The Improvement of Discrete Wavelet Transform*

Zhihua Zhang

Taishan Distinguished Professor, School of Mathematics, Shandong University, Jinan, China, 250100.

E-mail: zhangzhihua@sdu.edu.cn

Abstract. Discrete wavelet transform and discrete periodic wavelet transform have been widely used in image compression and data approximation. Due to discontinuity on the boundary of original data, the decay rate of the obtained wavelet coefficients is slow. In this study, we use the combination of polynomial interpolation and one-dimensional/two-dimensional discrete periodic wavelet transforms to mitigate boundary effects. The decay rate of the obtained wavelet coefficients in our improved algorithm is faster than that of traditional two-dimensional discrete wavelet transform. Moreover, our improved algorithm can be extended naturally to the higher-dimensional case.

Key words: Discrete wavelet transform; Boundary effects, Biorthonomal periodic wavelet; Fast approximation algorithm

1. Introduction

Discrete wavelet transform and discrete periodic wavelet transform have been widely used in signal processing and data compression [1-6]. For a discrete data $\{x_{m,n}\}_{n_1,n_2=0,1,...,2^{J-1}}$, one can pad $\{x_{m,n}\}_{n_1,n_2=0,1,...,2^{J-1}}$ with zeros and do the discrete wavelet transform, or one can extend $\{x_{m,n}\}_{n_1,n_2=0,1,...,2^{J-1}}$ into a periodic data and then do the discrete periodic wavelet transform. Due to discontinuity on the boundary (i.e., $n_1 = 0, 2^{J-1}$ or $n_2 = 0, 2^{J-1}$), the decay rate of the obtained wavelet coefficients is very slow.

In this study, we will improve discrete (periodic) wavelet transform in order to mitigate boundary effects.

*This research was partially supported by European Commissions Horizon2020 Framework Program No 861584 and Taishan Distinguished Professor Fund.

The core idea of our algorithm is to decompose the discrete data $\{x_{n_1,n_2}\}_{n_1,n_2=0,1,\dots,2^{J-1}}$ as

$$x_{n_1,n_2} = y^{(2)}_{n_1,n_2} - y^{(1)}_{n_1,n_2} + z_{n_1,n_2}, \tag{1.1}$$

where

$$\begin{aligned} y^{(1)}_{n_1,n_2} &= x_{0,0}(1-\tfrac{2n_1}{2^J})(1-\tfrac{2n_2}{2^J}) + x_{0,2^{J-1}}(1-\tfrac{2n_1}{2^J})(\tfrac{2n_2}{2^J}) + x_{2^{J-1},0}(\tfrac{2n_1}{2^J})(1-\tfrac{2n_2}{2^J}) + x_{2^{J-1},2^{J-1}}(\tfrac{2n_1}{2^J})(\tfrac{2n_2}{2^J}), \\ y^{(2)}_{n_1,n_2} &= x_{0,n_2}(1-\tfrac{2n_1}{2^J}) + x_{2^{J-1},n_2}(\tfrac{2n_1}{2^J}) + x_{n_1,0}(1-\tfrac{2n_2}{2^J}) + x_{n_1,2^{J-1}}(\tfrac{2n_2}{2^J}). \end{aligned}$$

The computation of $y^{(1)}_{n_1,n_2}$ depends on the values $\{x_{0,0}\}$, $\{x_{0,2^{J-1}}\}$, $\{x_{2^{J-1},0}\}$ and $\{x_{2^{J-1},2^{J-1}}\}$. The computation of $y^{(2)}_{n_1,n_2}$ depends on the boundary data $\{x_{n_1,0}\}$, $\{x_{n_1,2^{J-1}}\}$, $\{x_{0,n_2}\}$ and $\{x_{2^{J-1},n_2}\}$ which is further decomposed as

$$\begin{aligned} x_{n_1,0} &= x_{0,0}(1-\tfrac{2n_1}{2^J}) + x_{2^{J-1},0}(\tfrac{2n_1}{2^J}) + w^0_{n_1}, \\ x_{n_1,2^{J-1}} &= x_{0,2^{J-1}}(1-\tfrac{2n_1}{2^J}) + x_{2^{J-1},2^{J-1}}(\tfrac{2n_1}{2^J}) + w^1_{n_1}, \\ x_{0,n_2} &= x_{0,0}(1-\tfrac{2n_2}{2^J}) + x_{0,2^{J-1}}(\tfrac{2n_2}{2^J}) + v^0_{n_2}, \\ x_{2^{J-1},n_2} &= x_{2^{J-1},0}(1-\tfrac{2n_2}{2^J}) + x_{2^{J-1},2^{J-1}}(\tfrac{2n_2}{2^J}) + v^1_{n_2}, \end{aligned} \tag{1.2}$$

After that, we do an odd extension and then periodic extension for the data $\{w^0_{n_1}\}$, $\{w^1_{n_1}\}$, $\{v^0_{n_2}\}$, $\{v^1_{n_2}\}$ and $\{z_{n_1,n_2}\}$. By (1.1) and (1.2), it follows that for $n_1 = 0, 2^{J-1}$ or $n_2 = 0, 2^{J-1}$

$$z_{n_1,n_2} = 0, \qquad w^0_{n_1} = 0, \qquad w^1_{n_1} = 0, \qquad v^0_{n_2} = 0, \qquad v^1_{n_2} = 0,$$

so the above odd and periodic extension guarantees continuity and differentiability on the boundary of original data. Finally we do one-dimensional and two-dimensional discrete wavelet transform for these extension data, and the obtained wavelet coefficients decay fast. When the original data is smooth (e.g. CMIP6 data [13,14]), Formulas (1.1)-(1.2) will be replaced by (4.2)-(4.4) and (4.7) (see Algorithms in Section 4). Moreover, our improved wavelet algorithm can be extended naturally to the higher-dimensional case.

This paper is organized as follows. In Section 2, we state core theory on one- and two-dimensional biorthonomal wavelets and related discrete wavelet transform/discrete periodic wavelet transform. In Section 3, in order to explain the advantages of our algorithm over traditional wavelet methods, we discuss continuous version of our improved algorithm. Finally, in Section 4, we propose our improvement for traditional discrete wavelet transform/discrete periodic wavelet transform.

2. Preliminaries

In this section, we will state known results on one- and two-dimensional biorthonomal wavelets [1-2, 7-12]. Let $\psi(t)$ and $\widetilde{\psi}(t)$ be a pair of smooth real-valued biorthonomal wavelets in $L^2(\mathbb{R})$ generated by real-valued smooth scaling functions $\varphi(t)$ and $\widetilde{\varphi}(t)$. Take the tensor product of $\varphi(t)$ and $\psi(t)$:

$$\begin{aligned} &\varphi_0(t_1,t_2)=\varphi(t_1)\varphi(t_2), \qquad \psi_1(t_1,t_2)=\varphi(t_1)\psi(t_2),\\ &\psi_2(t_1,t_2)=\psi(t_1)\varphi(t_2), \qquad \psi_3(t_1,t_2)=\psi(t_1)\psi(t_2). \end{aligned} \tag{2.1}$$

Similarly, taking the tensor products of $\widetilde{\varphi}(t)$ and $\widetilde{\psi}(t)$, we get $\widetilde{\varphi}(t_1,t_2)$, $\widetilde{\psi}_1(t_1,t_2)$, $\widetilde{\psi}_2(t_1,t_2)$, and $\widetilde{\psi}_3(t_1,t_2)$. Then $\{\psi_\mu(t_1,t_2)\}_{\mu=1,2,3}$ and $\{\widetilde{\psi}_\mu(t_1,t_2)\}_{\mu=1,2,3}$ are a pair of two-dimensional biorthonomal wavelets of $L^2(\mathbb{R}^2)$.

Denote

$$\psi_{\mu,m,\mathbf{n}}(\mathbf{t}) =: 2^m\psi_\mu(2^m\mathbf{t}-\mathbf{n}) = 2^m\psi_\mu(2^m t_1-n_1, 2^m t_2-n_2) \qquad (\mathbf{t}=(t_1,t_2),\ \mathbf{n}=(n_1,n_2)).$$

Any smooth function $f\in C^l(\mathbb{R}^2)$ can be expanded into a biorthonomal periodic wavelet series:

$$f(\mathbf{t}) = \sum_{\mathbf{n}} c^w_{m,\mathbf{n}}\widetilde{\varphi}_{0,m,\mathbf{n}}(\mathbf{t}) + \sum_{\mu=1}^{3}\sum_{m=0}^{+\infty}\sum_{\mathbf{n}} d^w_{\mu,m,\mathbf{n}}\widetilde{\psi}_{\mu,m,\mathbf{n}}(\mathbf{t}),$$

where

$$c^w_{m,\mathbf{n}} = \int_{\mathbb{R}}\int_{\mathbb{R}} f(\mathbf{t})\overline{\varphi}_{0,m,\mathbf{n}}(\mathbf{t})\,d\mathbf{t}, \qquad d^w_{\mu,m,\mathbf{n}} = \int_{\mathbb{R}}\int_{\mathbb{R}} f(\mathbf{t})\overline{\psi}_{\mu,m,\mathbf{n}}(\mathbf{t})\,d\mathbf{t}. \tag{2.2}$$

Proposition 2.1.[1-2] If φ, $\widetilde{\varphi}$, ψ and $\widetilde{\psi}$ are real-valued and compactly supported, and $\varphi,\widetilde{\varphi},\psi,\widetilde{\psi}\in C^l(\mathbb{R})$, then for any smooth function $f\in C^l(\mathbb{R}^2)$, the following estimate holds

$$d^w_{\mu,m,\mathbf{n}} = O(2^{-m(l+1)})$$

Denote wavelet filter banks by $\{h_k\}$ and $\{\tau_k\}$:

$$\begin{aligned} h_k &= \sqrt{2}\int_{\mathbb{R}}\varphi(t)\widetilde{\varphi}(2t-k)dt,\\ \tau_k &= \sqrt{2}\int_{\mathbb{R}}\psi(t)\widetilde{\varphi}(2t-k)dt. \end{aligned} \tag{2.3}$$

The one-dimensional wavelet coefficients satisfy [1]:

$$\begin{aligned} c^w_{m-1,k} &= \sum_n h_{n-2k} c^w_{m,n}, \\ d^w_{m-1,k} &= \sum_n \tau_{n-2k} c^w_{m,n}, \end{aligned} \tag{2.4}$$

and the two-dimensional wavelet coefficients satisfy [1]

$$\begin{aligned} c^w_{m-1,k_1,k_2} &= \sum_{n_1,n_2} h_{n_1-2k_1}\ h_{n_2-2k_2} c^w_{m,n_1,n_2}, \\ d^w_{1,m-1,k_1,k_2} &= \sum_{n_1,n_2} h_{n_1-2k_1}\ \tau_{n_2-2k_2} c^w_{m,n_1,n_2}, \\ d^w_{2,m-1,k_1,k_2} &= \sum_{n_1,n_2} \tau_{n_1-2k_1}\ h_{n_2-2k_2} c^w_{m,n_1,n_2}, \\ d^w_{3,m-1,k_1,k_2} &= \sum_{n_1,n_2} \tau_{n_1-2k_1}\ \tau_{n_2-2k_2} c^w_{m,n_1,n_2}, \end{aligned} \tag{2.5}$$

Formulas (2.4)-(2.5) are called the one-dimensional and two-dimensional *Discrete Wavelet Transform*, respectively [1-2].

Next, we discuss biorthonomal periodic wavelets. Denote the periodization of any function $h(\mathbf{t})$ by

$$h^{per}(\mathbf{t}) =: \sum_{\mathbf{k}\in\mathbb{Z}^2} h(\mathbf{t}+\mathbf{k}).$$

The families generated by the periodization of φ, ψ, $\widetilde{\varphi}$ and $\widetilde{\psi}$

$$\psi^{per} = \{\varphi_0^{per}\} \bigcup \{\psi^{per}_{\mu,m,\mathbf{n}},\ \mu = 1,2,3,\ m = 0,1,2,...,\ n_1,n_2 = 0,1,...,2^m-1\},$$

$$\widetilde{\psi}^{per} = \{\widetilde{\varphi}_0^{per}\} \bigcup \{\widetilde{\psi}^{per}_{\mu,m,\mathbf{n}},\ \mu = 1,2,3,\ m = 0,1,2,...,\ n_1,n_2 = 0,1,...,2^m-1\}$$

are a pair of biorthonomal periodic wavelet basis for $L^2([-\frac{1}{2},\frac{1}{2}]^2)$ [1-2]. Any periodic function $f\in C^l([-\frac{1}{2},\frac{1}{2}]^2)$ can be expanded into biorthonomal periodic wavelet series:

$$f(\mathbf{t}) = c_{0,\mathbf{0}} + \sum_{\mu=1}^{3}\sum_{m=0}^{\infty}\sum_{n_1,n_2=0}^{2^m-1} d_{\mu,m,\mathbf{n}} \widetilde{\psi}^{per}_{\mu,m,\mathbf{n}}(\mathbf{t}),$$

where

$$c_{m,\mathbf{n}} = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\mathbf{t})\overline{\varphi}^{per}_{0,m,\mathbf{n}}(\mathbf{t})\,d\mathbf{t}, \qquad d_{\mu,m,\mathbf{n}} = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\mathbf{t})\overline{\psi}^{per}_{\mu,m,\mathbf{n}}(\mathbf{t})\,d\mathbf{t}. \tag{2.6}$$

Similar to the proof of Proposition 2.1, we can deduce

Proposition 2.2. If φ, $\widetilde{\varphi}$, ψ and $\widetilde{\psi}$ are real-valued and compactly supported, and $\varphi, \widetilde{\varphi}, \psi, \widetilde{\psi} \in C^l(\mathbb{R})$, then for any smooth periodic function $f \in C^l([-\frac{1}{2}, \frac{1}{2}]^2) \bigcap C^l(\mathbb{R}^2)$, the following estimate holds

$$d_{\mu,m,\mathbf{n}} = O(2^{-m(l+1)}).$$

Since $\varphi(t)$ and $\widetilde{\varphi}(t)$ are compactly supported, by (2.3), we can assume that for a positive integer N,

$$h_k = \tau_k = 0 \qquad (|k| \geq N).$$

Take $2^{m_0-1} > N$. Define $2^{m_0}-$periodic sequences $\{h_n^*\}$ and $\{\tau_n^*\}$ such that

$$h_k^* = h_k, \qquad \tau_k^* = \tau_k \qquad (|k| \leq 2^{m_0-1}),$$

$$h_{k+2^{m_0}}^* = h_k^*, \qquad \tau_{k+2^{m_0}}^* = \tau_k^* \qquad (k \in \mathbb{Z}).$$

For $m \geq m_0$, the one-dimensional periodic wavelet coefficients satisfy [1]:

$$\begin{aligned} c_{m-1,k} &= \sum_{n=0}^{2^m-1} h_{n-2k}^* c_{m,n}, \\ d_{m-1,k} &= \sum_{n=0}^{2^m-1} \tau_{n-2k}^* c_{m,n}, \end{aligned} \tag{2.7}$$

and the two-dimensional periodic wavelet coefficients satisfy [1]

$$\begin{aligned} c_{m-1,k_1,k_2} &= \sum_{n_1,n_2=0}^{2^m-1} h_{n_1-2k_1}^* \; h_{n_2-2k_2}^* c_{m,n_1,n_2}, \\ d_{1,m-1,k_1,k_2} &= \sum_{n_1,n_2=0}^{2^m-1} h_{n_1-2k_1}^* \; \tau_{n_2-2k_2}^* c_{m,n_1,n_2}, \\ d_{2,m-1,k_1,k_2} &= \sum_{n_1,n_2=0}^{2^m-1} \tau_{n_1-2k_1}^* \; h_{n_2-2k_2}^* c_{m,n_1,n_2}, \\ d_{3,m-1,k_1,k_2} &= \sum_{n_1,n_2=0}^{2^m-1} \tau_{n_1-2k_1}^* \; \tau_{n_2-2k_2}^* c_{m,n_1,n_2}, \end{aligned} \tag{2.8}$$

Formulas (2.7) and (2.8) are said to be one-dimensional and two-dimensional *Discrete Periodic Wavelet Transform*, respectively [1-2]. Since the convolution of finite length data is always computed approximately by circular convolution in signal processing and data analysis [15], there is no big difference between (2.4)-(2.5) and (2.7)-(2.8).

3. Continuous version of our improved algorithm

In order to explain the advantages of our algorithm over traditional wavelet methods, we discuss continuous version of our improved wavelet algorithm. Denote $D^{(\alpha,\beta)}f = \frac{\partial^{\alpha+\beta}}{\partial t_1^\alpha \partial t_2^\beta} f$. If $D^{(\alpha,\beta)}f$ is continuous on the region Ω for all $\alpha, \beta \le l$, we say $f \in C^l(\Omega)$. Let the fundamental polynomial $p_m(t)$ be a univariate polynomial of degree $2m+1$ satisfying

$$D^{(2\lambda)} p_m(0) = 0, \qquad D^{(2\lambda)} p_m(1) = \delta_{\lambda,m}, \tag{3.1}$$

where $\delta_{\lambda,m} = 0\,(\lambda \ne m)$ and $\delta_{\lambda,m} = 1\,(\lambda = m)$. Then $p_m(t)$ can be represented as follows:

$$p_m(t) = \frac{1}{(2m+1)!} t^{2m+1} + \sum_{k=0}^{m-1} c_k t^{2k+1},$$

where the coefficients $\{c_k\}_{k=0,\dots,m-1}$ satisfy

$$\sum_{k=j}^{m-1} \frac{(2k+1)!}{(2k-2j+1)!} c_k = -\frac{1}{(2m-2j+1)!} \qquad (j = 0, 1, ..., m-1),$$

For any function $f \in C^l([0, \frac{1}{2}])$, define its interpolation polynomial of f at the nodes 0 and $\frac{1}{2}$:

$$h(t) = \sum_{k=0}^{n} \frac{1}{2^{2k}} \left(D^{(2k)} f(0) p_k(1-2t) + D^{(2k)} f\left(\frac{1}{2}\right) p_k(2t) \right) \qquad (0 \le t \le \frac{1}{2}).$$

where $n = [\frac{l}{2}]$ and $[\cdot]$ represents the integral part. Then $h(t)$ satisfies

$$D^{(2j)} h(0) = D^{(2j)} f(0),\ D^{(2j)} h(\frac{1}{2}) = D^{(2j)} f(\frac{1}{2}) \qquad (j = 0, 1, ..., n).$$

Let $r(t) = f(t) - h(t)\,(t \in [0, \frac{1}{2}])$. Then

$$r^{(2j)}(0) = r^{(2j)}(\frac{1}{2}) = 0\,(j = 0, 1, ..., n). \tag{3.2}$$

For $r(t)$, we do odd extension, denoted by $r^o(t)$. Again do 1-periodic extension of $r^o(t)$, we get $\widetilde{r}(t)$. By (3.2), it follows that $\widetilde{r} \in C^l(\mathbb{R})$, i.e., the univariate function $f \in C^l([0, \frac{1}{2}])$ can be decomposed as:

$$f(t) = h(t) + \widetilde{r}(t),$$

where $h(t)$ is the interpolation polynomial of f at the nodes 0 and $\frac{1}{2}$, and $\widetilde{r}(t)$ is a 1-periodic odd function and $\widetilde{r} \in C^l(\mathbb{R})$.

For a bivariate smooth function $f \in C^l([0, \frac{1}{2}]^2)$, denote $\mathbf{t} = (t_1, t_2)$ and $n = \left[\frac{l}{2}\right]$. Define $\tau_1(\mathbf{t})$ as the interpolation polynomial of f at vertices of $[0, \frac{1}{2}]^2$:

$$\begin{aligned}\tau_1(\mathbf{t}) = \sum_{\alpha,\beta=0}^{n} 2^{-2\alpha-2\beta}[\, & D^{(2\alpha,2\beta)}f(0,0)p_\alpha(1-2t_1)p_\beta(1-2t_2) \\ & +D^{(2\alpha,2\beta)}f(0,\tfrac{1}{2})p_\alpha(1-2t_1)p_\beta(2t_2) \\ & +D^{(2\alpha,2\beta)}f(\tfrac{1}{2},0)p_\alpha(2t_1)p_\beta(1-2t_2) \\ & +D^{(2\alpha,2\beta)}f(\tfrac{1}{2},\tfrac{1}{2})p_\alpha(2t_1)p_\beta(2t_2)\,],\end{aligned} \tag{3.3}$$

such that for $0 \le \alpha, \beta \le n$ and $(t_1, t_2) = (0,0), (0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})$,

$$D^{(2\alpha,2\beta)}\tau_1(t_1, t_2) = D^{(2\alpha,2\beta)}f(t_1, t_2).$$

Define

$$\begin{aligned}\tau_2(\mathbf{t}) = & \sum_{\alpha=0}^{n} \tfrac{1}{2^{2\alpha}}[D^{(2\alpha,0)}f(0,t_2)p_\alpha(1-2t_1) + D^{(2\alpha,0)}f(\tfrac{1}{2},t_2)p_\alpha(2t_1)] \\ & + \sum_{\beta=0}^{n} \tfrac{1}{2^{2\beta}}[D^{(0,2\beta)}f(t_1,0)p_\beta(1-2t_2) + D^{(0,2\beta)}f(t_1,\tfrac{1}{2})p_\beta(2t_2)].\end{aligned} \tag{3.4}$$

Theorem 3.1. Let $f \in C^l([0, \frac{1}{2}]^2)$. Then the following decomposition formula holds:

$$f(\mathbf{t}) = \tau_2(\mathbf{t}) - \tau_1(\mathbf{t}) + r(\mathbf{t}) \qquad (\mathbf{t} \in [0, \frac{1}{2}]^2), \tag{3.5}$$

and $r(\mathbf{t})$ can be extended into a 1-periodic odd function $\widetilde{r}(\mathbf{t})$ and $\widetilde{r} \in C^l(\mathbb{R}^2)$.

Proof. By (3.1), we know that $D^{(2\mu)}p_\alpha(1) = \delta_{\alpha,\mu}$ and $D^{(2\mu)}p_\alpha(0) = 0$. Again by (3.3), for $0 \le \mu, \nu \le n$,

$$D^{(2\mu,2\nu)}(\tau_1(0,t_2)) = \sum_{\beta=0}^{n}[\, D^{(2\mu,2\beta)}f(0,0)D^{(2\nu)}p_\beta(1-2t_2) + D^{(2\mu,2\beta)}f(0,\frac{1}{2})D^{(2\nu)}p_\beta(2t_2)\,].$$

By (3.4), we deduce that for $0 \le \mu, \nu \le n$ and $0 \le t_2 \le \frac{1}{2}$,

$$D^{(2\mu,2\nu)}(\tau_2(0,t_2)) = D^{(2\mu,2\nu)}f(0,t_2) + \sum_{\beta=0}^{n}[\, D^{(2\mu,2\beta)}f(0,0)D^{(2\nu)}p_\beta(1-2t_2) + D^{(2\mu,2\beta)}f(0,\frac{1}{2})D^{(2\nu)}p_\beta(2t_2)\,].$$

Therefore, for $0 \le \mu, \nu \le n$,

$$D^{(2\mu,2\nu)}(\tau_2(0,t_2) - \tau_1(0,t_2)) = D^{(2\mu,2\nu)}f(0,t_2) \qquad (0 \le t_2 \le \frac{1}{2}).$$

By (3.5), it means that for $0 \le \mu, \nu \le n$,

$$D^{(2\mu,2\nu)} r(0, t_2) = 0 \qquad (0 \le t_2 \le \frac{1}{2}).$$

Similarly, we can deduce that for $0 \le \mu, \nu \le n$,

$$\begin{aligned} &D^{(2\mu,2\nu)} r(\tfrac{1}{2}, t_2) = 0 && (0 \le t_2 \le \tfrac{1}{2}), \\ &D^{(2\mu,2\nu)} r(t_1, 0) = D^{(2\mu,2\nu)} r(t_1, \tfrac{1}{2}) = 0 && (0 \le t_1 \le \tfrac{1}{2}). \end{aligned}$$

Finally

$$D^{(2\mu,2\nu)} r(\mathbf{t}) = 0 \qquad (\mathbf{t} \in \partial([0, \frac{1}{2}]^2),\ 0 \le \mu, \nu \le n). \tag{3.6}$$

For r, we do odd extension, denoted by r^o,

$$\begin{aligned} &r^o(t_1, t_2) = r(t_1, t_2) \quad (\mathbf{t} \in [0, \tfrac{1}{2}]^2), \\ &r^o(-t_1, t_2) = r^o(t_1, -t_2) = -r^o(t_1, t_2) \quad (\mathbf{t} \in [-\tfrac{1}{2}, \tfrac{1}{2}]^2). \end{aligned}$$

From this and (3.6), it follows that $r^o \in C^{(l)}([-\frac{1}{2}, \frac{1}{2}]^2)$. Again, do periodic extension, denoted by $\widetilde{r}$,

$$\widetilde{r}(\mathbf{t} + \mathbf{m}) = r^o(\mathbf{t}) \quad (\mathbf{t} \in [-\frac{1}{2}, \frac{1}{2}]^2,\ \mathbf{m} \in \mathbb{Z}^2),$$

so $\widetilde{r}$ is a 1-periodic odd function and $\widetilde{r} \in C^l(\mathbb{R}^2)$, and can be reconstructed well by its two-dimensional periodic wavelet coefficients.

By Propositions 2.1-2.2, the periodic wavelet coefficients of $\widetilde{r}$ decays as fast as $O(2^{-m(l+1)})$. Compared with this, if we directly compute traditional (periodic) wavelet coefficients of $f \in C^l([0, \frac{1}{2}]^2)$, due to discontinuity on the boundary $\partial([0, \frac{1}{2}]^2)$, the obtained (periodic) wavelet coefficients decays as fast as $O(2^{-m})$.

By (3.4), $\tau_2(\mathbf{t})$ is determined by four derivative functions of f on the boundary of $[0, \frac{1}{2}]^2$:

$$\{D^{(2\alpha,0)} f(0, t_2)\}_{\alpha=0,1\ldots,n}, \qquad \{D^{(2\alpha,0)} f(\frac{1}{2}, t_2)\}_{\alpha=0,1,\ldots,n},$$

$$\{D^{(0,2\beta)} f(t_1, 0)\}_{\beta=0,1,\ldots,n}, \qquad \{D^{(0,2\beta)} f(t_1, \frac{1}{2})\}_{\beta=0,1,\ldots,n}.$$

Similarly to the above process, these four functions can be decomposed as

$$
\begin{aligned}
D^{(2\alpha,0)}f(0,t_2) &= \sum_{\beta=0}^{n} \tfrac{1}{2^{2\beta}}\big(D^{(2\alpha,2\beta)}f(0,0)p_\beta(1-2t_2) + D^{(2\alpha,2\beta)}f(0,\tfrac{1}{2})p_\beta(2t_2)\big) + u_\alpha(t_2),\\
D^{(2\alpha,0)}f(\tfrac{1}{2},t_2) &= \sum_{\beta=0}^{n} \tfrac{1}{2^{2\beta}}\big(D^{(2\alpha,2\beta)}f(\tfrac{1}{2},0)p_\beta(1-2t_2) + D^{(2\alpha,2\beta)}f(\tfrac{1}{2},\tfrac{1}{2})p_\beta(2t_2)\big) + v_\alpha(t_2),\\
D^{(0,2\beta)}f(t_1,0) &= \sum_{\alpha=0}^{n} \tfrac{1}{2^{2\alpha}}\big(D^{(2\alpha,2\beta)}f(0,0)p_\alpha(1-2t_1) + D^{(2\alpha,2\beta)}f(\tfrac{1}{2},0)p_\alpha(2t_1)\big) + w_\beta(t_1),\\
D^{(0,2\beta)}f(t_1,\tfrac{1}{2}) &= \sum_{\alpha=0}^{n} \tfrac{1}{2^{2\alpha}}\big(D^{(2\alpha,2\beta)}f(0,\tfrac{1}{2})p_\alpha(1-2t_1) + D^{(2\alpha,2\beta)}f(\tfrac{1}{2},\tfrac{1}{2})p_\alpha(2t_1)\big) + \gamma_\beta(t_1).
\end{aligned} \tag{3.7}
$$

From this, we have

$$
\begin{aligned}
&u_\alpha^{(2\beta)}(0) = u_\alpha^{(2\beta)}(\tfrac{1}{2}) = 0, \qquad v_\alpha^{(2\beta)}(0) = v_\alpha^{(2\beta)}(\tfrac{1}{2}) = 0,\\
&w_\beta^{(2\alpha)}(0) = w_\beta^{(2\alpha)}(\tfrac{1}{2}) = 0, \qquad \gamma_\beta^{(2\alpha)}(0) = \gamma_\beta^{(2\alpha)}(\tfrac{1}{2}) = 0 \quad (\alpha,\beta = 0,1,...,n).
\end{aligned}
$$

After odd extensions and then 1-periodic extensions for u_α, v_α, w_β, and γ_β, we get four 1-periodic smooth odd functions u_α^*, v_α^*, w_β^*, $\gamma_\beta^* \in C^l(\mathbb{R})$. Similarly to the argument of Proposition 2.2, the one-dimensional periodic wavelet coefficients of these four periodic functions decay as fast as $O(2^{-m(l+1)})$. So $\tau_2(\mathbf{t})$ can be reconstructed well by the value of the derivative of f on four vertices of $[0,\frac{1}{2}]^2$ and one-dimensional periodic wavelet coefficients. By (3.3), $\tau_1(\mathbf{t})$ is can be reconstructed by the value of the derivative of f on four vertices of $[0,\frac{1}{2}]^2$. Therefore, f can be reconstructed well by the value of the derivative of f on four vertices of $[0,\frac{1}{2}]^2$ and one-dimensional and two-dimensional fast decaying periodic wavelet coefficients.

4. Discrete Version of our Improved Algorithm

In this section, we will improve traditional discrete wavelet transform such that boundary effects can be mitigated well. Our improvement of discrete (periodic) wavelet transform is stated as follows:

For $f \in C^l([0,\frac{1}{2}]^2)$, assume that the sampling of f on $[0,\frac{1}{2}]^2$ is given:

$$x_{n_1,n_2} = f\left(\frac{n_1}{2^J}, \frac{n_2}{2^J}\right) \ (n_1, n_2 = 0, 1, ..., 2^{J-1}),$$

The derivatives on the boundary of the square is denoted by

$$x_{n_1,n_2}^{(\alpha,\beta)} = D^{(2\alpha,2\beta)} f\left(\frac{n_1}{2^J}, \frac{n_2}{2^J}\right) \qquad (n_1 = 0, 2^{J-1} \text{ or } n_2 = 0, 2^{J-1};\ \alpha, \beta = 0, 1, ..., [l/2]). \tag{4.1}$$

By (3.3) and (3.4), define $y_{n_1,n_2}^{(1)}$ and $y_{n_1,n_2}^{(2)}$ as

$$\begin{aligned} y_{\mathbf{n}}^{(1)} = \tau_1(\tfrac{\mathbf{n}}{2^J}) = \sum_{\alpha,\beta=0}^{n} \tfrac{1}{2^{2\alpha+2\beta}} [&x_{0,0}^{(\alpha,\beta)} p_\alpha(1 - \tfrac{2n_1}{2^J}) p_\beta(1 - \tfrac{2n_2}{2^J}) + x_{0,2^{J-1}}^{(\alpha,\beta)} p_\alpha(1 - \tfrac{2n_1}{2^J}) p_\beta(\tfrac{2n_2}{2^J}) \\ &+ x_{2^{J-1},0}^{(\alpha,\beta)} p_\alpha(\tfrac{2n_1}{2^J}) p_\beta(1 - \tfrac{2n_2}{2^J}) + x_{2^{J-1},2^{J-1}}^{(\alpha,\beta)} p_\alpha(\tfrac{2n_1}{2^J}) p_\beta(\tfrac{2n_2}{2^J})] \\ &(\mathbf{n} = (n_1, n_2),\ n_1, n_2 = 0, 1, ..., 2^{J-1}), \end{aligned} \tag{4.2}$$

$$\begin{aligned} y_{\mathbf{n}}^{(2)} = \tau_2(\tfrac{\mathbf{n}}{2^J}) = &\sum_{\alpha=0}^{n} \tfrac{1}{2^{2\alpha}} [x_{0,n_2}^{(\alpha,0)} p_\alpha(1 - \tfrac{2n_1}{2^J}) + x_{2^{J-1},n_2}^{(\alpha,0)} p_\alpha(\tfrac{2n_1}{2^J})] \\ &+ \sum_{\beta=0}^{n} \tfrac{1}{2^{2\beta}} [x_{n_1,0}^{(0,\beta)} p_\beta(1 - \tfrac{2n_2}{2^J}) + x_{n_1,2^{J-1}}^{(0,\beta)} p_\beta(\tfrac{2n_2}{2^J})] \\ &(\mathbf{n} = (n_1, n_2),\ n_1, n_2 = 0, 1, ..., 2^{J-1}). \end{aligned} \tag{4.3}$$

The discrete data x_{n_1,n_2} can be decomposed as follows:

$$x_{n_1,n_2} = y_{n_1,n_2}^{(2)} - y_{n_1,n_2}^{(1)} + z_{n_1,n_2} \qquad (n_1, n_2 = 0, 1, ..., 2^{J-1}). \tag{4.4}$$

By (4.2)-(4.4), we have

$$z_{n_1,n_2} = 0, \qquad (n_1 = 0, 2^{J-1} \text{ or } n_2 = 0, 2^{J-1})$$

We do an odd extension for the data $\{z_{n_1,n_2}\}$ as follows:

$$z_{2^{J-1}+k,n_2}^{o} = -z_{2^{J-1}-k,n_2},\ z_{n_1,2^{J-1}+k}^{o} = -z_{n_1,2^{J-1}-k}^{o},\ z_{2^{J-1}+k,2^{J-1}+k}^{o} = z_{2^{J-1}-k,2^{J-1}-k}^{o} \tag{4.5}$$

where $n_1, n_2 = 0, 1, ..., 2^{J-1},\ k = 1, ..., 2^{J-1} - 1$. After that we do the 2^J-periodic extension for $\{z_{n_1,n_2}^{o}\}$ and get $\{z_{n_1,n_2}^{*}\}$.

Proposition 4.1. For a large J, the periodic wavelet coefficients $c_{J,\mathbf{n}}$ of $\widetilde{r}$ satisfy

$$c_{J,\mathbf{n}} \approx \frac{\lambda}{2^J} z_{\mathbf{n}}^{*} \qquad (n_1, n_2 = 0, 1, ..., 2^J - 1),$$

where $\lambda = (\int_{\mathbb{R}} \varphi(t)dt)^2$ and $\widetilde{r}$ is stated in Theorem 3.1.

Proof. Comparing (3.5) and (4.4), it follows that $\{z^*_{n_1,n_2}\}$ is just the sampling of $\widetilde{r}$. Noticing that $\varphi^{per}_{0,J,\mathbf{n}}(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^2} \varphi_{0,J,\mathbf{n}}(\mathbf{t}+\mathbf{k})$, by (2.6), it follows that

$$c_{J,\mathbf{n}} = \int_{[-\frac{1}{2},\frac{1}{2}]^2} \widetilde{r}(\mathbf{t}) \sum_{\mathbf{k}\in\mathbb{Z}^2} \varphi_{0,J,\mathbf{n}}(\mathbf{t}+\mathbf{k})d\mathbf{t} = \sum_{\mathbf{k}\in\mathbb{Z}^2} \int_{[-\frac{1}{2},\frac{1}{2}]^2+\mathbf{k}} \widetilde{r}(\mathbf{t})\varphi_{0,J,\mathbf{n}}(\mathbf{t})d\mathbf{t} = \int_{\mathbb{R}^2} \widetilde{r}(\mathbf{t})\varphi_{0,J,\mathbf{n}}(\mathbf{t})d\mathbf{t}.$$

Denote the compact support of φ_0 by Ω. Noticing that $\varphi_{0,J,\mathbf{n}}(\mathbf{t}) = 2^J\varphi_0(2^J\mathbf{t}-\mathbf{n})$ and $\widetilde{r} \in C^l(\mathbb{R}^2)$, we have

$$\begin{aligned}
c_{J,\mathbf{n}} &= 2^J \textstyle\int_{\mathbb{R}^2} \widetilde{r}(\mathbf{t})\varphi_0(2^J\mathbf{t}-\mathbf{n})d\mathbf{t} = 2^{-J} \int_{\mathbb{R}^2} \widetilde{r}(2^{-J}\mathbf{u})\varphi_0(\mathbf{u}-\mathbf{n})d\mathbf{u} \\
&= 2^{-J} \textstyle\int_{\mathbb{R}^2} \widetilde{r}(2^{-J}(\mathbf{u}+\mathbf{n}))\varphi_0(\mathbf{u})d\mathbf{u} = 2^{-J} \int_{\Omega} \widetilde{r}(2^{-J}(\mathbf{u}+\mathbf{n}))\varphi_0(\mathbf{u})d\mathbf{u} \\
&= 2^{-J} \left(\widetilde{r}(2^{-J}\mathbf{n}) \textstyle\int_{\mathbb{R}^2} \varphi_0(\mathbf{u})d\mathbf{u} + \int_{\Omega}(\widetilde{r}(2^{-J}(\mathbf{u}+\mathbf{n})) - \widetilde{r}(2^{-J}\mathbf{n}))\varphi_0(\mathbf{u})d\mathbf{u}\right) \\
&= 2^{-J}(\lambda\widetilde{r}(2^{-J}\mathbf{n}) + O(2^{-J})),
\end{aligned}$$

i.e.,

$$c_{J,\mathbf{n}} \approx \frac{\lambda}{2^J}\widetilde{r}(2^{-J}\mathbf{n}) = \frac{\lambda}{2^J} z^*_{\mathbf{n}} \qquad (n_1, n_2 = 0, 1, ..., 2^{J-1}),$$

where

$$\lambda = \int_{\mathbb{R}^2} \varphi_0(\mathbf{u})d\mathbf{u} = \int_{\mathbb{R}^2} \varphi_0(u_1,u_2)du_1du_2 = \int_{\mathbb{R}^2} \varphi(u_1)\varphi(u_2)du_1du_2 = \left(\int_{\mathbb{R}} \varphi(t)dt\right)^2.$$

□

By (2.8), $\{c_{J,\mathbf{n}}\}$ ($\mathbf{n} = (n_1,n_2)$, $n_1, n_2 = 0, 1, ..., 2^J - 1$) can be used further to compute two-dimensional periodic wavelet coefficients in finer resolution levels:

$$\{c_{m_0,k_1,k_2}\}_{k_1,k_2=0,1,...,2^{m_0}-1}, \qquad \{d_{\mu,m,k_1,k_2}\}_{\mu=1,2,3,k_1,k_2=0,1,...,2^m-1,\, m=m_0,m_0+1,...,J-1}. \tag{4.6}$$

Theorem 4.2. Suppose that φ and $\widetilde{\varphi}$ are symmetric at $t = 0$ and ψ and $\widetilde{\psi}$ are symmetric at $t = \frac{1}{2}$ and $t = -\frac{1}{2}$, respectively. Then periodic wavelet coefficients in (4.6) are symmetric:

$$\begin{aligned}
&c_{m,2^m-k_1,k_2} = -c_{m,k_1,k_2}, && c_{m,k_1,2^m-k_2} = -c_{m,k_1,k_2}, \\
&d_{1,m,2^m-k_1,k_2} = -d_{1,m,k_1,k_2}, && d_{1,m,k_1,2^m-k_2-1} = -d_{1,m,k_1,k_2}, \\
&d_{2,m,2^m-k_1-1,k_2} = -d_{2,m,k_1,k_2}, && d_{2,m,k_1,2^m-k_2} = -d_{2,m,k_1,k_2}, \\
&d_{3,m,2^m-k_1-1,k_2} = -d_{3,m,k_1,k_2}, && d_{3,m,k_1,2^m-k_2-1} = -d_{3,m,k_1,k_2}.
\end{aligned}$$

Proof. Since φ and $\widetilde{\varphi}$ are symmetric at $t=0$ and ψ and $\widetilde{\psi}$ are symmetric at $t=\frac{1}{2}$ and $t=-\frac{1}{2}$, respectively, by (2.3), it follows that

$$h^*_{-n}=h^*_n, \qquad \tau^*_{1+n}=\tau^*_{1-n}.$$

Again by (2.8) and (4.5), we easily get the symmetry property of wavelet coefficients. □

In (4.3), $\{y^{(2)}_{\mathbf{n}}\}$ is determined by four sequences $\{x^{(\alpha,0)}_{0,n_2}\}$, $\{x^{(\alpha,0)}_{2^{J-1},n_2}\}$, $\{x^{(0,\beta)}_{n_1,0}\}$ and $\{x^{(0,\beta)}_{n_1,2^{J-1}}\}$ which will be decomposed further as:

$$\begin{aligned}
x^{(\alpha,0)}_{0,n_2} &= \sum_{\beta=0}^{n} \tfrac{1}{2^{2\beta}}\big(\, x^{(\alpha,\beta)}_{0,0}p_\beta(1-\tfrac{2n_2}{2^J}) + x^{(\alpha,\beta)}_{0,2^{J-1}}p_\beta(\tfrac{2n_2}{2^J})\,\big) + u_{\alpha,n_2},\\
x^{(\alpha,0)}_{2^{J-1},n_2} &= \sum_{\beta=0}^{n} \tfrac{1}{2^{2\beta}}\big(\, x^{(\alpha,\beta)}_{2^{J-1},0}p_\beta(1-\tfrac{2n_2}{2^J}) + x^{(\alpha,\beta)}_{2^{J-1},2^{J-1}}p_\beta(\tfrac{2n_2}{2^J})\,\big) + v_{\alpha,n_2},\\
x^{(0,\beta)}_{n_1,0} &= \sum_{\alpha=0}^{n} \tfrac{1}{2^{2\alpha}}\big(\, x^{(\alpha,\beta)}_{0,0}p_\alpha(1-\tfrac{2n_1}{2^J}) + x^{(\alpha,\beta)}_{2^{J-1},0}p_\alpha(\tfrac{2n_1}{2^J})\big) + w_{\beta,n_1},\\
x^{(0,\beta)}_{n_1,2^{J-1}} &= \sum_{\alpha=0}^{n} \tfrac{1}{2^{2\alpha}}\big(\, x^{(\alpha,\beta)}_{0,2^{J-1}}p_\alpha(1-\tfrac{2n_1}{2^J}) + x^{(\alpha,\beta)}_{2^{J-1},2^{J-1}}p_\alpha(\tfrac{2n_1}{2^J})\,\big) + \gamma_{\beta,n_1},
\end{aligned} \tag{4.7}$$

After odd extensions and then 1-periodic extensions for u_{α,n_2}, v_{α,n_2}, w_{β,n_1}, γ_{β,n_1}, we get four 2^J-periodic smooth sequences u^*_{α,n_2}, v^*_{α,n_2}, w^*_{β,n_1}, γ^*_{β,n_1}. Again by (2.7), we can get one-dimensional periodic wavelet coefficients of u^*_{α,n_2}, v^*_{α,n_2}, w^*_{β,n_1}, γ^*_{β,n_1}:

$$\begin{aligned}
&\{c^u_{m_0,k}\}_{k=0,1,\dots,2^{m_0}-1}, &&\{d^u_{m,k}\}_{k=0,1,\dots,2^m-1,\, m=m_0,m_0+1,\dots,J-1}.\\
&\{c^v_{m_0,k}\}_{k=0,1,\dots,2^{m_0}-1}, &&\{d^v_{m,k}\}_{k=0,1,\dots,2^m-1,\, m=m_0,m_0+1,\dots,J-1}.\\
&\{c^w_{m_0,k}\}_{k=0,1,\dots,2^{m_0}-1}, &&\{d^w_{m,k}\}_{k=0,1,\dots,2^m-1,\, m=m_0,m_0+1,\dots,J-1}.\\
&\{c^\gamma_{m_0,k}\}_{k=0,1,\dots,2^{m_0}-1}, &&\{d^\gamma_{m,k}\}_{k=0,1,\dots,2^m-1,\, m=m_0,m_0+1,\dots,J-1}.
\end{aligned} \tag{4.8}$$

Similar to Theorem 4.2, the above periodic wavelet coefficients are also symmetric.

Finally, we summarize our improvement of discrete (periodic) wavelet transform for $\{x_{n_1,n_2}\}_{n_1,n_2=0,1,\dots,2^{J-1}}$, as follows:

Decomposition Algorithm.

Step 1. By (4.2)-(4.4), we can get $\{z_{n_1,n_2}\}_{n_1,n_2=0,1,\dots,2^{J-1}}$. After that, we do an odd extension and a periodic extension for $\{z_{n_1,n_2}\}_{n_1,n_2=0,1,\dots,2^{J-1}}$ and get $\{z^*_{n_1,n_2}\}_{n_1,n_2\in\mathbb{Z}^2}$. Finally, by Proposition 4.1 and (2.8), we can compute two-dimensional periodic wavelet coefficients in (4.6). Again by Theorem 4.2, the number of non-zero two-dimensional periodic wavelet coefficients that we need to store is $(2^{J-1}-1)^2$.

Step 2. By (4.7), we can get u_{α,n_2}, v_{α,n_2}, w_{β,n_1}, γ_{β,n_1}. After odd extensions and then periodic extensions, we get four 2^J-periodic smooth sequences u^*_{α,n_2}, v^*_{α,n_2}, w^*_{β,n_1}, γ^*_{β,n_1}. Again by (2.7), we can get one-dimensional periodic wavelet coefficients in (4.7). Similar to Theorem 4.2, due to symmetric property of periodic wavelet coefficients, the number of non-zero two-dimensional periodic wavelet coefficients that we need to store is $4\times(2^{J-1}-1)$.

Step 3. The following $4(n+1)^2$ values are stored:

$$\{x^{(\alpha,\beta)}_{0,0}\},\quad \{x^{(\alpha,\beta)}_{0,2^{J-1}}\},\quad \{x^{(\alpha,\beta)}_{2^{J-1},0}\},\quad \{x^{(\alpha,\beta)}_{2^{J-1},2^{J-1}}\}\qquad (\alpha,\beta=0,1,...,n),$$

Reconstruction Algorithm.

Step 1. By (4.2), using $4(n+1)^2$ values:

$$\{x^{(\alpha,\beta)}_{0,0}\},\quad \{x^{(\alpha,\beta)}_{0,2^{J-1}}\},\quad \{x^{(\alpha,\beta)}_{2^{J-1},0}\},\quad \{x^{(\alpha,\beta)}_{2^{J-1},2^{J-1}}\}\qquad (\alpha,\beta=0,1,...,n),$$

we get $\{y^{(1)}_{n_1,n_2}\}_{n_1,n_2=0,1,...,2^{J-1}}$.

Step 2. By $4(n+1)^2$ values in Step 1, (4.7)-(4.8) and one-dimensional inverse discrete periodic wavelet transform, we can get

$$\{x^{(\alpha,0)}_{0,n_2}\},\quad \{x^{(\alpha,0)}_{2^{J-1},n_2}\}\quad (n_2=0,1,...,2^{J-1},\ \alpha=0,1,...,n)$$

$$\{x^{(0,\beta)}_{n_1,0}\},\quad \{x^{(0,\beta)}_{n_1,2^{J-1}}\}\quad (n_1=0,1,...,2^{J-1},\ \beta=0,1,...,n),$$

by (4.3), we get $\{y^{(2)}_{n_1,n_2}\}_{n_1,n_2=0,1,...,2^{J-1}}$.

Step 3. By (4.6) and the two-dimensional inverse discrete periodic wavelet transform, we get $\{z^*_{n_1,n_2}\}$. It means that $\{z_{n_1,n_2}\}_{n_1,n_2=0,1,...,2^{J-1}}$ is computed

Step 4. By (4.2), we can reconstruct f:

$$x_{n_1,n_2}=y^{(2)}_{n_1,n_2}-y^{(1)}_{n_1,n_2}+z_{n_1,n_2}\qquad (n_1,n_2=0,1,...,2^{J-1}).$$

Remark. Assume that $\{x_{n_1,n_2}\}_{n_1,n_2=0,1,\ldots,2^{J-1}}$, are sampled from $f \in C^l([0,\frac{1}{2}]^2)$. By Propositions 2.1-2.2, due to smooth extension in our algorithm, the one-dimensional and two-dimensional periodic wavelet coefficients in our algorithm decay as fast as $O(2^{-m(l+1)})$. Compared with this, if we apply traditional discrete (periodic) wavelet transform for $f \in C^l([0,\frac{1}{2}]^2)$, due to discontinuity on the boundary, the corresponding wavelet coefficients of f decays as fast as $O(2^{-m})$. Even when we consider the simplest version $n=0$ in our algorithm, where no any derivative is involved, the corresponding wavelet coefficients in our algorithm still decay as fast as $O(2^{-3m})$, better than that of traditional discrete (periodic) wavelet transform (i.e., $O(2^{-m})$). Full version of our algorithm can be applied to compress smooth data, e.g. CMIP6 data [14]. The size of CMIP6 data is increasing sharply at the petabyte scale [13], but by now there is no good algorithm to compress it. Since CMIP6 data are output data from numerical solutions of fluid equations and energy equations governing the Earth's climate system [14], CMIP6 data is smooth and the needful derivative values on the data boundary are easily estimated, so our algorithm can compress this kind of data well.

References

[1] S. Mallat, A wavelet tour of signal processing, Academic Press, 2008.

[2] Z. Zhang, P.E.T. Jorgenson, Frame Theory in Data Science; Springer: Heidelberg, Germany, 2021

[3] C.K. Jha, M.H. Kolekar, Tunable Q-wavelet based ECG data compression with validation using cardiac arrhythmia patterns, Biomedical Signal Processing and Control, 66, 2021, 102464

[4] A. Otsuka, K. Muramatsu, F. Nagata, Data-driven generation of random skin model shapes by using wavelet transformation Computers & Industrial Engineering, 150, 2020, 106860.

[5] J. Chakraborty, A. Nandy, Discrete wavelet transform based data representation in deep neural network for gait abnormality detection, Biomedical Signal Processing and Control, 62, 2020, 102076.

[6] R. Kamgar, M. Dadkhah, H. Naderpour, Seismic response evaluation of structures using discrete wavelet transform through linear analysis Structures, 29, 2021, 863-882.

[7] E. Deriaz, M. Farge, K. Schneider, Craya decomposition using compactly supported biorthogonal wavelets Applied and Computational Harmonic Analysis, 28, 2010, 267-284.

[8] Q. Chen, H. Cao, Z. Shi, Construction and decomposition of biorthogonal vector-valued wavelets with compact support Chaos, Solitons & Fractals, 42, 2009, 2765-2778.

[9] Antolin, A.S.; Zalik, R.A. Compactly supported Parseval framelets with symmetry associated to E(2)d(Z) matrices. Appl. Math. Comput. 2018, 325, 179C190.

[10] Atreasa, N.; Karantzasb, N.; Papadakisb, M.; Stavropoulosc, T. On the design of multi-dimensional compactly supported Parseval framelets with directional characteristics. Linear Algebra Appl. 2019, 582, 1C36.

[11] D. E.Dutkay, P. E.T. Jorgensen, S. Silvestrov, Decomposition of wavelet representations and Martin, Journal of Functional Analysis, 262, 1043-1061.

[12] Z. Zhang, Splitting of Framelets and Framelet Packets, Mathematics, 2021, 9(7), Article ID 697

[13] D. Huisingh and Z. Zhang, Big data mining approaches for helping to reduce and prevent global warming, Int J Big Data Mining for Global Warming, 1, 2019, 1901001

[14] Z. Zhang, J. Li, Big Data Mining for Climate Change, Elsevier, 2020

[15] A. Najmi, T. Moon, Advanced Signal Processing, McGraw-Hill, 2020