Received: 00 Month 0000 | Revised: 00 Month 0000 | Accepted: 00 Month 0000
DOI: xxx/xxxx

ARTICLE TYPE

# Blow up for the solutions of the pressureless Euler-Poisson equations with time-dependent damping†

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**Summary**

The Euler-Poisson equations can be used to describe the important physical phenomena in many areas, such as semiconductor modeling and plasma physics. In this paper, we show the singularity formation mechanism for the solutions of the pressureless Euler-Poisson equations with time-dependent damping for the attractive forces in $\mathbb{R}^n$ $(n \geq 1)$ and the repulsive forces in $\mathbb{R}$. We obtain the blow up of the derivative of the velocity under the appropriate assumptions.

**KEYWORDS:**
Euler-Poisson equations, Damping, Singularity formation

## 1 | INTRODUCTION AND MAIN RESULTS

The compressible Euler-Poisson equations with time-dependent damping in $\mathbb{R}^n$ can be written as following:

$$\begin{cases} \rho_t + \nabla \cdot (\rho U) = 0, \\ \rho[U_t + U \cdot \nabla U] + \nabla p = -\delta \rho \nabla \phi - \frac{\mu}{(1+t)^\lambda} \rho U, \\ -\Delta \phi(t,x) = \rho, \\ t = 0 : \rho = \rho_0(x), U = U_0(x), \end{cases} \tag{1}$$

where $U = U(t,x)$, $\rho = \rho(t,x) \geq 0$, $p(t,x)$, $\phi = \phi(t,x)$ are the velocity, the density, pressure and potential function respectively. $\frac{\mu}{(1+t)^\lambda}$ with $\lambda \geq 0$, $\mu > 0$ is frictional coefficient and $\delta$ is a scaled physical constant.

When $\mu = 0$, the system (1) is the standard isentropic compressible Euler-Poisson equations. Furthermore, if $\delta = 1$, the Euler-Poisson equations can be used as semiconductor model with repulsive forces[8]. If $\delta = -1$, the system can be used to model the gaseous stars in astrophysics with attractive forces[3,7,10]. Jang, Li and Zhang[11] constructed smooth global solutions for the two dimensional isentropic compressible Euler-Poisson system. Tadmor and Wei[19] proved that the one dimensional isentropic compressible Euler-Poisson system admits global solutions for a large class of initial data. Yuen[21] obtained the $n$ dimensional isentropic compressible non-trivial solutions in radial symmetry of the Euler-Poisson equations with repulsive forces ($\delta = 1$) will blow up on or before the finite time $T = \frac{R^3}{2H_0}$ under the condition that the initial data has a compact support in $[0, R]$ and $H_0 = \int_0^R rV_0 dr > 0$. Hereafter, Yuen[22] studied the blowup problem of the non-trivial classical solutions in radial symmetry of the $n$ dimensional isentropic compressible Euler-Poisson system with attractive forces under the condition that the initial configurations have a compact support. Wang[20] obtained the finite time blow-up solutions of high dimensions full Euler-Poisson equations for $n \geq 3$ and the singularity of the isentropic Euler-Poisson equations for a large class of initial data, which is not required the initial data has a compact support.

†This is an example for title footnote.

When $\mu \neq 0, \lambda \neq 0$, the system (1) is the isentropic compressible Euler-Poisson equations with time-dependent damping. Li et al[15] used the time-weighted energy method to get the global smooth solutions of the one dimensional compressible Euler-Poisson equations with time-dependent damping effect $-\frac{\mu}{(1+t)^{\lambda}}$ for $-1 < \lambda < 1$.

In this paper, we will consider the blowup results of the isentropic pressureless Euler-Poisson equations with time-dependent damping. The **pressureless** Euler-Poisson equations with time-dependent damping in $\mathbb{R}^n$ is

$$\begin{cases} \rho_t + \nabla \cdot (\rho U) = 0, \\ \rho[U_t + U \cdot \nabla U] = -\delta \rho \nabla \phi - \frac{\mu}{(1+t)^{\lambda}} \rho U, \\ -\Delta \phi(t,x) = \rho, \\ t = 0 : \rho = \rho_0(x), U = U_0(x). \end{cases} \tag{2}$$

When $\mu = 0$, Liu and Tadmor[18] provided a complete description of the critical threshold phenomenon and gave precise explicit formula for the global solution and finite-time breakdown of two dimensional restricted Euler-Poisson equations. Chae and Tadmor[5] obtained that the $C^1$ solutions of the $n$-dimensional isentropic pressureless Euler-Poisson equations with attractive forces will break down if the initial data $(\rho_0, U_0)$ satisfies

$$S := \{x_0 \in \mathbb{R}^n | \ \rho_0(x_0) > 0, \Omega_0(x_0) = 0, divU_0(x_0) + \sqrt{-n\delta c} < 0\} \neq \emptyset,$$

where the rescaled vorticity matrix $\Omega = (\Omega_{ij}), \Omega_{ij} := \frac{1}{2}(\partial_i U^j - \partial_j U^i), i, j = 1, 2, ..., n$ and the background constant $c = \int \rho_0(x)dx$. Later, Cheng and Tadmor[9] improved the results and obtained that the $C^1$ solutions of n dimensional isentropic pressureless Euler-Poisson equations will breakdown in the condition of $divU_0(x) < \text{sgn}(\rho_0(x) - 1)\sqrt{nF(\rho_0(x))}$. Chae[6] also showed the blowup results of the n dimensional isentropic pressureless Euler-Poisson equations for the $C^1$ solutions under the condition that the initial data satisfies $S = \{x_0 \in \mathbb{R}^n | \ \Omega_0(x_0) = 0, -divU_0(x_0) \geq \sqrt{\frac{2\delta\rho_0(x_0)}{3}} > 0\} \neq \emptyset$. Kwong and Yuen[12] applied the generalized Hubble transformation to obtain the blowup of $C^2$ solutions for the isentropic pressureless Euler-Poisson system with attractive forces and repulsive forces for $\mathbb{R}^n$ $(n \geq 1)$. Lee[13] proved that Riccati system which is a two dimensional isentropic pressureless Euler-Poisson system with attractive forcing has global smooth solutions for a large set of initial configurations. Later, Lee[14] studied the global smooth solutions for the two dimensional isentropic pressureless Euler-Poisson equations with either attractive or repulsive forces under the appropriate assumptions of initial data.

When $\mu \neq 0$, Carrillo and Choi[4] studied the one dimensional isentropic pressureless Euler-Poisson equations with linear damping $(\lambda = 0)$ and non-local interaction forces to get the time-asymptotic behavior of classical solutions with the initial data in the subcritical region. Liu and Fang[17] obtained the blowup results of the solutions to the $n$ dimensional isentropic pressureless Euler-Poisson equations with damping for attractive Poisson forcing under the assumption of initial data satisfying a set of conditions. Bhatnagar and Liu[1] studied the one dimensional isentropic pressureless Euler-Poisson equations with variable background and damping under attractive forces to get the sufficient conditions of the global solutions and blowup results respectively. Furthermore, they also studied the critical threshold phenomenon for one dimensional damped, pressureless Euler-Poisson equations with electric forces to obtain that if the initial data is within the threshold region, the system will have a smooth solution, otherwise it will blowup[2]. Liu and Fang[16] applied the generalized Hubble transformation to obtain the blow up results to the solution of the $n$ dimensional isentropic pressureless Euler-Poisson equations with damping. In this paper, we further study the blowup mechanism for the solutions of the isentropic pressureless Euler-Poisson equations with time-dependent damping (2) with attractive forces in $\mathbb{R}^n (n \geq 1)$ and the repulsive forces in $\mathbb{R}$. The main results are as follows.

**Theorem 1** $(\delta = -1, n \geq 1)$**.** Denote the set $Q = \{x_0 \in \mathbb{R}^n | \ \rho_0(x_0) > 0, \ \Omega_0(x_0) = 0\} \neq \emptyset$, where $\Omega_0(x_0)$ is the vorticity matrix defined by $\Omega_{0ij}(x_0) = \frac{1}{2}[\partial_i U_0^j(x_0) - \partial_j U_0^i(x_0)]$, and let $divU_0(x_0) = \frac{n\dot{a}(0)}{a(0)} \triangleq \frac{na_1}{a_0}$, where $a_0 > 0$ and $a_1 < 0$.

(1) When $\lambda > 1$, if $\mu > 0$, then $divU(t, x_0(t))$ will blow up on or before the finite time $T_0 = -\frac{a_0}{a_1} e^{\frac{\mu}{\lambda - 1}}$.

(2) When $\lambda = 1$,

1° if $0 < \mu < 1$, then $divU(t, x_0(t))$ will blow up on or before the finite time $T_0 = [1 - \frac{a_0(1-\mu)}{a_1}]^{\frac{1}{1-\mu}} - 1$.

2° if $\mu = 1$, then $divU(t, x_0(t))$ will blow up on or before the finite time $T_0 = e^{-\frac{a_0}{a_1}} - 1$.

3° if $\mu > 1$ and $divU_0(x_0) = \frac{na_1}{a_0} < n(1-\mu)$, then $divU(t, x_0(t))$ will blow up on or before the finite time $T_0 = [1 - \frac{a_0(1-\mu)}{a_1}]^{\frac{1}{1-\mu}} - 1$.

(3) When $0 \leq \lambda < 1$, if $\mu > 0$, and $divU(x_0) = \frac{na_1}{a_0} < -\frac{n\mu}{1-\lambda} e^{\frac{\mu}{1-\lambda}}, x_0 \in Q$, then $divU(t, x_0(t))$ will blow up on or before the finite time $T_0 = -\frac{1-\lambda}{\mu} \ln[\frac{a_0 \mu}{a_1(1-\lambda)} + e^{-\frac{\mu}{1-\lambda}}] - 1$.

For the pressureless Euler-Poisson equations with time-dependent damping and the repulsive forces, we have the following result.

**Theorem 2** ($\delta = 1, n = 1$)**.** Denote the set $\mathcal{Q} = \{x_0 \in \mathbb{R} | \ \rho_0(x_0) > 0\} \neq \emptyset$, let $divU_0(x_0) = \frac{\dot{a}(0)}{a(0)} \triangleq \frac{a_1}{a_0}$ with $a_0 > 0$ and $a_1 < 0$.
(1) When $\lambda > 1$, if $\mu > 0$, and $divU_0(x_0) \leq -\sqrt{2\rho_0(x_0)e^{\frac{3\mu}{\lambda-1}}}, x_0 \in \mathcal{Q}$, then $divU(t, x_0(t))$ will blow up on or before the finite time $T_1 = \frac{-a_1 e^{-\frac{\mu}{\lambda-1}} - \sqrt{a_1^2 e^{-\frac{2\mu}{\lambda-1}} - 2\rho_0(x_0)a_0^2 e^{\frac{\mu}{\lambda-1}}}}{\rho_0(x_0)a_0 e^{\frac{\mu}{\lambda-1}}}$.
(2) When $\lambda = 1$,

$1^\circ$ if $\mu = 1$, and for any given $T_0 > 0$, $divU_0(x_0) \leq -\frac{1+\frac{1}{(\mu+1)}\rho_0(x_0)(T_0+\frac{T_0^2}{2})}{\ln(1+T_0)}, x_0 \in \mathcal{Q}$,

$2^\circ$ if $\mu \neq 1$, and for any given $T_0 > 0$, $divU_0(x_0) \leq -\frac{[1+\frac{1}{(\mu+1)}\rho_0(x_0)(T_0+\frac{T_0^2}{2})](1-\mu)}{(1+T_0)^{1-\mu}-1}, x_0 \in \mathcal{Q}$,

then the $C^1$ solutions of the pressureless Euler-Poisson equations with time-dependent damping (2) will blow up on or before the finite time $t = T_0$.
(3) When $0 \leq \lambda < 1$, if $\mu > 0$, and for any given $T_0 > 0$, $divU_0(x_0) \leq -\frac{\mu(1+(\frac{1-\lambda}{\mu})^2\rho_0(x_0)e^{\frac{\mu}{1-\lambda}(1+T_0)})}{(1-\lambda)(e^{-\frac{\mu}{1-\lambda}} - e^{-\frac{\mu}{1-\lambda}(1+T_0)})}, x_0 \in \mathcal{Q}$,
then the $C^1$ solutions of the pressureless Euler-Poisson equations with time-dependent damping (2) will blow up on or before the finite time $t = T_0$.

*Remark 1.* In this paper, we give the blowup mechanism of pressureless Euler-Poisson equations for the case of zero background with time-dependent damping. For the case of nonzero background $-\Delta\phi(t, x) = \rho - c$, where $c$ is given by the average mass $c = \int \rho(t, x)dx = \int \rho_0(x)dx$, we can get the blow up result in the case with $\delta = 1, n = 1$ similarly.

## 2 | THE PROOF OF OUR MAIN RESULTS

In this section, we will use the Hubble transformation to prove the blowup results of the pressureless Euler-Poisson equations with time-dependent damping. By the first equation of the system (2), we obtain

$$\frac{D\rho}{Dt} + \rho\nabla \cdot U = 0, \tag{3}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} U^i \frac{\partial}{\partial x_i}$ is the material derivative. Then, for the $C^1$ solutions, taking $x_0 \in \mathcal{Q}$, we obtain

$$\rho(t, x_0(t)) = \rho_0(x_0)e^{-\int_0^t div\ U(s,x_0(s))ds} > 0. \tag{4}$$

The second equation of system (2) can be recasted as

$$U_t + (U \cdot \nabla)U = -\delta\nabla\phi - \frac{\mu}{(1+t)^\lambda}U. \tag{5}$$

Taking a partial derivative of (5), we have

$$\partial_t V + (U \cdot \nabla)V + V^2 = -\delta\Phi - \frac{\mu}{(1+t)^\lambda}V, \tag{6}$$

where $V = (\partial_i U^j)$, and $\Phi = (\partial_i\partial_j\phi)$ is the Hessian of $\phi$. The symmetric part and the skew-symmetric part of (6) satisfy

$$\frac{D}{Dt}\mathcal{D} = -\mathcal{D}^2 - A^2 - \delta\Phi - \frac{\mu}{(1+t)^\lambda}\mathcal{D}, \tag{7}$$

and

$$\frac{D}{Dt}A = -\mathcal{D}A - A\mathcal{D} - \frac{\mu}{(1+t)^\lambda}A, \tag{8}$$

where $\mathcal{D} = \frac{1}{2}(V + V^T)$, $A = \frac{1}{2}(V - V^T) = \frac{1}{2}\Omega$. We consider evolution along the particle trajectory $\{x = X(t, x_0)\}$, which is defined by the solution of the following ODE,

$$\frac{\partial}{\partial t}X(t, x_0) = U(t, X(t, x_0)), X(0, x_0) = x_0 \in \mathcal{Q}.$$

Noting (8), we have

$$\frac{D}{Dt}|A| \leq 2|\mathcal{D}||A| + \frac{\mu}{(1+t)^\lambda}|A|,$$

where we used the matrix norm, $|M| := \sqrt{\sum_{i,j=1}^{n} M_{ij}^2}$. Then we obtain

$$|A(t, X(t, x_0))| \le |A_0(x_0)| e^{\int_0^t 2|\mathcal{D}| + \frac{\mu}{(1+\tau)^\lambda} d\tau}.$$

Since $A_0(x_0) = \frac{1}{2}\Omega(x_0) = 0, x_0 \in \mathcal{Q}$, then along the particle trajectory $\{x = X(t, x_0)\}$, we have $A(t, X(t, x_0)) = 0$. Hence, taking the trace of (7), we have

$$\frac{D}{Dt}(tr(\mathcal{D})) = -tr(\mathcal{D}^2) - \delta\Delta\phi - \frac{\mu}{(1+t)^\lambda} tr(\mathcal{D}).$$

we observe that $tr(\mathcal{D}) = divU(t, x)$, hence

$$\frac{D}{Dt} divU(t, x_0(t)) = -tr(\mathcal{D}^2) - \delta\Delta\phi - \frac{\mu}{(1+t)^\lambda} divU(t, x_0(t)) \tag{9}$$

Using the spectral dynamics techniques[5,9,12] and Schwarz inequality, we obtain

$$\frac{D}{Dt} div\ U(t, x_0(t)) + \frac{1}{n}[div\ U(t, x_0(t))]^2 \le \delta\rho(t, x_0(t)) - \frac{\mu}{(1+t)^\lambda} div\ U(t, x_0(t)). \tag{10}$$

By (4), we have

$$\frac{D}{Dt} div\ U(t, x_0(t)) + \frac{1}{n}[div\ U(t, x_0(t))]^2 \le \delta\rho_0(x_0) e^{-\int_0^t div\ U(s, x_0(s)) ds} - \frac{\mu}{(1+t)^\lambda} div\ U(t, x_0(t)). \tag{11}$$

By the generalized Hubble transformation $div\ U(t, x_0(t)) = \frac{n\dot{a}(t)}{a(t)}$, we can get

$$\frac{D}{Dt}\frac{n\dot{a}(t)}{a(t)} + \frac{1}{n}[\frac{n\dot{a}(t)}{a(t)}]^2 \le \delta\rho_0(x_0) e^{-\int_0^t \frac{n\dot{a}(s)}{a(s)} ds} - \frac{\mu}{(1+t)^\lambda}\frac{n\dot{a}(t)}{a(t)}. \tag{12}$$

It is not difficult to get

$$\frac{n\ddot{a}(t)}{a(t)} \le \delta\frac{\rho_0(x_0)a(0)^n}{a(t)^n} - \frac{\mu}{(1+t)^\lambda}\frac{n\dot{a}(t)}{a(t)}. \tag{13}$$

Note that the initial data satisfies $a(0) = a_0 > 0$ *and* $\dot{a}(0) = a_1 = \frac{a_0 div\ U_0(x_0)}{n} < 0$. In the following, we set $\beta := \frac{\rho_0(x_0)a(0)^n}{n}$, which is a positive number. Therefore, the inequality (13) becomes

$$\ddot{a}(t) + \frac{\mu}{(1+t)^\lambda}\dot{a}(t) \le \delta\frac{\rho_0(x_0)a(0)^n}{na(t)^{n-1}} = \frac{\delta\beta}{a(t)^{n-1}}. \tag{14}$$

Then we can get $a(t) > 0$ for all $t \in \mathbb{R}^+$. Otherwise, if there exists a finite time $T_0 > 0$, such that $a(T_0) \le 0$. Since $a(t)$ is a continuous function, then there exists a finite time $T_1 > 0$, such that $a(T_1) = 0$. Thus, $div\ U(t, x_0(t))$ will blow up on $t = T_1$.

In the following, we will give the proof of the singularity formation for system (2) case by case.

*The proof of Theorem 1.* For the case of attractive forces ($\delta = -1$) and $n \ge 1$, multiplying (14) by $e^{\int_0^t \frac{\mu}{(1+s)^\lambda} ds}$, we have

$$(e^{\int_0^t \frac{\mu}{(1+s)^\lambda} ds}\dot{a}(t))^{\cdot} \le -e^{\int_0^t \frac{\mu}{(1+s)^\lambda} ds}\frac{\beta}{a(t)^{n-1}} \le 0. \tag{15}$$

By integrating the above inequality, we can get

$$\dot{a}(t) \le a_1 e^{-\int_0^t \frac{\mu}{(1+s)^\lambda} ds}. \tag{16}$$

(1) When $\lambda > 1, \mu > 0$. Since

$$e^{-\frac{\mu}{\lambda-1}} \le e^{-\int_0^t \frac{\mu}{(1+s)^\lambda} ds} = e^{-\frac{\mu}{1-\lambda}[(1+t)^{1-\lambda}-1]} \le 1.$$

Thus

$$\dot{a}(t) \le a_1 e^{-\int_0^t \frac{\mu}{(1+s)^\lambda} ds} \le a_1 e^{-\frac{\mu}{\lambda-1}}. \tag{17}$$

Integrating the above inequality, we can get

$$a(t) \le a_1 e^{-\frac{\mu}{\lambda-1}} t + a_0. \tag{18}$$

Therefore, $divU(t, x_0(t))$ will blow up on or before the finite time $T_0 = -\frac{a_0}{a_1} e^{\frac{\mu}{\lambda-1}}$.

(2) When $\lambda = 1, \mu > 0$, since

$$e^{-\int_0^t \frac{\mu}{(1+s)} ds} = e^{-\mu\ln(1+t)} = (1+t)^{-\mu}.$$

Thus

$$\dot{a}(t) \le a_1(1+t)^{-\mu}. \tag{19}$$

Integrating the above inequality, we obtain

$$a(t) \le a_1 \int_0^t (1+s)^{-\mu} ds + a_0. \tag{20}$$

$1^\circ$ When $\mu = 1$, we have $a(t) \le \ln(1+t)a_1 + a_0$.
Therefore, $divU(t, x_0(t))$ will blow up on or before the finite time $T_0 = e^{-\frac{a_0}{a_1}} - 1$.
$2^\circ$ When $0 < \mu < 1$, we have $a(t) \le \frac{a_1}{1-\mu}(1+t)^{1-\mu} - \frac{a_1}{1-\mu} + a_0$.
Therefore, $divU(t, x_0(t))$ will blow up on or before the finite time $T_0 = [1 - \frac{a_0(1-\mu)}{a_1}]^{\frac{1}{1-\mu}} - 1$.
$3^\circ$ When $\mu > 1$, we have $a(t) \le \frac{a_1}{1-\mu}(1+t)^{1-\mu} - \frac{a_1}{1-\mu} + a_0$.
Therefore, if $divU_0(x_0) = \frac{na_1}{a_0} < n(1-\mu)$, then $divU(t, x_0(t))$ will blow up on or before the finite time $T_0 = [1 - \frac{a_0(1-\mu)}{a_1}]^{\frac{1}{1-\mu}} - 1$.

(3) When $0 \le \lambda < 1, \mu > 0$, we have

$$e^{-\frac{\mu}{1-\lambda}(1+t)} \le e^{-\int_0^t \frac{\mu}{(1+s)^\lambda} ds} \le 1.$$

Thus

$$\dot{a}(t) \le a_1 e^{-\frac{\mu}{1-\lambda}(1+t)}. \tag{21}$$

Integrating the above inequality, we can get

$$a(t) \le a_1 \frac{1-\lambda}{\mu}[e^{-\frac{\mu}{1-\lambda}} - e^{-\frac{\mu}{1-\lambda}(1+t)}] + a_0. \tag{22}$$

Therefore, if $divU(x_0) = \frac{na_1}{a_0} < -\frac{n\mu}{1-\lambda}e^{\frac{\mu}{1-\lambda}}$, then $divU(t, x_0(t))$ will blow up on or before the finite time $T_0 = -\frac{1-\lambda}{\mu}\ln[\frac{a_0\mu}{a_1(1-\lambda)} + e^{-\frac{\mu}{1-\lambda}}] - 1$. □

Then, we can get the conclusion of the system (2) with the attractive force.

*The proof of Theorem 2.* For the case of repulsive forcing ($\delta = 1$) and $n = 1$, by inequality (10), we can get

$$\frac{D}{Dt} div\, U(t, x_0(t)) + [div\, U(t, x_0(t))]^2 \le \rho(t, x_0(t)) - \frac{\mu}{(1+t)^\lambda} div\, U(t, x_0(t)). \tag{23}$$

Then, we have

$$\frac{D}{Dt}\frac{\dot{a}(t)}{a(t)} + [\frac{\dot{a}(t)}{a(t)}]^2 \le \rho_0(x_0)e^{-\int_0^t \frac{\dot{a}(s)}{a(s)} ds} - \frac{\mu}{(1+t)^\lambda}\frac{\dot{a}(t)}{a(t)}. \tag{24}$$

Hence, we can get

$$\ddot{a}(t) + \frac{\mu}{(1+t)^\lambda}\dot{a}(t) \le \rho_0(x_0)a_0. \tag{25}$$

Multiplying (25) by weight function $e^{\int_0^t \frac{\mu}{(1+s)^\lambda} ds}$ to get

$$(e^{\int_0^t \frac{\mu}{(1+s)^\lambda} ds}\dot{a}(t))^\cdot \le \rho_0(x_0)a_0 e^{\int_0^t \frac{\mu}{(1+s)^\lambda} ds}. \tag{26}$$

Integrating the above inequality, we have

$$e^{\int_0^t \frac{\mu}{(1+s)^\lambda} ds}\dot{a}(t) \le \rho_0(x_0)a_0 \int_0^t e^{\int_0^s \frac{\mu}{(1+\xi)^\lambda} d\xi} ds + a_1. \tag{27}$$

(1) For $\lambda > 1, \mu > 0$, we have

$$e^{\int_0^t \frac{\mu}{(1+s)^\lambda} ds} = e^{\frac{\mu}{1-\lambda}(1+t)^{1-\lambda} - \frac{\mu}{1-\lambda}} \le e^{\frac{\mu}{\lambda-1}}.$$

Then we can get

$$\dot{a}(t) \le \rho_0(x_0)a_0 e^{\frac{\mu}{\lambda-1}} t + a_1 e^{-\frac{\mu}{\lambda-1}}. \tag{28}$$

Thus

$$a(t) \le \frac{1}{2}\rho_0(x_0)a_0 e^{\frac{\mu}{\lambda-1}} t^2 + a_1 e^{-\frac{\mu}{\lambda-1}} t + a_0. \tag{29}$$

Therefore, as long as $divU_0(x_0) \le -\sqrt{2\rho_0(x_0)e^{\frac{3\mu}{\lambda-1}}}$, then $divU(t, x_0(t))$ will blow up on or before the finite time $T_0 = \frac{-a_1 e^{-\frac{\mu}{\lambda-1}} - \sqrt{a_1^2 e^{-\frac{2\mu}{\lambda-1}} - 2\rho_0(x_0)a_0^2 e^{\frac{\mu}{\lambda-1}}}}{\rho_0(x_0)a_0 e^{\frac{\mu}{\lambda-1}}}$.

(2) For $\lambda = 1, \mu > 0$, since

$$e^{\int_0^s \frac{\mu}{(1+\xi)} d\xi} = (1+s)^\mu, \quad \int_0^t (1+s)^\mu ds \leq \frac{1}{1+\mu}(1+t)^{1+\mu}.$$

From inequality (27), we have

$$\dot{a}(t) \leq \frac{1}{1+\mu}\rho_0(x_0)a_0(1+t) + a_1(1+t)^{-\mu}. \tag{30}$$

Then

$$a(t) \leq \frac{1}{(\mu+1)}\rho_0(x_0)a_0(t+\frac{t^2}{2}) + a_1\int_0^t (1+s)^{-\mu}ds + a_0. \tag{31}$$

1° If $\mu = 1$, then $a(t) \leq \frac{1}{(\mu+1)}\rho_0(x_0)a_0(t+\frac{t^2}{2}) + a_1\ln(1+t) + a_0$.

Therefore, for any given time $T_0 > 0$, as long as $divU_0(x_0) = \frac{a_1}{a_0} \leq -\frac{1+\frac{1}{(\mu+1)}\rho_0(x_0)(T_0+\frac{T_0^2}{2})}{\ln(1+T_0)}$, then $a(T_0) \leq 0$. Hence, $divU(t, x_0(t))$ will blow up on or before the finite time $T_0$.

2° If $\mu \neq 1$, we have $a(t) \leq \frac{1}{(\mu+1)}\rho_0(x_0)a_0(t+\frac{t^2}{2}) + \frac{a_1}{1-\mu}[(1+t)^{1-\mu} - 1] + a_0$.

Therefore, for any given time $T_0 > 0$, as long as $divU_0(x_0) = \frac{a_1}{a_0} \leq -\frac{[1+\frac{1}{(\mu+1)}\rho_0(x_0)(T_0+\frac{T_0^2}{2})](1-\mu)}{(1+T_0)^{1-\mu}-1}$, then $a(T_0) \leq 0$. Hence, $divU(t, x_0(t))$ will blow up on or before the finite time $T_0$.

(3) For $0 \leq \lambda < 1, \mu > 0$, we have

$$e^{\int_0^s \frac{\mu}{(1+\xi)^\lambda} d\xi} = e^{\frac{\mu}{1-\lambda}[(1+s)^{1-\lambda}-1]} \leq e^{\frac{\mu}{1-\lambda}(1+s)},$$

$$\int_0^t e^{\frac{\mu}{1-\lambda}(1+s)}ds \leq \frac{1-\lambda}{\mu}e^{\frac{\mu}{1-\lambda}(1+t)}.$$

Then

$$\dot{a}(t) \leq \frac{1-\lambda}{\mu}\rho_0(x_0)a_0e^{\frac{\mu}{1-\lambda}(1+t)} + a_1e^{-\frac{\mu}{1-\lambda}(1+t)}. \tag{32}$$

Integrating inequality (32), we can get

$$a(t) \leq [\frac{1-\lambda}{\mu}]^2\rho_0(x_0)a_0e^{\frac{\mu}{1-\lambda}(1+t)} + a_1\frac{1-\lambda}{\mu}[e^{-\frac{\mu}{1-\lambda}} - e^{-\frac{\mu}{1-\lambda}(1+t)}] + a_0. \tag{33}$$

Therefore, for any given time $T_0 > 0$, as long as $divU_0(x_0) = \frac{a_1}{a_0} \leq -\frac{\mu(1+(\frac{1-\lambda}{\mu})^2\rho_0(x_0)e^{\frac{\mu}{1-\lambda}(1+T_0)})}{(1-\lambda)(e^{-\frac{\mu}{1-\lambda}}-e^{-\frac{\mu}{1-\lambda}(1+T_0)})}$, then $a(T_0) \leq 0$. Hence, $divU(t, x_0(t))$ will blow up on or before the finite time $T_0$.

In conclusion, we can get the proof of the Theorem 2. □

## CONFLICT OF INTEREST

This work does not have any conflicts of interest.

## FUNDING INFORMATION

This work was supported by National Natural Science Foundation of China under Grant No. 11771274 and Natural Science Foundation of Shanghai under Grant No. 20ZR1419400.

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