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RESEARCH ARTICLE

C-controllability of stochastic semilinear systems

Agamirza E. Bashirov

Department of Mathematics, Eastern Mediterranean University, Gazimagusa, Mersin 10, Tiurkey

Correspondence
Agamirza E. Bashirov, Department of Mathematics, Eastern Mediterranean University, Gazimagusa, Mersin 10, Turkey. Email: agamirza.bashirov@emu.edu.tr

Present Address
Department of Mathematics, Eastern Mediterranean University, Gazimagusa, Mersin 10, Turkey

Abstract

It is difficult to prove a capable sufficient condition for the exact controllability of systems containing nonlinearities and randomness. As a result, scientists are investigating the concept of approximate controllability for such systems. In this paper, we handle the so-called C-controllability, which was suggested as a weaker analog of the exact controllability at the beginning of the period when controllability issue oversteps to stochastic systems. We prove a sufficient condition of C-controllability for a semilinear stochastic system driven by a Wiener process. This sufficient condition is verified on examples. Two ways of improvement of this sufficient condition are discussed.

KEYWORDS:
Exact controllability; C-controllability; stochastic system, semilinear system

1 | INTRODUCTION

The concept of controllability was defined by Kalman[1] in 1960 for finite dimensional deterministic control systems. The motivation was to find conditions under which control systems are able to move any point to any point in the state space for a finite time. This is very important in engineering applications. A simple example is a robot which has to move an object in some area to some another location in the same area. Such a robot is functional if the control system, describing the movement of its arm, is controllable. Another example is a network which has to communicate every input to every output.

Further studies in the area of controllability demonstrated that it is suitable to consider stronger and weaker versions of controllability called, respectively, exact and approximate controllability. The reason for this was the fact that many infinite dimensional control systems are not exactly controllable although they are approximately controllable[2,3]. The necessary and sufficient conditions of exact and approximate controllability for deterministic linear systems are almost completely studied and presented in books[4,5,6,7,8].

Many applied control systems are nonlinear. Just sufficient conditions of controllability have been studied for different kinds of nonlinear systems including first order[9,10,11], second order[12,13,14,15], fractional order[16,17], impulsive[18,19,20], constrained[21], with memory[22], etc., systems. Different fixed-point theorems form a basic method of study for deterministic nonlinear systems[23,24,25,26,27].

The extension of controllability concepts to stochastic systems adds a new element to the issue. Now, the terminal states are random. Therefore, two factors lead up to the wideness of the attainability set. The first one is the wideness of the state space as in the deterministic case and the second one is the randomness of the values. Some authors raise both these factors and define exact controllability of a stochastic system, subject to the filtration $\{\mathcal{F}_t : 0 \le t \le T\}$, as steering of any $\mathcal{F}_0$-measurable L_2-random variable to any $\mathcal{F}_T$-measurable L_2-random variable for a finite time T. At first glance, such a definition looks acceptable. But unlike quantum physics in which positions of electrons are naturally random, engineering deals with objects which have well-determined positions in nature so that a possible randomness is due to the luck of knowledge. On the example of the control system describing the motion of the robot's arm, mentioned at the beginning, consider a possible random state η which has two different values a and b with the equal probabilities 1/2. We can raise the following question: Is it important to know that the robot is able to steer an initial state to η? What is wanted from the robot's arm is reaching all possible nonrandom values including a and b with probability 1, but reaching η is a completely needless ability. This example clearly demonstrates that in the stochastic case, the wideness of the state space should be raised while the randomness damped.

Another argument supporting this idea is an example[28,29], which demonstrates that a linear stochastic control system driven by a Wiener process never steers every $\mathcal{F}_0$-measurable L_2-random variable to every $\mathcal{F}_T$-measurable L_2-random variable for a finite time T unless the system remains essentially stochastic (does not reduce to deterministic equation in which $\mathcal{F}_0$ and $\mathcal{F}_T$ are trivial). This fact essentially extends to nonlinear stochastic systems because their controllability significantly depends on the controllability of their linear part.

Note that damping the randomness in the definitions of controllability was considered previously as well. In such a way, the C- and S-controllability concepts were defined as extensions of the exact and approximate controllability, respectively, to stochastic systems[30,31] . Partial versions of these concepts have also been investigated[29,32]. In this paper, we handle C-controllability. Fixed-point theorems are not appropriate for dealing with this concept because they require coerciveness of stochastic controllability operator, which is non-coercive indeed unless the system remains essentially stochastic[28,29]. Therefore, we apply an alternative method, avoiding fixed-point theorems. The basic idea of this method is a construction of steering controls piece-wisely based on the properties of linear systems. Originally, this method was introduced for deterministic systems[33] and was successfully applied in the context of approximate controllability[18,19,20,28]. A modification of this method to the context of exact controllability has already been done for deterministic systems[34]. In this paper, we use a combination of the piecewise construction method in the both contexts and prove the C-controllability of a stochastic semilinear system under consideration.

The rest of the paper is organised as follows. In the next section, we introduce general notation used in this paper. Section 3 contains a description of the stochastic semilinear control system under consideration. In Section 4, we discuss the C-controllability. We prove a sufficient condition of C-controllability for the stochastic system under consideration in Section 5. Finally, Section 6 discusses this sufficient condition on examples and Section 7 concludes the paper.

2 | NOTATION

One major notation is that we prefer to write the arguments of functions in subscripts, for example, x_t instead of $x(t)$. This allows to make shorter long mathematical expressions. $\mathbb{R}$ denotes the system of real numbers. The norm and scalar product in all Hilbert and Banach spaces will be denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. In ambiguous cases, we will mention shortly the space, to which they correspond.

Let X and Y be separable Hilbert spaces. We denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators, and by $\mathcal{L}_2(X,Y)$ the space of all Hilbert–Schmidt operators from X to Y. If $X=Y$, then we use the short symbols $\mathcal{L}(X)$ and $\mathcal{L}_2(X)$ for them. The space $\mathcal{L}(X,Y)$ is a Banach space and $\mathcal{L}_2(X,Y)$ is a Hilbert space. I and 0 are identity and zero operators, respectively. The adjoint of a linear operator A is denoted by A^*.

If $Q\in\mathcal{L}(X)$ is such that $Q^*=Q$ and $\langle Qx,x\rangle\ge 0$ for all $x\in X$, then we write $Q\ge 0$. Note that for $Q\ge 0$, $Q^{1/2}$ exists. If $Q\in\mathcal{L}(X)$ is such that $Q^*=Q$ and $\langle Qx,x\rangle>0$ for all $x\in X$ with $x\neq 0$, then we write $Q>0$. Note that for $Q>0$, Q^{-1} exists but maybe unbounded linear operator. If $Q\in\mathcal{L}(X)$ is such that $Q^*=Q$ and there exists $c>0$ satisfying $\langle Qx,x\rangle\ge c\|x\|^2$ for all $x\in X$, then Q is said to be coercive. If Q is coercive with the preceding inequality, then Q^{-1} exists as a bounded linear operator and $\|Q^{-1}\|\le 1/c$.

For a nuclear operator W on X with $W\ge 0$, we denote by $\mathcal{L}_W(X,Y)$ the collection of all (bounded or unbounded) linear operators A from the range of $W^{1/2}$ to Y such that $AW^{1/2}\in\mathcal{L}_2(X,Y)$. Note that $\mathcal{L}_W(X,Y)$ is a Hilbert space with

$$\langle A,B\rangle_{\mathcal{L}_W(X,Y)}=\langle AW^{1/2},BW^{1/2}\rangle_{\mathcal{L}_2(X,Y)}.$$

We assume that a complete probability space $(\Omega,\Sigma,\mathbf{P})$ is given. $\mathbf{E}\xi$ denotes the expectation of ξ. A Wiener process w over this probability space is called standard if $w_0=0$, $\mathbf{E}w_t=0$ and $\operatorname{cov} w_t=Wt$, where W is a nuclear operator if w takes values in an infinite-dimensional Hilbert space and $W=I$ if w is finite dimensional.

$C(0,T;X)$ denotes the space of continuous functions from $[0,T]$ to X. $L_2(0,T;X)$ is the space of measurable and square integrable in the Lebesque sense X-valued functions on $[0,T]$. $L_2(\Omega,X)$ is the space of Σ-measurable and square integrable

X-valued random variables. For a sub-σ-field Σ_0 of Σ, we let

$$L_2(\Omega, \Sigma_0, X) = \{\xi \in L_2(\Omega, X) : \xi \text{ is } \Sigma_0\text{-measurable}\}.$$

For a given filtration $\mathcal{F} = \{\mathcal{F}_t : 0 \le t \le T\}$, we denote

$$L_2^{\mathcal{F}}(0,T;X) = \{x \in L_2(0,T;L_2(\Omega, X)) : x \text{ is } \mathcal{F}_t\text{-adapted}\},$$

and

$$C^{\mathcal{F}}(0,T;X) = \{x \in C(0,T;L_2(\Omega, X)) : x \text{ is } \mathcal{F}_t\text{-adapted}\}.$$

3 | DESCRIPTION OF THE SYSTEM

Consider the stochastic semilinear control system

$$dx_t = (Ax_t + B_t u_t + f(t,u_t))\,dt + g(t,x_t,u_t)\,dw_t \tag{1}$$

on the interval $[0,T]$ with $T > 0$, where x and u are state and control processes. We assume that the following conditions hold.

(A) X, Y, and U are separable Hilbert spaces.

(B) A is a closed linear operator on X with the dense domain $D(A) \subseteq X$, generating a strongly continuous semigroup e^{At}, $t \ge 0$.

(C) $B \in \mathcal{L}(U,X)$.

(D) w is a Y-valued standard Wiener process on $[0,T]$ with cov $w_t = Wt$, $0 \le t \le T$, generating a complete and continuous filtration $\mathcal{F} = \{\mathcal{F}_t : 0 \le t \le T\}$.

(E) $f : [0,T] \times U \to X$ and $g : [0,T] \times X \times U \to \mathcal{L}_W(Y,X)$ are functions with properties

- f and g are Lebesgue measurable in t,
- g are Lipschitz continuous in x,
- f and g are continuous in u,
- f and g are bounded.

We let $U_{\text{ad}} = L_2^{\mathcal{F}}(0,T;U)$ and call it as a set of admissible controls. The above conditions imply that for every initial state $x_0 \in L_2(\Omega, \mathcal{F}_0, X)$ and control $u \in U_{\text{ad}}$, the equation

$$x_t = e^{At}x_0 + \int_0^t e^{A(t-s)}(Bu_s + f(s,u_s))\,ds + \int_0^t e^{A(t-s)} g(s,x_s,u_s)\,dw_s \tag{2}$$

admits a unique solution in $C^{\mathcal{F}}(0,T;X)$[35,36]. This solution is called a mild solution of (1). Note that if x_τ is given for $0 \le \tau < T$, then (2) can also be written as

$$x_t = e^{A(t-\tau)}x_\tau + \int_\tau^t e^{A(t-s)}(Bu_s + f(s,u_s))\,ds + \int_\tau^t e^{A(t-s)} g(s,x_s,u_s)\,dw_s \tag{3}$$

for all $\tau \le t \le T$. We will denote by $x^{\xi,u}$ the mild solution of (1) corresponding to the initial state $x_0 = \xi \in L_2(\Omega, \mathcal{F}_0, X)$ and the control $u \in U_{\text{ad}}$.

If $f(t,u) = 0$ and $g(t,x,u) = 0$, then the system (1) becomes linear. The controllability operator Q_t associated with this linear system is defined by

$$Q_t = \int_0^t e^{As}BB^* e^{A^*s}\,ds,\ 0 \le t \le T. \tag{4}$$

In addition to (A)–(E), we will need in the following condition as well:

(F) The operator Q_t, defined by (4), is coercive for all $0 < t \le T$ and there exists $0 \le \alpha < 1$ such that $t^{1+\alpha}\|Q(t)^{-1}\|$ is bounded on $(0,T]$.

4 | C-CONTROLLABILITY

The C-controllability is an analog of exact controllability for stochastic systems on the basis of raising wideness of the state space and damping randomness [30,31]. It was defined for partially observable systems. Since the system in (1) is completely observable, we will adapt the definition of C-controllability to this case.

Definition 1. Let $x^{\xi,u}$ be the state process of the stochastic control system (1), corresponding to the initial value $\xi \in L_2(\Omega, \mathcal{F}_0, X)$ and the control $u \in U_{\text{ad}}$. Denote

$$C_T^{\xi} = \bigcap_{\varepsilon>0,\, 0\le p<1} \{\eta \in X : \exists u \in U_{\text{ad}} \text{ such that } \mathbf{E}x^{\xi,u} = \eta \text{ and } \mathbf{P}(\|x_T^{\xi,u} - \eta\|^2 > \varepsilon) \le 1-p\}.$$

The system in (1) is said to be

- C-controllable (for the time T) if $C_T^{\xi} = X$ for all $\xi \in L_2(\Omega, \mathcal{F}_0, X)$;
- C-controllable to $D(A)$ (for the time T) if $D(A) \subseteq C_T^{\xi}$ for all $\xi \in L_2(\Omega, \mathcal{F}_0, X)$.

For a substantial discussion of C-controllability, we refer to [30], noticing the result [29] that the system (1) is C-controllable if and only if for every $\xi \in L_2(\Omega, \mathcal{F}_0, X)$ and $\eta \in X$, there is a sequence of controls $\{u_n\}$ in U_{ad} such that

- x_T^{ξ,u_n} converges to η with probability 1;
- $\mathbf{E}x_T^{\xi,u_n} = \eta$ for all n.

In other words, $\{u_n\}$ should provide a sequence $\{x_T^{\xi,u_n}\}$ of terminal random values which converges to η with probability 1 being centralized on the limit. If the second item drops, then the system is called S-controllable which is an analog of approximate controllability for stochastic systems. So, the second item moves the controllability from the approximate level to the exact level.

Here, C is the abbreviation of the word "combined". Letting $f(t,u) = 0$ and $g(t,x,u) = 0$ in (1), we obtain a linear system, which can be decomposed into the sum of two linear systems: deterministic and purely stochastic. Then C-controllability becomes a combination of exact controllability of the deterministic component and approximate null-controllability of the purely stochastic component in the sense of convergence with probability 1. At the same time, the term "exact" in the stochastic case is used for a controllability in the sense of moving every $x_0 \in L_2(\Omega, \mathcal{F}_0, X)$ to every $\eta \in L_2(\Omega, \mathcal{F}_T, X)$. Whilst it fails for stochastic systems [29], there are many past papers devoting to this concept. In order to remove a possible confusion, we prefer the term "C-controllability" although "exact controllability" suites Definition 1 better. The arguments supporting C- and S-controllability as sustainable extensions of exact and approximate controllability to stochastic systems are coming from the linear stochastic systems. According to [30], a linear stochastic system is

- C-controllable for the time T for every $T > 0$ if and only if its deterministic component is exactly controllable for the time T for every $T > 0$;
- S-controllable for the time T for every $T > 0$ if and only if its deterministic component is approximately controllable for the time T for every $T > 0$.

A sufficient condition of S-controllability for semilinear stochastic systems has already been proved [28]. The same problem for C-controllability is still open because it is difficult to prove a sufficient condition of C-controllability for semilinear systems.

In this paper, we prove a sufficient condition of C-controllability to $D(A)$ for the system (1). In applications, we are mostly interested in reaching the points from $D(A)$. The points from $X \setminus D(A)$ are secondary, that is, reaching these points is not an important ability of the systems. The methods of study of the exact controllability by fixed-point theorems, require so strong conditions that they cover the points in $D(A)$ together with the points of $X \setminus D(A)$. In this paper. we use a delicate proof method which differentiates the points of $D(A)$ and $X \setminus D(A)$. This method is a combination of two previously developed methods. The first one was suggested for a construction of a control steering any point in X to any point in $D(A)$ for a deterministic semilinear system [34]. The second one was initiated for proving approximate controllability of deterministic semilinear systems [33] and then it was extended to stochastic semilinear systems for proving S-controllability [28].

5 | MAIN RESULT

The following theorem states the main result of this paper.

Theorem 1. Under the conditions (A)–(F), the stochastic system (1) is C-controllable to $D(A)$ on the interval $[0, T]$.

Proof. Our proof consists of three steps.

Step 1: This step points out a well-known controllability result, which will be used in the next step. Letting $f(t, u) = 0$ and $g(t, x, u) = 0$ in (3), we obtain the linear system

$$y_t = e^{A(t-\tau)} y_\tau + \int_\tau^t e^{A(t-s)} B v_s \, ds, \; 0 \le \tau \le t \le T. \tag{5}$$

Under the conditions (A)–(C), the linear system (5) is exactly controllable on the interval $[\tau, T]$ if and only if $Q_{T-\tau}$ is coercive. Moreover, a control steering the initial state $y_\tau = \xi \in X$ to the final state $\eta \in X$ can be defined by

$$v_t = B^* e^{A^*(T-t)} Q_{T-\tau}^{-1} (\eta - e^{A(T-\tau)} \xi), \; \tau \le t \le T. \tag{6}$$

If $\xi \in L_2(\Omega, \mathcal{F}_\tau, X)$, then v defined by (6) belongs to $L_2(\tau, T; L_2(\Omega, \mathcal{F}_\tau, U))$ and steers the random variable ξ to nonrandom $\eta \in X$. For $\xi \in X$, this theorem is proved in many sources[4]. Then for $\xi \in L_2(\Omega, \mathcal{F}_\tau, X)$, it is immediate.

Step 2: Now, we let just $g(t, x, u) = 0$ in (3) and obtain the system

$$z_t = e^{A(t-\tau)} z_\tau + \int_\tau^t e^{A(t-s)} (B v_s + f(t, v_s)) \, ds, \; 0 \le \tau \le t \le T. \tag{7}$$

It is already proved[34] that for every $z_\tau = \xi \in X$ and $\eta \in D(A)$, there exists a control $v \in L_2(0, T; U)$ steering ξ to η along the system in (7). We claim that every $\xi \in L_2(\Omega, \mathcal{F}_\tau, X)$ can be steered to every nonrandom $\eta \in D(A)$ by some control $v \in L_2(\tau, T; L_2(\Omega, \mathcal{F}_\tau, U))$. The proof is essentially based on the proof method from[34]. We will go over basic items of this proof.

Fix $\xi \in L_2(\Omega, \mathcal{F}_\tau, X)$ and $\eta \in D(A)$ and construct a control v steering ξ to η along the system in (7) in the following recursive way. Let $t_0 = \tau$, let $\theta_k = (T - \tau)/2^k$, and let

$$t_n = \tau + \sum_{k=1}^n \theta_k, \; n = 1, 2, \dots$$

Clearly, $\tau = t_0 < t_1 < \cdots < t_n < \cdots < T$ with $\lim_{n\to\infty} t_n = T$. Consider the system (7) on $[t_0, t_1]$. According to Step 1, the control

$$v_t^1 = B^* e^{A^*(t_1-t)} Q_{\theta_1}^{-1} e^{A\theta_1} (\eta - \xi), \; t_0 \le t \le t_1,$$

steers ξ at the instant t_0 to $e^{A\theta_1}\eta$ at the instant t_1 along the system in (5). Therefore,

$$e^{A\theta_1}\eta = e^{A\theta_1}\xi + \int_{t_0}^{t_1} e^{A(t_1-s)} B v_s^1 \, ds.$$

We let $v_t = v_t^1$ for $t_0 \le t \le t_1$. Clearly, v_t is $\mathcal{F}_\tau$-measurable for all $t_0 \le t \le t_1$ since ξ has the same property. Then

$$z_1 = z_{t_1}^{\xi,v} = e^{A\theta_1}\eta + \int_{t_0}^{t_1} e^{A(t_1-s)} f(s, v_s) \, ds \in L_2(\Omega, \mathcal{F}_\tau, X),$$

where $z^{\xi,v}$ is defined by (7) for the initial value $z_\tau = \xi$ and control v.

Next, assume that v is defined on $[t_0, t_{n-1}]$ with $z_{n-1} = z_{t_{n-1}}^{\xi,v} \in L_2(\Omega, \mathcal{F}_\tau, X)$ and continue v to $(t_{n-1}, t_n]$ by letting

$$v_t^n = B^* e^{A^*(t_n-t)} Q_{\theta_n}^{-1} e^{A\theta_n} (\eta - z_{n-1}), \; t_{n-1} \le t \le t_n. \tag{8}$$

According to Step 1, v^n steers z_{n-1} at the instant t_{n-1} to $e^{A\theta_n}\eta$ at the instant t_n so that

$$e^{A\theta_n}\eta = e^{A\theta_n} z_{n-1} + \int_{t_{n-1}}^{t_n} e^{A(t_n-s)} B v_s^n \, ds.$$

Define v on the interval $(t_{n-1}, t_n]$ as $v_t = v_t^n$. Then

$$z_n = z_{t_n}^{\xi,v} = e^{A\theta_n}\eta + \int_{t_{n-1}}^{t_n} e^{A(t_n-s)} f(s, v_s)\, ds \in L_2(\Omega, \mathcal{F}_\tau, X).$$

Thus, the function $v : [\tau, T) \to L_2(\Omega, \mathcal{F}_\tau, U)$ is defined by recursion. We can let v_T be any value because it does not change anything in (7). So, v is defined on $[\tau, T]$.

For estimation purposes, introduce the constants

$$K = \sup_{[0,T]} \|e^{At}\|, \; M = \sup_{[0,T]\times U} \|f(t, v)\|.$$

Then

$$\|z_n - \eta\| \le \|e^{A\theta_n}\eta - \eta\| + \left\| \int_{t_{n-1}}^{t_n} e^{A(t_n-s)} f(s, v_s)\, ds \right\| \le \|e^{A\theta_n}\eta - \eta\| + KM\theta_n.$$

Here, the right side is nonrandom and converges to 0 since $\lim_{n\to\infty} \theta_n = 0$. So, $\lim_{n\to\infty} z_n = \eta$ uniformly, which implies the convergence in the mean square sense as well.

Let us prove that $v \in L_2(\tau, T; L_2(\Omega, \mathcal{F}_\tau, U))$. By construction, v takes values in $L_2(\Omega, \mathcal{F}_\tau, U)$. Moreover, by (8),

$$\begin{aligned}
\int_{t_{n-1}}^{t_n} \mathbf{E}\,\|v_t^n\|^2\, dt &= \int_{t_{n-1}}^{t_n} \mathbf{E}\,\|B^* e^{A^*(t_n-t)} Q_{\theta_n}^{-1} e^{A\theta_n}(\eta - z_{n-1})\|^2\, dt \\
&= \int_{t_{n-1}}^{t_n} \mathbf{E}\,\langle e^{A(t_n-t)} BB^* e^{A^*(t_n-t)} Q_{\theta_n}^{-1}\zeta_n, Q_{\theta_n}^{-1}\zeta_n\rangle\, dt \\
&= \mathbf{E}\,\langle \zeta_n, Q_{\theta_n}^{-1}\zeta_n\rangle \le K^2 \|Q_{\theta_n}^{-1}\|\,\mathbf{E}\,\|z_{n-1} - \eta\|^2 \\
&\le 2K^2 \|Q_{\theta_n}^{-1}\| (\|e^{A\theta_{n-1}}\eta - \eta\|^2 + K^2M^2\theta_{n-1}^2),
\end{aligned}$$

where $\zeta_n = e^{A\theta_n}(\eta - z_{n-1})$. Therefore,

$$\begin{aligned}
\int_0^T \mathbf{E}\|v_t\|^2\, dt &= \sum_{n=0}^{\infty} \int_{t_n}^{t_{n+1}} \mathbf{E}\|v_t^{n+1}\|^2\, dt \\
&\le 2K^2 \sum_{n=0}^{\infty} \|Q_{\theta_{n+1}}^{-1}\| (\|e^{A\theta_n}\eta - \eta\|^2 + K^2M^2\theta_n^2) \\
&\le 2K^2 \sum_{n=0}^{\infty} \theta_{n+1}^{1+\alpha} \|Q_{\theta_{n+1}}^{-1}\| \left(\frac{\theta_n}{\theta_{n+1}}\right)^{1+\alpha} \theta_n^{1-\alpha} \left(\left\| \frac{e^{A\theta_n}\eta - \eta}{\theta_n} \right\|^2 + K^2M^2 \right).
\end{aligned}$$

Here, $\theta_{n+1}^{1+\alpha}\|Q_{\theta_{n+1}}^{-1}\|$ is bounded by condition (F), $(\theta_n/\theta_{n-1})^{1+\alpha} = 2^{1+\alpha}$ and K^2M^2 are constants and $\|(e^{A\theta_n}\eta - \eta)/\theta_n\|$ is also bounded since $\eta \in D(A)$ and its limit exists. Then, there is $c > 0$ such that

$$\int_0^T \mathbf{E}\|v_t\|^2\, dt \le c \sum_{n=0}^{\infty} \theta_n^{1-\alpha} = c(T - \tau) \sum_{n=0}^{\infty} \left(\frac{1}{2^{1-\alpha}}\right)^n. \tag{9}$$

Since $0 < 1 - \alpha \le 1$, the geometric series in the right side of (9) converges, proving that $v \in L_2(\tau, T; L_2(\Omega, \mathcal{F}_\tau, U))$. Consequently, $z^{\xi,v} \in C(\tau, T; L_2(\Omega, \mathcal{F}_\tau, X))$. Therefore, we can calculate $z_T^{\xi,v}$ as

$$z_T^{\xi,v} = \lim_{t\to T} z_t^{\xi,v} = \lim_{n\to\infty} z_n = \eta,$$

where both limits are in the mean square sense.

Step 3: Finally, we consider the system (1) (or (2)) by itself. To construct a sequence $\{u^n\}$ of controls in $L_2^T(0, T; U)$ for proving its C-controllability, we employ method from[33].

Take any $\xi \in L_2(\Omega, \mathcal{F}_0, X)$ and $\eta \in D(A)$. Let $\{T_n\}$ be a strictly increasing sequence in $[0, T]$ with $\lim_{n\to\infty} T_n = T$. Define u^n on $[0, T_n]$ to be constant, say $u_t^n = 0$, $0 \le t \le T_n$. Denote $x_n = x_{T_n}^{\xi,u^n}$ and complete the definition of u^n, selecting it on $(T_n, T]$

as a control which steers x_n to η along the system (7) with $\tau = T_n$. The existence of such a control is proved in Step 2, according to which u_t is $\mathcal{F}_0$-measurable for $t \in [0, T_n]$ and $\mathcal{F}_{T_n}$-measurable for $t \in (T_n, T]$. So, $u_n \in L_2^{\mathcal{F}}(0, T; U)$. Then

$$\eta = e^{A(T-T_n)}x_n + \int_{T_n}^{T} e^{A(T-t)}(Bu_t^n + f(t, u_t^n))\,dt.$$

Therefore,

$$x_T^{\xi,u^n} = \eta + \int_{T_n}^{T} g(t, u_t^n, x_t^{\xi,u^n})\,dw_t.$$

This implies

$$\mathbf{E}\,x_T^{\xi,u^n} = \eta + \mathbf{E}\left(\int_{T_n}^{T} g(t, u_t^n, x_t^{\xi,u^n})\,dw_t\right) = \eta,\ n = 1, 2, \ldots$$

and

$$\begin{aligned}\mathbf{E}\|x_T^{\xi,u^n} - \eta\|^2 &= \mathbf{E}\left\|\int_{T_n}^{T} g(t, u_t^n, x_t^{\xi,u^n})\,dw_t\right\|^2 \\ &\le \int_{T_n}^{T} \mathbf{E}\|g(t, u_t^n, x_t^{\xi,u^n})\|_{\mathcal{L}_W}^2\,dt \le L^2(T - T_n) \to 0,\ n \to \infty.\end{aligned}$$

Here, L is a positive constant coming from the boundedness of g. Thus, x_T^{ξ,u^n} converges to η in the mean square sense. Therefore, x_T^{ξ,u^n} converges to η in probability. Finally, there is $\{n_k\}$ such that $\{x_T^{\xi,u^{n_k}}\}$ converges to η with probability 1.

This completes the proof of the theorem. □

Remark 1. One can observe that the method of construction of u^n in the proof of Theorem 1 accepts not only $\eta \in D(A)$, but also $\eta \in L_2(\Omega, \mathcal{F}_{T_1}, D(A))$, where T_1 maybe arbitrarily close to T but $T_1 < T$. Consequently, if $\{\mathcal{F}_t\}$ is left-continuous at T, that is $\bigcup_{0\le t<T}\mathcal{F}_t = \mathcal{F}_T$, which is the case for a Wiener process if $\{\mathcal{F}_t\}$ is its natural filtration, this proof is valid for more general C'-controllability as defined next: The system (1) is said to be C'-controllable to $D(A)$ on $[0, T]$ if for every $\xi \in L_2(\Omega, \mathcal{F}_0, X)$ and $\eta \in L_2(\Omega, \mathcal{F}_T, D(A))$, there is a sequence of controls $\{u^n\}$ in $L_2^{\mathcal{F}}(0, T; U)$ such that

- x_T^{ξ,u_n} converges to η with probability 1;
- $\mathbf{E}x_T^{\xi,u_n} = \mathbf{E}\eta$ for all n.

With all that, it should be noted that this is not a steering of any initial $\xi \in L_2(\Omega, \mathcal{F}_0, X)$ to any final $\eta \in L_2(\Omega, \mathcal{F}_T, X)$, which is indeed not possible unless the system remains essentially stochastic.

6 | EXAMPLES

The concept of C-controllability to $D(A)$ is mostly suitable for the cases with $D(A) = X$. These are cases when A is a bounded linear operator, for example, A is an integral operator or $X = \mathbb{R}^n$.

Consider the following simple one-dimensional semilinear stochastic system

$$dx_t = (ax_t + u_t + f(t, u_t))dt + g(t, x_t, u_t)dw_t \tag{10}$$

with $a \ne 0$. Its controllability operator, which turns to be a real-valued function, has the form

$$Q_t = \int_0^t e^{2as}\,ds = \frac{e^{2at} - 1}{2a}.$$

We calculate

$$\|Q_t\| = \frac{e^{2at} - 1}{2a} > 0 \text{ and } \|Q_t^{-1}\| = \frac{2a}{e^{2at} - 1}.$$

Since

$$\lim_{t\to 0+} t\|Q_t^{-1}\| = \lim_{t\to 0+} \frac{2at}{e^{2at}-1} = 1,$$

$t\|Q_t^{-1}\|$ is bounded on $(0,T]$. So, the condition (F) holds with $\alpha = 0$. If f and g satisfy condition (E), the system described by (10) is C-controllable by Theorem 1.

Consider a delay version of the system in (10):

$$dx_t = (ax_t + bx_{t-\varepsilon} + u_t + f(t,x_t,u_t))dt + g(t,x_t,u_t)dw_t, \tag{11}$$

where $\varepsilon > 0$ and $b \neq 0$. Although the process x in (11) is one-dimensional, the system (11) in the whole is infinite dimensional because of the presence of a delay and governed by a first order partial differential equation. To demonstrate this issue, assume the initial conditions

$$x_0 = \xi^1 \text{ and } x_\theta = \xi^2_\theta,\ -\varepsilon \le \theta < 0,$$

and consider the state space $X = \mathbb{R} \times L_2(-\varepsilon, 0; \mathbb{R})$. Define the linear operator A by

$$A\begin{bmatrix}\xi^1\\ \xi^2\end{bmatrix} = \begin{bmatrix} a\xi^1 + b\xi^2_{-\varepsilon} \\ \frac{d}{d\theta}(\xi^2 - \xi^1)\end{bmatrix},$$

with the domain

$$D(A) = \left\{ \begin{bmatrix}\xi^1\\ \xi^2\end{bmatrix} \in X : \frac{d}{d\theta}\xi^2 \in L_2(-\varepsilon,0;\mathbb{R}),\ \xi^2(0) = \xi^1 \right\}.$$

Now extend the solution x if (11) in the form

$$\tilde{x}_t = \begin{bmatrix} x_t \\ \bar{x}_t \end{bmatrix},$$

where $\bar{x}$ is the ε-past of x defined by

$$[\bar{x}_t]_\theta = \begin{cases} x_{t+\theta} & \text{if } t+\theta > 0, \\ \xi^2_{t+\theta} & \text{if } t+\theta \le 0. \end{cases}$$

Then

$$d\tilde{x}_t = (A\tilde{x}_t + Bu_t + F(t,u_t))dt + G(t,x_t,u_t)dw_t, \tag{12}$$

where

$$\tilde{x}_0 = \begin{bmatrix}\xi^1\\ \xi^2\end{bmatrix},\ B = \begin{bmatrix}1\\0\end{bmatrix},\ F(t,u) = \begin{bmatrix} f(t,u) \\ 0\end{bmatrix},\ G(t,\tilde{x},u) = \begin{bmatrix} g(t,x,u)\\ 0\end{bmatrix}.$$

Therefore, the system (11) is completed to the form (12) without delay and fits to the form of (1).

C-controllability of (11) can be deduced without referring to (12) as well. Consider the greatest integer n with $n\varepsilon < T$. It suffices to manage the system (11) on $[n\varepsilon, T]$. On this interval, $x_{t-\varepsilon}$ is a known function of t. We can add it to the nonlinear drift term f and obtain a new nonlinear drift term

$$f_1(t,u) = x_{t-\varepsilon} + f(t,u),$$

noticing that f_1 is still independent on x. If f is a bounded function of t and u then so is f_1 because x is continuous on the compact interval $[(n-1)\varepsilon, n\varepsilon]$ implying boundedness of x on that interval. So, the function f_1 satisfies condition (F) if so is f. Then the system (11) on $[n\varepsilon, T]$ becomes C-controllable since the system (10) has the same property. This implies its C-controllability in the wider interval $[0,T]$.

Following to the second example, C-controllability of the finite dimensional impulsive systems with delays or not can be proved under the conditions (A)–(F). Theorem 1 is not applicable to semilinear systems governed by heat equation because the coercivity of Q_t drops. Calculations demonstrate that for semilinear systems governed by wave equation, the boundedness of $t^3\|Q_t^{-1}\|$ in condition (F) is required. In this regard, Theorem 1 does not cover such systems as well.

7 | CONCLUSION

In this paper, a sufficient condition for C-controllability of a semilinear stochastic control system is proved. C-controllability is an analog of exact controllability for stochastic systems, suggested at the beginning of the period when controllability issue oversteps to stochastic systems. Generally speaking, it is difficult to prove sufficient conditions of exact controllability or its analogs for systems containing nonlinearity and randomness although it is relatively easy for approximate controllability. This is the reason that there is no any capable sufficient condition of exact controllability or its analogs proven in the existing literature for such systems. The sufficient condition from this paper covers many systems, but still needs improvements. It does not cover important second order systems as it was mentioned at the end of Section 6. Also, the independence on x of the nonlinear drift term f in (1) could be considered as a point for improvement. It should be noted in the deterministic case, that is, if $g(t, x, u) = 0$, the dependence of f on x does not create any difficulty[34]. Summarizing, we see essentially two points for improvement of the conditions of Theorem 1:

- The condition (F) requires the boundedness of $t^{1+\alpha}\|Q_t^{-1}\|$, $0 < t \le T$, for $0 \le \alpha \le 1$. In order to cover important second order systems, this condition should be weakened up to boundedness of $t^3\|Q_t^{-1}\|$, $0 < t \le T$.
- The independence of f on x was used in the third step of proof of Theorem 1. It seems there should be a way of modifying this proof for the case when f depends on x as well. An argument, supporting this idea, comes from the deterministic case.

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