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Stepanov-like pseudo anti-periodicity and applications to semi-linear parabolic boundary differential equations

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This paper is mainly devoted to the existence of pseudo anti-periodic solutions of parabolic boundary differential equations by the measure theory. A new class of functions called Stepanov-like (μ_0, ν_0)-pseudo anti-periodic functions is proposed, which generalizes the classical weighted pseudo anti-periodic functions in Stepanov sense. The completeness of the space composed of these functions is proved. Translation invariance and two composition theorems are also established. As an application different from parabolic equations with linear boundary conditions, one shows that semi-linear parabolic evolution equations with inhomogeneous boundary conditions admit a (μ_0, ν_0)-pseudo anti-periodic solution in interpolation and extrapolation spaces. An example is presented to verify the existence of pseudo anti-periodic solution. The Copyright © 2021 John Wiley & Sons, Ltd.

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1. Introduction

Since Poincaré researched the periodic solution of three-body problem,[1–3] periodicity has always been one of the central topics in the study of dynamic systems so far. With further research of periodicity, it is found that generally non-coercive evolution equations cannot be shown to have a classical periodic solution, but possesses an anti-periodic solution, such as $g(t) \in \partial\phi(y(t)) + y'(t)$, a. e. $t \in \mathbb{R}$ in paper.[4] By the virtue of maximal monotone operator theory and self-adjoint mapping, Aftabizadeh, Aizicovici and Pavel[5,6] discussed the anti-periodic solutions of second-order and higher order anti-periodic boundary value problems, respectively. Afterward, the investigation of the existence of anti-periodic solutions attaches much attention because of its applications in physics, control theory, engineering and other subjects (see studies[7–13]). Using semigroup theory, Liu[14] studied anti-periodic solutions of semi-linear evolution equations in Banach space. Many other researchers established some theorems about anti-periodic solutions of different systems and one refers readers to references.[15–26] Among these references, anti-periodic functions are investigated with various boundary conditions and most of them are linear type. The generalized anti-periodic boundary value problem of impulsive fractional differential equations was researched by Li et al. in the paper.[27]

As we all know, weights play an increasingly important role in the study of anti-periodic functions, which results in the properties of weighted pseudo anti-periodic functions with values in Banach spaces are more complicated than the general anti-periodic functions, such as S-asymptotically anti-periodic functions. Moreover, anti-periodic functions with weight have uncertain due

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to the diversity of weighted functions. By weight, Al-Islam et al.[28] proposed weighted pseudo anti-periodic functions in 2012. Zhou and Shao[29] researched weighted pseudo-anti-periodic SICNNs with mixed delays. To discuss unbounded or discontinuous functions, Alvarez et al.[30] further put forward weighted pseudo anti-periodic functions in Stepanov sense, which are by far the widely used functions of anti-periodic type. However, it appears that little discussion is devoted to (μ_0, ν_0)-pseudo anti-periodic functions in Stepanov sense, let alone studying (μ_0, ν_0)-pseudo anti-periodic solutions of semi-linear parabolic evolution equations with inhomogeneous boundary conditions.

Using exponential dichotomy, this paper is devoted to investigate the (μ_0, ν_0)-pseudo anti-periodic solutions of the following equation

$$\begin{cases} y'(t) = B_h y(t) + g(t, y(t)) & t \in \mathbb{R}, \\ Ly(t) = \eta(t, y(t)) & t \in \mathbb{R}, \end{cases} \tag{1}$$

where $(B_h, D(B_h))$ is a densely defined closed linear operator on Banach space Y and ∂Y is a boundary space such that $L : D(B_h) \to \partial Y$ composes a linear bounded boundary operator. Here, functions $g : \mathbb{R} \times Y_{\alpha_1} \to Y$ and $\eta : \mathbb{R} \times Y_{\alpha_1} \to \partial Y$ are (μ_0, ν_0)-pseudo anti-periodic functions in Stepanov sense, where Y_{α_1}, $0 < \alpha_1 < 1$ are certain continuous interpolation spaces about the linear operator $B := B_h|\mathrm{ker}L$. The theory of semilinear initial-boundary value problems is based on the theory of strongly continuous semigroups, which is designed to solve issues arising in differential equations with delay and mathematical biology. Note that retarded differential equations, boundary control systems and population equations can be abstracted as these equations which are formed by partial differential equations with semilinear terms at the boundary.

Similar to,[31] the boundary equations (1) turns equivalently into semi-linear evolution equations given by

$$y'(t) = B_{\alpha_2-1} y(t) + g(t, y(t)) + (\mu - B_{\alpha_2-1}) L_\mu \eta(t, y(t)) \text{ for } t \in \mathbb{R}, \tag{2}$$

where B_{α_2-1}, $0 < \alpha_2 < 1$, constitutes the continuous extension of $B := B_h|\mathrm{ker}L$ to the extrapolated Banach spaces Y_{α_2-1} of Y_{α_2}. As Greiner's assumptions, the operator $L_\mu := (L|\mathrm{ker}(\mu - B_h))^{-1}$ from ∂Y to Y is bounded and the semi-linear term $G(t, y(t)) = g(t, y(t)) + (\mu - B_{\alpha_2-1}) L_\mu \eta(t, y(t))$ belongs to Y_{α_2-1}.

For the above equation, Baroun, Maniar, NGuerékata studied almost periodic and almost automorphic solutions in the paper.[32] Baroun, Ezzinbi, Khalil, Maniar researched pseudo almost periodic solutions in reference.[33] It is well known that the existence of anti-periodic solutions plays a key function in characterizing the behavior of nonlinear differential equations. For example, similar to the periodic Lasalle oscillation theorem of Lasalle in 1950, Wu[34] in 2007 presented an anti-periodic Lasalle oscillation theorem. Recently, Edgardo Alvarez et al.[30] researched Stepanov-like weighted pseudo anti-periodic functions for fractional integro-differential equations. And solutions to some equations describing the propagation of heat or waves in solid state physics are often expected to have the Bloch type periodicity in paper,[35] which includes both periodicity and anti-periodicity. Moreover, anti-periodic boundary conditions have been considered for the Schrödinger and Hill differential operator.[36,37] Note that these anti-periodic boundary conditions appear in physics in a variety of situations. In order to study deeper complex dynamic behavior, can one investigate Stepanov-like (μ_0, ν_0)-pseudo anti-periodic functions for the equation (2)? And the natural question is raised: what are asymptotic properties of mild solutions about equation (2) provided that the nonlinear term g satisfies Stepanov-like (μ_0, ν_0)-pseudo anti-periodic condition? That is to say, is the solution y of equation (2) more regular than the function g? Besides, can one find a pseudo anti-periodic solution in extrapolation and interpolation spaces instead of in the usual Banach space? This thought constitutes the motivation of this paper. Note that the decomposition of weighted pseudo anti-periodic functions in the classical sense are not unique, which results that the weighted pseudo anti-periodic solution based on the completion is not true.

To fill these gaps, one investigates the weighted pseudo anti-periodic function in Stepanov sense by adopting a new approach proposed by Blot in paper.[38] Precisely, one defines an ergodic function by the measure theory and explores some interesting properties of it including composition theorems. And one also studies the applications of Stepanov-like weighted pseudo anti-periodic functions to semi-linear parabolic boundary differential equations in interpolation and extrapolation spaces. And all these results give a positive response to the above questions. Based on our conclusions, one believes that the generalized Stepanov-like (μ_0, ν_0)-pseudo anti-periodic function not only paves a way for the research of pseudo anti-periodic solutions of integro-differential equations and fractional differential equations, but also helps investigating (μ_0, ν_0)-pseudo S-asymptotic periodic functions in Stepanov sense.

The rest of the paper is structured as follows: In Section 2, one reviews some theory of the extrapolation and interpolation of generators, then one makes estimates on the dichotomy in the extrapolated spaces which play a key role to show our results. In Section 3, one shows the completion of Banach space in Stepanov sense and some composition theorems. Further, one shows that some parabolic evolution equations admit a unique pseudo anti-periodic solution. In Section 4, one devotes to the (μ_0, ν_0)-pseudo anti-periodic solution to (1) under Greiner's assumptions and presents an illustrating application. In Section 5, one gives some conclusions and discussions.

2. Notations and preliminaries

Note that $\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{R}$ and $(Y, \|\cdot\|)$ are correspondingly the set of natural numbers, positive natural numbers, real numbers and Banach space. Let $C(\mathbb{R}, Y)$ (resp. $C(\mathbb{R}\times Y, Y)$) be the set of continuous functions from $\mathbb{R}$ to Y(resp. from $\mathbb{R}\times Y$ to Y) and $(BC(\mathbb{R}, Y), \|\cdot\|_\infty)$ (resp. $BC(\mathbb{R}\times Y, Y)$) be the Banach space (set) of bounded continuous functions from $\mathbb{R}$ to Y (resp. from $\mathbb{R}\times Y$ to Y), where the normal $\|\cdot\|_\infty$ is given by $\|\psi\|_\infty = \sup\limits_{t\in\mathbb{R}} \|\psi(t)\|$. Denote by $\|\cdot\|_p = (\int_r^{r+1} \|f(\sigma)\|^p \, d\sigma)^{\frac{1}{p}}$. By $L^p(\mathbb{R}, Y)$ (resp. $L^p_{loc}(\mathbb{R}, Y)$), one represents the space of all equivalent classes of measurable functions f from $\mathbb{R}$ to Y so that $\|f(\cdot)\|^p$ is integrable (resp. locally integrable).

Now, let us review some basic results on interpolation and extrapolation spaces for certainly associated semigroup. One refers the reader to references[39,40] for more details. Set $R(\omega_0, B) = (B - \omega_0)^{-1}$. Denote that $(B, D(B))$ is a closed linear operator defined on a Banach space Y, which is sectorial i.e. there are constants $\omega_0 \in \mathbb{R}$, $\theta_0 \in (\frac{\pi}{2}, \pi)$ and $M_0 > 0$ satisfying

$$\text{for all } \mu \in \Sigma_{\theta_0} := \{\mu \in \mathbb{C} : \mu \neq 0, |\arg(\mu)| \leq \theta_0\} \subset \rho(B - \omega_0), \tag{3}$$

$$\|\mu R(\mu, B - \omega_0)\|_{\mathcal{L}(Y)} \leq M_0 \ . \tag{4}$$

The conditions (3) and (4) imply that $(B, D(B))$ produces an analytic semigroup $(\Phi(t))_{t\geq 0}$ on Y.

Set $\alpha_2 \in (0, 1)$. One utilizes the real interpolation space

$$Y_{\alpha_2} := \overline{D(B)}^{\|\cdot\|_{\alpha_2}},$$

with $\|y\|_{\alpha_2} := \sup\limits_{\mu>0} \|\mu^{\alpha_2}(B - \omega_0)R(\mu, B - \omega_0)y\|$ for all $y \in Y$. Therefore one supposes that $\omega_0 > \omega_1(\Phi(t)_{t\geq 0})$ with $\omega_1(\Phi(t)_{t\geq 0})$ being the growth bound of $(\Phi(t))_{t\geq 0}$. That means $\omega_0 \in \rho(B)$ and the norms $\|\cdot\|_{\alpha_2}$ are equivalent for any other $\omega' \in \rho(B)$ according to the resolvent equation. Then $(Y_{\alpha_2}, \|\cdot\|_{\alpha_2})$ composes a Banach space.

Further, one writes

- $Y_0 := Y$, $Y_1 := D(B)$ and $\|y\|_0 = \|y\|$;
- $\|y\|_1 = \|(B - \omega_0)y\|$;
- $\widehat{Y} := \overline{D(B)}$;
- the norm $\|y\|_{-1} = \|R(\omega_0, B)y\|$ for $y \in Y$.

Then, the completion of $(\widehat{Y}, \|\cdot\|_{-1})$ is said to be the extrapolation space of Y about B and will be described as Y_{-1}, which means that B admits a unique extension $B_{-1} : \widehat{Y} \to Y_{-1}$. Because $\Phi(t)$ commutes with the resolvent operator $R(\omega_0, B)$ for each $t \geq 0$, the extension of $\Phi(t)$ to Y_{-1} exists and B_{-1} generates an analytic semigroup $(\Phi(t))_{t\geq 0}$. Naturally, one can give the space

$$Y_{\alpha_2 - 1} := (Y_{-1})_{\alpha_2} = \overline{\widehat{Y}}^{\|\cdot\|_{\alpha_2 - 1}},$$

where $\|y\|_{\alpha_2-1} := \sup\limits_{\mu>0} \|\mu^{\alpha_2} R(\mu, B_{-1} - \omega_0)y\|$. The restriction $B_{\alpha_2-1} : Y_{\alpha_2} \to Y_{\alpha_2-1}$ of B_{-1} produces the analytic semigroup $(\Phi_{\alpha_2-1}(t))_{t\geq 0}$ on Y_{α_2-1} such that it is the extension of $\Phi(t)$ to Y_{α_2-1}. One will make use of the continuous embeddings

$$D(B) \hookrightarrow Y_{\alpha_1} \hookrightarrow Y_{\alpha_2} \hookrightarrow Y, \quad Y \hookrightarrow Y_{\alpha_1-1} \hookrightarrow Y_{\alpha_2-1} \hookrightarrow Y_{-1},$$

for all $0 < \alpha_2 < \alpha_1 < 1$.

Denote that $\mathcal{B}$ is the Lebesgue σ-field on $\mathbb{R}$. Note that $\mathcal{M}$ is the set of all positives measures μ_0 on $\mathcal{B}$ fulfilling $\mu_0(\mathbb{R}) = +\infty$ and $\mu_0([b,c]) < \infty$, for all $b, c \in \mathbb{R}(b \leq c)$.

(H_1) Let $\mu_0, \nu_0 \in \mathcal{M}$ satisfy

$$\limsup_{S\to\infty} \frac{\mu_0([-S,S])}{\nu_0([-S,S])} < \infty;$$

(H_2) For all $\tau \in \mathbb{R}$, there are a bounded interval I_0 and a number $\beta > 0$ so that $\mu_0(b+\tau : b \in D) \leq \beta\mu_0(D)$ for $D \in \mathcal{B}$ fulfilling $D \bigcap I_0 = \emptyset$.

One first gives a condition before presenting hyperbolic semigroups about the generator B and its spectrum $\sigma(B)$.
(H) $(\Phi(t))_{t\geq 0}$ is an analytic semigroup satisfying $\sigma(B) \cap i\mathbb{R} = \emptyset$.

Theorem 2.1 (Baroun et al.[32]) *Let $(\Phi(t))_{t\geq 0}$ be a semigroup satisfying (H) and let $0 < \alpha_2 \leq 1$ and $\varepsilon > 0$ satisfy $0 < \alpha_2 - \varepsilon < 1$. Then the operators P_s and P_u have continuous extensions $P_u^{\alpha_2-1} : Y_{\alpha_2-1} \to Y$ and $P_s^{\alpha_2-1} : Y_{\alpha_2-1} \to Y_{\alpha_2-1}$ respectively. Moreover, one has the following assertions:*

- (a_1) $P_u^{\alpha_2-1} Y_{\alpha_2-1} = P_u Y$;
- (a_2) $\Phi_{\alpha_2-1}(t) P_s^{\alpha_2-1} = P_s^{\alpha_2-1} \Phi_{\alpha_2-1}(t)$;
- (a_3) $\Phi_{\alpha_2-1}(t) : P_u^{\alpha_2-1} Y_{\alpha_2-1} \to P_u^{\alpha_2-1} Y_{\alpha_2-1}$ *is invertible with inverse* $\Phi_{\alpha_2-1}(-t)$ *for* $t \geq 0$;
- (a_4) *for* $0 < \alpha_2 - \varepsilon < 1$, *one has*

$$\begin{aligned} \left\|\Phi_{\alpha_2-1}(t) P_s^{\alpha_2-1} y\right\| &\leq m t^{\alpha_2-1-\varepsilon} e^{-\gamma t} \|y\|_{\alpha_2-1} \quad \text{for } y \in Y_{\alpha_2-1} \text{ and } t \geq 0, \\ \left\|\Phi_{\alpha_2-1}(t) P_u^{\alpha_2-1} y\right\| &\leq c e^{\delta t} \|y\|_{\alpha_2-1} \quad \text{for } y \in Y_{\alpha_2-1} \text{ and } t \leq 0. \end{aligned}$$

Theorem 2.2 (Baroun et al.[32]) *Let $y \in Y_{\alpha_2-1}$, $0 \leq \alpha_1 \leq 1$, $0 < \alpha_2 \leq 1$ and $\varepsilon > 0$ satisfy $0 < \alpha_1 + \varepsilon < \alpha_2$ and $0 < \alpha_2 - \varepsilon < 1$. Then, statements $(a_1) - (a_2)$ hold:*

- (a_1) *there is a constant* $c(\alpha_1, \alpha_2)$ *satisfying*

$$\left\|\Phi_{\alpha_2-1}(t) P_u^{\alpha_2-1} y\right\|_{\alpha_1} \leq c(\alpha_1,\alpha_2) e^{\delta t} \|y\|_{\alpha_2-1}, \quad \text{for} \quad t \leq 0;$$

- (a_2) *there is a constant* $m(\alpha_1, \alpha_2)$ *so that for* $0 < \alpha_2 - \varepsilon < 1$

$$\left\|\Phi_{\alpha_2-1}(t) P_s^{\alpha_2-1} y\right\|_{\alpha_1} \leq m(\alpha_1,\alpha_2) t^{\alpha_2-\alpha_1-1-\varepsilon} e^{\gamma t} \|y\|_{\alpha_2-1}, \quad \text{for} \quad t \geq 0.$$

Now, one reviews some notations, basic knowledge about measures and results of (pseudo) anti-periodic type functions.

Provided that $\mu_0, \nu_0 \in \mathcal{M}$, one then defines

$$PAP_0(Y, \mu_0, \nu_0) := \left\{ g \in BC(\mathbb{R}, Y) : \lim_{S\to\infty} \frac{1}{\nu_0([-S,S])} \int_{[-S,S]} \|g(\sigma)\| \, d\mu_0(\sigma) = 0 \right\}.$$

Definition 2.1 *A function $g \in C(\mathbb{R}, Y)$ is said to be anti-periodic provided that $g(t+\omega) = -g(t)$ for all $t \in \mathbb{R}$. Let $P_{ap}(\mathbb{R}, Y)$ represent the set of these functions.*

Definition 2.2 *Set $\mu_0, \nu_0 \in \mathcal{M}$. A function $g \in C(\mathbb{R}, Y)$ is defined as (μ_0, ν_0)-pseudo anti-periodic provided that it can be decomposed into $g = g_1 + g_2$, where $g_1 \in P_{ap}(\mathbb{R}, Y)$ and $g_2 \in PAA_0(\mathbb{R}, Y, \mu_0, \nu_0)$. Let $PP_{ap}(\mathbb{R}, Y, \mu_0, \nu_0)$ represent the collect of these functions.*

One now introduces Stepanov functions and related properties, which are from literatures.[41,42] A function $g \in L^p_{loc}(\mathbb{R}, Y)$ with $1 \leq p < \infty$ is called bounded in Stepanov sense provided that

$$\sup_{\zeta\in\mathbb{R}} \left(\int_{[\zeta,\zeta+1]} \|g(r)\|^p \, dr \right)^{\frac{1}{p}} < \infty.$$

For the same function g, the Bochner transform of it for all $t \in \mathbb{R}$ given by

$$(g^b(t))(r) = g(t+r) \text{ for } r \in [0,1],$$

equipped with the norm $\|g\|_{S^p} := \sup\limits_{\zeta\in\mathbb{R}} \|g(\zeta+\cdot)\|_{L^p(0,1;Y)} = \sup\limits_{\zeta\in\mathbb{R}} \left(\int_\zeta^{\zeta+1} \|g(\tau)\|^p \, d\tau\right)^{\frac{1}{p}}$, these Stepanov bounded functions compose a Banach space.

Now, one gives some definitions of (μ_0, ν_0)-pseudo anti-periodicity in Stepanov sense.

Definition 2.3 *A function $g \in BS^p(\mathbb{R}, Y)$ is defined as Stepanov anti-periodic if $g^b \in P_{ap}(\mathbb{R}, L^p(0,1;Y))$. One remembers the set of all such functions as $P_{ap}S^p(\mathbb{R}, Y)$.*

Note that the preceding definition implies

$$\sup_{\zeta\in\mathbb{R}} \left(\int_\zeta^{\zeta+1} \|g(r+\omega)+g(r)\|^p \, dr\right)^{\frac{1}{p}} = 0,$$

which means that $g(r+\omega) = -g(r)$ a. e. $r \in \mathbb{R}$; that is

$$\|g(r+\omega)+g(r)\|_p = 0.$$

Definition 2.4 *A function $g \in BS^p(\mathbb{R}, Y)$ is defined as Stepanov (μ_0, ν_0)-pseudo anti-periodic provided that it is decomposed into $g = g_1 + g_2$, where $g_1^b \in P_{ap}(\mathbb{R}, L^p(0,1;Y))$ and $g_2^b \in PAA_0(\mathbb{R}, L^p(0,1;Y), \mu_0, \nu_0)$.*

Definition 2.5 (Blot et al.[38]) *Set ν_1 and ν_2 belong to $\mathcal{M}$. ν_1 and ν_2 are said to be equivalent provided that there are constants $\alpha_0 > 0$ and $\beta_0 > 0$ and a bounded interval I_0 (eventually $I_0 = \emptyset$) fulfilling $\alpha_0\mu_1(D) \le \mu_2(D) \le \beta_0\mu_1(D)$, for $D \in \mathcal{B}$ and $D \bigcap I_0 = \emptyset$.*

Setting $\nu_0 \in \mathcal{M}$ and $\tau \in \mathbb{R}$, let $\nu_{0,\tau}$ represent the positive measure on $(\mathbb{R}, \mathcal{B})$ expressed as,

$$\nu_{0,\tau}(D) = \nu_0(\{a+\tau : a \in D\}), \text{for } D \in \mathcal{B}.$$

Lemma 2.1 (Blot et al.[38]) *Let $\nu_0 \in \mathcal{M}$ satisfy (H_2). Then for all $\tau \in \mathbb{R}$ the measures ν_0 and $\nu_{0,\tau}$ are equivalent.*

In the following, one presents an example to illustrate that ρ-weighted pseudo anti-periodic functions are ν-pseudo anti-periodic functions.

Example 2.1 *Let ν be given as: $\nu(B) = \nu^1(B) + \nu^2(B)$ for all $B \in \mathcal{B}$, where ν^2 is Lebesgue measure on $(\mathbb{R}, \mathcal{B})$ and ν^1 the measure on $(\mathbb{R}, \mathcal{B})$ expressed as*

$$\begin{cases} \nu^1(B) = card(B \cap \mathbb{Z}) & \text{when } (B \cap \mathbb{Z}) \text{ is finite,} \\ \nu^1(B) = \infty & \text{when } (B \cap \mathbb{Z}) \text{ is infinite.} \end{cases}$$

$d\nu^1 = \rho dt$, where ρ represents the Radon-Nikodym derivative.

The Radon-Nikodym derivative ρ of the measure ν^1 expresses as

$$\begin{cases} \rho(s) = e^{-\theta s}, & \text{when } \theta > 0,\ s \ge 0, \\ \rho(s) = 1, & \text{when } s < 0, \end{cases}$$

for the Lebesgue measure on $\mathbb{R}$.

One obtains that

$$\nu([-S,S]) = \nu^1([-S,S]) + \nu^2([-S,S]) = \int_{-S}^{S} \rho(s)ds + card([-S,S]\cap\mathbb{Z})$$
$$= S + 2[S] + 1 + \frac{1}{\theta} - \frac{1}{\theta}e^{-\theta S},$$

where $[\cdot]$ *denotes the greatest integer function. Since*

$$\lim_{S\to+\infty} \nu([-S,S]) = \lim_{S\to+\infty} S + 2[S] + 1 + \frac{1}{\theta} - \frac{1}{\theta}e^{-\theta S} = +\infty,$$

then $\nu \in \mathcal{M}$.

Next, one gives two examples to say that the (μ_0,ν_0)-pseudo anti-periodicity in Stepanov sense is an generalization of classical weighted anti-periodicity in Stepanov sense.

Example 2.2 *Define* $g:\mathbb{R}^+\to\mathbb{R}$ *by*

$$g(t) = \begin{cases} \dfrac{(-1)^{n+1}2}{7^k} & \text{for } k \le t \le k + \dfrac{1}{4\times 7^k} \text{ with } k\in\mathbb{N}^+, n\in\mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

When $p=1$, *one has*

$$\begin{aligned}\frac{1}{2S}\int_{-S}^{S}\int_{\zeta}^{\zeta+1} |g(r+\omega)+g(r)|drd\zeta &\le \frac{1}{2S}\int_{1}^{+\infty}\int_{\zeta}^{\zeta+1}(|g(r+\omega)|+|g(r)|)drd\zeta \\ &\le \frac{1}{S}\int_{1}^{+\infty}\int_{[\zeta]}^{[\zeta]+2}|g(r)|drd\zeta \\ &\le \frac{1}{S}\sum_{k\ge1}\frac{1}{7^{2k}} \to 0 \text{ as } S\to\infty.\end{aligned}$$

Since g *is not continuous,* $g \notin PP_{ap}(Y)$.

Example 2.3 *Define* $h:\mathbb{R}^+\to\mathbb{R}$ *by*

$$h(t) = \begin{cases} -n^6\left(t - n^4 - \dfrac{1}{n}\right)^2 + n^4, & t\in\left[n^4,\ n^4+\dfrac{2}{n}\right], n\in\mathbb{N}, \\ 0, & \text{otherwise.}\end{cases}$$

Obviously, h *is unbounded. For any* $N_1>0$, *there is an integer* n_0 *large enough such that* $h(t+\omega)=0$ *for all* $n\ge n_0$ *and* $t\in[n^4, n^4+\frac{2}{n}]$. *Set* $p=1$. S *is large enough and* k_0 *is the largest integer fulfilling* $n_0^4+\frac{2}{n_0}\le k_0^4+\frac{2}{k_0}\le S$.

According to Fubini theorem, one gains that

$$\begin{aligned}\frac{1}{2S}\int_{-S}^{S}\left(\int_{\zeta}^{\zeta+1}\|h(\sigma+\omega)+h(\sigma)\|\,d\sigma\right)d\zeta &= \frac{1}{2S}\int_{-S}^{S}\left(\int_{0}^{1}\|h(\sigma+\zeta+\omega)+h(\sigma+\zeta)\|\,d\zeta\right)d\sigma \\ &= \int_0^1\left(\frac{1}{2S}\int_{-S}^{S}\|h(\sigma+\zeta+\omega)+h(\sigma+\zeta)\|\,d\sigma\right)d\zeta \\ &\ge \frac{1}{2}\int_0^1 \frac{1}{(k_0+1)^4+\frac{2}{k_0+1}}\left\{\sum_{n=n_0}^{k_0}\int_{n^4}^{n^4+\frac{2}{n}}\left[-n^6\left(\zeta-n^4-\frac{1}{n}\right)^2+n^4\right]d\zeta\right\}d\sigma \\ &= \frac{1}{2}\frac{1}{(k_0+1)^4+\frac{2}{k_0+1}}\sum_{n=n_0}^{k_0}2n^3 \\ &= \frac{1}{2}\frac{1}{(k_0+1)^4+\frac{2}{k_0+1}}\left(\frac{k_0(k_0+1)}{2}\right)^2 \to \frac{1}{8}(k_0\to\infty).\end{aligned}$$

This means that $g \notin PP_{ap}S^p(Y)$.

Taking $d\mu_0(t) = \rho_1(t)dt = \frac{1}{t^4}dt$, $d\nu_0(t) = \rho_2(t)dt = \frac{1}{t^6}dt$, *by Fubini theorem, one has*

$$\begin{aligned}
&\lim_{S\to\infty} \frac{1}{\nu_0(S,\rho_2)} \int_{-S}^{S} \rho_1(\zeta) \left(\int_{\zeta}^{\zeta+1} \|g(r+\omega)+g(r)\|\, dr \right) d\zeta \\
&= \lim_{S\to\infty} \int_0^1 \left(\frac{1}{\nu_0(S,\rho_2)} \int_{-S}^{S} \rho_1(\zeta)\, \|g(r+\omega+\zeta)+g(r+\zeta)\|\, d\zeta \right) dr \\
&\leq \lim_{N_1\to\infty} 2\int_0^1 \frac{1}{\int_{N_1^{-4}}^{N_1^4} \zeta^{-6} d\zeta} \left\{ \sum_{n=1}^{N_1} \int_{n^4}^{n^4+\frac{2}{n}} \frac{1}{\zeta^4} \left[-n^6 \left(\zeta - n^4 - \frac{1}{n} \right)^2 + n^4 \right] d\zeta \right\} dr \\
&\leq \lim_{N_1\to\infty} \frac{5N_1^{20}}{-1+N_1^{40}} \sum_{n=1}^{N_1} \frac{1}{3n^{12}} = 0,
\end{aligned}$$

which means that $g \in PP_{ap}S^p(Y)$.

Now, one gives a lemma.

Lemma 2.2 *Let* $\mu_0, \nu_0 \in \mathcal{M}$ *satisfy conditions* $(H_1)-(H_2)$ *and* I_0 *is a bounded interval (eventually* $I_0 = \emptyset$*). Assume that* $g \in BS^p(\mathbb{R}, Y)$*. Then* $g^b \in PAA_0(\mathbb{R}, L^p(0,1;Y), \mu_0, \nu_0)$ *if and only if for every* $\varepsilon > 0$, C_ε *is a* (μ_0,ν_0)*-ergodic zero set; that is* $\lim\limits_{S\to\infty} \frac{\mu_0([-S,S]\setminus I_0\cap C_\varepsilon)}{\nu_0([-S,S]\setminus I_0)} = 0$*, where* $C_\varepsilon := \{t \in \mathbb{R} : \|g(t+\cdot)\|_p \geq \varepsilon\}$, $\|\cdot\|_p$ *is the norm of* $L^p(0,1;Y)$.

Proof. "Sufficiency". Let $A = \mu_0(I_0)$. For any $\varepsilon > 0$.

$$\begin{aligned}
&\frac{1}{\nu_0([-S,S])} \int_{[-S,S]} \left(\int_{\zeta}^{\zeta+1} \|g(r)\|^p\, dr \right)^{\frac{1}{p}} d\mu_0(\zeta) \\
&= \frac{1}{\nu_0([-S,S])} \int_{[-S,S]\setminus C_\varepsilon} \left(\int_{\zeta}^{\zeta+1} \|g(r)\|^p\, dr \right)^{\frac{1}{p}} d\mu_0(\zeta) + \frac{1}{\nu_0([-S,S])} \int_{[-S,S]\cap C_\varepsilon} \left(\int_{\zeta}^{\zeta+1} \|g(r)\|^p\, dr \right)^{\frac{1}{p}} d\mu_0(\zeta) \\
&\leq \frac{\varepsilon}{\nu_0([-S,S])} \int_{[-S,S]\setminus C_\varepsilon} d\mu_0(\zeta) + \|g\|_{S^p} \frac{1}{\nu_0([-S,S])} \int_{[-S,S]\cap C_\varepsilon} d\mu_0(\zeta) \\
&\leq \frac{\varepsilon}{\nu_0([-S,S])} \int_{[-S,S]\setminus C_\varepsilon} d\mu_0(\zeta) + \|g\|_{S^p} \frac{1}{\nu_0([-S,S])} \int_{[-S,S]\setminus I_0\cap C_\varepsilon} d\mu_0(\zeta) + \|g\|_{S^p} \frac{1}{\nu_0([-S,S])} \int_{[-S,S]\cap I_0\cap C_\varepsilon} d\mu_0(\zeta) \\
&\leq \frac{\varepsilon\mu_0([-S,S]\setminus C_\varepsilon)}{\nu_0([-S,S])} + \|g\|_{S^p} \frac{\nu_0([-S,S]\setminus I_0)}{\nu_0([-S,S])} \cdot \frac{\mu_0([-S,S]\setminus I_0\cap C_\varepsilon)}{\nu_0([-S,S]\setminus I_0)} + \|g\|_{S^p} \frac{A}{\nu_0([-S,S])}.
\end{aligned}$$

If C_ε is a (μ_0,ν_0)-ergodic zero set, that is $\lim\limits_{S\to\infty} \frac{\mu_0([-S,S]\setminus I_0\cap C_\varepsilon)}{\nu_0([-S,S]\setminus I_0)}=0$. Combining the arbitrariness of ε and I_0 is a bounded interval, one gains

$$\lim_{S\to\infty} \frac{1}{\nu_0([-S,S])} \int_{[-S,S]} \left(\int_{\zeta}^{\zeta+1} \|g(r)\|^p\, dr \right)^{\frac{1}{p}} d\mu_0(\zeta) = 0,$$

which means that $g^b \in PAA_0(\mathbb{R}, L^p(0,1;Y), \mu_0, \nu_0)$.

"Necessity" If $g^b \in PAA_0(\mathbb{R}, L^p(0,1;Y), \mu_0, \nu_0)$, one obtains that

$$\begin{aligned}
0 &= \frac{1}{\nu_0([-S,S])} \int_{[-S,S]} \left(\int_{\zeta}^{\zeta+1} \|g(r)\|^p\, dr \right)^{\frac{1}{p}} d\mu_0(\zeta) \\
&= \frac{1}{\nu_0([-S,S])} \int_{[-S,S]\setminus C_\varepsilon} \left(\int_{\zeta}^{\zeta+1} \|g(r)\|^p\, dr \right)^{\frac{1}{p}} d\mu_0(\zeta) + \frac{1}{\nu_0([-S,S])} \int_{[-S,S]\cap C_\varepsilon} \left(\int_{\zeta}^{\zeta+1} \|g(r)\|^p\, dr \right)^{\frac{1}{p}} d\mu_0(\zeta) \\
&\geq \frac{1}{\nu_0([-S,S])} \int_{[-S,S]\setminus C_\varepsilon} \left(\int_{\zeta}^{\zeta+1} \|g(r)\|^p\, dr \right)^{\frac{1}{p}} d\mu_0(\zeta) + \frac{\varepsilon}{\nu_0([-S,S])} \int_{[-S,S]\cap C_\varepsilon} d\mu_0(\zeta) \\
&\geq \frac{\varepsilon}{\nu_0([-S,S])} \int_{[-S,S]\cap C_\varepsilon} d\mu_0(\zeta) \geq \frac{\nu_0([-S,S]\setminus I_0)}{\nu_0([-S,S])} \cdot \frac{\varepsilon}{\nu_0([-S,S]\setminus I_0)} \int_{[-S,S]\setminus I_0\cap C_\varepsilon} d\mu_0(\zeta),
\end{aligned}$$

which proves that $\lim\limits_{S\to\infty} \frac{1}{\nu_0([-S,S]\setminus I_0)} \int_{[-S,S]\setminus I_0\cap C_\varepsilon} d\mu_0(\zeta) = 0$; it means that for every $\varepsilon > 0$, C_ε is a (μ_0,ν_0)-ergodic zero set.

3. Stepanov-like (μ_0, ν_0)-pseudo anti-periodicity

In this section, one first gives two lemmas to prove that $(PP_{ap}S^p(\mathbb{R}, Y, \mu_0, \nu_0), \|\cdot\|_{S^p})$ is a Banach space.

Lemma 3.1 *$(P_{ap}S^p(\mathbb{R}, Y), \|\cdot\|_{S^p})$ composes a Banach space for all $1 \le p < \infty$.*

Proof. Let $1 \le p < \infty$. Then $P_{ap}S^p(\mathbb{R}, Y)$ is a linear subspace of $BS^p(\mathbb{R}, Y)$. To conclude it is enough to show that $P_{ap}S^p$ is complete in $BS^p(\mathbb{R}, Y)$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $P_{ap}S^p(\mathbb{R}, Y)$ satisfying $g_n \to g$ as $n \to \infty$ in $BS^p(\mathbb{R}, Y)$. Therefore, for all $\varepsilon > 0$, there is $\omega \in \mathbb{R}$ such that

$$\left(\int_\zeta^{\zeta+1} \|g_n(s+\omega) + g_n(s)\|^p \, ds\right)^{\frac{1}{p}} = 0 \text{ for all } \zeta \in \mathbb{R}.$$

Based on the triangle inequality, one has

$$\begin{aligned} 0 \le \left(\int_\zeta^{\zeta+1} \|g(s+\omega) + g(s)\|^p \, ds\right)^{\frac{1}{p}} &\le \left(\int_\zeta^{\zeta+1} \|g(s+\omega) - g_n(s+\omega)\|^p \, ds\right)^{\frac{1}{p}} \\ &\quad + \left(\int_\zeta^{\zeta+1} \|g_n(s+\omega) + g_n(s)\|^p \, ds\right)^{\frac{1}{p}} + \left(\int_\zeta^{\zeta+1} \|g_n(s) - g(s)\|^p \, ds\right)^{\frac{1}{p}}. \end{aligned}$$

Therefore,

$$0 \le \left(\int_\zeta^{\zeta+1} \|g(s+\omega) + g(s)\|^p \, ds\right)^{\frac{1}{p}} < \frac{2\varepsilon}{3} + \left(\int_\zeta^{\zeta+1} \|g_n(s+\omega) + g_n(s)\|^p \, ds\right)^{\frac{1}{p}}$$

for all $n \in \mathbb{N}$, which is obtained by $g_n \to g$ as $n \to \infty$ in $BS^p(\mathbb{R}, Y)$. Further, since $(g_n)_{n\in\mathbb{N}}$ is in $P_{ap}S^p(\mathbb{R}, Y)$, one derives

$$0 \le \left(\int_\zeta^{\zeta+1} \|g(s+\omega) + g(s)\|^p \, ds\right)^{\frac{1}{p}} < \frac{2}{3}\varepsilon \text{ for all } \zeta \in \mathbb{R}.$$

Due to the arbitrary of ε, it follows that $\left(\int_\zeta^{\zeta+1} \|g(s+\omega) + g(s)\|^p \, ds\right)^{\frac{1}{p}} = 0$. Thus $g \in P_{ap}S^p(\mathbb{R}, Y)$.

Lemma 3.2 *Let $\mu_0, \nu_0 \in \mathcal{M}$. Assume that conditions (H_1) and (H_2) are satisfied. Then $PAA_0(\mathbb{R}, L^p(0, 1; Y), \mu_0, \nu_0)$ composes a closed linear subspace of $BS^p(\mathbb{R}, Y)$.*

Proof. Set $1 \le p < \infty$. Because $PAA_0(\mathbb{R}, L^p(0, 1; Y), \mu_0, \nu_0)$ composes a linear subspace of $BS^p(\mathbb{R}, Y)$, one only needs to show that $PAA_0(\mathbb{R}, L^p(0, 1; Y), \mu_0, \nu_0)$ is closed in $BS^p(\mathbb{R}, Y)$. Take $(g_n)_{n\in\mathbb{N}}$ as a sequence in $PAA_0(\mathbb{R}, L^p(0, 1; Y), \mu_0, \nu_0)$ satisfying $g_n \to g$ as $n \to \infty$ in $BS^p(\mathbb{R}, Y)$. For sufficiently large $S > 0$

$$\begin{aligned} &\int_{[-S,S]} \left(\int_{[\zeta,\zeta+1]} \|g_n(r)\|^p \, dr\right)^{\frac{1}{p}} d\mu_0(\zeta) \\ &\le \frac{(\mu_0([-S,S]))^{\frac{1}{q}}}{(\nu_0([-S,S]))^{\frac{1}{q}}} (\nu_0([-S,S])) \left[\int_{[\zeta,\zeta+1]} \frac{1}{\nu_0([-S,S])} \int_{[-S,S]} \|g_n(r)\|^p \, d\mu_0(\zeta) dr\right]^{\frac{1}{p}}. \end{aligned}$$

Therefore, according to the Lebesgue dominated convergence theorem and $g_n \in PAA_0(\mathbb{R}, L^p(0, 1; Y), \mu_0, \nu_0)$, it yields that

$$\begin{aligned} &\lim_{S\to\infty} \frac{1}{\nu_0([-S,S])} \int_{[-S,S]} \left(\int_{[\zeta,\zeta+1]} \|g_n(s)\|^p \, ds\right)^{\frac{1}{p}} d\mu_0(\zeta) \\ &= \limsup_{S\to\infty} \frac{(\mu_0([-S,S]))^{\frac{1}{q}}}{(\nu_0([-S,S]))^{\frac{1}{q}}} \left[\int_{[\zeta,\zeta+1]} \lim_{S\to\infty} \frac{1}{\nu_0([-S,S])} \int_{[-S,S]} \|g_n(s)\|^p \, d\mu_0(\zeta) ds\right]^{\frac{1}{p}} = 0. \end{aligned} \tag{5}$$

On the other hand,

$$
\begin{aligned}
&\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_{[\zeta,\zeta+1]}\|g(r)\|^p\,dr\right)^{\frac{1}{p}}d\mu_0(\zeta)\\
&\leq \frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_{[\zeta,\zeta+1]}\|g(r)-g_n(r)\|^p\,dr\right)^{\frac{1}{p}}d\mu_0(\zeta)+\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_{[\zeta,\zeta+1]}\|g_n(r)\|^p\,dr\right)^{\frac{1}{p}}d\mu_0(\zeta),
\end{aligned}
$$

hence, it follows from (5) and $g_n \to g$ as $n \to \infty$ in $BS^p(\mathbb{R}, Y)$ that $g \in PAA_0(\mathbb{R}, L^p(0,1;Y), \mu_0, \nu_0)$.

From Lemmas 3.1 and 3.2, one can obtain that for all $1 \leq p < \infty$, $(PP_{ap}S^p(\mathbb{R}, Y, \mu_0, \nu_0), \|\cdot\|_{S^p})$ composes a Banach space equipped with the norm

$$
\|g\|_{S^p} = \|g_1\|_{S^p} + \|g_2\|_{S^p}
$$

where $g = g_1 + g_2$ with $g_1^b \in P_{ap}(\mathbb{R}, L^p(0,1;Y))$ and $g_2^b \in PAA_0(\mathbb{R}, L^p(0,1;Y), \mu_0, \nu_0)$.

To show the translation invariance of Stepanov (μ_0, ν_0)-pseudo anti-periodicity, one presents the following proposition.

Proposition 3.1 *Assume that $(H_1) - (H_2)$ hold. If $\mu_0 \sim \mu_1, \nu_0 \sim \nu_1$ for $\mu_i, \nu_i \in \mathcal{M}$ $(i = 0, 1)$, then $PP_{ap}S^p(\mathbb{R}, Y, \mu_0, \nu_0) = PP_{ap}S^p(\mathbb{R}, Y, \mu_1, \nu_1)$.*

Proof. Since $\mu_0 \sim \mu_1$, $\nu_0 \sim \nu_1$ and $\mathcal{B}$ is the Lesbegue σ-field, according to Definition 2.5, there exist $\theta_i > 0, \gamma_i > 0, (i = 1, 2)$ satisfying $\theta_1\mu_1(A) \leq \mu_0(A) \leq \gamma_1\mu_1(A)$, $\theta_2\nu_1(A) \leq \nu_0(A) \leq \gamma_2\nu_1(A)$. Then for all interval $I \in \mathcal{B}$ and $I \subset [-S, S]$ fulfilling $\nu_1([-S, S]\backslash I) > 0$, one has

$$
\begin{aligned}
&\frac{\theta_1}{\gamma_2}\times\frac{\mu_1\left(\left\{\zeta\in[-S,S]\backslash I:\left(\int_\zeta^{\zeta+1}\|g(r)\|^p\,dr\right)^{\frac{1}{p}}>\varepsilon\right\}\right)}{\nu_1([-S,S]\backslash I)}\\
&\leq\frac{\mu_0\left(\left\{\zeta\in[-S,S]\backslash I:\left(\int_\zeta^{\zeta+1}\|g(r)\|^p\,dr\right)^{\frac{1}{p}}>\varepsilon\right\}\right)}{\nu_0([-S,S]\backslash I)}\\
&\leq\frac{\gamma_1}{\theta_2}\times\frac{\mu_1\left(\left\{\zeta\in[-S,S]\backslash I:\left(\int_\zeta^{\zeta+1}\|g(r)\|^p\,dr\right)^{\frac{1}{p}}>\varepsilon\right\}\right)}{\nu_1([-S,S]\backslash I)}.
\end{aligned}
$$

According to Lemma 2.2, one deduces that $PAA_0(\mathbb{R}, L^p(0,1;Y), \mu_0, \nu_0) = PAA_0(\mathbb{R}, L^p(0,1;Y), \mu_1, \nu_1)$. In the light of the definition of Stepanov-like (μ_0, ν_0)-pseudo anti-periodic function, it follows that $PP_{ap}S^p(\mathbb{R}, Y, \mu_0, \nu_0) = PP_{ap}S^p(\mathbb{R}, Y, \mu_1, \nu_1)$.

Now one gives the translation invariance of Stepanov (μ_0, ν_0)-pseudo anti-periodicity.

Proposition 3.2 *If (H_2) holds and $g \in PP_{ap}S^p(\mathbb{R}, Y, \mu_0, \nu_0)$, then $g(\cdot - \tau) \in PP_{ap}S^p(\mathbb{R}, Y, \mu_0, \nu_0)$.*

Proof. One divides the proof into two parts. To begin with, for $g \in P_{ap}S^p$, one proves that $g(\cdot - \tau) \in P_{ap}S^p$. For $\nu_0 \in \mathcal{M}$, because $\nu_0(R) = +\infty$, there is $S_0 > 0$ satisfying $\nu_0([-S - |\tau|, S + |\tau|]) > 0$ for all $S > S_0$. One always assumes that $S > S_0$ in this proof. Set $\tau^+ = \max(\tau, 0)$, $\tau^- = \max(-\tau, 0)$, then it yields $|\tau| + \tau = 2\tau^+$, $|\tau| - \tau = 2\tau^-$, so

$$
[-S-|\tau|+\tau, S+|\tau|+\tau] = [-S-2\tau^-, S+2\tau^+]. \tag{6}
$$

For $S > S_0$ and $\tau \in \mathbb{R}$, one gains

$$
\begin{aligned}
&\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_\zeta^{\zeta+1}\|g(\sigma-\tau+\omega)+g(\sigma-\tau)\|^p\,d\sigma\right)^{\frac{1}{p}}d\mu_0(\zeta) \qquad (7)\\
&\leq\frac{1}{\nu_0([-S,S])}\int_{[-S-2\tau^-,S+2\tau^+]}\left(\int_\zeta^{\zeta+1}\|g(\sigma-\tau+\omega)+g(\sigma-\tau)\|^p\,d\sigma\right)^{\frac{1}{p}}d\mu_0(\zeta)\\
&\leq\frac{\nu_0([-S-2\tau^-,S+2\tau^+])}{\nu_0([-S,S])}\Phi_\tau(S),
\end{aligned}
$$

where

$$\Phi_\tau(S) = \frac{\int_{[-S-2\tau^-, S+2\tau^+]} \left(\int_\zeta^{\zeta+1} \|g(\sigma-\tau+\omega)+g(\sigma-\tau)\|^p \, d\sigma\right)^{\frac{1}{p}} d\mu_0(\zeta)}{\nu_0([-S-2\tau^-, S+2\tau^+])}.$$

By (H_2) and (6), one has

$$\begin{aligned}\Phi_\tau(S) =& \frac{\int_{[-S-|\tau|+\tau, S+|\tau|+\tau]} \left(\int_\zeta^{\zeta+1} \|g(\sigma-\tau+\omega)+g(\sigma-\tau)\|^p \, d\sigma\right)^{\frac{1}{p}} d\mu_0(\zeta)}{\nu_0([-S-|\tau|+\tau, S+|\tau|+\tau])}\\ =& \frac{\int_{[-S-|\tau|, S+|\tau|]} \left(\int_\zeta^{\zeta+1} \|g(\sigma+\omega)+g(\sigma)\|^p \, d\sigma\right)^{\frac{1}{p}} d\mu_{0,\tau}(\zeta)}{\nu_{0,\tau}([-S-|\tau|, S+|\tau|])}.\end{aligned}$$

Note that by Lemma 2.1 it yields $\mu_0 \sim \mu_{0,\tau}$, $\nu_0 \sim \nu_{0,\tau}$. Additionally, by Proposition 3.1 and $g \in PP_{ap}(\mathbb{R}, Y, \mu_{0,\tau}, \nu_{0,\tau})$, one has

$$\lim_{S\to\infty} \Phi_\tau(S) = 0.$$

By (7), one has

$$\lim_{S\to\infty} \frac{1}{\nu_0([-S,S])} \int_{[-S,S]} \left(\int_\zeta^{\zeta+1} \|g(\sigma-\tau+\omega)+g(\sigma-\tau)\|^p \, d\sigma\right)^{\frac{1}{p}} d\mu_0(\zeta) = 0,$$

that is, $g(\cdot-\tau) \in P_{ap}S^p(\mathbb{R}, Y)$ for all $\tau \in \mathbb{R}$. Now one will show that the second part is that if $g^b(\cdot) \in PAA_0(\mathbb{R}, L^p(0,1;Y), \mu_0, \nu_0)$, then $g^b(\cdot-\tau) \in PAA_0(\mathbb{R}, L^p(0,1;Y), \mu_0, \nu_0)$ for all $\tau \in \mathbb{R}$. Set $g \in PAA_0(\mathbb{R}, L^p(0,1;Y), \mu_0, \nu_0)$ and $\tau \in \mathbb{R}$. For $S > S_0$ and $\tau \in \mathbb{R}$, one obtains that

$$\begin{aligned}&\frac{1}{\nu_0([-S,S])} \int_{[-S,S]} \left(\int_\zeta^{\zeta+1} \|g(\sigma-\tau)\|^p \, d\sigma\right)^{\frac{1}{p}} d\mu_0(\zeta)\\ &\leq \frac{\nu_0([-S-2\tau^-, S+2\tau^+])}{\nu_0([-S,S])} \Theta_\tau(S)\end{aligned}$$

with

$$\Theta_\tau(S) = \frac{\int_{[-S-2\tau^-, S+2\tau^+]} \left(\int_\zeta^{\zeta+1} \|g(\sigma-\tau)\|^p \, d\sigma\right)^{\frac{1}{p}} d\mu_0(\zeta)}{\nu_0([-S-2\tau^-, S+2\tau^+])}$$

By (H_2) and (6), one has

$$\begin{aligned}\Theta_\tau(S) =& \frac{\int_{[-S-|\tau|+\tau, S+|\tau|+\tau]} \left(\int_\zeta^{\zeta+1} \|g(\sigma-\tau)\|^p \, d\sigma\right)^{\frac{1}{p}} d\mu_0(\zeta)}{\nu_0([-S-|\tau|+\tau, S+|\tau|+\tau])}\\ =& \frac{\int_{[-S-|\tau|, S+|\tau|]} \left(\int_\zeta^{\zeta+1} \|g(\sigma)\|^p \, d\sigma\right)^{\frac{1}{p}} d\mu_{0,\tau}(\zeta)}{\nu_{0,\tau}([-S-|\tau|, S+|\tau|])}.\end{aligned}$$

By Lemma 2.1, it follows that $\mu_0 \sim \mu_{0,\tau}$, $\nu_0 \sim \nu_{0,\tau}$. Further, in the light of Proposition 3.1 and $f^b \in PAA_0(\mathbb{R}, L^p(0,1;Y), \mu_{0,\tau}, \nu_{0,\tau})$, it follows that $\lim_{S\to\infty} \Theta_\tau(S) = 0$. By (7), one has

$$\lim_{S\to\infty} \frac{1}{\nu_0([-S,S])} \int_{[-S,S]} \left(\int_\zeta^{\zeta+1} \|g(\sigma-\tau)\|^p \, d\sigma\right)^{\frac{1}{p}} d\mu_0(\zeta) = 0,$$

it means that $g^b(\cdot-\tau) \in PAA_0(\mathbb{R}, L^p(0,1;Y), \mu_0, \nu_0)$ for all $\tau \in \mathbb{R}$. Hence $g(\cdot-\tau) \in PP_{ap}S^p(\mathbb{R}, Y, \mu_0, \nu_0)$. This ends the proof.

The next two theorems show that the composition of two (μ_0, ν_0)-pseudo anti-periodic function in Stepanov sense is still an (μ_0, ν_0)-pseudo anti-periodic function in Stepanov sense.

Theorem 3.1 *Suppose that $G : \mathbb{R} \times Y \to Y$ is a BS^p-bounded function that satisfy*

(a) *there is $\omega > 0$ satisfying $G(t+\omega, -y) = -G(t, y)$ for a. e. $t \in \mathbb{R}$ and for all $y \in Y$;*
(b) *there is a function $L_G(\cdot) \in BS^p(\mathbb{R}, \mathbb{R})$ for all $p \geq 1$ so that for all $t \in \mathbb{R}$ and $y_1, y_2 \in Y$*

$$\|G(t, y_1) - G(t, y_2)\| \leq L_G(t) \|y_1 - y_2\|;$$

(c) *$v \in P_{ap}S^p(\mathbb{R}, Y)$.*

Then $G(s, v(s)) \in P_{ap}S^p(\mathbb{R}, Y)$.

Proof. According conditions $(a) - (c)$, one has

$$\begin{aligned}
0 &\leq \left(\int_{[\zeta,\zeta+1]} \|G(r+\omega, v(r+\omega)) + G(r, v(r))\|^p \, dr\right)^{\frac{1}{p}} \\
&\leq \left(\int_{[\zeta,\zeta+1]} \|G(r+\omega, v(r+\omega)) - G(r+\omega, -v(r))\|^p \, dr\right)^{\frac{1}{p}} \\
&\quad + \left(\int_{[\zeta,\zeta+1]} \|G(r+\omega, -v(r)) + G(r, v(r))\|^p \, dr\right)^{\frac{1}{p}} \\
&\leq \sup_{\zeta\in\mathbb{R}} |L_G^b(\zeta)|_{L^p(0,1;\mathbb{R})} \left(\int_{[\zeta,\zeta+1]} \|v(r+\omega) + v(r)\|^p \, dr\right)^{\frac{1}{p}} \\
&\quad + \left(\int_{[\zeta,\zeta+1]} \|G(r+\omega, -v(r)) + G(r, v(r))\|^p \, dr\right)^{\frac{1}{p}} = 0.
\end{aligned}$$

Therefore $G(r+\omega, v(r+\omega)) = -G(r, v(r))$ a. e. $r \in \mathbb{R}$ and consequently $G(\cdot, v(\cdot)) \in PP_{ap}S^p(\mathbb{R}, Y)$.

Theorem 3.2 *Let $\mu_0, \nu_0 \in \mathcal{M}$ satisfy condition $(H_1) - (H_2)$, $p \geq 1$, $g = g_1 + g_2 \in PP_{ap}S^p(\mathbb{R} \times Y, Y, \mu_0, \nu_0)$ with $g_1^b \in P_{ap}(\mathbb{R} \times Y, L^p(0, 1; Y))$ and $g_2^b \in PAA_0(\mathbb{R} \times Y, L^p(0, 1; Y), \mu_0, \nu_0)$. Assume that*

(a_1) *there is $\omega > 0$ such that $g_1(t+\omega, -y) = -g_1(t, y)$;*
(a_2) *there exist $L_g(t), L_{g_1}(t) \in BS^p(\mathbb{R}, \mathbb{R})$ such that*

$$\|g(t, y_1) - g(t, y_2)\| \leq L_g(t) \|y_1 - y_2\|, \|g_1(t, y_1) - g_1(t, y_2)\| \leq L_{g_1}(t) \|y_1 - y_2\|, \ t \in \mathbb{R}, \ y_1, \ y_2 \in Y;$$

(a_3) *$h = \alpha + \beta \in PP_{ap}S^p(\mathbb{R}, Y, \mu_0, \nu_0)$ with $\alpha^b \in P_{ap}(\mathbb{R}, L^p(0, 1; Y))$ and $\beta^b \in PAA_0(\mathbb{R}, L^p(0, 1; Y), \mu_0, \nu_0)$ is such that the set $K := \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact in Y. Then $g(\cdot, h(\cdot)) \in PP_{ap}S^p(\mathbb{R}, Y, \mu_0, \nu_0)$.*

Proof. $g(t, h(t))$ can be decomposed into

$$g(t, h(t)) = g_1(t, \alpha(t)) + g(t, h(t)) - g(t, \alpha(t)) + g_2(t, \alpha(t)).$$

Let $G_1(t) = g_1(t, \alpha(t))$, $G(t) = g(t, h(t)) - g(t, \alpha(t))$, $G_2(t) = g_2(t, \alpha(t))$. Since $\alpha \in P_{ap}S^p(\mathbb{R}, Y)$ and $g_1 \in P_{ap}S^p(\mathbb{R} \times Y, Y)$, by Theorem 3.1, one obtains that $G_1^b(t) \in P_{ap}(\mathbb{R}, L^p(0, 1; Y))$. Next one shows that $G^b(t) \in PAA_0(\mathbb{R}, L^p(0, 1; Y), \mu_0, \nu_0)$. Indeed

$$\begin{aligned}
&\int_\zeta^{\zeta+1} \|G(r)\|^p \, dr = \int_\zeta^{\zeta+1} \|g(r, h(r)) - g(r, \alpha(r))\|^p \, dr \\
&\leq \int_\zeta^{\zeta+1} L_g^p(r) \|h(r) - \alpha(r)\|^p \, dr = \int_\zeta^{\zeta+1} L_g^p(r) \|\beta(r)\|^p \, dr.
\end{aligned}$$

Then

$$\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_\zeta^{\zeta+1}\|G(r)\|^p\,dr\right)^{\frac{1}{p}}d\mu_0(\zeta)$$
$$\leq h_0\limsup_{S\to\infty}\frac{(\mu_0([-S,S]))^{\frac{1}{q}}}{(\nu_0([-S,S]))^{\frac{1}{q}}}\left[\int_0^1\lim_{S\to\infty}\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\|\beta(r+\zeta)\|^p\,d\mu_0(\zeta)dr\right]^{\frac{1}{p}}.$$

where $h_0=\sup_{\zeta\in\mathbb{R}}|L_g^b(\zeta)|_{L^p(0,1;\mathbb{R})}$. Since $\beta^b(\cdot)\in PAA_0(\mathbb{R},L^p(0,1;Y),\mu_0,\nu_0)$, one obtains that $G^b(\cdot)\in PAA_0(\mathbb{R},L^p(0,1;Y),\mu_0,\nu_0)$. Next, one proves that $G_2^b(\cdot)\in PAA_0(\mathbb{R},L^p(0,1;Y),\mu_0,\nu_0)$. Since $g_2^b\in PAA_0(\mathbb{R}\times Y,L^p(0,1;Y),\mu_0,\nu_0)$, then for any $\varepsilon>0$ there exist $r_0>0$ such that $r>r_0$ implies that

$$\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_\zeta^{\zeta+1}\|g_2(s,u)\|^p\,ds\right)^{\frac{1}{p}}d\mu_0(\zeta)<\varepsilon,\ (u\in Y)$$

Since K is compact i.e. $P\{\alpha(t)\in K_\varepsilon\}\geq 1-\varepsilon$, one can find a finite sequence $y_1,y_2,\cdots,y_m$ such that

$$K_\varepsilon\subset\cup_{i=1}^m B\left(y_i,\frac{\varepsilon}{\|L_f+L_g\|_{S^p}}\right)$$

By Minkowski inequality, for $r>r_0$ one has

$$\begin{aligned}
&\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_\zeta^{\zeta+1}\|g_2(r,\alpha(r))\|^p\,dr\right)^{\frac{1}{p}}d\mu_0(\zeta)\\
&\leq\min_{1\leq i\leq m}\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_\zeta^{\zeta+1}\|g_2(r,\alpha(r))-g_2(r,y_i)\|^p\,dr\right)^{\frac{1}{p}}d\mu_0(\zeta)\\
&\quad+\max_{1\leq i\leq m}\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_\zeta^{\zeta+1}\|g_2(r,y_i)\|^p\,dr\right)^{\frac{1}{p}}d\mu_0(\zeta)\\
&\leq\min_{1\leq i\leq m}\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_\zeta^{\zeta+1}\|g(r,\alpha(r))-g(r,y_i)\|^p\,dr\right)^{\frac{1}{p}}d\mu_0(\zeta)\\
&\quad+\min_{1\leq i\leq m}\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_\zeta^{\zeta+1}\|g_1(r,\alpha(r))-g_1(r,y_i)\|^p\,dr\right)^{\frac{1}{p}}d\mu_0(\zeta)\\
&\quad+\max_{1\leq i\leq m}\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_\zeta^{\zeta+1}\|g_2(r,y_i)\|^p\,dr\right)^{\frac{1}{p}}d\mu_0(\zeta)\\
&\leq(\sup_{t\in\mathbb{R}}|L_g^b(t)|+\sup_{t\in\mathbb{R}}|L_{g_1}^b(t)|)\frac{(\mu_0([-S,S]))^{\frac{1}{q}}}{(\nu_0([-S,S]))^{\frac{1}{q}}}\varepsilon+\max_{1\leq i\leq m}\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_\zeta^{\zeta+1}\|g_2(r,y_i)\|^p\,dr\right)^{\frac{1}{p}}d\mu_0(\zeta).
\end{aligned}$$

Then

$$\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_\zeta^{\zeta+1}\|g_2(r,\alpha(r))\|^p\,dr\right)^{\frac{1}{p}}d\mu_0(\zeta)$$
$$\leq(\sup_{t\in\mathbb{R}}|L_g^b(\zeta)|+\sup_{t\in\mathbb{R}}|L_{g_1}^b(\zeta)|)\frac{(\mu_0([-S,S]))^{\frac{1}{q}}}{(\nu_0([-S,S]))^{\frac{1}{q}}}\varepsilon+\varepsilon(r>r_0).$$

Therefore, $H^b(\cdot)\in PAA_0(\mathbb{R},L^p(0,1;Y),\mu_0,\nu_0)$. Whence $g(\cdot,h(\cdot))\in PP_{ap}S^p(\mathbb{R},Y,\mu_0,\nu_0)$.

4. Applications to semi-linear parabolic boundary differential equations

Now, one presents the existence and uniqueness of (μ_0,ν_0)-pseudo anti-periodic solutions of equation (1).

(H_3) There exists $\omega > 0$ satisfying $g_1(t+\omega,-y) = -g_1(t,y)$ and the set $K := \{g_1(t) : t \in \mathbb{R}\}$ is compact in Y_{α_2-1};

(H_4) There exist constants $L_g(t), L_{g_1}(t) \in BS^p$ such that

$$\|g(t,y_1) - g(t,y_2)\|_{\alpha_2-1} \le L_g(t)\,\|y_1 - y_2\|_{\alpha_1},$$
$$\|g_1(t,y_1) - g_1(t,y_2)\|_{\alpha_2-1} \le L_{g_1}(t)\,\|y_1 - y_2\|_{\alpha_1};$$

(H_5) The family of closed linear operator $B(t)$ for $t \in \mathbb{R}$ on Y with domain $D(B(t))$ satisfy Acquistapace-Terreni conditions and condition (H) is satisfied.

Definition 4.1 *A function $y : \mathbb{R} \to Y_{\alpha_1}$ of continuous is defined as a mild solution corresponding to (1) provided that it holds for*

$$y(t) = \Phi(t-r)y(r) + \int_r^t \Phi_{\alpha_2-1}(t-s)g(s,y(s))ds, \qquad t \ge r. \tag{8}$$

Theorem 4.1 *Let $\mu_0,\nu_0 \in \mathcal{M}$, $p \ge 1$, $g : \mathbb{R}\times Y_{\alpha_1} \to Y_{\alpha_2-1}$, $0 \le \alpha_1 < \alpha_2$ and $g = g_1 + g_2 \in PP_{ap}S^p(\mathbb{R}\times Y_{\alpha_1}, Y_{\alpha_2-1}, \mu_0, \nu_0)$ be given. Suppose that conditions $(H_1) - (H_5)$ hold, then there exists pseudo anti-periodic solution.*

Proof. One shows that $y(t)$ is (μ_0,ν_0)-pseudo anti-periodic solution. Take into account the operator $\Upsilon : PP_{ap}(\mathbb{R}, Y_{\alpha_1}, \mu_0, \nu_0) \to PP_{ap}(\mathbb{R},\, Y_{\alpha_1}, \mu_0, \nu_0)$ defined by

$$(\Upsilon y)(\zeta) = \int_{-\infty}^{\zeta} \Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}g(s,y(s))ds - \int_{\zeta}^{+\infty} \Phi_{\alpha_2-1}(\zeta-s)P_u^{\alpha_2-1}g(s,y(s))ds.$$

First, one shows that $\Upsilon(PP_{ap}(\mathbb{R},Y_{\alpha_1})) \subset PP_{ap}(\mathbb{R},Y_{\alpha_1})$. Set $h(t) = g(t,y(t))$. By Theorem 3.2 one knows that $h \in PP_{ap}S^p(\mathbb{R},Y_{\alpha_2-1},\mu_0,\nu_0)$. Now denote $h = \phi + \psi$, where $\phi \in P_{ap}S^p(\mathbb{R},Y_{\alpha_2-1})$, $\psi^b \in PAA_0(\mathbb{R}, L^p(0,1;Y_{\alpha_2-1}),\mu_0,\nu_0)$. Consider the integrals

$$\begin{aligned} u_n(\zeta) &= \int_{\zeta-n}^{\zeta-n+1} \Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}h(s)ds \\ &= \int_{\zeta-n}^{\zeta-n+1} \Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}\phi(s)ds + \int_{\zeta-n}^{\zeta-n+1} \Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}\psi(s)ds, \ \ n = 1,2,\cdots, \end{aligned}$$

and set

$$X_{n,1}(\zeta) = \int_{\zeta-n}^{\zeta-n+1} \Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}\phi(s)ds, \ \ X_{n,2}(\zeta) = \int_{\zeta-n}^{\zeta-n+1} \Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}\psi(s)ds.$$

First, one proves that $X_{n,1} \in P_{ap}(\mathbb{R},X_{\alpha_1})$. Fix $n \in \mathbb{N}$ and $t \in \mathbb{R}$, using equation (8) and the fact that the projection $P_s^{\alpha-1}$ are necessary periodic, one obtains that

$$\begin{aligned} \|X_{n,1}(\zeta+\omega) + X_{n,1}(\zeta)\|_{\alpha_1} &= \left\| \int_{\zeta+\omega-n}^{\zeta+\omega-n+1} \Phi_{\alpha_2-1}(\zeta+\omega-s)P_s^{\alpha_2-1}\phi(s)ds + \int_{\zeta-n}^{\zeta-n+1} \Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}\phi(s)ds \right\|_{\alpha_1} \\ &= \left\| \int_{\zeta-n}^{\zeta-n+1} \Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}(\phi(s+\omega)+\phi(s))ds \right\|_{\alpha_1} \\ &\le m(\alpha_1,\alpha_2)\,(q\gamma)^{\frac{-\alpha_2+\alpha_1+\varepsilon}{q}}\,(\Gamma(q(\alpha_2-\alpha_1-\varepsilon)))^{\frac{1}{q}} \left(\int_{\zeta-n}^{\zeta-n+1} \|\phi(s+\omega)+\phi(s)\|_{\alpha_2-1}^p\, ds \right)^{\frac{1}{p}} = 0, \end{aligned}$$

so $X_{n,1} \in P_{ap}(\mathbb{R},Y_{\alpha_1})$ for $n \in \mathbb{N}$. By the Hölder inequality, one has

$$\|X_{n,1}(\zeta)\|_{\alpha_1} \le \int_{\zeta-n}^{\zeta-n+1} \left\|\Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}\phi(s)\right\| ds$$

$$\leq m(\alpha_1,\alpha_2)\int_{\zeta-n}^{\zeta-n+1} e^{-\gamma(\zeta-s)}(\zeta-s)^{\alpha_2-\alpha_1-\tilde{\varepsilon}-1}\left\|\phi(s)\right\|_{\alpha_2-1}ds$$
$$\leq m(\alpha_1,\alpha_2)\left(\frac{2}{q\gamma}\right)^{\frac{\alpha_2-\alpha_1-\varepsilon}{q}}\left(\Gamma(q(\alpha_2-\alpha_1-\varepsilon))\right)^{\frac{1}{q}}\left(\int_{\zeta-n}^{\zeta-n+1} e^{-\frac{p\gamma(\zeta-s)}{2}}\left\|\phi(s)\right\|^p_{\alpha_2-1}ds\right)^{\frac{1}{p}}$$
$$\leq m(\alpha_1,\alpha_2)\left\|\phi\right\|_{S^p_{\alpha_2-1}}\left(\frac{2}{q\gamma}\right)^{\frac{\alpha_2-\alpha_1-\varepsilon}{q}}\left(\Gamma(q(\alpha_2-\alpha_1-\varepsilon))\right)^{\frac{1}{q}}\left(1-e^{-\frac{p\gamma}{2}}\right)^{-\frac{1}{p}}<\infty.$$

Let $X_1(\zeta)=\sum\limits_{n=1}^{\infty}X_{n,1}(\zeta), \zeta\in\mathbb{R}$, then

$$X_1(\zeta)=\int_{-\infty}^{\zeta}\Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}\phi(s)ds, \zeta\in\mathbb{R}.$$

Dealing with like Lemma 3.1, one gets $X_1(\zeta)=\sum\limits_{n=1}^{\infty}X_{n,1}(\zeta)\in P_{ap}(\mathbb{R},Y_{\alpha_1})$. For the rest, one just needs to show that $X_{n,2}\in PAA_0(\mathbb{R},Y_{\alpha_1},\mu_0,\nu_0)$. With a similar view above, one obtains that $\sum\limits_{n=1}^{\infty}X_{n,2}(\zeta)$ is convergent on $\mathbb{R}$ uniformly. Let $X_2(\zeta)=\int_{-\infty}^{\zeta}\Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}\psi(s)ds=\sum\limits_{n=1}^{\infty}X_{n,2}(\zeta)$, then

$$X_{n,2}(\zeta)=\int_{\zeta-n}^{\zeta-n+1}\Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}\psi(s)ds, \zeta\in\mathbb{R}.$$

Obviously, $X_2\in BC(\mathbb{R},Y_{\alpha_1})$. Next one needs to prove that

$$\lim_{S\to\infty}\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left\|X_2(\zeta)\right\|_{\alpha_1}d\mu_0(\zeta)=0.$$

In fact, by the Hölder inequality, one has

$$\left\|X_{n,2}(\zeta)\right\|_{\alpha_1}\leq\int_{\zeta-n}^{\zeta-n+1}\left\|\Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}\psi(s)\right\|_{\alpha_1}ds$$
$$\leq m(\alpha_1,\alpha_2)\int_{\zeta-n}^{\zeta-n+1} e^{-\gamma(\zeta-s)}(\zeta-s)^{\alpha_2-\alpha_1-\tilde{\varepsilon}-1}\left\|\psi(s)\right\|_{\alpha_2-1}ds$$
$$\leq D_0\left(\int_{\zeta-n}^{\zeta-n+1}\left\|\psi(s)\right\|^p_{\alpha_2-1}ds\right)^{\frac{1}{p}},$$

where $D_0=m(\alpha_1,\alpha_2)\,(q\gamma)^{\frac{-\alpha_2+\alpha_1+\varepsilon}{q}}\left(\Gamma(q(\alpha_2-\alpha_1-\varepsilon))\right)^{\frac{1}{q}}$, then

$$\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left\|X_{n,2}(\zeta)\right\|_{\alpha_1}d\mu_0(\zeta)\leq\frac{D_0}{\nu_0([-S,S])}\int_{[-S,S]}\left(\int_{\zeta-n}^{\zeta-n+1}\left\|\psi(s)\right\|^p_{\alpha_2-1}ds\right)^{\frac{1}{p}}d\mu_0(\zeta).$$

Since $\psi^b\in PAA_0(\mathbb{R},L^p(0,1;Y_{\alpha_2-1}),\mu_0,\nu_0)$, $X_{n,2}\in PAA_0(\mathbb{R},Y_{\alpha_1},\mu_0,\nu_0)$. From $X_{n,2}\in PAA_0(\mathbb{R},Y_{\alpha_1},\mu_0,\nu_0)$ and

$$\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left\|X_2(\zeta)\right\|_{\alpha_1}d\mu_0(\zeta)$$
$$\leq\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left\|X_2(\zeta)-\sum_{n=1}^{N}X_{n,2}(\zeta)\right\|_{\alpha_1}d\mu_0(\zeta)+\sum_{n=1}^{N}\frac{1}{\nu_0([-S,S])}\int_{[-S,S]}\left\|X_{n,2}\right\|_{\alpha_1}d\mu_0(\zeta),$$

it follows from $X_2\in PAA_0(\mathbb{R},Y_{\alpha_1},\mu_0,\nu_0)$ that one has $\int_{-\infty}^{\zeta}\Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}h(s)ds\in PP_{ap}(\mathbb{R},Y_{\alpha_1},\mu_0,\nu_0)$. Next one shows $\int_{\zeta}^{\infty}\Phi_{\alpha_2-1}(\zeta-s)P_u^{\alpha_2-1}h(s)ds\in PP_{ap}(\mathbb{R},Y_{\alpha_1},\,\mu_0,\nu_0)$. Let

$$Y_n(\zeta)=\int_{\zeta+n-1}^{\zeta+n}\Phi_{\alpha_2-1}(\zeta-s)P_u^{\alpha_2-1}\phi(s)ds+\int_{\zeta+n-1}^{\zeta+n}\Phi_{\alpha_2-1}(\zeta-s)P_u^{\alpha_2-1}\psi(s)ds, n=1,2,\cdots,$$

and set

$$Y_{n,1}(\zeta) = \int_{\zeta+n-1}^{\zeta+n} \Phi_{\alpha_2-1}(\zeta - s)P_u^{\alpha_2-1}\phi(s)ds, \qquad Y_{n,2}(\zeta) = \int_{\zeta+n-1}^{\zeta+n} \Phi_{\alpha_2-1}(\zeta - s)P_u^{\alpha_2-1}\psi(s)ds.$$

Now, one proves that $Y_{n,1} \in P_{ap}(\mathbb{R}, Y_{\alpha_1})$. Fix $n \in \mathbb{N}$ and $\zeta \in \mathbb{R}$, it follows that

$$\begin{aligned}
&\|Y_{n,1}(\zeta+\omega) + Y_{n,1}(\zeta)\|_{\alpha_1}\\
&= \left\| \int_{\zeta+\omega+n-1}^{\zeta+\omega+n} \Phi_{\alpha_2-1}(\zeta+\omega-s)P_u^{\alpha_2-1}\phi(s)ds + \int_{\zeta+n-1}^{\zeta+n} \Phi_{\alpha_2-1}(\zeta-s)P_u^{\alpha_2-1}\phi(s)ds \right\|_{\alpha_1}\\
&= \left\| \int_{\zeta+n-1}^{\zeta+n} \Phi_{\alpha_2-1}(\zeta-s)P_u^{\alpha_2-1}(\phi(s+\omega)+\phi(s))ds \right\|_{\alpha_1}\\
&\le c(\alpha_1,\alpha_2) \int_{\zeta+n-1}^{\zeta+n} e^{\delta(\zeta-s)} \|\phi(s+\omega)+\phi(s)\|_{\alpha_2-1} ds\\
&\le c(\alpha_1,\alpha_2) \left(\int_{\zeta+n-1}^{\zeta+n} e^{q\delta(\zeta-s)} ds \right)^{\frac{1}{q}} \left(\int_{\zeta+n-1}^{\zeta+n} \|\phi(s+\omega)+\phi(s)\|_{\alpha_2-1}^p ds \right)^{\frac{1}{p}}\\
&\le c(\alpha_1,\alpha_2) (q\delta)^{-\frac{1}{q}} \left(\int_{\zeta+n-1}^{\zeta+n} \|\phi(s+\omega)+\phi(s)\|_{\alpha_2-1}^p ds \right)^{\frac{1}{p}} = 0,
\end{aligned}$$

so $Y_{n,1} \in P_{ap}(\mathbb{R}, Y_{\alpha_2-1})$ for $n \in \mathbb{N}$. According to Hölder inequality, one gets

$$\begin{aligned}
\|Y_{n,2}(t)\|_{\alpha_1} \le& \int_{\zeta+n-1}^{\zeta+n} \left\|\Phi_{\alpha_2-1}(\zeta-s)P_u^{\alpha_2-1}\phi(s)\right\| ds\\
\le& c(\alpha_1,\alpha_2) \int_{\zeta+n-1}^{\zeta+n} e^{\delta(\zeta-s)} \|\phi(s)\|_{\alpha_2-1} ds\\
\le& c(\alpha_1,\alpha_2) \|\phi\|_{S^p_{\alpha_2-1}} \left(\frac{2}{q\delta}\right)^{\frac{1}{q}} \left(1-e^{-\frac{p\delta}{2}}\right)^{-\frac{1}{p}} < \infty.
\end{aligned}$$

For the rest part, similar to the process of $X_2(\zeta) \in PAA_0(\mathbb{R}, Y_{\alpha_2-1}, \mu_0, \nu_0)$, one can gain $Y_2(\zeta) \in PAA_0(\mathbb{R}, Y_{\alpha_2-1}, \mu_0, \nu_0)$, whence $\Upsilon y \in PP_{ap}(\mathbb{R}, Y_{\alpha_2-1}, \mu_0, \nu_0)$.

Next, one will prove that equation (1) admits a unique solution.

Theorem 4.2 *Let $\mu_0, \nu_0 \in \mathcal{M}$, $g : \mathbb{R} \times Y_{\alpha_1} \to Y_{\alpha_2-1}$, $0 \le \alpha_1 < \alpha_2$ and $\varepsilon > 0$. Suppose that (H_2), (H_4) and (H) hold. Besides, it also satisfies:*

(a_1) *$0 < \alpha_2 - \varepsilon < 1$ and $0 < \alpha_1 + \varepsilon < \alpha_2$,*

(a_2) *for each $y \in Y_{\alpha_1}$, $g(\cdot, y) \in PP_{ap}S^p(\mathbb{R}, Y_{\alpha_2-1}, \mu_0, \nu_0)$ satisfies (H_4) with*

$$\|L_g\|_{BS^p} \le \left(a_0 \left(1-e^{-\frac{p\gamma}{2}}\right)^{-\frac{1}{p}} + c(\alpha_1,\alpha_2)(q\delta)^{-\frac{1}{q}} \left(1-e^{-\frac{p\delta}{2}}\right)^{-\frac{1}{p}} \right)^{-1},$$

where

$$a_0 = m(\alpha_1,\alpha_2) \left(\frac{2}{qr}\right)^{\frac{\alpha_2-\alpha_1-\varepsilon}{q}} (\Gamma(q(\alpha_2-\alpha_1-\varepsilon)))^{\frac{1}{q}}.$$

Thus equation (1) admits a unique solution $y \in PP_{ap}(\mathbb{R}, Y_{\alpha_1}, \mu_0, \nu_0)$.

Proof. Define an operator Υ as

$$(\Upsilon y)(\zeta) := \int_{-\infty}^{\zeta} \Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}g(s,y(s))ds - \int_{\zeta}^{+\infty} \Phi_{\alpha_2-1}(\zeta-s)P_u^{\alpha_2-1}g(s,y(s))ds.$$

One can see that $\Upsilon(PP_{ap}(\mathbb{R}, Y_{\alpha_1}, \mu_0, \nu_0)) \subset PP_{ap}(\mathbb{R}, Y_{\alpha_1}, \mu_0, \nu_0)$. Next, one shows that Υ composes a contraction operator. Indeed, for every $u_1, u_2 \in PP_{ap}(\mathbb{R}, Y_{\alpha_1}, \mu_0, \nu_0)$ one has

$$
\begin{aligned}
\|(\Upsilon u_1)(\zeta) - (\Upsilon u_2)(\zeta)\|_{\alpha_1} \leq & m(\alpha_1, \alpha_2) \int_{-\infty}^{\zeta} e^{-\gamma(\zeta - s)} (\zeta - s)^{\alpha_2 - \alpha_1 - \varepsilon - 1} \|g(s, u_1(s)) - g(s, u_2(s))\|_{\alpha_2 - 1} ds \\
& + c(\alpha_1, \alpha_2) \int_{\zeta}^{+\infty} e^{-\delta(\zeta - s)} \|g(s, u_1(s)) - g(s, u_2(s))\|_{\alpha_2 - 1} ds \\
& \leq \|L_g\|_{BS^p} \left(m(\alpha_1, \alpha_2) \left(\frac{2}{q\gamma}\right)^{\frac{\alpha_2 - \alpha_1 - \varepsilon}{q}} (\Gamma(q(\alpha_2 - \alpha_1 - \widetilde{\varepsilon})))^{\frac{1}{q}} \left(1 - e^{-\frac{p\gamma}{2}}\right)^{-\frac{1}{p}} \right. \\
& \left. + c(\alpha_1, \alpha_2) \left(\frac{2}{q\delta}\right)^{\frac{1}{q}} \left(1 - e^{-\frac{p\delta}{2}}\right)^{-\frac{1}{p}} \right) \|u_1 - u_2\|_{\infty}.
\end{aligned}
$$

Thus, Υ admits a unique fixed point of $PP_{ap}(\mathbb{R}, Y_{\alpha_1}, \mu_0, \nu_0)$.

Next one takes into account the (μ_0, ν_0)-pseudo anti-periodic solution for semi-linear boundary differential equations. Take into consideration the following equation

$$
\begin{cases}
y'(t) = B_h y(t) + g(t, y(t)) \text{ for } t \in \mathbb{R}, \\
Ly(t) = \eta(t, y(t)) \qquad\quad \text{for } t \in \mathbb{R},
\end{cases}
\tag{9}
$$

where $(B_h, D(B_h))$ is a linear densely operator defined on a Banach space Y, $L : D(B_h) \to \partial Y$, composes a linear boundary operator, $g : \mathbb{R} \times Y_h \to Y$ and $\eta : \mathbb{R} \times Y_h \to \partial Y$ are certain given functions. One shall have the below assumptions given by G.Greiner:[43]

(E_1) There is a new norm $|\cdot|$ that is better than $\|\cdot\|_Y$ so that the space $X_h = (D(A_h), |\cdot|_h)$ is complete, i. e. Y_h is continuously embeded in Y and $B_h \in \mathcal{C}(Y_h, Y)$;

(E_2) The operator $B := B_h|\ker(L)$ composes a closed sectorial operator satisfying $\sigma(B) \cap i\mathbb{R} = \emptyset$;

(E_3) The operator $L : Y_h \to \partial Y$ is surjective and bounded, i. e. $ih(L) = \partial Y$;

(E_4) Y_h is continuously embeded on Y_{α_2}. i. e., $Y_h \to Y_{\alpha_2}$ for certain $0 < \alpha_2 < 1$;

(E_5) The function $g : \mathbb{R} \times Y_{\alpha_1} \to Y$ and $\eta : \mathbb{R} \times Y_{\alpha_1} \to \partial Y$ are locally integrable for the first variable and continuous for the second one concerning $0 \leq \alpha_1 < \alpha_2$.

Definition 4.2 *A continuous function $y : \mathbb{R} \to Y_{\alpha_1}$ is defined as a solution of (1) provided that it fulfills:*

(a_1) $\int_s^t y(r)dr \in Y_h$;

(a_2) $y(t) - y(s) = B_h \int_s^t y(r)dr + \int_s^t g(r, y(r))dr$;

(a_3) $L \int_s^t y(r)dr = \int_s^t \eta(r, y(r))dr$.

As in[44] one changes (1) to the equivalent equation given by

$$
y'(t) = B_{\alpha_2 - 1} y(t) + g(t, y(t)) + B_{\alpha_2 - 1} L_0 \eta(t, y(t)) \text{ for all } t \in \mathbb{R},
\tag{10}
$$

where $L_0 = (L|\ker(B_h))^{-1}$.

Now, one presents our main results:

Theorem 4.3 *Let $\mu_0, \nu_0 \in \mathcal{M}$ satisfy (H_1) and (H_2). Suppose that $(E_1) - (E_5)$ hold and the functions $g = g_1 + g_2 \in PP_{ap}S^p(\mathbb{R} \times Y_{\alpha_1}, Y, \mu_0, \nu_0)$, $\eta = \eta_1 + \eta_2 \in PP_{ap}S^p(\mathbb{R} \times Y_{\alpha_1}, \partial Y, \mu_0, \nu_0)$ satisfy*

$$
\begin{aligned}
&\|g(t, y_1) - g(t, y_2)\| \leq L_g \|y_1 - y_2\|, \quad && \|\eta(t, y_1) - \eta(t, y_2)\| \leq L_\eta \|y_1 - y_2\|, \\
&g_1(t + \omega, -y) = -g_1(t, y), && \eta_1(t + \omega, -y) = -\eta_1(t, y),
\end{aligned}
$$

where L_g, L_η are small constants and $g_1 \in P_{ap}S^p(\mathbb{R}\times Y_{\alpha_1}, Y)$, $\eta_1 \in P_{ap}S^p(\mathbb{R}\times Y_{\alpha_1}, \partial Y)$, $g_2^b \in PAA_0(\mathbb{R}\times Y_{\alpha_1}, L^p(0,1;Y), \mu_0, \nu_0)$, $\eta_2^b \in PAA_0(\mathbb{R}\times Y_{\alpha_1}, L^p(0,1;\partial Y), \mu_0, \nu_0)$. Further, g_1 and η_1 satisfy that the sets $K_1 := \{g_1(t) : t \in \mathbb{R}\}$ and $K_2 := \{\eta_1(t) : t \in \mathbb{R}\}$ are compact in Y_{α_2-1}. Then, (10) admits a unique solution $y \in PP_{ap}(\mathbb{R}, Y_{\alpha_1}, \mu_0, \nu_0)$ satisfy the following formula

$$
\begin{aligned}
y(\zeta) &= \int_{-\infty}^{\zeta} \Phi(\zeta-s)P^s g(s,y(s))ds - \int_{\zeta}^{+\infty} \Phi(\zeta-s)P^u g(s,y(s))ds \\
&\quad - B\left[\int_{-\infty}^{\zeta} \Phi(\zeta-s)P^s L_0\eta(s,y(s))ds - \int_{\zeta}^{+\infty} \Phi(\zeta-s)P^u L_0\eta(s,y(s))d\zeta\right], \zeta \in \mathbb{R}.
\end{aligned} \tag{11}
$$

Proof. Because $B_h \hookrightarrow Y_{\alpha_2-1}$, we obtain that the operator $B_{\alpha_2-1}L_0 \in L(\partial Y, Y_{\alpha_2-1})$. Since $g \in PP_{ap}S^p(\mathbb{R}\times Y_{\alpha_1}, Y, \mu_0, \nu_0)$ and $\eta \in PP_{ap}S^p(\mathbb{R}\times Y_{\alpha_1}, \partial Y, \mu_0, \nu_0)$ and through the injection $Y \hookrightarrow Y_{\alpha_2-1}$, the function $G(t,y) = g(t,y) + B_{\alpha_2-1}L_0\eta(t,y)$ belongs to $PP_{ap}S^p(\mathbb{R}\times Y_{\alpha_1}, Y_{\alpha_2-1}, \mu_0, \nu_0)$. Moreover, the function g satisfies with constant of Lipschitz $L_G = L_g + A_{\alpha-1}L_0L_\eta$, then, by choosing L_g and L_η appropriately, L_G can be sufficiently small. Therefore, according to last Theorems 4.1 and 4.2, it follows that equation (1) admits a unique mild solution $y \in PP_{ap}(\mathbb{R}, Y_{\alpha_1})$ fulfilling

$$
y(\zeta) = \int_{-\infty}^{\zeta} \Phi_{\alpha_2-1}(\zeta-s)P_s^{\alpha_2-1}G(s,y(s))ds - \int_{\zeta}^{+\infty} \Phi_{\alpha_2-1}(\zeta-s)P_u^{\alpha_2-1}G(s,y(s))ds
$$

for all $\zeta \in \mathbb{R}$. By replacing g with G and then it follows that the formula (11) and that $y \in PP_{ap}(\mathbb{R}, Y_{\alpha_1})$ is the unique mild solution of (1).

Now, one presents the following example to illustrate the effectiveness of our results.

Example 4.1 *One considers the equation:*

$$
\begin{cases}
\dfrac{\partial v(t,y)}{\partial t} = \triangle v(t,y) + av(t,y(t)) \text{ for } t \in \mathbb{R} \text{ and } y \in \Omega \\
\dfrac{\partial v(t,y)}{\partial n} = \Phi_1(t, n(y)v(t,y)) \text{ for } t \in \mathbb{R} \text{ and } y \in \partial\Omega,
\end{cases} \tag{12}
$$

where $a \in \mathbb{R}$, Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and n is a C^1 function. The function $\overline{\psi} : \mathbb{R}\times\partial Y \to \partial Y$ defined by

$$
\overline{\psi}(t,\varphi)(y) = \Phi_1(t, n(y)\varphi(y)) = \frac{l\cdot d(t)}{4+|n(y)\varphi(y)|}
$$

where $d(t) = d_1(t) + d_2(t)$ with $d_1(t) = \sum\limits_{n=1}^{\infty} \frac{sin((2n+1)t)}{n^2}$ and $d_2(t) = \frac{1}{1+t^2}$. Let $R_1(t,\varphi)(y) = \frac{l\cdot d_1(t)}{4+|n(y)\varphi(y)|}$ and $R_2(t,\varphi)(y) = \frac{l\cdot d_2(t)}{4+|n(y)\varphi(y)|}$. Note that $R_1(t+\pi,\varphi)(y) = -R_1(t,\varphi)(y)$ for all t. Hence $R_1(t,\varphi) \in P_{ap}S^p(\mathbb{R}\times Y, Y)$ with $\omega = \pi$. On the other hand, by,[30] *one knows that $R_2 \in PAA_0(\mathbb{R}\times Y_{\alpha_2-1}, Y_{\alpha_1}, \mu_0, \nu_0)$.*

$$
\begin{aligned}
&\int_{\Omega} |\overline{\psi}(t,\varphi_1)(y) - \overline{\psi}(t,\varphi_2)(y)|^2 dy \\
&= (l\cdot d(t))^2 \int_{\Omega} \left|\frac{1}{4+|n(y)\varphi_1(y)|} - \frac{1}{4+|n(y)\varphi_2(y)|}\right|^2 dy \\
&\le (l\cdot d(t))^2 |n|_\infty^2 \, \|\varphi_1-\varphi_2\|^2 \ \text{ for all } t \in \mathbb{R}.
\end{aligned}
$$

Since $d \in PP_{ap}S^p(\mathbb{R}\times Y_{\alpha_2-1}, Y_{\alpha_1})$, one deduces that $\overline{\psi}$ satisfies Lipschitz with $L_\psi = (ld(\cdot))|n|_\infty$.

5. Conclusion and discussion

One introduces a new class of functions named Stepanov-like (μ_0, ν_0)-pseudo anti-periodic functions via the measure theory, which generalizes the classical weighted pseudo anti-periodic function in Stepanov sense. On the other hand, there are kinds of literatures on various equations with anti-periodic boundary value. However, there is little work on anti-periodic solutions for evolution equations with inhomogeneous boundary conditions. Note that anti-periodic functions with weight are more complex and uncertain due to the diversity of weighted functions. Moreover, the uniqueness of decomposition of the vector-valued functions is one of the keys for deep analysis of these functions and their applications to various equations. By exponential dichotomy, one is devoted to the existence of (μ_0, ν_0)-pseudo anti-periodic solutions to semi-linear parabolic equations with inhomogeneous boundary conditions in interpolation and extrapolation spaces. These results generalize the results in the related literature.[14]

Despite the results for Stepanov-like (μ_0, ν_0)-pseudo anti-periodic boundary condition in this article, there is still room for improvement. For example, apply the theoretical results obtained above to other equations such as integro-differential (or fractional) equations so that one can better understand the dynamic behavior of them. Moreover, can one find a unified framework to study functions which include both S^p-pseudo S-asymptotic periodic function in paper[45] and S-asymptotically Bloch type periodic functions in literature[46]? Furthermore, can one find some interesting properties of such functions? This is worth our studying.

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References

1. Poincaré H. Les méthodes nouvelles de la mécanique céleste, vol. I. Gauthiers-Villars, Paris; 1892.
2. Poincaré H. Les méthodes nouvelles de la mécanique céleste, vol. II. Gauthiers-Villars, Paris; 1893.
3. Poincaré H. Les méthodes nouvelles de la mécanique céleste, vol. II. Gauthiers-Villars, Paris; 1899.
4. Okochi H. On the existence of anti-periodic solutions to a nonlinear evolution equation associated with odd subdifferential operators. J.Func.Anal. 1990;91(3):246–258.
5. Aftabizadeh A.R, Aizciovici S, Pavel N.H. On a class of seconder-order anti-periodic boundary value problems. J.Math.Anal.Appl. 1992;18(3):253–267.
6. Aftabizadeh A.R, Aizciovici S, Pavel N.H. Anti-periodic boundary value problems for higher order differential equations in Hilbert spaces. Nonlinear Anal-Theor. 1992;171(2):301–320.
7. Aftabizadeh A.R, Pavel N.H, Huang Y.K. Anti-periodic oscillations of some second-order differential equations and optimal control problems. J.Comput.Appl.Math. 1994;52(1-2):3–21.
8. Chen Y.Q, Nieto J.J, O'Regan D. Anti-periodic solutions for fully nonlinear first-order differential equations. Math.Comput.Model. 2007;46(2):1183–1190.
9. Pan L.J, Cao J.D. Anti-periodic solution for delayed cellular neural networks with impulsive effects. Nonlinear Anal-Real. 2011;12(6):3014-3027.
10. Shi P.L, Dong L.Z. Existence and exponential stability of anti-periodic solutions of Hopfield neural networks with impulses. Appl.Math.Comput. 2010;216(2):623-630.
11. Shao J.Y. Anti-periodic solutions for shunting inhibitory cellular neural networks with time-varying delays. Phys.Lett.A. 2008;372(30):5011-5016.
12. Kong F.C, Nieto J.J, Fu X.Y. Stability analysis of anti-periodic solutions of the time-varying delayed Hematopoiesis model with discontinuous harvesting terms. Acta Appl. Math. https://doi.org/10.1007/s10440-020-00328-8.
13. Andres J. Multiple anti-periodic solutions of implicit differential inclusions on tori. J.Differ.Equations. 2021;273(5):1-13.
14. Liu Q. Existence of anti-periodic mild solutions for semilinear evolution equations. J.Math.Anal.Appl. 2011;377(1):110-120.
15. Cao J.F, Yang Q.G, Huang Z.T. Existence of anti-periodic mild solutions for a class of semilinear fractional differential equations. Commun. Nonlinear Sci. 2012;17(1):277-283.
16. Chen Y, O'Regan D, Agarwal B.P. Anti-periodic solutions for semilinear evolution equations in Banach spaces. J.Appl.Math.Comput. 2012;38(5):63-70.
17. Aizicovici S, Pavel N.H. Anti-periodic solutions to a class of nonlinear differential equations in Hilbert space. J.Func.Anal. 1991;99(1):387-408.
18. Aizicovici S, Mckibben M, Reich S. Anti-periodic solutions to nonmonotone evolution equations with discontinuous nonlinearities. Nonlinear Anal-Theor. 2001;43(3):233-251.
19. Wang R, Chen D. Anti-periodic problems for semilinear partial neutral evolution equations. Electron.J.Qual.Theo. 2013;16(8):1-16.
20. Liu J.H, Cheng S.H, Zhang L.T. Anti-periodic mild solutions of semilinear fractional differential equations. J.Appl.Math.Comput. 2015;48(1-2):381-393.
21. Liua X, Liub Y. Existence of anti-periodic solutions for a class of first order evolution inclusions. Filomat. 2014;28(6):1167-1180.

22. Li Y.K, Zhang T.W. Existence and uniqueness of anti-periodic solution for a kind of forced Rayleigh equation with state dependent delay and impulses. Commun.Nonlinear Sci. 2010;15(12):4076-4083.
23. Liu B.W. Anti-periodic solutions for forced Rayleigh-type equations. Nonlinear Anal-Real. 2009;10(5):2850-2856.
24. Agarwal R.P, Ahmadb B. Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions. Comput. Math. Appl. 2011;62(3):1200-1214.
25. Zhao J, Liu J, Fang L.J. Anti-periodic boundary value problems of second-order functional differential equations. B. Malays. Math. Sci. 2014;2(2):311-320.
26. Franco D, Nieto J.J, O'Regan D. Anti-periodic boundary value problem for nonlinear first order ordinary differential equations. Math.Inequal.Appl. 2003;6(3):477-485.
27. Li X.P, Chen F.L, Li X.Z. Generalized anti-periodic boundary value problems of impulsive fractional differential equations. Commun.Nonlinear Sci. 2013;18(1):28-41.
28. Al-Islam N.S, Alsulami S.M, Diagana T. Existence of weighted pseudo anti-periodic solutions to some non-autonomous differential equations. Appl.Math.Comput. 2012;218(11):6536-6548.
29. Zhou Q.Y, Shao J.Y. Weighted pseudo-anti-periodic SICNNs with mixed delays. Neural Comput. Appl. 2018;29(10):865-872.
30. Alvarez E, Lizama C, Ponce R. Weighted pseudo anti-periodic solutions for fractional integro-differential equations in Banach spaces. Appl.Math.Comput. 2015;259(11):164-172.
31. Boulite S, Maniar L, N'Guérékata G.M. Almost automorphic solutions for semilinear boundary differential equations. Proc. Amer. Math. Soc. 2006;134(12):3613-3624.
32. Baroun M, Maniar L, NGuerekata G.M. Almost periodic and almost automorphic solutions to semilinear parabolic boundary differential equations. Nonlinear Anal. 2008;69(7):2114-2124.
33. Baroun M, Ezzinbi K, Khalil K, Maniar L. Pseudo almost periodic solutions for some parabolic evolution equations with Stepanov-like pseudo almost periodic forcing terms. J.Math.Anal.Appl. 2018;462(1):233-262.
34. Wu R. An anti-periodic LaSalle oscillation theorem. Math. Appl. Lett. 2008;21(9):928-933.
35. Li Q, Wang W. Ground states for fractional Schrodinger equations with electromagnetic fields and critical growth. Acta Math. Sci. 2020;40(B)(1):59-74.
36. Djakov P, Mityagin B. Simple and double eigenvalues of the Hill operator with a two-term potential. J. Approx. Theory. 2005;135(1):70-104.
37. Djakov P, Mityagin B. Spectral gaps of the periodic Schrodinger operator when its potential is an entire function. Adv. Appl. Math. 2003;31(3):562–596.
38. Blot J, Cieutat P, Ezzinbi K. New approach for weighted pseudo almost periodic functions under the light of measure theory, basic results and applications. Appl.Anal. 2013;92(3):493-526.
39. Engel R, Nagel K.J. One-Parameter semigroups for linear evolution equations. vol.194, Springer-Verlag, New York; 2000.
40. Lunardi A. Analytic semigroups and optimal regularity in parabolic problems. Birkhauser, Basel, Boston, Berlin; 1995.
41. Es-sebbar B, Ezzinbi K. Stepanov ergodic perturbations for some neutral partial functional differential equations. Math.Method.Appl.Sci 2016;39(8):1945-1963.
42. Diagana T, Mophou G.M, N'Guérékata G.M. Existence of weighted pseudo-almost periodic solutions to some classes of differential equations with S^p-weighted pseudo-almost periodic coefficients. Nonlinear Anal-Theor. 2010;72(1):430-438.
43. Greiner G. Perturbing the boundary conditions of a generator. Houston Journal of Mathematics. 1987;13(5):213-229.
44. Boulite S, Maniar L, N'Guerekata G.M. Almost automorphic solutions for hyperbolic semilinear evolution equations. Semigroup Forum. 2005;71(2):231-240.
45. He B, Wang Q.R, Cao J.F. Weighted S^p-pseudo S-asymptotic periodicity and applications to Volterra integral equations. Appl.Math.Comput. 2019;380(1):125275.
46. Chang Y.K, Wei Y.Y. S-asymptotically Bloch type periodic solutions to some semi-linear evolution equations in Banach spaces. Acta.Math.Sci. 2021;41(2):413-425.