$ -condition and $C_k(\mathcal{J}, 0) \neq C_k(\mathcal{J}, \infty)$ for some $k \in \mathbb{N}$, then \mathcal{J} has a nontrivial critical point.

Proposition 2.2. (see [8, Proposition 2.1]) Assume that \mathcal{I} has a critical point $u = 0$ with $\mathcal{I}(0) = 0$. Suppose that \mathcal{I} has a local linking at 0 with respect to $Y = U \oplus W$, $k = \dim U < \infty$, that is, there exists $\rho > 0$ small such that

$$\begin{cases} \mathcal{I}(u) \leq 0, & \text{for } u \in U, \quad \|u\| \leq \rho, \\ \mathcal{I}(u) > 0, & \text{for } u \in W, \quad 0 < \|u\| \leq \rho. \end{cases}$$

Then $C_k(\mathcal{I}, 0) \neq 0$.

We are ready to state our main results.

Theorem 2.1. Let $t \in (0, 1)$, $2t + 4s > 3$. Assume that (V) , $(f_1) - (f_4)$ hold. Then systems (1.1) has a nontrivial solution.

We know that if the quadratic form Q is indefinite, it will always be more difficult to prove the boundedness of Palais-Smale (PS) sequences. In [6] this is done by taking advantage of the compact embedding. In present paper, the related Sobolev embedding $H^s(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ is not compact. To overcome this difficulty, we use the method in [7].

As in [6], if $f(x, t)$ is odd, we can gain an unbounded sequence of solutions.

Theorem 2.2. *Let $t \in (0, 1)$, $2t + 4s > 3$. Assume that (V) , $(f_1) - (f_4)$ hold. Moreover, if $f(x, \cdot)$ is odd, then systems (1.1) has a sequence of solutions such that $I(u_n, \phi_n) \rightarrow +\infty$.*

The paper is organized as follows. In section 2 we recall the results in Morse theory [4] and outline the variational framework. In section 3 we prove that I satisfies the $(PS)_c$ -condition. After setting up the $(PS)_c$ -condition, the proof of Theorem 2.2 is similar to that of (see [6, Theorem 1.2]), so we will not repeat it.

3 Some technical lemmas

Note that with the equivalent norm on E mentioned in Section 2, functional I can be rewritten as

$$I(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\mathbb{R}^3} F(x, u) dx, \quad (3.1)$$

where u^\pm denotes the orthogonal projection of u on E^\pm .

Lemma 3.1. *Assume (V) , $(f_1) - (f_4)$ hold, let $\{u_n\}$ be a $(PS)_c$ -sequence of I , that is,*

$$I(u_n) \rightarrow c, \quad I'(u_n) \rightarrow 0.$$

Then $\{u_n\}$ is bounded in E .

Proof. By contradiction, we assume $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Let $w_n = \frac{u_n}{\|u_n\|}$. Then

$$w_n = w_n^+ + w_n^- \rightharpoonup w = w^+ + w^- \in E, \quad w_n^\pm, w^\pm \in E^\pm.$$

If $w = 0$, then $w_n^- \rightarrow w^- = 0$ because $\dim E^- < \infty$. Since

$$\|w_n^+\|^2 + \|w_n^-\|^2 = 1,$$

for n large enough we have

$$\|w_n^+\|^2 - \|w_n^-\|^2 \geq \frac{1}{2}.$$

Now, by (f_3) , we deduce that for n large enough,

$$\begin{aligned} & 1 + \sup_n |I(u_n)| + \|u_n\| \\ & \geq I(u_n) - \frac{1}{4} \langle I'(u_n), u_n \rangle \\ & = \frac{1}{2} \left([u_n]_s^2 + \int_{\mathbb{R}^3} V(x) |u_n|^2 dx \right) + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \int_{\mathbb{R}^3} F(x, u_n) dx \\ & \quad - \frac{1}{4} [u_n]_s^2 - \frac{1}{4} \int_{\mathbb{R}^3} V(x) u_n^2 dx - \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} f(x, u_n) u_n dx \\ & = \frac{1}{4} \|u_n\|^2 (\|w_n^+\|^2 - \|w_n^-\|^2) + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(x, u_n) u_n - F(x, u_n) \right) dx \\ & \geq \frac{1}{8} \|u_n\|^2, \end{aligned}$$

this is a contradiction when $\|u_n\| \rightarrow \infty$.

If $w \neq 0$, then the set $\Theta = \{w \neq 0\}$ has positive Lebesgue measure. According to (2.8), for $x \in \Theta$ we have $|u_n(x)| \rightarrow \infty$ and

$$\frac{F(x, u_n(x))}{u_n^4} w_n^4(x) \rightarrow +\infty.$$

Then using Fatou lemma, we have

$$\int_{w \neq 0} \frac{F(x, u_n)}{u_n^4} w_n^4 dx \rightarrow +\infty. \quad (3.2)$$

On the other hand, for n large enough,

$$\begin{aligned} \int_{w \neq 0} \frac{F(x, u_n)}{u_n^4} w_n^4 &= \frac{1}{\|u_n\|^4} \int_{w \neq 0} F(x, u_n) dx \\ &\leq \frac{1}{\|u_n\|^4} \int_{\mathbb{R}^3} F(x, u_n) dx \\ &= \frac{1}{\|u_n\|^4} \left(\frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n}^t(x) u_n^2 dx - I(u_n) \right) \\ &\leq 1 + \frac{1}{4\|u_n\|^4} \int_{\mathbb{R}^3} \phi_{u_n}^t(x) u_n^2 dx \\ &\leq 1 + C_7, \end{aligned}$$

a contradiction to (3.2).

Therefore, we obtain that the $(PS)_c$ -sequence $\{u_n\}$ is bounded. \square

In order to obtain a convergent subsequence of the $(PS)_c$ -sequence, we need some compact properties of operator involving ϕ_u^t in Lemma 2.1. Consider the \mathcal{C}^1 -function $N : E \rightarrow \mathbb{R}$,

$$N(u) = \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx.$$

So, for all $u, v \in E$, we have

$$\langle N'(u), v \rangle = \int_{\mathbb{R}^3} \phi_u^t u v dx.$$

Lemma 3.2. Assume (V) , $(f_1 - f_4)$ hold, I satisfies the $(PS)_c$ -condition.

Proof. Let $\{u_n\}$ be a $(PS)_c$ -sequence. We know from Lemma 3.1 that $\{u_n\}$ is bounded in E . Going if necessary to a subsequence, we can assume that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } E, \\ u_n &\rightarrow u \text{ in } L_{\text{loc}}^p(\mathbb{R}^3), \quad 2 \leq p < 2^*, \\ u_n &\rightarrow u \text{ a.e. in } \mathbb{R}^3. \end{aligned} \quad (3.3)$$

Therefore, we have

$$\int_{\mathbb{R}^3} [(-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} u + V(x) u_n u] dx \rightarrow [u]_s^2 + \int_{\mathbb{R}^3} V(x) |u|^2 dx.$$

Consequently, we have

$$\begin{aligned}
o(1) &= \langle I'(u_n), u_n - u \rangle \\
&= \int_{\mathbb{R}^3} [(-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} (u_n - u) + V(x) u_n (u_n - u)] dx \\
&\quad + \int_{\mathbb{R}^3} \phi_{u_n}^t u_n (u_n - u) dx - \int_{\mathbb{R}^3} f(x, u_n) (u_n - u) dx \\
&= \left([u_n]_s^2 + \int_{\mathbb{R}^3} V(x) |u_n|^2 dx \right) - \int_{\mathbb{R}^3} [(-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} u + V(x) u_n u] dx \\
&\quad + \int_{\mathbb{R}^3} \phi_{u_n}^t u_n (u_n - u) dx - \int_{\mathbb{R}^3} f(x, u_n) (u_n - u) dx \\
&= \left([u_n]_s^2 + \int_{\mathbb{R}^3} V(x) |u_n|^2 dx \right) - \left([u]_s^2 + \int_{\mathbb{R}^3} V(x) |u|^2 dx \right) \\
&\quad + \int_{\mathbb{R}^3} \phi_{u_n}^t u_n (u_n - u) dx - \int_{\mathbb{R}^3} f(x, u_n) (u_n - u) dx + o(1) \\
&= (\|u_n^+\|^2 - \|u_n^-\|^2) - (\|u^+\|^2 - \|u^-\|^2) + \int_{\mathbb{R}^3} \phi_{u_n}^t u_n (u_n - u) dx \\
&\quad - \int_{\mathbb{R}^3} f(x, u_n) (u_n - u) dx + o(1).
\end{aligned}$$

Due to $\dim E^- < \infty$, we have $u_n^- \rightarrow u^-$ and so $\|u_n^-\| \rightarrow \|u^-\|$. We gain

$$\|u_n^+\|^2 - \|u^+\|^2 \leq o(1) + \int_{\mathbb{R}^3} f(x, u_n) (u_n - u) dx - \int_{\mathbb{R}^3} \phi_{u_n}^t u_n (u_n - u) dx. \quad (3.4)$$

By assumption (f_4) , we obtain that

$$\overline{\lim}_{n \rightarrow \infty} \int_{\mathbb{R}^3} f(x, u_n) (u_n - u) dx \leq 0. \quad (3.5)$$

In fact, let $\varepsilon > 0$. For $r \geq 1$, we have

$$\begin{aligned}
\int_{\{x: |u_n| \geq r\}} f(x, u_n) (u_n - u) dx &\leq 2C_6 \int_{\{x: |u_n| \geq r\}} |u_n|^{p-1} |u_n - u| dx \\
&\leq 2C_6 r^{p-2_s^*} \int_{\{x: |u_n| \geq r\}} |u_n|^{2_s^*-1} |u_n - u| dx \\
&\leq 2C_6 r^{p-2_s^*} |u_n|_{2_s^*}^{2_s^*-1} |u_n - u|_{2_s^*}.
\end{aligned}$$

Since $p < 2_s^*$, we can fix r large enough such that

$$\int_{\{x: |u_n| \geq r\}} f(x, u_n) (u_n - u) dx \leq \frac{\varepsilon}{3} \quad (3.6)$$

for all n . Moreover, according to (f_4) there exists $R > 0$ such that

$$\int_{\{|x| \geq R, |u_n| \leq r\}} f(x, u_n) (u_n - u) dx \leq |u_n|_2 |u_n - u|_2 \sup_{|t| \leq r, |x| \geq R} \frac{|f(x, t)|}{|t|} \leq \frac{\varepsilon}{3} \quad (3.7)$$

for all n . Note that by (f_1) and (f_2) for every $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ such that

$$|f(x, t)| \leq \varepsilon |t| + K(\varepsilon) |t|^{p-1} \text{ for } x \in \mathbb{R}^3, t \in \mathbb{R}. \quad (3.8)$$

Finally, due to $u_n \rightarrow u$ in $L^p(B_R(0))$ for $p \in [2, 2_s^*)$, we may use (3.8) to obtain

$$\int_{\{|x| \leq R, |u_n| \leq r\}} f(x, u_n)(u_n - u) dx \leq \frac{\varepsilon}{3} \quad (3.9)$$

for n large enough. Combining (3.6), (3.7) and (3.9) we obtain that (3.5) holds.

Now, we claim that

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} \phi_{u_n}^t u_n (u_n - u) dx \geq 0. \quad (3.10)$$

In fact, $\{u_n\}$ is bounded and $u_n \rightharpoonup u$ in E , according to weak lower semi-continuity, we have

$$\liminf_{n \rightarrow \infty} N(u_n) \geq N(u), \quad \lim_{n \rightarrow \infty} \langle N'(u_n), u \rangle = \langle N'(u), u \rangle.$$

Then,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} \phi_{u_n}^t u_n (u_n - u) dx &= \liminf_{n \rightarrow \infty} \left(\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \int_{\mathbb{R}^3} \phi_{u_n}^t u_n u dx \right) \\ &= \liminf_{n \rightarrow \infty} (4N(u_n) - \langle N'(u_n), u \rangle) \\ &\geq 4N(u) - \langle N'(u), u \rangle = 0. \end{aligned}$$

Therefore, it follows from (3.4), (3.5) and (3.10) that

$$\begin{aligned} &\overline{\lim}_{n \rightarrow \infty} (\|u_n^+\|^2 - \|u^+\|^2) \\ &\leq \overline{\lim}_{n \rightarrow \infty} \left(\int_{\mathbb{R}^3} f(x, u_n)(u_n - u) dx - \int_{\mathbb{R}^3} \phi_{u_n}^t u_n (u_n - u) dx \right) \\ &\leq \overline{\lim}_{n \rightarrow \infty} \int_{\mathbb{R}^3} f(x, u_n)(u_n - u) dx - \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} \phi_{u_n}^t u_n (u_n - u) dx \\ &\leq \overline{\lim}_{n \rightarrow \infty} \int_{\mathbb{R}^3} f(x, u_n)(u_n - u) dx \leq 0. \end{aligned}$$

Combining this with the weakly lower semi-continuity of the norm functional $u \mapsto \|u\|$, we have

$$\|u^+\|^2 \leq \liminf_{n \rightarrow \infty} \|u_n^+\|^2 \leq \overline{\lim}_{n \rightarrow \infty} \|u_n^+\|^2 \leq \|u^+\|^2.$$

That is, $\|u_n^+\|^2 \rightarrow \|u^+\|^2$. On the other hand, $\|u_n^-\| \rightarrow \|u^-\|$, so we obtain $\|u_n\| \rightarrow \|u\|$, i.e., $u_n \rightarrow u$ in E . \square

Since I satisfies $(PS)_c$ -condition, the critical group $C_*(I, \infty)$ of I at infinity makes sense. To investigate $C_*(I, \infty)$, we need the following lemma.

Lemma 3.3. *There exists $A > 0$ such that, if $I(u) \leq -A$, then*

$$\left. \frac{d}{d\xi} \right|_{\xi=1} I(\xi u) < 0.$$

Proof. By contradiction, we can assume that exists a sequence $\{u_n\} \subset E$ such that $I(u_n) \leq -n$ but

$$\langle I'(u_n), u_n \rangle = \frac{d}{d\xi} \Big|_{\xi=1} I(\xi u_n) \geq 0. \quad (3.11)$$

Consequently, according to (f_3) , we have

$$\begin{aligned} \|u_n^+\|^2 - \|u_n^-\|^2 &\leq (\|u_n^+\|^2 - \|u_n^-\|^2) + \int_{\mathbb{R}^3} f(x, u_n) u_n - 4F(x, u_n) dx \\ &= 4I(u_n) - \langle I'(u_n), u_n \rangle \leq -4n. \end{aligned} \quad (3.12)$$

Let $w_n = \frac{u_n}{\|u_n\|}$ and w_n^\pm be the orthogonal projection of w_n on E^\pm . Then due to $\dim E^- < \infty$, $w_n^- \rightarrow w^-$ for some $w^- \in E^-$.

If $w^- \neq 0$, then we have $w_n \rightharpoonup w$ in E for some $w \in X \setminus \{0\}$. Similar to (3.2), by (f_3) , we have

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{f(x, u_n) u_n}{\|u_n\|^4} dx &\geq 4 \int_{\mathbb{R}^3} \frac{F(x, u_n)}{\|u_n\|^4} dx \\ &\geq 4 \int_{w \neq 0} \frac{F(x, u_n)}{u_n^4} w_n^4 dx \rightarrow +\infty. \end{aligned}$$

Therefore, by (3.11), we obtain

$$\begin{aligned} 0 &\leq \frac{\langle I'(u_n), u_n \rangle}{\|u_n\|^4} \\ &= \frac{\|u_n^+\|^2 - \|u_n^-\|^2}{\|u_n\|^4} + \frac{1}{\|u_n\|^4} \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \int_{\mathbb{R}^3} \frac{f(x, u_n) u_n}{\|u_n\|^4} dx \\ &\leq 1 + C_7 - \int_{\mathbb{R}^3} \frac{f(x, u_n) u_n}{\|u_n\|^4} dx \rightarrow -\infty, \end{aligned}$$

which is a contradiction.

So $w^- = 0$, from

$$\|w_n^+\|^2 + \|w_n^-\|^2 = 1,$$

we obtain $\|w_n^+\| \rightarrow 1$. Consequently, for n large enough,

$$\|u_n^+\| = \|u_n\| \|w_n^+\| \geq \|u_n\| \|w_n^-\| = \|u_n^-\|,$$

violating (3.12). □

Lemma 3.4. $C_q(I, \infty) = 0$ for all $q = 0, 1, 2, \dots$

Proof. Like [12], let

$$B = \{w \in E \mid \|w\| \leq 1\}, S = \partial B$$

be the unit sphere in E , and $A > 0$ be the number given in Lemma 3.3. Without loss of generality, we can assume that

$$-A < \inf_{\|u\| \leq 2} I(u).$$

By (2.8), for any $w \in S$,

$$\begin{aligned} I(\tau w) &= \frac{\tau^2}{2} (\|w^+\|^2 - \|w^-\|^2) + \frac{\tau^4}{4} \int_{\mathbb{R}^3} \phi_w^t w^2 dx - \int_{\mathbb{R}^3} F(x, \tau w) dx \\ &= \tau^4 \left\{ \frac{\|w^+\|^2 - \|w^-\|^2}{2\tau^2} + \frac{1}{4} \int_{\mathbb{R}^3} \phi_w^t w^2 dx - \int_{\mathbb{R}^3} \frac{F(x, \tau w)}{\tau^4} dx \right\} \rightarrow -\infty \end{aligned}$$

as $\tau \rightarrow +\infty$. So there is $\tau_w > 0$ such that $I(\tau_w w) = -A$. Set $u = \tau_w w$, then a direct computation together with Lemma 3.3 gives

$$\left. \frac{d}{d\tau} \right|_{\tau=\tau_w} I(\tau u) = \frac{1}{\tau_w} \left. \frac{d}{d\xi} \right|_{\xi=1} I(\xi u) < 0.$$

According the implicit function theorem, $T : w \rightarrow \tau_w$ is a continuous function on S . Using the function T , as in [17], we can construct a strong deformation retract $\eta : E \setminus B \rightarrow I_{-A}$,

$$\eta(u) = \begin{cases} u, & \text{if } I(u) \leq -A, \\ T\left(\frac{u}{\|u\|}\right) \frac{u}{\|u\|}, & \text{if } I(u) > -A \end{cases}$$

and get

$$C_q(I, \infty) = H_q(E, I_{-A}) \cong H_q(E, E \setminus B) = 0, \quad q = 0, 1, 2, \dots$$

□

4 Proof of Theorem 2.1

Now, we prove our results.

Proof of Theorem 2.1. Using (V) and (f_2) , we have

$$I(u) = \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) + o(\|u\|^2) \text{ as } \|u\| \rightarrow 0.$$

So there exists $\delta > 0$ such that $I > 0$ on $(E^+ \setminus \{0\}) \cap B_\delta$ and $I < 0$ on $(E^- \setminus \{0\}) \cap B_\delta$. That is I has a local linking at 0, $E = E^+ \oplus E^-$. According Proposition 2.2, we have

$$C_k(I, 0) \neq 0,$$

where $k = \dim X^-$. Using Lemma 3.4 and Proposition 2.1, we obtain that I has a nontrivial critical point.

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