MULTIPLICITY OF SOLUTIONS TO CLASS OF NONLOCAL ELLIPTIC PROBLEMS WITH CRITICAL EXPONENTS

E. JUÁREZ HURTADO, O. H. MIYAGAKI, R. S. RODRIGUES

Abstract. In this paper, we establish existence of infinitely many weak solutions for a class of quasilinear stationary Kirchhoff type equations, which involves a general variable exponent elliptic operator with critical growth. Precisely, we study the following nonlocal problem

$$\begin{cases} -M(\mathscr{A}(u))\,\mathrm{div}\Big(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\Big) = \lambda f(x,u) + |u|^{s(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, with homogeneous Dirichlet boundary conditions on $\partial\Omega$, the nonlinearity $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a continuous function, $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a function of the class C^1, $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function, whose properties will be introduced later, λ is a positive parameter and $p, s \in C(\overline{\Omega})$. We assume that $\mathscr{C} = \{x \in \Omega : s(x) = \gamma^*(x)\} \neq \emptyset$, where $\gamma^*(x) = N\gamma(x)/(N - \gamma(x))$ is the critical Sobolev exponent. We will prove that the problem has infinitely many solutions and also we obtain the asymptotic behavior of the solution as $\lambda \to 0^+$. Furthermore, we emphasize that a difference with previous researches is that the conditions on $a(\cdot)$ are general overall enough to incorporate some interesting differential operators. Our work covers a feature of the Kirchhoff's problems, that is, the fact that the Kirchhoff's function M in zero is different from zero, it also covers a wide class of nonlocal problems for $p(x) > 1$, for all $x \in \overline{\Omega}$. The main tool to find critical points of the Euler Lagrange functional associated with this problem is through a suitable truncation argument, concentration-compactness principle for variable exponent found in [9], and the genus theory introduced by Krasnoselskii. The result of this paper extends or complements, or else completes recent papers and is new in several directions for the stationary Kirchhoff equations involving the $p(x)-$Laplacian type operators.

1. Introduction

The present research deals with the existence and multiplicity of weak solutions of the nonlocal boundary value problems of a class of quasilinear Kirchhoff type equations involving a critical nonlinearity, in the framework of function spaces with variable exponents and, therefore, we need more delicate estimates. Motivated by the little literature in the critical case for nonlocal operators of elliptic equations with variable exponents, the objective of this research is to demonstrate the existence of infinitely many solutions of the critical problem $(\mathscr{P}_\lambda)$, by using the Krasnoselski genus theory. In this sense, our result is new even in the $p-$Laplacian operator's context, nonlocal type problems and problems involving critical terms with exponents variables.

Kirchhoff established in [1883] (see [39]) a model for his celebrated equation

$$\rho\partial_{tt}^2 u - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^{2L} |\partial_x u(x)|^2\,dx\right)\partial_{xx}^2 u = 0,$$

as a nonlinear extension of D'Alembert's wave equation by considering the small vertical vibrations of a stretched elastic string when the tension is variable and the ends of the string are fixed. Here $u = u(x,t)$ is the transverse string displacement at the space coordinate x and time t, L is the length of the string, h is the area of the cross section, E is Young's modulus of the material (also referred to as the elastic modulus) it measures the string's resistance to being deformed elastically, ρ is the mass density, and ρ_0 is the initial tension.

2000 *Mathematics Subject Classification.* 35A15, 35A16,35J70.

Key words and phrases. Variable exponent Lebesgue Sobolev spaces, critical point theory, Mountain Pass Theorem, concentration-compactness principle, boundary value problems.

First author was supported by CAPES/Brazil, the second author named was partially supported by INCTMAT/CNPq/Brazil and CNPq/Brasil 304015/2014-8, while third author was supported by Grant 2015/11912-6, São Paulo Research Foundation (FAPESP).

These types of problems have received a lot of attention of many researchers only after Jacques-Louis Lions [42], proposed a framework to attack it. Several papers are dedicated to these kinds of problems, for example, see [2], [6], [7], [14], [15], [20], and references therein.

In the last decades, the study of nonlocal problems has generated a lot of interest, by their structure, by theoretical aspects, and by the presence of the $p(x)-$Laplacian type operators, these mathematical complications motivate our work. Such problems are being studied intensively in recent years due to its mathematics difficulties, and also by the applications in the motion of electrorheological fluids, elastic mechanic, image restoration, mathematical biology, polycrystal plasticity, non-Newtonian fluids with thermo-convective effects, we refer the reader to [5], [19], [23], [36], [44], [46], [47], and references therein for more applications. However, such generalizations are nontrivial, since $p(x)-$Laplacian type operators possess more complicated structures than the $p-$Laplacian type operators, for instance, the $p(x)-$Laplacian operator is nonhomogeneous.

The goal of this research is to complete, complement, extend, and improve the study of quasilinear stationary Kirchhoff equation involving $p(x)-$Laplacian type operators for a wide class of problems since elliptic problems wrapping variable exponent with critical growth still is new. We will need to overcome various difficulties arising from the fact that the problem is nonlocal and $M(0) > 0$.

Some authors obtained infinitely many solutions in the critical case, in the case $M \equiv 1$, for instance, we refer to [37] for the classical Laplacian operator, to [8] for the $p-$Laplacian, to [35] for the $(p, q)-$Laplacian operator, and for a class of $p\&q$ elliptic problems when Ω is a bounded domain of $\mathbb{R}^N$ we cite [32], for the $p(x)-$Laplacian operator we cite [9]. For the $(p, q)-$Laplacian operator defined in all $\mathbb{R}^N$ we cite [18] and for a class of $p\&q$ elliptic problems on $\mathbb{R}^N$ we refer to [31].

Concerning these types of nonlocal problems with critical growth, for example, we refer to [1] and [30] for Kirchhoff problem involving the classical Laplace operator, to [16], [17], and references therein for the Kirchhoff type problem involving the $p(x)-$Laplacian operator.

On the other hand, there has been an increasing interest in problems involving critical exponents, we mention the famous work of Brezis and Nirenberg [11], since then there have been several types of research and generalizations. Furthermore, the novelty of this research is to consider the operator $div(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u)$ with $p(x) > 1$, for $x \in \overline{\Omega}$.

More precisely, we study the following variable exponent nonlocal elliptic problem:

$$\begin{cases} -M(\mathscr{A}(u)))\,\mathrm{div}\Big(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\Big) = \lambda f(x,u) + |u|^{s(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{$\mathscr{P}_\lambda$}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a bounded domain with smooth boundary $\partial\Omega$, and $\lambda > 0$ is a real parameter, $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are continuous function on $\overline{\Omega}$ satisfying the folllowing inequalities

$$\begin{gathered} 1 < r^- \leqslant r(x) \leqslant r^+ < p^- \leqslant p(x) \leqslant p^+ < q^- \leqslant q(x) \leqslant q^+ < N, \\ \text{and} \\ 1 < r^- \leqslant r(x) \leqslant r^+ < \gamma^- \leqslant \gamma(x) \leqslant \gamma^+ < s^- \leqslant s(x) \leqslant \gamma^*(x) < +\infty, \end{gathered} \tag{1.1}$$

for all $x \in \overline{\Omega}$, where $p^- := \min_{x\in\overline{\Omega}} p(x)$, $p^+ := \max_{x\in\overline{\Omega}} p(x)$, and analogously to r^-, r^-, q^-, q^+, γ^-, γ^+, s^-, and s^+, with $\gamma(x) = (1 - \mathcal{H}(\kappa_3))p(x) + \mathcal{H}(\kappa_3)q(x)$, and the variable critical exponent $\gamma^*(x)$ defined by

$$\gamma^*(x) = \begin{cases} \frac{N\gamma(x)}{N-\gamma(x)}, & \text{se } \gamma(x) < N, \\ +\infty, & \text{se } \gamma(x) \geqslant N, \end{cases}$$

for all $x \in \overline{\Omega}$, where $\mathcal{H} : \mathbb{R}_0^+ \to \{0, 1\}$ is given by

$$\mathcal{H}(\kappa) = \begin{cases} 1, & \text{if} \quad \kappa > 0, \\ 0, & \text{if} \quad \kappa = 0. \end{cases}$$

Furthermore, we consider the set $\mathscr{C} = \{x \in \Omega : s(x) = \gamma^*(x)\} \neq \emptyset$. The operator $\mathscr{A} : X \to \mathbb{R}$ is defined by

$$\mathscr{A}(u) = \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})\, dx,$$

where $A(\cdot)$ is the function $A(t) = \int_0^t a(\sigma) d\sigma$ and the function $a(\cdot)$ is described in the hypothesis (a_0).

We assume that the function $a : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ verify the following hypotheses:

(a_0) The function $a(\cdot)$ is of class C^1;

(a_1) There exist positive constants, κ_0, κ_1, and κ_2, and a nonnegative constant κ_3, such that

$$\kappa_0 + \mathcal{H}(\kappa_3)\kappa_2 \tau^{\frac{q(x)-p(x)}{p(x)}} \leqslant a(\tau) \leqslant \kappa_1 + \kappa_3 \tau^{\frac{q(x)-p(x)}{p(x)}},$$

for all $\tau \geqslant 0$ and for all $x \in \overline{\Omega}$;

(a_2) There exists $c > 0$ such that

$$\min\left\{a(\tau^{p(x)})\tau^{p(x)-2}, a(\tau^{p(x)})\tau^{p(x)-2} + \tau\frac{\partial(a(\tau^{p(x)})\tau^{p(x)-2})}{\partial\tau}\right\} \geqslant c\tau^{p(x)-2},$$

for almost every $x \in \Omega$ and for all $\tau > 0$;

(a_3) There is a positive constant α such that

$$A(\tau) \geqslant \frac{1}{\alpha}a(\tau)\tau \text{ with } \frac{r^+}{p^+} < \frac{\gamma^+}{p^+} \leqslant \alpha < \frac{s^-}{p^+},$$

for all $\tau \geqslant 0$.

The problem $(\mathscr{P}_\lambda)$ is called nonlocal because of the presence of the nonlocal Kirchhoff function $M(\mathscr{A}(\cdot))$ which implies that the quasilinear partial differential equation $(\mathscr{P}_\lambda)$ is no longer a pointwise identity.

In terms of Kirchhoff's function, we assume general conditions on $M : \mathbb{R}_0^+ \to \mathbb{R}$. We consider that M is a continuous function that satisfies the conditions:

$(\mathcal{M}_1)$ There exists $m_0 > 0$ such that $M(t) \geqslant m_0 = M(0) > 0$ for all $t \in \mathbb{R}^+$;

$(\mathcal{M}_2)$ The function M is increasing.

In the present research, we denote by $\mathscr{M}(t) := \int_0^t M(\tau)d\tau$ for all $t \in \mathbb{R}_0^+$.

The standard Kirchhoff function introduced in [39] is given by

$$M(s) = as + m_0, \text{ with } a,\, m_0 \geqslant 0.$$

For such M the problem $(\mathscr{P}_\lambda)$ is degenerate if $m_0 = 0$ and $a > 0$, and non-degenerate when $m_0 > 0$ and $a \geqslant 0$. Finally, when $m_0 > 0$ and $a = 0$, the Kirchhoff function M is simply a constant and $(\mathscr{P}_\lambda)$ reduces to a local quasilinear elliptic Dirichlet problem.

We assume that the nonlinearity $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying:

(f_1) f is odd in the second variable, that is,

$$f(x, -t) = -f(x, t) \text{ for any } (x, t) \in \overline{\Omega} \times \mathbb{R};$$

(f_2) There exist positive constant a_1, a_2 and a function $r \in C^+(\overline{\Omega})$ such that

$$a_1 t^{r(x)-1} \leqslant f(x, t) \leqslant a_2 t^{r(x)-1} \text{ for all } (x, t) \in \overline{\Omega} \times \mathbb{R}_0^+.$$

A simple example for the nonlinearity f is given by $f(x, t) = \vartheta(x)|t|^{r(x)-2}t$, where $\vartheta : \overline{\Omega} \to \mathbb{R}$ is a continuous nonnegative function.

Note that the assumption (f_1) implies that the function $u = 0$ is solution of the problem $(\mathscr{P}_\lambda)$.

The main result in this paper establishes the existence of infinitely many solutions to problem $(\mathscr{P}_\lambda)$ under the above hypotheses.

Theorem 1.1. *Assume that the functions a, M, and f satisfy the conditions $(a_0) - (a_3)$, $(\mathcal{M}_1) - (\mathcal{M}_2)$, and $(f_1) - (f_2)$. Then, there exists $\overline{\lambda} > 0$, such that problem $(\mathscr{P}_\lambda)$ has infinitely many weak solutions for each $\lambda \in (0, \overline{\lambda})$. Moreover, if u_λ is a solution of problem $(\mathscr{P}_\lambda)$ then*

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0.$$

In order to prove our principal result we follow the main ideas of [30], [32], [33], and [35]. The hypothesis $(\mathcal{M}_1)$ provides only a positive lower bounded for M near zero, this creates serious technical difficulties, we will need to do a truncation on the Kirchhoff function to obtain priori estimates of the boundedness from above, and thus obtain a new auxiliary problem, afterward, we do another truncation to control the Euler-Lagrange's functional associated with the auxiliary problem. Note that there are serious difficulties to prove the existence of solutions for canonical variational methods. For instance, a difficulty that arises is to prove the Palais-Smale condition due to the lack of compactness of the embedding of $W_0^{1,p(\cdot)}(\Omega)$ into $L^{p^*(\cdot)}(\Omega)$, and, to overcome this difficulty we use the concentration compactness principle (see [9]). Then, to obtain infinitely many solutions to the auxiliary problem for any λ sufficiently small, we will use genus theory as in [8], and, due to the truncation on M, we obtain infinitely many solutions to problem $(\mathscr{P}_\lambda)$.

Finally, we would like to emphasize that our result improves, complements and generalizes some recent results including the following aspects:

- $(\mathscr{N}_1)$ We extended the results in three directions: the equations involving the $p(x)-$Laplacian type operators or a more general nonlocal operator, and study of the existence of infinitely many solutions, we refer to [8], [16], [17], [30], [35], and [45]. Moreover, observe that the Theorem 1.1 improves the main results of the articles mentioned above, with slight differences in the nonlinearities, where the authors consider only the case where $p(x)$ is a constant, or an operator that is included in our general operator, so we show a general result for a large class of operators. In some important cases, our result covers the nonlocal problems involving for instance the $p-$Lapalacian, the $p(x)-$Laplacian, the $(p(x), q(x))-$Laplacian, the $p(x)-$Laplacian-Like operator, among other types of differential operators.
- $(\mathscr{N}_2)$ In the study of quasilinear Kirchhoff equations type involving critical growth. In particular, our results cover a wide class of problems with variable exponent, which also has a difficulty due to Kirchhoff's function be different from zero in zero.
- $(\mathscr{N}_3)$ We emphasize that in our work, we cover the case $p(x) > 1$ for all $x \in \overline{\Omega}$, for a broad class of operators compare, see [30], [31], and [37].

Now, we are going to illustrate the kind of problems that surrounds our operator. We set some examples that are of mathematical interest, as well as of great physical interest due to applications in different areas. The following operators satisfy the hypotheses $a(\cdot)$.

1.1. **Some examples.**

The following operators satisfy the hypotheses $(a_0) - (a_3)$:

(a) If $a \equiv 1$, then
$$-div(|\nabla u|^{p(x)-2}\nabla u),$$
which coincides with the usual $p-$Laplacian when $p(x) = p$ and with the Laplacian when $p(x) = 2$.

(b) If $a(\tau) = 1 + \tau^{\frac{q(x)-p(x)}{p(x)}}$, we get
$$-div(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u) = -\Delta_{p(x)}u - \Delta_{q(x)}u,$$
when $p(x) = p$ and $q(x) = q$ it is the known $(p, q)-$Laplacian operator.

(c) If $a(\tau) = 1 + \frac{\tau}{\sqrt{1+\tau^2}}$, we have
$$-div(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u) = -div\left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right),$$
which is called $p(x)$-Laplacian like operator.

This paper is organized as follows. In the Section 2 we fix the notation and recall the definition of the variable exponent spaces that will be used throughout our research. In the Section 3, we will give some necessary definitions and properties of genus theory. In Section 4 we discuss the variational formulation of the problem $(\mathscr{P}_\lambda)$. In Section 5 we introduce the truncated auxiliary problem. Section 6 is dedicated to the local Palais Smale condition for the Euler - Lagrange's functional associated to the truncated auxiliary problem. In Section 7 we prove some technical results that imply the existence of infinitely many solutions for the auxiliary truncated problem and therefore infinitely many solutions for the original problem $(\mathscr{P}_\lambda)$, and finally get the asymptotic behavior for these solutions when $\lambda \to 0^+$. Finally in the last section we present some fundamental properties of the operator that will be useful in all our work.

2. Preliminaries

In this section, we recall some results about the Lebesgue and Sobolev space with variable exponent and properties, the spaces $L^{h(\cdot)}(\Omega)$, $W^{1,h(\cdot)}(\Omega)$, $W_0^{1,h(\cdot)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$, which are used throughout all our work. For details, we refer to [22], [24], [26], [27], [28], [29], and [40].

Let X be a Banach space, we denote by X' its topological dual and by $\langle \cdot, \cdot \rangle$ the duality brackets between X' and X. The following notations will be used in the present work. C and C_i will denote generic positive constants. We set
$$C^+(\overline{\Omega}) = \left\{h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\right\}.$$

For each $h \in C^+(\overline{\Omega})$, we define

$$h^+ = \max_{x\in\overline{\Omega}} h(x) \text{ and } h^- = \min_{x\in\overline{\Omega}} h(x).$$

Also, we define the variable exponent Lebesgue space $L^{h(\cdot)}(\Omega)$

$$L^{h(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable: } \int_\Omega |u(x)|^{h(x)}\, dx < +\infty \right\}$$

which is equiped with the Luxemburg norm

$$\|u\|_{h(\cdot)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{h(x)} dx \leqslant 1 \right\}.$$

Note that, for a constant function h, the Luxemburg norm $\|\cdot\|_{h(\cdot)}$ coincides with the standard norm $\|\cdot\|_h$ of the Lebesgue space $L^h(\Omega)$. As usual, we denote by $h'(x) = \frac{h(x)}{h(x)-1}$ the conjugate exponent $h(x)$. Since $0 < |\Omega| < +\infty$, the following result holds.

Lemma 2.1. *(*[25, Proposition 2.1]*)*

(i) *The space $\left(L^{h(\cdot)}(\Omega), \|\cdot\|_{h(\cdot)}\right)$ is a separable, uniform convex Banach space, and its conjugate space is $L^{h'(\cdot)}(\Omega)$, where $\frac{1}{h(x)} + \frac{1}{h'(x)} = 1$. For any $u \in L^{h(\cdot)}(\Omega)$ and $v \in L^{h'(\cdot)}(\Omega)$, we have*

$$\left| \int_\Omega uv\, dx \right| \leqslant \left(\frac{1}{h^-} + \frac{1}{(h')^-} \right) \|u\|_{h(\cdot)} \|v\|_{h'(\cdot)},\ \forall u \in L^{h(\cdot)}(\Omega),\ v \in L^{h'(\cdot)}(\Omega).$$

(ii) *If $h_1, h_2 \in C^+(\overline{\Omega})$, $h_1(x) \le h_2(x)$ for any $x \in \overline{\Omega}$, then the embedding $L^{h_2(\cdot)}(\Omega) \hookrightarrow L^{h_1(\cdot)}(\Omega)$ is continuous.*

We introduce an important function, so called $\rho(\cdot)-$modular of the $L^{h(\cdot)}(\Omega)$ space, that will help us in handling of generalized Lebesgue-Sobolev spaces, which is the convex function $\rho_{h(\cdot)} : L^{h(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{h(\cdot)}(u) = \int_\Omega |u|^{h(x)}\, dx,\ \forall u \in L^{h(\cdot)}(\Omega).$$

Proposition 2.2. *(*[25, Proposition 2.3]*) For $u \in L^{h(\cdot)}(\Omega)$ and $\{u_n\}_{n\in\mathbb{N}} \subset L^{h(\cdot)}(\Omega)$. Then, we have*

1. *If $u \neq 0$, $\|u\|_{h(\cdot)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$.*
2. $\|u\|_{h(\cdot)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$.
3. *If $\|u\|_{h(\cdot)} > 1$ then $\|u\|_{h(\cdot)}^{h_-} \leqslant \rho(u) \leqslant \|u\|_{h(\cdot)}^{h_+}$.*
4. *If $\|u\|_{h(\cdot)} < 1$ then $\|u\|_{h(\cdot)}^{h_+} \leqslant \rho(u) \leqslant \|u\|_{h(\cdot)}^{h_-}$.*
5. $\lim_{n\to+\infty} \|u_n\|_{h(\cdot)} = 0 \Leftrightarrow \lim_{n\to+\infty} \rho(u_n) = 0$.
6. $\lim_{n\to+\infty} \|u_n\|_{h(\cdot)} = +\infty \Leftrightarrow \lim_{n\to+\infty} \rho(u_n) = +\infty$.

Lemma 2.3. *(*[24, Lemma 2.1]*) Let $h \in L^\infty(\Omega)$ be such that $1 \leqslant h(x)p(x) \leqslant +\infty$ for almost everywhere (a.e.) $x \in \Omega$. Let $u \in L^{hp(\cdot)}(\Omega)$, $u \not\equiv 0$. Then*

$$\|u\|_{hp(\cdot)}^{h^-} \leqslant \||u|^{h(\cdot)}\|_{p(\cdot)} \leqslant \|u\|_{hp(\cdot)}^{h^+},\ \text{if } \|u\|_{hp(\cdot)} \geqslant 1,$$
$$\|u\|_{hp(\cdot)}^{h^+} \leqslant \||u|^{h(\cdot)}\|_{p(\cdot)} \leqslant \|u\|_{hp(\cdot)}^{h^-},\ \text{if } \|u\|_{hp(\cdot)} \leqslant 1.$$

In particular, if h is a constant function, then

$$\||u|^h\|_{hp(\cdot)} = \|u\|_{hp(\cdot)}^h.$$

The following lemma is a version of the well-known Brezis-Lieb Lemma for variable exponents, the proof follows the same steps of the constant case, the reader can see for instance [34].

Lemma 2.4. *Suppose $\{u_n\}_{n\in\mathbb{N}}$ bounded in $L^{h(\cdot)}(\Omega)$ and $u_n(x) \to u(x)$ a.e. in Ω. Then, $u \in L^{h(\cdot)}(\Omega)$ and*

$$\lim_{n\to+\infty} \left(\int_\Omega |u_n|^{h(x)}\, dx - \int_\Omega |u_n - u|^{h(x)}\, dx \right) = \int_\Omega |u|^{h(x)}\, dx.$$

The variable exponent Sobolev space $W^{1,h(\cdot)}(\Omega)$ is defined by

$$W^{1,h(\cdot)}(\Omega)=\Big\{u\in L^{h(\cdot)}(\Omega):|\nabla u|\in L^{h(\cdot)}(\Omega)\Big\},$$

endowed with the norm

$$\|u\|_{1,h(\cdot)}=\|u\|_{h(\cdot)}+\|\nabla u\|_{h(\cdot)}.$$

The space $W_0^{1,h(\cdot)}(\Omega)$ is defined as the closure of $C_0^{+\infty}(\Omega)$ in $W^{1,h(\cdot)}(\Omega)$ with respect to the above norm.

Proposition 2.5. *(*[25, Proposition 2.5]*) Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$ and $h\in C(\overline{\Omega})$ with $h(x)<N$ for all $x\in\overline{\Omega}$.*

(i) *$W^{1,p(\cdot)}(\Omega)$ and X are separable reflexive Banach spaces.*
(ii) *If $q\in C^+(\overline{\Omega})$ and $q(x)<p^*(x)$ for any $x\in\overline{\Omega}$, then the embedding $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$ is compact and continuous.*
(iii) *There is a constant $C>0$, such that*

$$\|u\|_{p(\cdot)}\leqslant C\|\nabla u\|_{p(\cdot)},\ \forall u\in X.$$

Note that, by (iii) of Proposition 2.5, we know that $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms on X. We will use $\|\nabla u\|_{p(\cdot)}$ to replace $\|u\|_{1,p(\cdot)}$ in the following discussions.

Theorem 2.6. *(*[27, Theorem 3.1]*) Consider the mapping $L:W_0^{1,q(\cdot)}(\Omega)\to(W_0^{1,q(\cdot)}(\Omega))'$ defined by*

$$\langle L(u),v\rangle=\int_\Omega|\nabla u|^{q(x)-2}\nabla u\nabla v dx.$$

Then, the mapping L is of the type (S_+), that is, if $u_n\rightharpoonup u$ weakly in $W_0^{1,q(\cdot)}(\Omega)$ and $\limsup\limits_{n\to\infty}\langle L(u_n),u_n-u\rangle\leqslant 0$, it follow that $u_n\to u$ strongly in $W_0^{1,q(\cdot)}(\Omega)$.

Proposition 2.7. *(*[9]*) Let $p(x)$, $q(x)$ be two continuous functions such that*

$$1<\inf_{x\in\Omega}p(x)\leqslant\sup_{x\in\Omega}p(x)<N \text{ and } 1\leqslant q(x)\leqslant p^*(x) \text{ in } \Omega.$$

Let $\{u_n\}_{n\in\mathbb{N}}$ be a weakly convergent sequence in $W^{1,p(\cdot)}(\Omega)$ with weak limit u and such that:

$$|\nabla u_n|^{p(x)}\rightharpoonup\mu \text{ weakly-* in the sense of measures.}$$
$$|u_n|^{q(x)}\rightharpoonup\nu \text{ weakly-* in the sense of measures.}$$

In addition we assume that $\mathscr{C}:=\{x\in\Omega:q(x)=p^*(x)\}$ is nomempty. Then, for some countable index set $\mathcal{J}$, we have:

$$\nu=|u|^{q(x)}+\sum_{j\in\mathcal{J}}\nu_j\delta_{x_j},\ \nu_j>0,$$
$$\mu\geqslant|\nabla u|^{p(x)}+\sum_{j\in\mathcal{J}}\mu_j\delta_{x_j},\ \mu_j>0,$$
$$S\nu_j^{\frac{1}{p^*(x_j)}}\leqslant\mu_j^{\frac{1}{p(x_j)}},\ \forall j\in\mathcal{J}.$$

Where $\{x_j\}_{j\in\mathcal{J}}\subset\mathscr{C}$ and S is the best constant in the Gagliardo-Nirenberg-Sobolev inequality for variable exponents, namely

$$S=S_q(\Omega):=\inf_{v\in C_0^{+\infty}(\Omega)}\frac{\|\nabla v\|_{p(\cdot)}}{\|v\|_{q(\cdot)}}.$$

Lemma 2.8. *(Deformation Lemma)(*[4, Lemma 1.3]*) Let X be a Banach space, $\varphi\in C^1(X,\mathbb{R})$ and satisfy the (PS) condition. Let $c\in\mathbb{R}$ and N be any neighborhood of $K_c:=\{u\in X:\varphi(u)=c,\ \varphi'(u)=0\}$. Then there exists $\eta(t,x)\equiv\eta_t(x)\in C([0,1]\times X,X)$ and positive constants $\overline{\varepsilon}>\varepsilon$ such that:*

(1) *$\eta_0(x)=x$ for all $x\in X$;*
(2) *$\eta_t(x)=x$ for all $x\notin\varphi^{-1}[c-\overline{\varepsilon},c-\overline{\varepsilon}]$ and all $t\in[0,1]$;*
(3) *η_t is a homeomorphism of X onto X for all $t\in[0,1]$;*
(4) *$\varphi(\eta_t(x))\leqslant\varphi(x)$ for all $x\in X$, $t\in[0,1]$;*
(5) *$\eta_1(A_{c+\varepsilon}-N)\subset A_{c-\varepsilon}$, where $A_c=\{x\in X:\varphi(x)\leqslant c\}$ for any $c\in\mathbb{R}$;*

(6) *if* $K_c = \emptyset$, $\eta_1(A_{c+\varepsilon}) \subset A_{c-\varepsilon}$;
(7) *if* φ *is even,* η_t *is odd in* x.

Remark 2.9. *The Lemma 2.8 remains valid if* φ *satisfies the* $(PS)_c$ *condition for* $c < 0$.

3. Krasnoselskii's genus

In order to prove the multiplicity results stated in Theorem 1.1, our main tool is given by the classical Krasnoselskii's genus theory. We will present some basic notions on Krasnoselskii's genus and, for more information about this subject, we refer [3], [4], [12], [13], [41], and [43].

Let E be a real Banach space and let us denote by $\mathfrak{A}$ the class of all closed subsets $A \subset E \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$. Let $A \in \mathfrak{A}$. The Krasnoselskii's genus $\gamma(A)$ of A is defined as being the least positive integer k such that there is an odd mapping $\phi \in C(A, \mathbb{R}^k)$ such that $\phi(x) \neq 0$ for every $x \in A$. If such a k does not exist, we set $\gamma(A) = +\infty$. Furthermore, we set $\gamma(\emptyset) = 0$.

Theorem 3.1. *Let* $E = \mathbb{R}^k$ *and* $\partial\Omega$ *be the boundary of an open, symmetric and bounded subset* $\Omega \subset \mathbb{R}^k$ *with* $0 \in \Omega$. *Then* $\gamma(\partial\Omega) = k$.

Corollary 3.2. $\gamma(\mathbb{S}^{k-1}) = k$ *where* $\mathbb{S}^{k-1}$ *is a unit sphere of* $\mathbb{R}^k$.

As a consequence, we have that, if E is a separable infinite dimensional vector space and $\mathbb{S}$ is the unit sphere in E, then $\gamma(\mathbb{S}) = +\infty$.

Proposition 3.3. *If* $K \in \mathfrak{U}$, $0 \notin K$, *and* $\gamma(K) \geqslant 2$, *then* K *has infinitely many points.*

Proposition 3.4. *(*[43, Proposition 7.5] *and* [3, Lemma 10.4]*) Let* $A, B \in \mathfrak{A}$. *Then*

(1) *Mapping property: If there exists* $\varphi \in C(A, B)$, *odd, then* $\gamma(A) \leqslant \gamma(B)$;
(2) *If there exists an odd homeomorphism between* A *and* B, *then* $\gamma(A) = \gamma(B)$;
(3) *Monotonicity property: If* $A \subset B$, *then* $\gamma(A) \leqslant \gamma(B)$;
(4) *Subadditivity: If* $\gamma(A \cup B) \leqslant \gamma(A) + \gamma(B)$;
(5) *If* $\gamma(B) < +\infty$, *then* $\gamma(\overline{A \setminus B}) \geqslant \gamma(A) - \gamma(B)$;
(6) *Continuity property: If* A *is compact, then* $\gamma(A) < \infty$, *and there exists* $\delta > 0$ *such that* $\gamma(A) = \gamma(N_\delta(A))$ *where* $N_\delta(A) = \{x \in X : d(x, A) \leqslant \delta\}$;
(7) *if* $\eta \in C(A, E)$ *is odd then* $\gamma(A) \leqslant \gamma(\eta(A))$.

4. Variational framework

The problem $(\mathscr{P}_\lambda)$ has a variational structure and the natural space to look for solutions are variable exponent Sobolev spaces.

Definition 4.1. *We say that* $u \in X$ *is a weak solution of the problem* $(\mathscr{P}_\lambda)$ *if verifies*

$$M(\mathscr{A}(u)) \int_\Omega a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u \nabla v \, dx = \lambda \int_\Omega f(x,u) v \, dx + \int_\Omega |u|^{s(x)-2} u v \, dx \tag{4.1}$$

for all $v \in X$, *where* X *is the Banach space*

$$X := W_0^{1,p(\cdot)}(\Omega) \cap W_0^{1,\gamma(\cdot)}(\Omega),$$

endowed with the norm

$$\|u\| = \|\nabla u\|_{p(\cdot)} + \mathcal{H}(\kappa_3)\|\nabla u\|_{q(\cdot)}.$$

Thanks to our hypotheses on a, M, f, Ω, $p(\cdot)$, and $s(\cdot)$ the integrals in (4.1) are well defined if $u, v \in X$.

The weak solutions of $(\mathscr{P}_\lambda)$ coincides with the critical points of the C^1- functional $\Phi_\lambda : X \to \mathbb{R}$ given by

$$\Phi_\lambda(u) = \mathscr{M}(\mathscr{A}(u)) - \lambda \int_\Omega F(x,u)\, dx - \int_\Omega \frac{1}{s(x)} |u|^{s(x)}\, dx,$$

for all $u \in X$. Note that the functional Φ_λ is Fréchet differentiable in $u \in X$ with Fréchet derivative

$$\langle \Phi_\lambda'(u), v\rangle = M(\mathscr{A}(u)) \int_\Omega a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u \nabla v \, dx - \lambda \int_\Omega f(x,u) v\, dx - \int_\Omega |u|^{s(x)-2} u v\, dx,$$

for all $v \in X$. Hence, the critical points of Φ_λ are weak solutions for $(\mathscr{P}_\lambda)$.

5. The auxiliary problem

One of the difficulties of the problem $(\mathscr{P}_\lambda)$ is the presence of the critical term which makes it not simple to show that the functional Φ_λ verifies the Palais-Samle condition. The genus Krasnoselskii theory requires that the functional Φ_λ is bounded from below. Moreover, since, we only have the information of that M is an increasing function bounded from below, it complicates to prove that the functional $\Phi_{\omega,\lambda}$ verifies the Palais-Smale condition. In order to prove Theorem 1.1, we will make a truncation in the function M and get an truncated problem $(\mathscr{M}_\lambda)$. Therefore, we will obtain infinity many solutions to the problem $(\mathscr{M}_\lambda)$, which will be proven that are solutions to the original problem $(\mathscr{P}_\lambda)$.

Since $\gamma^+ < s^-$, there exists $\sigma \in \mathbb{R}$ such that $\sigma \in (\gamma^+, s^-)$. Furthermore, by $(\mathcal{M}_2)$, exist $t_0 > 0$ such that $m_0 = M(0) < \omega = M(t_0) < \frac{m_0\sigma}{p^+\alpha}$

$$M_\omega(t) = \begin{cases} M(t) & \text{if } 0 \leqslant t \leqslant t_0, \\ \omega & \text{if } t \geqslant t_0. \end{cases} \tag{5.1}$$

Now, we introduce the auxiliary truncated problem

$$\begin{cases} -M_\omega(\mathscr{A}(u))\,\mathrm{div}\Big(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\Big) = \lambda f(x,u) + |u|^{s(x)-2}u & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{M}_\lambda$}$$

with a, f, and λ defined as in problem $(\mathscr{P}_\lambda)$. Note that, due to (5.1), we have

$$M_\omega(t) \leqslant \omega \text{ for all } t \geqslant 0. \tag{5.2}$$

Theorem 5.1. *Assume $(a_0) - (a_3)$, and $(\mathcal{M}_1) - (\mathcal{M}_2)$, and $(f_1) - (f_2)$. Then, there exists $\overline{\lambda} > 0$, such that Problem $(\mathcal{M}_\lambda)$ has infinitely many weak solutions in X for each $\lambda \in (0, \overline{\lambda})$.*

5.1. **Variational formulation of the auxiliary problem.**

We observe that problem $(\mathscr{M}_\lambda)$ has a variational structure, the Euler-Lagrange functional $\Phi_{\omega,\lambda} : X \to \mathbb{R}$ is defined as follows

$$\Phi_{\omega,\lambda}(u) = \mathscr{M}_\omega(\mathscr{A}(u)) - \lambda \int_\Omega F(x,u)\,dx - \int_\Omega \frac{1}{s(x)}|u|^{s(x)}\,dx$$

where $\mathscr{M}_\omega(t) = \int_0^t M_\omega(s)\,ds$. Moreover, by standard arguments the functional $\Phi_{\omega,\lambda} \in C^1(X,\mathbb{R})$, and

$$\langle \Phi'_{\omega,\lambda}(u), v\rangle = M_\omega(\mathscr{A}(u)) \int_\Omega a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u \nabla v\,dx - \lambda \int_\Omega f(x,u)v\,dx - \int_\Omega |u|^{s(x)-2}uv\,dx, \tag{5.3}$$

$u \in X$ and for any $v \in X$.

6. The Palais-Smale Condition

In order to use variational methods, we give some results related to the Palais-Smale condition. We say that a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ is a Palais-Smale sequence for the functional Φ_λ at level c if

$$\Phi_\lambda(u_n) \to c \text{ and } \Phi'_\lambda(u_n) \to 0 \text{ in } X', \text{ as } n \to +\infty. \tag{6.1}$$

If (6.1) implies the existence of a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ which converges in X, we say that Φ_λ satisfies the Palais-Smale condition. Moreover, if this strongly convergent subsequence exits only for some c values, we say that Φ_λ satisfies a local Palais-Smale condition.

We will discuss the property of compactness of the auxiliary functional $\Phi_{\omega,\lambda}$, given by the Palais-Smale condition. In order to overcome the lack of compactness will use the compactness principle of variable exponent Sobolev spaces (see Proposition 2.7). However, the Palais-Smale condition is not hold at any level. In fact, we will see that this occurs just below an adequate level, which depends on the best constant in the inequality Gagliardo-Nirenberg-Sobolev for exponents variables, namely,

$$S = S_q(\Omega) = \inf_{v \in C_0^\infty(\Omega)} \frac{\|\nabla v\|_{p(\cdot)}}{\|v\|_{q(\cdot)}}. \tag{6.2}$$

Lemma 6.1. *Let $\{u_n\}_{n\in\mathbb{N}} \subset X$ be a Palais-Smale sequence for $\Phi_{\omega,\lambda}$, with energy level c_λ, then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X.*

Proof. Fix $\lambda > 0$. Let $\{u_n\}_{n\in\mathbb{N}} \subset X$ be a Palais-Smale sequence with energy level c_λ (see (6.1)). Hence, there exists $C > 0$ such that $\left|\Phi_{\omega,\lambda}(u_n)\right| \leqslant C$ for any $n \in \mathbb{N}$. Therefore, by $(\mathcal{M}_1)$, (a_3), (5.2), and remembering that $m_0 < \omega < \frac{m_0\sigma}{p^+\alpha}$, we obtain

$$
\begin{aligned}
C\left(1+\|u_n\|\right) \geqslant & \Phi_{\omega,\lambda}(u_n) - \frac{1}{\sigma}\langle \Phi'_{\omega,\lambda}(u_n), u_n\rangle \\
\geqslant & \mathscr{M}_\omega(\mathscr{A}(u_n)) - \frac{1}{\sigma} M_\omega(\mathscr{A}(u_n)) \int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)} dx + \lambda\left(\frac{a_1}{\sigma} - \frac{a_2}{r^-}\right)\int_\Omega |u_n|^{r(x)} dx \\
\geqslant & \frac{m_0}{p^+}\int_\Omega A(|\nabla u_n|^{p(x)})\, dx - \frac{\omega}{\sigma}\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)} dx + \lambda\left(\frac{a_1}{\sigma} - \frac{a_2}{r^-}\right)\int_\Omega |u_n|^{r(x)} dx \\
\geqslant & \left(\frac{m_0}{p^+\alpha} - \frac{\omega}{\sigma}\right)\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\, dx + \lambda\left(\frac{a_1}{\sigma} - \frac{a_2}{r^-}\right)\int_\Omega |u_n|^{r(x)}\, dx.
\end{aligned}
$$

Then, we have

$$
C\left(1+\|u_n\|\right) + \lambda\left(\frac{a_2}{r^-} - \frac{a_1}{\sigma}\right)\int_\Omega |u_n|^{r(x)}\, dx \geqslant \left(\frac{m_0}{p^+\alpha} - \frac{\omega}{\sigma}\right)\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\, dx.
$$

Furthermore, by (a_1), there are positive constants C_7, C_8, and C_9 such that

$$
C(1+\|u_n\|) + C_7 \max\left\{\|u_n\|^{r^+}, \|u_n\|^{r^-}\right\} \geqslant C_8\left(\int_\Omega |\nabla u_n|^{p(x)}\, dx\right) + C_9\mathcal{H}(\kappa_3)\left(\int_\Omega |\nabla u_n|^{q(x)}\, dx\right). \tag{6.3}
$$

Thus, if $\kappa_3 = 0$, suppose by contradiction that, up to a subsequence, $\|u_n\| \to +\infty$, then, we get from (6.3) that

$$
C(1+\|u_n\|) + C_7\|u_n\|^{r^+} \geqslant C_8\|u_n\|^{p^-},
$$

which it is a contradiction, because $1 < r^+ < p^-$. So, we conclude that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X.

If $\kappa_3 > 0$, suppose by contradiction that, up to a subsequence, $\|u_n\| \to +\infty$. Then, we analyze the following cases:

(i) $\|\nabla u_n\|_{p(\cdot)} \to +\infty$ and $\|\nabla u_n\|_{q(\cdot)} \to +\infty$ as $n \to +\infty$;
(ii) $\|\nabla u_n\|_{p(\cdot)} \to +\infty$ and $\|\nabla u_n\|_{q(\cdot)}$ is bounded;
(iii) $\|\nabla u_n\|_{p(\cdot)}$ is bounded and $\|\nabla u_n\|_{q(\cdot)} \to +\infty$.

In the case (i), for n large enough, $\|\nabla u_n\|_{q(\cdot)}^{q^- - p^-} \geqslant 1$, this is $\|\nabla u_n\|_{q(\cdot)}^{q^-} \geqslant \|\nabla u_n\|_{q(\cdot)}^{p^-}$. Hence, follows from (6.3) that

$$
\begin{aligned}
C(1+\|u_n\|) + C_7\|u_n\|^{r^+} &\geqslant C_8\|\nabla u_n\|_{p(\cdot)}^{p^-} + C_9\mathcal{H}(\kappa_3)\|\nabla u_n\|_{q(\cdot)}^{q^-} \\
&\geqslant C_8\|\nabla u_n\|_{p(\cdot)}^{p^-} + C_9\mathcal{H}(k_3)\|\nabla u_n\|_{q(\cdot)}^{p^-} \\
&\geqslant C_{10}\left(\|\nabla u_n\|_{p(\cdot)} + \mathcal{H}(k_3)\|\nabla u_n\|_{q(\cdot)}\right)^{p^-} \\
&= C_{10}\|u_n\|^{p^-},
\end{aligned}
$$

therefore, taking limit as $n \to +\infty$, we obtain an absurd.

In the case (ii), we achieve by (6.3) that

$$
\begin{aligned}
C(1+\|u_n\|) + K(\|\nabla u_n\|_{p(\cdot)}^{r^+} + \mathcal{H}(\kappa_3)\|\nabla u_n\|_{q(\cdot)}^{r^+}) &\geqslant C(1+\|u_n\|) + C_7\|u_n\|^{r^+} \\
&\geqslant C_{11}\|\nabla u_n\|_{p(\cdot)}^{p^-},
\end{aligned}
$$

so, it follows

$$
C\left(\frac{1}{\|\nabla u_n\|_{p(\cdot)}^{p^-}} + \frac{1}{\|\nabla u_n\|_{p(\cdot)}^{p^- -1}} + \frac{\|\nabla u_n\|_{q(\cdot)}}{\|\nabla u_n\|_{p(\cdot)}^{p^-}}\right) + K\left(\frac{1}{\|\nabla u_n\|_{p(\cdot)}^{p^- - r^+}} + \frac{\|\nabla u_n\|_{q(\cdot)}^{r^+}}{\|\nabla u_n\|_{p(\cdot)}^{p^-}}\right) \geqslant C_{11} > 0.
$$

Since $p^- > r^+ > 1$, taking limit as $n \to +\infty$, we obtain a contradiction.

The case (iii) is similar to case (ii).

Then, we conclude that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X. □

In the following lemma, we obtain a local Palais-Smale condition for the truncated functional $\Phi_{\omega,\lambda}$, the main result of this section.

Lemma 6.2. *Assume* (a_0), (a_1), (a_2), (a_3), $(\mathcal{M}_1)$, $(\mathcal{M}_2)$, (f_1), *and* (f_2). *Let* $\{u_n\}_{n\in\mathbb{N}}$ *be a Palais-Smale sequence at level* c_λ *with*

$$c_\lambda < \left(\frac{1}{\sigma} - \frac{1}{s^-_{\mathscr{C}}}\right)(\overline{m_0\kappa}S)^N - \mathcal{K}\min\left\{\lambda^{\frac{\left(\frac{s}{r}\right)^-}{\left(\frac{s}{r}\right)^- - 1}}, \lambda^{\frac{\left(\frac{s}{r}\right)^+}{\left(\frac{s}{r}\right)^+ - 1}}\right\}, \tag{6.4}$$

where $\overline{m_0\kappa} = \min\left\{(m_0((1-\mathcal{H}(\kappa_3))\kappa_0\mu_j + \mathcal{H}(\kappa_3)\kappa_2\mu_j))^{\frac{1}{\gamma^-}}, (m_0((1-\mathcal{H}(\kappa_3))\kappa_0\mu_j + \mathcal{H}(\kappa_3)\kappa_2\mu_j))^{\frac{1}{\gamma^+}}\right\}$ *and* $\mathcal{K}$ *is a positive constant independent of* λ. *Then, up to subsequence,* $\{u_n\}_{n\in\mathbb{N}}$ *is strongly convergent in* X.

Proof. We have by Lemma 6.1 that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X. Thus, as X is reflexive, by *Eberlein – Šmulian* Theorem ([10], Theorem 3.19), up to a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, there is $u \in X$, such that

$$\begin{gathered} u_n \rightharpoonup u \text{ weakly in } X, \\ u_n \to u \text{ in } L^{t(\cdot)}(\Omega), \text{ for all } 1 < t^- \leqslant t(x) \leqslant t^+ < \gamma^*(x), \\ u_n \to u \text{ a.e. in } \Omega. \end{gathered}$$

From the concentration compactness principle of Lions for variable exponents (see Proposition 2.7), we have that there exist two nonnegative measures $\mu, \nu \in \mathscr{M}(\Omega)$, a countable set $\mathcal{J}$, points $\{x_j\}_{j\in\mathcal{J}} \subset \mathscr{C} \subset \Omega$ and sequences $\{\mu_j\}_{j\in\mathcal{J}}, \{\nu_j\}_{j\in\mathcal{J}} \subset [0, +\infty)$, such that

$$\begin{gathered} |\nabla u_n|^{\gamma(x)} \rightharpoonup \mu \quad \text{(weak*-sense of measures) in } \mathscr{M}(\Omega), \\ |u_n|^{s(x)} \rightharpoonup \nu \quad \text{(weak*-sense of measures) in } \mathscr{M}(\Omega), \\ \nu = |u|^{s(x)} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j}, \quad \nu_j > 0, \\ \mu \geqslant |\nabla u|^{\gamma(x)} + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j}, \quad \mu_j > 0, \end{gathered} \tag{6.5}$$

where δ_{x_j} is the Dirac's delta measure supported on $x_j \in \Omega$ and

$$S\nu_j^{\frac{1}{\gamma^*(x_j)}} \leqslant \mu_j^{\frac{1}{\gamma(x_j)}} \ \forall j \in \mathcal{J}, \tag{6.6}$$

where S is the best positive constant of the Gagliardo-Nirenberg-Sobolev embedding (6.2).
Claim 1. We claim that the set $\mathcal{J}$ is empty. Consequently, $\{u_n\}_{n\in\mathbb{N}}$ converge strongly to u in $L^{s(\cdot)}(\Omega)$.
Arguing by contradiction, suppose $\mathcal{J} \neq \emptyset$. Fix $j \in \mathcal{J}$. Then, we prove that

$$\nu_j \geqslant (\overline{m_0\kappa}S)^N.$$

We consider a cutoff function $\psi \in C_0^\infty(\Omega, [0,1])$ such that $\psi \equiv 1$ in $B(0,1)$ and $\psi \equiv 0$ in $\mathbb{R}^N \setminus B(0,2)$, and $|\nabla\psi|_\infty \leqslant 2$. We define, for all $\varepsilon > 0$, the function

$$\psi_{\varepsilon,j}(x) := \psi\left(\frac{x - x_j}{\varepsilon}\right),$$

thus $\psi_{\varepsilon,j} \in C_0^{+\infty}(\mathbb{R}^N)$, $0 \leqslant \psi_{\varepsilon,j}(x) \leqslant 1$, for all $x \in \mathbb{R}^N$, $|\nabla\psi_{\varepsilon,j}|_\infty \leqslant \frac{2}{\varepsilon}$, and

$$\psi_{\varepsilon,j}(x) = \begin{cases} 1 & \text{if } x \in B(x_j, \varepsilon), \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B(x_j, 2\varepsilon). \end{cases}$$

Clearly by Lemma 2.1, Proposition 2.2, and Lemma 2.3 the sequence $\{\psi_{\varepsilon,j}u_n\}_{n\in\mathbb{N}}$ is bounded in X. Thus, as $\{u_n\}_{n\in\mathbb{N}}$ is a Palais-Smale sequence, it follows that $\langle \Phi'_{\omega,\lambda}(u_n), \psi_{\varepsilon,j}u_n\rangle \to 0$ as $n\to+\infty$, that is,

$$\begin{aligned} M_\omega(\mathscr{A}(u_n)) &\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n u_n \nabla\psi_{\varepsilon,j}\,dx \\ &= - M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\psi_{\varepsilon,j}\,dx + \lambda\int_\Omega f(x,u_n)u_n\psi_{\varepsilon,j}\,dx \\ &\quad + \int_\Omega \psi_{\varepsilon,j}|u_n|^{s(x)}\,dx + o_n(1), \end{aligned} \tag{6.7}$$

where $o_n(1)\to 0$ as $n\to+\infty$.

We obtain by (a_1), $(\mathcal{M}_1)$, and (6.7) that

$$\begin{aligned} M_\omega(\mathcal{A}(u_n)) &\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n u_n\nabla\psi_{\varepsilon,j}\,dx \\ &\leqslant - m_0\left(\int_\Omega \kappa_0|\nabla u_n|^{p(x)}\psi_{\varepsilon,j}\,dx + \mathcal{H}(\kappa_3)\kappa_2\int_\Omega |\nabla u_n|^{q(x)}\psi_{\varepsilon,j}\,dx\right) \\ &\quad + \lambda\int_\Omega f(x,u_n)u_n\psi_{\varepsilon,j}\,dx + \int_\Omega \psi_{\varepsilon,j}|u_n|^{s(\cdot)}\,dx + o_n(1). \end{aligned} \tag{6.8}$$

Since that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X, we have, up to subsequence, that $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in $L^{r(\cdot)}(\Omega)$ and that there exists $h\in L^{r(\cdot)}(\Omega)$ such that $|u_n(x)|\leqslant h(x)$ and $u_n(x)\to u(x)$, for a.e. in Ω (see [21, Proposition 2.67]). From (f_2) we get

$$|f(x,u_n)u_n\psi_{\varepsilon,j}| \leqslant a_2|h(x)|^{r(x)} \in L^1(\Omega).$$

Hence, applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{n\to+\infty}\lambda\int_\Omega f(x,u_n(x))u_n(x)\psi_{\varepsilon,j}(x)\,dx = \lambda\int_\Omega f(x,u(x))u(x)\psi_{\varepsilon,j}(x)\,dx. \tag{6.9}$$

Again, by applying the the Lebesgue Dominated Convergence Theorem, we get

$$\begin{gathered} \lim_{\varepsilon\to0^+}\int_\Omega |\nabla u(x)|^{p(x)}\psi_{\varepsilon,j}(x)\,dx = 0,\ \lim_{\varepsilon\to0^+}\int_\Omega f(x,u(x))u(x)\psi_{\varepsilon,j}(x)\,dx = 0, \\ \lim_{\varepsilon\to0^+}\int_\Omega \psi_{i,\varepsilon}d\mu = \mu_i\psi(0),\ \text{and}\ \lim_{\varepsilon\to0^+}\int_\Omega \psi_{i,\varepsilon}d\nu = \nu_i\psi(0). \end{gathered} \tag{6.10}$$

Futhermore, we achieve by (6.5) that

$$\lim_{n\to+\infty}\int_\Omega |u_n(x)|^{s(x)}\psi_{\varepsilon,j}(x)\,dx = \int_\Omega \psi_{\varepsilon,j}(x)d\nu. \tag{6.11}$$

From (a_1), and since $M_\omega(\cdot)$ is bounded, we argue like in [8] to obtain that

$$\lim_{\varepsilon\to0^+}\left[\limsup_{n\to+\infty}\left(M_\omega(\mathscr{A}(u_n)\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n u_n\nabla\psi_{\varepsilon,j}\,dx\right)\right] = 0, \tag{6.12}$$

Now, we will use the expressions (6.8), (6.9), (6.10), (6.11), and (6.12) to analyze the following cases:

(I) When $\kappa_3 = 0$, we have

$$\begin{aligned} 0 &\leqslant \lim_{\varepsilon\to0^+}\left[\int_\Omega \psi_{\varepsilon,j}d\nu - m_0\kappa_0\int_\Omega \psi_{\varepsilon,j}d\mu\right] \\ &\leqslant \lim_{\varepsilon\to0^+}\left[\int_{B(x_j,2\varepsilon)} \psi_{\varepsilon,j}d\nu - m_0\kappa_0\int_{B(x_j,2\varepsilon)}\psi_{\varepsilon,j}d\mu\right] \\ &\leqslant \nu_j - m_0\kappa_0\mu_j, \end{aligned}$$

then, we achieve

$$m_0\kappa_0\mu_j \leqslant \nu_j. \tag{6.13}$$

(II) When $\kappa_3 > 0$, we have

$$\begin{aligned} 0 \leqslant & \lim_{\varepsilon\to 0^+}\left[\int_\Omega \psi_{\varepsilon,j} d\nu - m_0\mathcal{H}(\kappa_3)\kappa_2\int_\Omega \psi_{\varepsilon,j} d\mu\Big)\right] \\ \leqslant & \lim_{\varepsilon\to 0^+}\left[\int_{B(x_j,2\varepsilon)} \psi_{\varepsilon,j} d\nu - m_0\kappa_0\int_{B(x_j,2\varepsilon)} \psi_{\varepsilon,j} d\mu\right] \\ \leqslant & \nu_j - m_0\mathcal{H}(\kappa_3)\kappa_2\mu_j, \end{aligned}$$

thus, we obtain

$$m_0\mathcal{H}(\kappa_3)\kappa_2\mu_j \leqslant \nu_j. \tag{6.14}$$

Consequently, by combining (6.13) and (6.14), we get

$$m_0((1-\mathcal{H}(\kappa_3))\kappa_0\mu_j + \mathcal{H}(\kappa_3)\kappa_2\mu_j) \leqslant \nu_j. \tag{6.15}$$

Therefore, by using (6.6) and (6.15), we obtain

$$(\overline{m_0\kappa}S)^N \leqslant \nu_j, \tag{6.16}$$

where

$$\overline{m_0\kappa} = \min\left\{ (m_0((1-\mathcal{H}(\kappa_3))\kappa_0 + \mathcal{H}(\kappa_3)\kappa_2))^{\frac{1}{\gamma^-}}, (m_0((1-\mathcal{H}(\kappa_3))\kappa_0 + \mathcal{H}(\kappa_3)\kappa_2))^{\frac{1}{\gamma^+}} \right\}.$$

Now, we claim that the inequality (6.16) cannot hold, and hence the set $\mathcal{J}$ is empty. Indeed, remembering that $m_0 \leqslant M_\omega(t) < \frac{\sigma m_0}{p^+\alpha}$, for all $t \in \mathbb{R}$, we get

$$\begin{aligned} c_\lambda =& \Phi_{\omega,\lambda}(u_n) - \frac{1}{\sigma}\langle \Phi'_{\omega,\lambda}(u_n), u_n\rangle + o_n(1) \\ \geqslant & m_0\int_\Omega \frac{1}{p(x)}A(|\nabla u_n|^{p(x)})\,dx - \lambda\int_\Omega F(x,u_n)\,dx - \int_\Omega \frac{1}{s(x)}|u_n|^{s(x)}\,dx \\ & - \frac{m_0}{p^+\alpha}\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\,dx + \frac{\lambda}{\sigma}\int_\Omega f(x,u_n)u_n\,dx + \frac{1}{\sigma}\int_\Omega |u_n|^{s(x)}\,dx + o_n(1). \end{aligned}$$

However, by (a_3), we have

$$\int_\Omega \frac{1}{p(x)}A(|\nabla u_n|^{p(x)})\,dx - \frac{1}{p^+\alpha}\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\,dx \geqslant 0,$$

for all $n \in \mathbb{N}$. Hence, since $0 \leqslant \psi_{\varepsilon,j} \leqslant 1$ and by using (f_1) and (f_2), we obtain

$$\begin{aligned} c_\lambda \geqslant & -\lambda a_2\left(\frac{1}{\sigma}+\frac{1}{r^-}\right)\int_\Omega |u_n|^{r(x)}\,dx + \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\int_\Omega |u_n|^{s(x)}\,dx + o_n(1) \\ \geqslant & -\lambda a_2\left(\frac{1}{\sigma}+\frac{1}{r^-}\right)\int_\Omega |u_n|^{r(x)}\,dx + \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\int_\Omega \psi_{\varepsilon,j}|u_n|^{s(x)}\,dx + o_n(1). \end{aligned} \tag{6.17}$$

Taking limit in (6.17) as $n \to +\infty$ and by using (6.5) and Lebesgue Dominated Convergence Theorem, we get

$$\begin{aligned} c_\lambda \geqslant & \lim_{n\to+\infty}\left(-\lambda a_2\left(\frac{1}{\sigma}+\frac{1}{r^-}\right)\int_\Omega |u_n|^{r(x)}\,dx + \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\int_\Omega \psi_{\varepsilon,j}|u_n|^{s(x)}\,dx \right) \\ \geqslant & -\lambda a_2\left(\frac{1}{\sigma}+\frac{1}{r^-}\right)\int_\Omega |u|^{r(x)}\,dx + \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\int_\Omega \psi_{\varepsilon,j}|u|^{s(x)}\,dx + \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\nu_j. \end{aligned}$$

On the other hand, observe that

$$|u(x)|^{s(x)}\psi_{\varepsilon,j}(x)\chi_{B(x_j,2\varepsilon)} \to |u(x)|^{s(x)} \text{ as } \varepsilon \to +\infty, \text{ a.e. in } \Omega,$$

and

$$|u(x)|^{s(x)}\psi_{\varepsilon,j}(x)\chi_{B(x_j,2\varepsilon)} \leqslant |u(x)|^{s(x)} \text{ a.e. in } \Omega,$$

thus, by applying the Lebesgue Dominated Convergence Theorem, as $\varepsilon \to 0^+$, we obtain

$$c_\lambda \geqslant -\lambda a_2\left(\frac{1}{\sigma}+\frac{1}{r^-}\right)\int_\Omega |u|^{r(x)}\,dx + \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\int_\Omega |u|^{s(x)}\,dx + \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\nu_j.$$

Then, by using (6.16) and Hölder inequality (Lemma 2.1), we achieve

$$\begin{aligned} c_\lambda \geqslant & -\lambda a_2\left(\frac{1}{\sigma}+\frac{1}{r^-}\right)\int_\Omega |u|^{r(x)}\,dx+\left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\int_\Omega |u|^{s(x)}\,dx+\left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\left(\overline{m_0}\kappa S\right)^N \\ \geqslant & -\lambda a_2\left(\frac{1}{\sigma}+\frac{1}{r^-}\right)\left(\frac{1}{(\frac{s}{r})^-}+\frac{1}{(\frac{s}{s-r})^-}\right)\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}\|1\|_{\frac{s(\cdot)}{s(\cdot)-r(\cdot)}}+\left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\int_\Omega |u|^{s(x)}\,dx \\ & +\left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\left(\overline{m_0}\kappa S\right)^N. \end{aligned} \tag{6.18}$$

Now, we will examine the possible cases:

(i) If $\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}} \geqslant 1$, by Proposition 2.2, we have

$$\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}^{\left(\frac{s}{r}\right)^-} \leqslant \int_\Omega |u|^{r(x)\cdot\frac{s(x)}{r(x)}}\,dx \leqslant \||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}^{\left(\frac{s}{r}\right)^+}. \tag{6.19}$$

Hence, by (6.18) and (6.19), we get

$$c_\lambda \geq c_1\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}^{(\frac{s}{r})^-}-\lambda c_2\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}+c_3,$$

where $c_1=\left(\frac{1}{\sigma}-\frac{1}{s^-}\right)$, $c_2=a_2\left(\frac{1}{\sigma}+\frac{1}{r^-}\right)\left(\frac{1}{(\frac{s}{r})^-}+\frac{1}{(\frac{s}{s-r})^-}\right)\|1\|_{\frac{s(\cdot)}{s(\cdot)-r(\cdot)}}$, and $c_3=\left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\left(\overline{m_0}\kappa S\right)^N$.

Defining the function $\mathcal{E}_1:\mathbb{R}^+\to\mathbb{R}$ given by $\mathcal{E}_1(t)=c_1t^{(\frac{s}{r})^-}-\lambda c_2 t$ (observe that $(\frac{s}{r})^->1$), attains its absolute minimum at the point

$$t^*=\left(\frac{\lambda c_2}{c_1(s/r)^-}\right)^{\frac{1}{(s/r)^--1}}>0.$$

Moreover, note that $r^-\leqslant r(x)\leqslant r^+<s^-$ and we have

$$\begin{aligned}\mathcal{E}_1(t^*)&=c_1\left(\frac{\lambda c_2}{c_1(s/r)^-}\right)^{\frac{(s/r)^-}{(s/r)^--1}}-\lambda c_2\left(\frac{\lambda c_2}{c_1(s/r)^-}\right)^{\frac{1}{(s/r)^--1}}\\ &=c_1\left(\frac{\lambda c_2}{c_1(s/r)^-}\right)^{\frac{(s/r)^-}{(s/r)^--1}}\left(1-\left(\frac{s}{r}\right)^-\right)\\ &=-\lambda^{\frac{(s/r)^-}{(s/r)^--1}}\mathcal{K},\end{aligned}$$

where $\mathcal{K}$ is a positive constant independent of λ.

(ii) If $\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}<1$, by Proposition 2.2, we obtain

$$\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}^{\left(\frac{s}{r}\right)^+} \leqslant \int_\Omega |u|^{r(x)\cdot\frac{s(x)}{r(x)}}\,dx \leqslant \||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}^{\left(\frac{s}{r}\right)^-}. \tag{6.20}$$

Then, by combining (6.18) and (6.20), we achieve

$$c_\lambda \geq c_1\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}^{(\frac{s}{r})^+}-\lambda c_2\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}+c_3,$$

where $c_1=\left(\frac{1}{\sigma}-\frac{1}{s^-}\right)$, $c_2=a_2\left(\frac{1}{\sigma}+\frac{1}{r^-}\right)\left(\frac{1}{(\frac{s}{r})^-}+\frac{1}{(\frac{s}{s-r})^-}\right)\|1\|_{\frac{s(\cdot)}{s(\cdot)-r(\cdot)}}$, and $c_3=\left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\left(\overline{m_0}\kappa S\right)^N$. Therefore, defining the function $\mathcal{E}_2:\mathbb{R}^+\to\mathbb{R}$ given by $\mathcal{E}_2(t)=c_1t^{(\frac{s}{r})^+}-\lambda c_2t$ (observe that $(\frac{s}{r})^+>1$), admits absolute minimum at the point

$$t_*=\left(\frac{\lambda c_2}{(\frac{s}{r})^+c_1}\right)^{\frac{1}{(s/r)^+-1}}>0.$$

Furthermore, analogously to the previous case (i), we get

$$\mathcal{E}_2(t_*)=-\lambda^{\frac{(s/r)^+}{(s/r)^+-1}}\mathcal{K},$$

where $\mathcal{K}$ is a positive constant independent of λ.

Therefore, by using (i) and (ii), we get

$$c_\lambda \geqslant \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)(\overline{m_0\kappa}S)^N-\mathcal{K}\min\left\{\lambda^{\frac{\left(\frac{s}{r}\right)^-}{\left(\frac{s}{r}\right)^- -1}},\lambda^{\frac{\left(\frac{s}{r}\right)^+}{\left(\frac{s}{r}\right)^+ -1}}\right\}.$$

Then, due to (6.4), we obtain that $\mathcal{J}$ is empty, and consequently $\rho_{s(\cdot)}(u_n)\to\rho_{s(\cdot)}(u)$ as $n\to+\infty$. Thus, from Lemma 2.4 and Proposition 2.2, we conclude that the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in $L^{s(\cdot)}(\Omega)$ as $n\to+\infty$. Thus, we conclude the proof of **Claim 1**.

Claim 2. We affirm that $u_n\to u$ strongly in X as $n\to+\infty$.

In fact. We have that $\Phi'_{\omega,\lambda}(u_n)\to 0$ in X', as $n\to+\infty$, and the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X, then

$$\begin{aligned}0&=\lim_{n\to+\infty}\langle\Phi'_{\omega,\lambda}(u_n),u_n-u\rangle\\&=\lim_{n\to+\infty}\left\{M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx\right.\\&\qquad\left.-\lambda\int_\Omega f(x,u_n)(u_n-u)\,dx-\int_\Omega|u_n|^{s(x)-2}u_n(u_n-u)\,dx\right\}.\end{aligned}\tag{6.21}$$

But, as $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X, by (a_1), we have that $\{\mathscr{A}(u_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Thus, by applying the Bolzano-Weierstrass Theorem, there exists $\widehat{t}\in\mathbb{R}^+$ such that $\mathscr{A}(u_n)\to\widehat{t}$ as $n\to+\infty$, and, as M is continuous,

$$\lim_{n\to+\infty}M_\omega(\mathscr{A}(u_n))=M_\omega(\widehat{t})\geq m_0.$$

By using (f_2) and the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to+\infty}\left|\lambda\int_\Omega f(x,u_n(x))(u_n(x)-u(x))\,dx\right|=0.$$

Also, by applying the Hölder's inequality and since $u_n\to u$ in $L^{s(\cdot)}(\Omega)$ as $n\to+\infty$, we achieve

$$\lim_{n\to+\infty}\left|\int_\Omega|u_n(x)|^{s(x)-2}u_n(x)(u_n(x)-u(x))\,dx\right|=0.$$

Therefore, follows from (6.21) that

$$\lim_{n\to+\infty}\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx=0.$$

Consequently, follows from [38, Lemma 8.2] that $u_n\to u$ strongly in $W_0^{1,p(\cdot)}(\Omega)$ as $n\to+\infty$, that is,

$$\lim_{n\to+\infty}\|\nabla u_n-\nabla u\|_{p(\cdot)}=0.\tag{6.22}$$

In particular, we point out that by the Hölder's inequality and (6.22) follows that

$$\lim_{n\to+\infty}\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)dx=0.\tag{6.23}$$

Note that, if $\kappa_3=0$, we get from (6.22) that $u_n\to u$ strongly in X as $n\to+\infty$. Now, let us consider $\kappa_3>0$. Then, since that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded Palais-Smale sequence, by using (a_1), $(\mathcal{M}_1)$, (6.21), (6.22), and (6.23), we obtain

$$\begin{aligned}o_n(1)&=\langle\Phi'_{\omega,\lambda}(u_n),u_n-u\rangle\\&=M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)\,dx+o_n(1)\\&\geqslant C\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)\,dx+o_n(1)\\&\geqslant C\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)+|\nabla u_n|^{q(x)-2}\nabla u_n\nabla(u_n-u)\,dx+o_n(1)\\&=C\int_\Omega|\nabla u_n|^{q(x)-2}\nabla u_n\nabla(u_n-u)\,dx+o_n(1),\end{aligned}$$

so, we get

$$\limsup_{n\to+\infty}\int_\Omega |\nabla u_n|^{q(x)-2}\nabla u_n \nabla(u_n-u)\,dx \leqslant 0,$$

Therefeore, by Theorem 2.6, we obtain $u_n \to u$ strongly in $W_0^{1,q(\cdot)}(\Omega)$ as $n\to+\infty$, that is,

$$\lim_{n\to+\infty}\|\nabla u_n-\nabla u\|_{q(\cdot)}=0. \tag{6.24}$$

Then, we conclude, by (6.22) and (6.24), that $u_n\to u$ strongly in X as $n\to+\infty$.

This concludes the proof of the **Claim 2**.

□

7. A Truncation Argument

We note that the Euler-Lagrange functional $\Phi_{\omega,\lambda}$ is not bounded from below in X as shown in the following lemma.

Lemma 7.1. *The Euler-Lagrange functional $\Phi_{\omega,\lambda}$ associated with $(\mathscr{P}_\lambda)$ is not bounded from below in X.*

Proof. Let us take $\upsilon\in X\setminus\{0\}$. Then, by using (f_2) and (5.2), we have for each $t>1$ that

$$\begin{aligned}\Phi_{\omega,\lambda}(t\upsilon) \leqslant& \int_0^{\mathscr{A}(t\upsilon)} M_\omega(t)\,dt-\lambda\int_\Omega F(x,t\upsilon)\,dx-\int_\Omega\frac{1}{s(x)}|t\upsilon|^{s(x)}\,dx\\ \leqslant&\frac{\omega\kappa_1}{p^-}\int_\Omega|\nabla(t\upsilon)|^{p(x)}\,dx+\frac{\omega\kappa_3}{q^-}\int_\Omega|\nabla(t\upsilon)|^{q(x)}\,dx\\ &+\frac{C}{r^+}\int_\Omega|t\upsilon|^{r(x)}\,dx-\frac{1}{s^+}\int_\Omega|t\upsilon|^{s(x)}\,dx\\ \leqslant&\frac{\omega\kappa_1}{p^-}t^{p^+}\int_\Omega|\nabla\upsilon|^{p(x)}\,dx+\frac{\kappa_3}{q^-}t^{q^+}\int_\Omega|\nabla\upsilon|^{q(x)}\,dx\\ &+\frac{C}{r^+}t^{r^+}\int_\Omega|\upsilon|^{r(x)}\,dx-\frac{1}{s^+}t^{s^-}\int_\Omega|\upsilon|^{s(x)}\,dx.\end{aligned}$$

Therefore, since $1<r^-\leqslant r(x)\leqslant r^+<\gamma^-\leqslant\gamma(x)\leqslant\gamma^+<s^-$ for all $x\in\overline{\Omega}$, we get

$$\lim_{t\to+\infty}\Phi_{\omega,\lambda}(t\upsilon)=-\infty.$$

□

Now, we will follow the ideas of Alonso and Azorero (see [8]). We will use an argument of truncation to obtain a new functional that will be bounded from below.

Remembering (a_1), $(\mathcal{M}_1)$, (f_2), and Proposition 2.2, we have

$$\begin{aligned}\Phi_{\omega,\lambda}(u)&=\mathscr{M}_\omega(\mathscr{A}(u))-\lambda\int_\Omega F(x,u)\,dx-\int_\Omega\frac{1}{s(x)}|u|^{s(x)}\,dx\\ &=\int_0^{\mathscr{A}(u)}M_\omega(t)\,dt-\lambda\int_\Omega F(x,u)\,dx-\int_\Omega\frac{1}{s(x)}|u|^{s(x)}\,dx\\ &\geqslant m_0\int_\Omega\left(\frac{\kappa_0}{p(x)}|\nabla u|^{p(x)}+\mathcal{H}(\kappa_3)\frac{\kappa_2}{q(x)}|\nabla u|^{q(x)}\right)dx-\frac{a_2}{r^-}\int_\Omega|u|^{r(x)}\,dx-\frac{1}{s^-}\int_\Omega|u|^{s(x)}\,dx.\end{aligned}$$

Note that

$$\min\{\|\nabla u\|_{p(\cdot)}^{p^-},\|\nabla u\|_{p(\cdot)}^{p^+}\}+\mathcal{H}(\kappa_3)\min\{\|\nabla u\|_{q(\cdot)}^{q^-},\|\nabla u\|_{q(\cdot)}^{q^+}\}\geqslant\min\{\|\nabla u\|_{\gamma(\cdot)}^{\gamma^-},\|\nabla u\|_{\gamma(\cdot)}^{\gamma^+}\}. \tag{7.1}$$

Also, applying the Gagliardo-Nirenberg-Sobolev inequality (6.2), we have

$$\int_\Omega|u|^{r(x)}\,dx\leqslant\|u\|_{r(\cdot)}^{r^-}+\|u\|_{r(\cdot)}^{r^+},\quad \|u\|_{r(\cdot)}S_r\leqslant\|\nabla u\|_{\gamma(\cdot)},$$

and

$$\int_\Omega|u|^{r(x)}\,dx\leqslant\frac{1}{S_r^{r^-}}\|\nabla u\|_{\gamma(\cdot)}^{r^-}+\frac{1}{S_r^{r^+}}\|\nabla u\|_{\gamma(\cdot)}^{r^+}.$$

Now, we consider $\|\nabla u\|_{\gamma(\cdot)} \leqslant 1$, to build the truncated functional $\overline{\Phi}_{\omega,\lambda}$ associated with the truncated auxiliary problem $(\mathscr{M}_\lambda)$. Hence, we achieve

$$\Phi_{\omega,\lambda}(u) \geqslant \frac{m_0\mathscr{K}}{q^+}\|\nabla u\|_{\gamma(\cdot)}^{\gamma^+} - \frac{\lambda}{S_r^{r^-}}\frac{a_2}{r^-}\|\nabla u\|_{\gamma(\cdot)}^{r^-} - \frac{\lambda}{S_r^{r^+}}\frac{a_2}{r^-}\|\nabla u\|_{\gamma(\cdot)}^{r^+} - \frac{1}{S_s^{s^-}s^-}\|\nabla u\|_{\gamma(\cdot)}^{s^-} - \frac{1}{S_s^{s^+}s^-}\|\nabla u\|_{\gamma(\cdot)}^{s^+}.$$

Moreover, there is a positive constant C_r such that

$$\frac{1}{S_r^{r^-}}\|\nabla u\|_{\gamma(\cdot)}^{r^-} + \frac{1}{S_r^{r^+}}\|\nabla u\|_{\gamma(\cdot)}^{r^+} \leqslant C_r(\|\nabla u\|_{\gamma(\cdot)}^{r^-} + \|\nabla u\|_{\gamma(\cdot)}^{r^+}),$$

and, since $\|\nabla u\|_{\gamma(\cdot)} \leqslant 1$, we have that $\|\nabla u\|_{\gamma(\cdot)}^{r^+} \leqslant \|\nabla u\|_{\gamma(\cdot)}^{r^-}$ and $\|\nabla u\|_{\gamma(\cdot)}^{s^+} \leqslant \|\nabla u\|_{\gamma(\cdot)}^{s^-}$, thus

$$\Phi_{\omega,\lambda}(u) \geqslant \frac{m_0\mathscr{K}}{q^+}\|\nabla u\|_{\gamma(\cdot)}^{\gamma^+} - \lambda\overline{C}_r\|\nabla u\|_{\gamma(\cdot)}^{r^-} - \overline{C}_s\|\nabla u\|_{\gamma(\cdot)}^{s^-} := \mathscr{G}_\lambda(\|\nabla u\|_{\gamma(\cdot)}), \tag{7.2}$$

where $\mathscr{G}_\lambda : [0,+\infty[\to \mathbb{R}$ given by

$$\mathscr{G}_\lambda(t) = \frac{m_0\mathscr{K}}{q^+}t^{\gamma^+} - \lambda\overline{C}_r t^{r^-} - \overline{C}_s t^{s^-}.$$

We will show that there is $\lambda^* > 0$ such that $\mathscr{G}_\lambda$ assumes positive values for each $\lambda \in (0,\lambda^*)$. Indeed, since $\gamma^+ < s^-$, we can take $\bar{t}$ small enough such that

$$\frac{m_0\mathscr{K}}{q^+}\bar{t}^{\gamma^+} - \overline{C}_s\bar{t}^{s^-} > 0,$$

and, we define

$$\overline{\lambda} = \frac{1}{2\overline{C}_r}\frac{1}{\bar{t}^{r^-}}\left(\frac{m_0\mathscr{K}}{q^+}\bar{t}^{\gamma^+} - \overline{C}_s\bar{t}^{s^-}\right).$$

Thus, for each $0 < \lambda < \overline{\lambda}$, follows that $\mathscr{G}_\lambda(\bar{t}) \geqslant \mathscr{G}_{\lambda^*}(\bar{t}) > 0$, this is, $\mathscr{G}_\lambda$ assumes positive values.

On the other hand, taking $\underline{t} < \left(\frac{\lambda q^+\overline{C}_r}{m_0\mathscr{K}}\right)^{\frac{1}{\gamma^+ - r^-}}$, we obtain $\frac{m_0\mathscr{K}}{q^+}t^{\gamma^+} - \lambda\overline{C}_r t^{r^-} < 0$, for all $0 < t < \underline{t}$. Since $\overline{C}_s t^{s^-} > 0$, we have

$$\mathscr{G}_\lambda(t) < \frac{m_0\mathscr{K}}{q^+}t^{\gamma^+} - \lambda\overline{C}_r t^{r^-} < 0$$

for all $t < \underline{t}$. Then, as $\mathscr{G}_\lambda$ assumes positive and negative values and $\lim_{t\to+\infty}\mathscr{G}_\lambda(t) = -\infty$, we conclude, by the Rolle's Theorem, that the function $\mathscr{G}_\lambda$ has exactly two roots, namely, $0 < \mathscr{R}_0(\lambda) < \mathscr{R}_1(\lambda)$.

The following lemma is fundamental to build our truncated functional.

Lemma 7.2. *Assume* $(\mathcal{M}_1)$, $(\mathcal{M}_2)$, (f_1), (f_2), (a_0), (a_1), *and* (1.1). *Then*

$$\lim_{\lambda\to 0^+}\mathscr{R}_0(\lambda) = 0.$$

Proof. Since $\mathscr{G}_\lambda(\mathscr{R}_0(\lambda)) = 0$ and $\mathscr{G}'_\lambda(\mathscr{R}_0(\lambda)) > 0$, we have

$$\frac{m_0\mathscr{K}}{q^+} = \frac{\lambda}{r^-}\overline{C}_r\mathscr{R}_0^{r^- - \gamma^+} + \frac{1}{s^-}\overline{C}_s\mathscr{R}_0^{s^- - \gamma^+} \tag{7.3}$$

and

$$\frac{\gamma^+ m_0\mathscr{K}}{q^+} > \lambda\overline{C}_r\mathscr{R}_0^{r^- - \gamma^+} + \overline{C}_s\mathscr{R}_0^{s^- - \gamma^+}, \tag{7.4}$$

for all $\lambda \in (0,\lambda^*)$. Hence, combining (7.3) and (7.4), we get

$$0 < \mathscr{R}_0 < \lambda^{\frac{1}{s^- - r^-}}\left[\frac{\left(\frac{\gamma^+\overline{C}_r}{r^-} - \overline{C}_r\right)}{\left(\overline{C}_s - \frac{\gamma^+\overline{C}_s}{s^-}\right)}\right]^{\frac{1}{s^- - r^-}}.$$

Therefore, as $s^- > r^-$, taking limit as $\lambda \to 0^+$, we obtain

$$\lim_{\lambda\to 0^+}\mathscr{R}_0(\lambda) = 0.$$

$\square$

Remark 7.3. *From Lemma 7.2, we can consider $\lambda^* \in (0,\overline{\lambda})$ such that*

$$\left(\frac{1}{\sigma}-\frac{1}{s^-}\right)(\overline{m_0}\kappa S)^N - \mathcal{K}\min\left\{\lambda^{\frac{(\frac{s}{r})^-}{(\frac{s}{r})^- -1}}, \lambda^{\frac{(\frac{s}{r})^+}{(\frac{s}{r})^+ -1}}\right\} > 0$$

and $\mathscr{R}_0(\lambda) < \min\{1, t_0\}$, for each $\lambda \in (0,\lambda^*)$. In particular, we have $\mathscr{R}_0(\lambda) < \min\{\mathscr{R}_1(\lambda), t_0, 1\}$.

We consider a function $\tau : \mathbb{R}_0^+ \to [0,1]$, $\tau \in C_0^{+\infty}(\mathbb{R}_0^+, [0,1])$, nonincreasing such that

$$\tau(t) = \begin{cases} 1 & \text{if } t \leqslant \mathscr{R}_0(\lambda), \\ 0 & \text{if } t \geqslant \min\{\mathscr{R}_1(\lambda), 1\}. \end{cases}$$

We define the truncated functional $\overline{\Phi}_{\omega,\lambda} : X \to \mathbb{R}$ given by

$$\overline{\Phi}_{\omega,\lambda}(u) = \mathscr{M}_\omega(\mathscr{A}(u)) - \lambda\int_\Omega F(x,u)\,dx - \tau(\|u\|)\int_\Omega \frac{1}{s(x)}|u|^{s(x)}\,dx,$$

for all $\lambda \in (0,\overline{\lambda})$.

Note that if $\|u\| \leqslant \mathscr{R}_0(\lambda)$, then $\|\nabla u\|_{\gamma(\cdot)} \leqslant \mathscr{R}_0(\lambda)$, and, consequently $\overline{\Phi}_{\omega,\lambda}(u) = \Phi_{\omega,\lambda}(u)$. Moreover, if $\|u\| \geqslant \|\nabla u\|_{\gamma(\cdot)} \geqslant \max\{\mathscr{R}_1(\lambda), 1\}$, then

$$\overline{\Phi}_{\omega,\lambda}(u) = \mathscr{M}_\omega(\mathscr{A}(u)) - \lambda\int_\Omega F(x,u)\,dx.$$

Lemma 7.4. *The functional $\overline{\Phi}_{\omega,\lambda}$ is bounded from below in X.*

Proof. Consider $\|u\| \geqslant \|\nabla u\|_{\gamma(\cdot)} \geqslant 1$. Following as in (7.2) and since that the norms $\|\cdot\|$ and $\|\cdot\|_\gamma$ are equivalent in X there are positive constants $\mathscr{K}_1$, $\mathscr{K}_2$ such that

$$\begin{aligned}\overline{\Phi}_{\omega,\lambda}(u) &= \mathscr{M}_\omega(\mathscr{A}(u)) - \lambda\int_\Omega F(x,u)\,dx \\ &\geqslant \mathscr{K}_1\|u\|^{\gamma^-} - \lambda\mathscr{K}_2\|u\|^{r^+},\end{aligned}$$

consequently, as $1 < r^+ < \gamma^-$, it follows that $\lim_{\|u\|\to+\infty}\overline{\Phi}_{\omega,\lambda}(u) = +\infty$. Then, we have $\overline{\Phi}_{\omega,\lambda}$ bounded from below in X.

□

Now, we will prove a local Palais-Smale condition and a topological result for the truncated functional $\overline{\Phi}_{\omega,\lambda}$.

Lemma 7.5. *If $\overline{\Phi}_{\omega,\lambda}(u) < 0$, then $\|\nabla u\|_{\gamma(\cdot)} < \mathscr{R}_0$ and $\Phi_{\omega,\lambda}(v) = \overline{\Phi}_{\omega,\lambda}(v)$ for all v in a small enough neighborhood of u. Moreover, $\overline{\Phi}_{\omega,\lambda}$ verifies a local Palais-Smale condition for $c_\lambda < 0$.*

Proof. Assume that $\overline{\Phi}_{\omega,\lambda}(u) < 0$. Supposing by contradiction that $\|\nabla u\|_{\gamma(\cdot)} \geqslant \mathscr{R}_0$, we obtain by the construction of truncated functional that $0 > \overline{\Phi}_{\omega,\lambda}(u) \geqslant \mathscr{G}_\lambda(\|\nabla u\|_{\gamma(\cdot)}) \geqslant 0$, which is a contradiction.Thus, we conclude that $\|\nabla u\|_{\gamma(\cdot)} < \mathscr{R}_0$ and $\Phi_{\omega,\lambda}(u) = \overline{\Phi}_{\omega,\lambda}(u)$. Remembering that the norms $\|\cdot\|_{\gamma(\cdot)}$ and $\|\cdot\|$ are equivalent, we get, for each $u \in B(0,\mathscr{R}_0)$, that there exists $\varepsilon > 0$ such that $B(u,\varepsilon) \subset B(0,\mathscr{R}_0)$ and $\overline{\Phi}_{\omega,\lambda}(v) = \Phi_{\omega,\lambda}(v)$ for all $v \in B(u,\varepsilon)$ once that $\|\nabla v\|_{\gamma(\cdot)} < \mathscr{R}_0$.

Now, we will prove a local Palais-Smale condition for $\overline{\Phi}_{\omega,\lambda}$ at level $c_\lambda < 0$. Consider $\{u_n\}_{n\in\mathbb{N}}$ a Palais-Smale sequence at level $c_\lambda < 0$. Then, for n large enough, $\Phi_{\omega,\lambda}(u_n) = \overline{\Phi}_{\omega,\lambda}(u_n) \to c_\lambda < 0$ and $\Phi'_{\omega,\lambda}(u_n) = \overline{\Phi'}_{\omega,\lambda}(u_n) \to 0$ in X. Moreover, since $\overline{\Phi}_{\omega,\lambda}$ is coercive, we get $\{u_n\}_{n\in\mathbb{N}}$ bounded in X. Also, by Remark 7.3, we have

$$\left(\frac{1}{\sigma}-\frac{1}{s^-}\right)(\overline{m_0}\kappa S)^N - \mathcal{K}\min\left\{\lambda^{\frac{(\frac{s}{r})^-}{(\frac{s}{r})^- -1}}, \lambda^{\frac{(\frac{s}{r})^+}{(\frac{s}{r})^+ -1}}\right\} > 0 > c_\lambda,$$

for each $\lambda \in (0,\lambda^*)$. Therefore, from Lemma 6.2, up to a subsequence, $\{u_n\}_{n\in\mathbb{N}}$ is strongly convergent in X. □

We will construct an appropriate minimax sequence of negative critical values for the functional $\overline{\Phi}_{\omega,\lambda}$.

Lemma 7.6. *For every $k \in \mathbb{N} \setminus \{0\}$ there exists $\varepsilon(k) > 0$ such that*

$$\gamma(\overline{\Phi}_{\omega,\lambda}^{-\varepsilon}) \geqslant k,$$

where $\lambda \in (0, \lambda^*)$, $\overline{\Phi}_{\omega,\lambda}^{-\varepsilon} = \{u \in X : \overline{\Phi}_{\omega,\lambda}(u) \leqslant -\varepsilon\}$, and γ is Krasnoselskii's genus.

Proof. Fix $k \in \mathbb{N}$. Since $C_0^{+\infty}(\Omega) \subset X$ has infinite dimension, we can consider $\mathfrak{X}_k$ ($\mathfrak{X}_k \subset C_0^{+\infty}(\Omega)$) a k-linear subspace of X. Note that $\mathscr{R}_0 < \min\{\mathscr{R}_1(\lambda), t_0, 1\}$, then, for any $u \in X$ with $\|u\| = 1$ and $0 < t < \mathscr{R}_0$, we get

$$\begin{aligned}
\overline{\Phi}_{\omega,\lambda}(tu) \leqslant & \frac{\omega\kappa_1}{p^-} t^{p^-} \int_\Omega |\nabla u|^{p(x)}\, dx + \frac{\omega\kappa_3}{q^-} t^{q^-} \int_\Omega |\nabla u|^{p(x)}\, dx \\
& - \lambda \frac{a_1 t^{r^+}}{r^+} \int_\Omega |u|^{r(x)}\, dx - \frac{t^{s^+}}{s^+} \int_\Omega |u|^{s(x)}\, dx \\
\leqslant & \frac{\omega\kappa_1}{p^-} t^{p^-} + \frac{\omega\kappa_3}{q^-} t^{q^-} - \frac{\lambda a_1}{r^+} t^{r^+} \alpha_k \\
\leqslant & \left(\frac{\omega\kappa_1}{p^-} + \frac{\omega\kappa_3}{q^-} \right) t^{p^-} - \frac{\lambda a_1}{r^+} t^{r^+} \alpha_k,
\end{aligned}$$

where

$$\alpha_k := \inf \left\{ \int_\Omega |u|^{r(x)}\, dx : u \in \mathfrak{X}_k, \|u\| = 1 \right\} > 0.$$

Since $\mathfrak{X}_k$ is k-dimensional, then the norms $\|\cdot\|$ and $\|\cdot\|_{r(\cdot)}$ are equivalent on $\mathfrak{X}_k$. Therefore, there is a positive constant ρ_k such that

$$0 < \rho_k < \min \left\{ \mathscr{R}_0, \left[\frac{\frac{\lambda a_1 \alpha_k}{r^+}}{\omega \left(\frac{\kappa_1}{p^-} + \frac{\kappa_3}{q^-} \right)} \right]^{\frac{1}{p^- - r^+}} \right\}. \tag{7.5}$$

Let us define

$$\mathbb{S}_{\rho_k} = \{u \in \mathfrak{X}_k : \|u\| = \rho_k\}.$$

We know that $\mathbb{S}_{\rho_k}$ is homeomorphic to $\mathbb{S}^{k-1}$, then, by Corollary 3.2, we have $\gamma(\mathbb{S}_{\rho_k}) = k$. Also, as $r^+ < p^-$, for any $u \in \mathbb{S}_{\rho_k,k}$ and by (7.5), we obtain

$$\begin{aligned}
\overline{\Phi}(u)_{\omega,\lambda} = & \overline{\Phi}_{\omega,\lambda}(\rho_k \frac{u}{\|u\|}) \\
\leqslant & \left(\frac{\omega\kappa_1}{p^-} + \frac{\omega\kappa_3}{q^-} \right) \rho_k^{p^-} - \lambda \frac{a_1}{r^+} \rho_k^{r^+} \alpha_k \\
\leqslant & \rho_k^{r^+} \left[\left(\frac{\omega\kappa_1}{p^-} + \frac{\omega\kappa_3}{q^-} \right) \rho_k^{p^- - r^+} - \frac{\lambda a_1}{r^+} \alpha_k \right] < 0.
\end{aligned}$$

Thus, we conclude that there exists a positive constant ε such that

$$\overline{\Phi}_{\omega,\lambda}(u) < -\varepsilon \text{ for any } u \in \mathbb{S}_{\rho_k}.$$

Hence, we achieve $\mathbb{S}_{\rho_k} \subset \overline{\Phi}_{\omega,\lambda}^{-\varepsilon}$ and $\gamma(\overline{\Phi}_{\omega,\lambda}^{-\varepsilon}) \geqslant \gamma(\mathbb{S}_{\rho_k}) = k$. □

We define, for any $k \in \mathbb{N} \setminus \{0\}$, the set

$$\Gamma_k = \{C \subset X \setminus \{0\} : C \text{ is closed, } C = -C \text{ and } \gamma(C) \geqslant k\},$$

and the number

$$c_k^\lambda = \inf_{C \in \Gamma_k} \sup_{u \in C} \overline{\Phi}_{\omega,\lambda}(u).$$

Lemma 7.7. *For all $k \in \mathbb{N} \setminus \{0\}$ and $\lambda \in (0, \lambda^*)$, the number c_k^λ is negative.*

Proof. Let $\lambda \in (0, \lambda^*)$ and $k \in \mathbb{N}$. Due to Lemma 7.6, for each $k \in \mathbb{N}$, there exists $\varepsilon > 0$ such that $\gamma(\overline{\Phi}_\lambda^{-\varepsilon}) \geqslant k$. We know that $\overline{\Phi}_{\omega,\lambda}$ is continuous and even, consequently $\overline{\Phi}_{\omega,\lambda}^{-\varepsilon} \in \Gamma_k$ and

$$\sup_{u \in \overline{\Phi}_{\omega,\lambda}^{-\varepsilon}} \overline{\Phi}_{\omega,\lambda}(u) \leqslant -\varepsilon.$$

Therefore, since $\overline{\Phi}_{\omega,\lambda}$ is bounded from below, we obtain

$$-\infty < c_k^\lambda = \inf_{C\in\Gamma_k}\sup_{u\in C}\overline{\Phi}_{\omega,\lambda}(u) \leqslant \sup_{u\in\overline{\Phi}^{-\varepsilon}_{\omega,\lambda}}\overline{\Phi}_{\omega,\lambda}(u) \leqslant -\varepsilon < 0.$$

□

The following lemma shows the existence of critical points for the functional $\overline{\Phi}_{\omega,\lambda}$.

Lemma 7.8. *Let $k \in \mathbb{N}\setminus\{0\}$ and $\lambda \in (0,\lambda^*)$. If $c_\lambda = c_k^\lambda = c_{k+1}^\lambda = \ldots = c_{k+l}^\lambda$, for some $l \in \mathbb{N}$, then*

$$\gamma(K_{c_\lambda}) \geqslant l+1,$$

where $K_{c_\lambda} := \{u \in X : \overline{\Phi}_{\omega,\lambda}(u) = c,\ \overline{\Phi}'_{\omega,\lambda}(u) = 0\}$. In particular, each c_k^λ is a critical value of $\overline{\Phi}_{\omega,\lambda}$.

Proof. Let $\lambda \in (0,\lambda^*)$ and $k,l \in \mathbb{N}$. We affirm that K_{c_λ} is compact. Indeed, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in K_{c_λ}. Due to Lemma 7.7, we know that $c_\lambda = c_k^\lambda = c_{k+1}^\lambda = \ldots = c_{k+l}^\lambda$ is negative, and, by Lemma 7.5, we have the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X and $\overline{\Phi}_{\omega,\lambda}(u_n) = \Phi_{\omega,\lambda}(u_n)$, for all $n\in\mathbb{N}$. By Lemma 6.2, the functional $\overline{\Phi}_{\omega,\lambda} \equiv \Phi_{\omega,\lambda}$ at level $c_\lambda < 0$ satisfies the Palais-Smale condition in K_{c_λ}. Hence, it follows that K_{c_λ} is compact. Furthermore, since $\overline{\Phi}_{\omega,\lambda}$ is even, $K_{c_\lambda} = -K_{c_\lambda}$.

Suppose by contradiction that $\gamma(K_{c_\lambda}) \leqslant l$, since K_{c_λ} is compact, by the Proposition 3.4, there exists a closed and symmetric set U, with $K_{c_\lambda} \subset U$ such that $\gamma(U) = \gamma(K_{c_\lambda}) \leqslant l$. Note that we can choose $U \subset \overline{\Phi}^0_{\omega,\lambda}$, since $c_\lambda < 0$, $K_{c_\lambda} \subset \overline{\Phi}^0_{\omega,\lambda}$. Since $\overline{\Phi}_{\omega,\lambda} \equiv \Phi_{\omega,\lambda}$ at any level $c_\lambda < 0$, by the Deformation Lemma 2.8, we have an odd homeomorphism $\eta : X \to X$ such that

$$\eta(\overline{\Phi}^{c_\lambda+\delta}_{\omega,\lambda} - \mathring{U}) \subset \overline{\Phi}^{c_\lambda-\delta}_{\omega,\lambda}, \tag{7.6}$$

for some $\delta \in (0,-c_\lambda)$. Note that $\overline{\Phi}^{c_\lambda+\delta}_{\omega,\lambda} \subset \overline{\Phi}^0_{\omega,\lambda}$, because $\delta + c_\lambda < 0$. By definition of $c_\lambda = c_{k+l}^\lambda = \inf\limits_{C\in\Gamma_{k+l}}\sup\limits_{u\in C}\overline{\Phi}_{\omega,\lambda}(u)$, there exists $A \in \Gamma_{k+l}$ such that $\sup_{u\in A}\overline{\Phi}_{\omega,\lambda}(u) < c_\lambda + \delta$, which implies that $A \subset \overline{\Phi}^{c_\lambda+\delta}_{\omega,\lambda}$. Therefore, by (7.6), we achieve

$$\eta(A - \mathring{U}) \subset \eta(\overline{\Phi}^{c_\lambda+\delta}_{\omega,\lambda} - \mathring{U}) \subset \overline{\Phi}^{c_\lambda-\delta}_{\omega,\lambda}. \tag{7.7}$$

We have $A \subset \overline{(A-U)} \cup U$, so, it follows by the Proposition 3.4 that

$$\gamma(\overline{A-U}) \geqslant \gamma(A) - \gamma(U) \geqslant (k+l) - l = k.$$

Thus, since η is odd, by Proposition 3.4, we obtain

$$\gamma(\eta(\overline{A-U})) \geqslant \gamma(\overline{A-U}) \geqslant k.$$

Note that $\eta(\overline{A-U})$ is closed and symmetrical, then $\eta(\overline{A-U}) \in \Gamma_k$, Hence, we get

$$\sup_{u\in\eta(\overline{A-U})}\overline{\Phi}_{\omega,\lambda}(u) \geqslant c_k^\lambda = c_\lambda. \tag{7.8}$$

On the other hand, since $\overline{\Phi}^{c_\lambda-\delta}_{\omega,\lambda}$ is closed and η is a homeomorphism, by (7.7), we have

$$\eta(\overline{A-U}) \subset \overline{\Phi}^{c_\lambda-\delta}_{\omega,\lambda},$$

which is a contradiction with (7.8). Then, we conclude that

$$\gamma(K_{c_\lambda}) \geqslant l+1.$$

In particular, we obtain $K_{c_\lambda} \neq \emptyset$, that is, c_λ is a critical value of $\overline{\Phi}_{\omega,\lambda}$. □

7.1. **Proof of Theorem 5.1.** Let λ^* be the constant given in Remark 7.3 and let $\lambda \in (0, \lambda^*)$. Note that, from Lemma 7.7, we have

$$c_1^\lambda \leqslant c_2^\lambda \leqslant c_3^\lambda \leqslant \cdots < \overline{\Phi}_{\omega,\lambda}(0) = 0.$$

We will consider two cases.

Firstly, if $c_j^\lambda \neq c_{j'}^\lambda$ for all $j, j' \in \mathbb{N}$, $j \neq j'$, this is, $-\infty < c_1^\lambda < c_2^\lambda < \ldots < c_k^\lambda < \ldots < \overline{\Phi}_{\omega,\lambda}(0) = 0$. Moreover, due to Lemma 7.8, each c_k^λ is a critical value of $\overline{\Phi}_{\omega,\lambda}$. Consequently, we obtain infinitely many critical points for $\overline{\Phi}_{\omega,\lambda}$. Hence, the problem $(\mathscr{M}_\lambda)$ has infinitely many solutions.

Now, if for some $k \in \mathbb{N} \setminus \{0\}$ there is $l \in \mathbb{N} \setminus \{0\}$ such that $c_k^\lambda = c_{k+l}^\lambda$, this is,

$$c_\lambda = c_k^\lambda = c_{k+1}^\lambda = \ldots = c_{k+l}^\lambda,$$

hence, from Lemma 7.8, we achieve

$$\gamma(K_{c_\lambda}) \geqslant l + 1 \geqslant 2.$$

Thus, from Proposition 3.3 the compact set K_{c_λ} has infinitely many points, which are critical points for $\overline{\Phi}_{\omega,\lambda}$. Hence, the problem $(\mathscr{M}_\lambda)$ has infinitely many solutions. □

7.2. **Proof of Theorem 1.1.** Let us consider $\lambda \in (0, \lambda^*)$ and u_λ a nontrivial solution of Problem $(\mathscr{M}_\lambda)$ (see Theorem 5.1). Note that, by Lemma 7.5, we have

$$\|\nabla u_\lambda\|_{\gamma(\cdot)} \leqslant \mathscr{R}_0 < t_0 \text{ and } \overline{\Phi}_{\omega,\lambda}(u_\lambda) = \Phi_{\omega,\lambda}(u_\lambda) < 0.$$

We have the followign cases:

(I) If $\kappa_3 = 0$, by Lemma 7.2, we have $\lim_{\lambda\to 0^+} \|\nabla u_\lambda\|_{p(\cdot)} = 0$. Thus, by (a_1) and changing λ^* by other smaller, if necessary, we have

$$\mathscr{A}(u) \leqslant \frac{\kappa_1}{p^-}\|\nabla u_\lambda\|_{p(\cdot)}^{p^-} < t_0, \forall \lambda \in (0, \lambda^*).$$

(II) If $\kappa_3 > 0$, by Lemma 7.2, we obtain $\lim_{\lambda\to 0^+} \|\nabla u\|_{q(\cdot)} = 0$. Thus, by (a_1), $W_0^{1,q(\cdot)}(\Omega) \subset W_0^{1,p(\cdot)}(\Omega)$, $p^- < q^-$, and changing λ^* by other smaller, if necessary, there is a positive constant C, such that

$$\begin{aligned}\mathscr{A}(u) &\leqslant \frac{\kappa_1}{p^-}\|\nabla u_\lambda\|_{p(\cdot)}^{p^-} + \frac{\kappa_3}{q^-}\|\nabla u_\lambda\|_{q(\cdot)}^{q^-}\\ &\leqslant \frac{\kappa_1}{p^-}C\|\nabla u_\lambda\|_{q(\cdot)}^{p^-} + \frac{\kappa_3}{q^-}\|\nabla u_\lambda\|_{q(\cdot)}^{q^-}\\ &\leqslant \left(\frac{\kappa_1}{p^-}C + \frac{\kappa_3}{q^-}\right)\|\nabla u_\lambda\|_{q(\cdot)}^{p^-} < t_0, \forall \lambda \in (0, \lambda^*).\end{aligned}$$

Therefore, by **(I)** and **(II)**, we conclude that

$$M_\omega(\mathscr{A}(u)) = M(\mathscr{A}(u)), \forall \lambda \in (0, \lambda^*).$$

Consequently, u_λ is solution of the problem $(\mathscr{P}_\lambda)$ for each $(0, \lambda^*)$. Since the problem $(\mathscr{M}_\lambda)$ has infinite many solutions for each $\lambda \in (0, \overline{\lambda})$, it follows that the problem $(\mathscr{P}_\lambda)$ has infinite many solutions for each $\lambda \in (0, \overline{\lambda})$.

Now, we will study the asymptotic behaviour of solutions to problem $(\mathscr{P}_\lambda)$. We remember that

$$\|u_\lambda\| = \|\nabla u_\lambda\|_{p(\cdot)} + \mathcal{H}(\kappa_3)\|\nabla u_\lambda\|_{q(\cdot)}. \tag{7.9}$$

We analyze the following cases:

(K_1) If $\kappa_3 = 0$, we know by Lemma 7.2 and (7.9) that

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = \lim_{\lambda\to 0^+} \|\nabla u_\lambda\|_{p(\cdot)} = 0.$$

(K_2) If $\kappa_3 > 0$, we obtain from the Sobolev embedding $W_0^{1,q(\cdot)}(\Omega) \subset W_0^{1,p(\cdot)}(\Omega)$ and (7.9) that

$$\begin{aligned}\|u_\lambda\| &= \|\nabla u_\lambda\|_{p(\cdot)} + \|\nabla u_\lambda\|_{q(\cdot)}\\ &\leqslant (C+1)\|\nabla u_\lambda\|_{q(\cdot)}.\end{aligned}$$

Thus, by Lemma 7.2 follows that

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

Hence, by (K_1) and (K_2), we conclude

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

□

References

[1] Alves CO, Corrêa FJSA, Figueiredo GM, *On a class of nonlocal elliptic problems with critical growth,* Differ. Equ. Appl. **2** (2010), 409-417.

[2] Alves CO, Corrêa FJSA, Ma TF, *Positive solutions for a quasilinear elliptic equation of Kirchhoff type,* Comput. Math. Appl. **49** (2005), 85-93.

[3] Ambrosetti A, Malchiodi A, *Nonlinear analysis and semilinear elliptic problems,* Cambridge University Press, New York, 2007.

[4] Ambrosetti A, Rabinowitz PH, *Dual variational methods in critical point theory and apllications,* J. Functional Analysis **14** (1973), 349-381.

[5] Antonsev SN, Rodrigues JF, *On stationary thermo-rheological viscous flows,* Ann. Univ. Ferrara Sez. VII Sci. Mat. **52** (2006), 19-36.

[6] Autuori G, Colasuonno F, Pucci P, *Lifespan estimates for solutions of polyharmonic Kirchhoff systems,* Math. Models Methods Appl. Sci. **22** (2012), 1150009, 36 pp.

[7] Autuori G, Colasuonno F, Pucci P, *On the existence of stationary solutions for higher–order p-Kirchhoff problems,* Commun. Contemp. Math. **16** (2014), 1450002, 43 pp.

[8] Azorero JG, Alonso IP, *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term,* Trans. Amer. Math. Soc. **323** (1991), 877-895.

[9] Bonder JF, Silva A, *Concentration-compactness principle for variable exponent spaces and applications,* Electron. J. Diff. Equ. **141** (2010), 1-18.

[10] Brezis H, *Functional Analysis, Sobolev spaces and Partial Differential Equations,* New York : Springer, 2011.

[11] Brezis H, Nirenberg L, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents,* Comm. Pure Appl. Math **36** (1983), 437-477.

[12] Castro A, *Metodos variacionales y analisis functional no linear,* X Coloquio colombiano de Matematicas, 1980.

[13] Clark DC, *A variant of the Lusternik-Schnirelman theory,* Indiana Univ. Math. J. **22** (1973), 65–74.

[14] Colasuonno F, Pucci P, *Multiplicity of solutions for p(x)-polyharmonic elliptic Kirchhoff equations,* Nonlinear Anal. **74** (2011), 5962-5974.

[15] Corrêa FJSA, Figueiredo GM, *On an elliptic equation of p-Kirchhoff type via variational methods,* Bull. Aust. Math. Soc. **74** (2006), 263-277.

[16] Corrêa FJSA, dos Reis Costa CA, *On a $p(x)-$Kirchhoff equation with critical exponent and an additional nonlocal term,* Funkcial. Ekvac. **58** (2015), 321-345.

[17] Corrêa FJSA, dos Reis Costa CA, *On a $p(x)-$Kirchhoff equation with critical exponent and an additional nonlocal term via truncation argument,* Math. Nachr. **288** (2015), 1226-1240.

[18] Chaves MF, Ercole G, Miyagaki O, *Existence of a nontrivial solutio for a (p,q)-Laplacian equation with p-critical exponent,* Bound. Value Probl. 2014, 2014:236, 15 pp.

[19] Chen Y, Levine S, Rao M, *Variable exponent, linear growth functionals in image restoration,* SIAM J. Appl. Math. **66** (2006), 1383-1406.

[20] Chipot M, Rodrigues J, *On a class of non-local non-linear elliptic problems*, RAIRO Modél. Math. Anal. Num. **26** (1992), 447-467.

[21] Cruz DV, Fiorenza A, *Variable Lebesgue spaces,* Springer, Birkhäuser, 2013.

[22] Diening L, Harjulehto P, Hasto P, Růžička M, *Lebesgue and Sobolev spaces with variable exponents,* Springer, Heidelberg, 2011.

[23] Diening L, *Theoretical and numerical results for electrorheological fluids,* Ph.D. thesis, Univ. Freiburg im Breisgau, Mathematische Fakultät, 156 p., 2002.

[24] Edmunds DE, Ráskosník J, *Sobolev embeddings with variable exponent*, Studia Math., **143** (2000), 267-293.

[25] Fan XL, Zhang QH, *Existence of solutions for p(x)-Laplacian Dirichlet problem,* Nonlinear Anal., **52** (2003), 1843-1852.

[26] Fan XL, Zhao D, *On the spaces $L^{p(x)}$ and $W^{m,p(x)}$*, J. Math. Anal. Appl. **263** (2001), 424-446.

[27] Fan X, Zhang Q, *Existence of solutions for $p(x)$-Laplacian Dirichlet problem,* Nonlinear Anal. **52** (2003), 1843-1852.

[28] Fan X, Shen JS, Zhao D, *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$*, J. Math. Anal. Appl. **62** (2001), 749-760.

[29] Fan X, Shen JS, Zhao D, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), 424-446.

[30] Figueiredo GM, *Existence of a solution for a Kirchhoff problem type with critical growth via truncation argument,* J. Math. Anal. Appl. **401** (2013), 706-713.

[31] Figueiredo GM, *Existence of positive solutions for a class of p&q elliptic problems with critical growth on $\mathbb{R}^N$*, J. Math. Anal. Appl. **378** (2011), 507-518.

[32] Figueiredo GM, *Existence and multiplicity of solutions for a class of p&q elliptic problems with critical exponent,* Math. Nachr. **286** (2013), 1129-1141.

[33] Figueiredo GM, Santos Junior JR, *Multiplicty of solutions for a Kirchhoff equation with subcritical or critical growth,* Differential and Integral Equations **25** (2012), 853-868.

[34] Fu YQ, *The principle of concentration compactness in $L^{p(x)}$ spaces and its application,* Nonlinear Anal. **71** (2009), 1876-1892.

[35] Gongbao L, Guo Z, *Multiple Solutions for the p&q-Laplacian problem with critical exponent,* Acta Mathematatica Scientia **29B** (2009), 903-918.

[36] Halsey TC, *Electrorheological fluids,* Science **258** (1992), 761-766.

[37] Hongzhuo F, Yaotian S, *Existence of infinitely many solutions for elliptic problems with critical exponent,* Acta Mathematica Scientia **24B**(3) (2004), 395-402.

[38] Hurtado EJ, Miyagaki OH, da Silva Rodrigues R, *Existence and asymptotic behaviour for a Kirchhoff type equation with variable critical growth exponent*, Milan J. Math., 85 (2017), 71–102.

[39] Kirchhoff GR, *Vorlesungen über Mathematische Physik: Mechanik,* Teubner, Leipzig, 1883.

[40] Kovăcik O, Răkosnik J, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$,* Czechoslovak Math. J., **41(116)** (1991), 592-618.

[41] Krasnoselskii MA, *Topological methods in the theory of nonlinear integral equations,* MacMillan, New York, 1964.

[42] Lions JL, *On some questions in boundary value problems of mathematical physics (Contemporary developments in continuum mechanics and partial differential equations),* North-Holland Mathematics Studies **30** (1978), 284-346.

[43] Rabinowitz PH, *Minimax methods in critical point theory with applications to differential equations,* Expository Lectures from the CBMS Regional Conference held at the University of Miami, American Mathematical Society, Providence, RI, 1984.

[44] Rajagopal KR, Ružička M, *Mathematical modeling of electrorheological materials,* Cont. Mech. Term, **13** (2001), 59-78.

[45] Rodrigues MM, *Multiplicity of Solutions on a Nonlinear Eigenvalue Problem for p(x)-Laplacian-like Operators,* Mediterr. J. Math. **9** (2012), 211-223.

[46] Ružička M, *Electrorheological Fluids Modeling and Mathematical Theory,* Springer-Verlag, Berlin, 2002.

[47] Zhikov VV, *On the density of smooth functions in Sobolev-Orlicz spaces,* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **226** (2004), 67-81.

Elard Juarez Hurtado
Department of Mathematics, Universidade de Brasília, CEP 70910-900, Distrito Federal-DF, Brazil
Email address: `elardjh2@gmail.com`

Olimpio Hiroshi Miyagaki
Department of Mathematics, Universidade Federal de São Carlos, CEP 13565-905, São Carlos-SP, Brazil
Email address: `ohmiyagaki@gmail.com`

Rodrigo da Silva Rodrigues
Department of Mathematics, Universidade Federal de São Carlos, CEP 13565-905, São Carlos-SP, Brazil
Email address: `rodrigo@dm.ufscar.br`