Spacetime estimates and scattering theory for quasilinear Schrödinger equations in arbitrary space dimension

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Abstract

In this paper, we consider Cauchy problem of a quasilinear Schrodinger equation which has general form containing potential term, power type nonlinearity and Hartree type nonlinearity. The space dimension is arbitrary, that is, it is larger than or equals to one.

First, we establish the local wellposedness of the solution and discuss the condition on the global existence of the solution.

Next, we establish some conservation laws such as mass conservation law, energy conservation law, pseudoconformal conservation law of the solution. Based on these conservation laws, we give Morawetz type estimates, spacetime bounds for the global solution.

Last, we take two ideas to establish scattering theory for the global solution in different functional spaces. The first idea is that we take different admissible pairs in Strichartz estimates for different terms on the right side of Duhamel's formula in order to keep each term independent, another one is that we factitiously let a continuous function be the sum of two piecewise functions and choose different admissible pairs in Strichartz estimates for the terms containing these functions.

Keywords: Quasilinear Schrödinger equation; Spacetime estimate; Morawetz type estimate; Scattering.

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Contents.

1 Introduction

In this paper, we consider the following Cauchy problem:

$$\begin{cases} iu_t = \Delta u + 2uh'(|u|^2)\Delta h(|u|^2) + V(x)u + F(|u|^2)u + (W * |u|^2)u, \ x \in \mathbb{R}^N, \ t > 0 \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^N. \end{cases} \tag{1.1}$$

Here $h(s)$, $F(s)$, $V(x)$ and $W(x)$ are some real functions, $W(x)$ is even, $N \geq 1$. (1.1) can be used to model a lot of physical phenomena, such as the superfluid film equation in plasma physics if $h(s) = s$, physics phenomenon in dissipative quantum mechanics if $h(s) = \sqrt{s}$ and the self-channelling of a high-power ultra short laser in matter if $h(s) = \sqrt{1+s}$. It also appears in condensed matter theory and nonlinear optical theory, see [3, 5, 6, 24, 31, 33, 34, 36, 41, 42, 43, 44]. There are many interesting topics on (1.1), such as local wellposedness, global wellposeness, decay rate and scattering phenomenon for the global solution.

First, we need to deal with the local wellposedness of the solution to (1.1). In convenience, we always assume that $h(s) \geq 0$ for $s \geq 0$, $V(x) \leq 0$ and $W(x) \leq 0$ for $x \in \mathbb{R}^N$ in this paper. We say that (1.1) is in defocusing case if $F(s) \leq 0$ for $s \geq 0$, while we say that (1.1) is in combined defocusing and focusing case if $F(s) \geq 0$ for $s \geq 0$ or changes sign. Other assumptions on $V(x)$ and $W(x)$ are as follows:

(**WV1**) If $h(s) \equiv 0$ for $s \geq 0$, we require that $V(x) \in L^{p_1}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $p_1 > \max(1, \frac{N}{2})$ and $W(x) \in L^{p_2}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $p_2 > \max(1, \frac{N}{4})$

or

(**WV2**) If $h(s) \geq 0, \not\equiv 0$ for $s \geq 0$, we require that $V(x) \in \mathfrak{B}^\infty(\mathbb{R}^N)$, and $W(x) \in L^1(\mathbb{R}^N) \cap \{L^{p_2}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)\}$ for some $p_2 > \max(1, \frac{N}{4})$. Here $\mathfrak{B}^\infty(\mathbb{R}^N)$ denotes the space of all functions in $C^\infty(\mathbb{R}^N)$ such that all partial derivatives are bounded in $\mathbb{R}^N$.

We will prove that: Besides the assumptions on $V(x)$ and $W(x)$, under certain conditions on $F(s)$, (1.1) posses a unique solution $u \in X$, where

$$X = \{w \in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |\nabla h(|w|^2)|^2 dx < +\infty\}. \tag{1.2}$$

The asymptotic behavior and scattering phenomenon are very important and interesting topics on the study of nonlinear Schödinger equation. Pseudoconformal conservation law is essential for the study of the asymptotic behavior for the solution, Morawetz estimate is an important tool to construct scattering operator on the energy space, see [4, 12, 14, 20, 21, 22, 23, 37, 38, 39].

However, two more interesting questions are as follows: 1. What is the relation between pseudoconformal conservation law and Morawetz estimate? 2. How to establish the link between pseudoconformal conservation law and spacetime estimate?

The first motivation of this paper is to obtain the answers of the two questions above. To do this, we will establish Morawetz type estimates and weighted spacetime bounds based on pseudoconformal conservation law, which reveals the relation among pseudoconformal conservation law, Morawetz type estimates and spacetime bounds. These results are also very interesting discover in the study of quasilinear Schödinger equation in the following sense: To our best knowledge, there are few results on Morawetz type estimates and weighted spacetime bounds for the solution of (1.1) which contains potential and more general nonlinearities.

The second motivation of this paper is to show some applications of spacetime estimates for the global solution. To do this, one thing is to consider the asymptotic behavior for the solution of (1.1) as $t \to +\infty$, another one is to establish scattering theory for (1.1) in the case of (WV1), i.e.,

$$\begin{cases} iu_t = \Delta u + V(x)u + F(|u|^2)u + (W * |u|^2)u, \ x \in \mathbb{R}^N, \ t > 0, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

Many authors obtained scattering results on (1.3) when at least one of $V(x) \equiv 0$, $F(|u|^2)u \equiv 0$ and $W(x) \equiv 0$ holds. We can refer to [2, 7, 8, 10, 12, 14, 15, 16, 17, 18, 19, 20, 22, 25, 29, 30, 35, 38, 45, 46, 47, 48, 49, 50, 51] and the references therein. Especially, in Chapter 7 of the book [12], Cazenave introduced systematically the scattering results on the Cauchy problem of $iu_t = \Delta u + |u|^\alpha u$. There are also many scattering results on the Cauchy problem of Schröodinger equation containing either power type potential or Hartree nonlinearity, see [1, 9, 11, 26, 28, 32]. However, to our best knowledge, there are few scattering results on the following special case of (1.3), i.e., containing both power type potential and Hartree nonlinearity,

$$\begin{cases} iu_t = \Delta u - \frac{a}{|x|^m}u - b|u|^{2\beta}u - (\frac{c}{|x|^n} * |u|^2)u, \quad x \in \mathbb{R}^N \setminus \{\mathbf{0}\}, \ t > 0 \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

$a \neq 0$, $b \neq 0$ and $c \neq 0$, let alone in the general case of containing both $V(x) \not\equiv 0$, $F(|u|^2)u \not\equiv 0$ and $W(x) \not\equiv 0$.

Since we will establish several theorems, we will state and prove them in the corresponding sections and don't state the precise expressions of them here. However, we would like to say something about them roughly below.

1. About the conditions on global existence of solution to (1.1) in the case of (WV2), if $F(s) = F_1(s) - F_2(s)$ in the combined defocusing and focusing case, $F_1(s) \geq 0$ and $F_2(s) \geq 0$ for $s \geq 0$, $G_1(s) = \int_0^s F_1(\eta)d\eta$, then a criterion is to find

$$0 < \gamma < 1, \quad \gamma' > 1 \quad \text{satisfying} \quad \frac{2^*(1-\gamma)}{2(\gamma'-\gamma)} \leq 1$$

such that

$$[|G_1(s)|]^\gamma \leq c_1 s, \quad [|G_1(s)|]^{\gamma'} \leq c_1'[s^{\frac{1}{2}} + h(s)]^{2^*} \quad \text{when} \quad h(s) \neq 0.$$

2. We will establish pseuduconformal conservation law, which is essential for the study of the asymptotic behavior for the global solution of (1.1). Based on it, we give Morawetz type

estimates, which reveals the relation between pseoduconformal conservation law and Morawetz type estimate.

3. About the decay rate of the solution to (1.1), we obtain

$$\int_{\mathbb{R}^N}[|\nabla h(|u|^2)|^2+|V(x)||u|^2+|G_1(|u|^2)|+|G_2(|u|^2)+\frac{1}{2}(|W|*|u|^2)|u|^2]dx \le \frac{C}{t^\iota}$$

for some $0<\iota\le 2$ and asymptotic behavior

$$|\int_{\mathbb{R}^N}|\nabla u(x,t)|^2dx-2E(u_0)|\le\frac{C}{t^\iota},\quad \lim_{t\to+\infty}\int_{\mathbb{R}^N}|\nabla u|^2dx=2E(u_0).$$

under certain conditions.

4. Under certain assumptions, we establish Morawetz type estimates such as

$$\int_0^{+\infty}\int_{\mathbb{R}^N}\frac{\left[|\nabla h(|u|^2)|^2+|V(x)||u|^2+|G_1(|u|^2)|+|G_2(|u|^2)|+\frac{1}{2}(|W|*|u|^2)|u|^2\right]^\theta}{a(x,t)}dxdt$$
$$\le M_1(u_0,\theta),$$

and weighted spacetime bounds such as

$$\|G_1(|u|^2)\|_{L^q_w(\mathbb{R}^+)L^r_w(\mathbb{R}^N)}=\left(\int_0^{+\infty}\left(\int_{\mathbb{R}^N}w(x,t)[|G_1(|u|^2)|]^rdx\right)^{\frac{q}{r}}dt\right)^{\frac{1}{q}}\le C(u_0,r,q).$$

5. Under certain assumptions, we establish classic scattering theory for (1.3) with general $V(x)$, $F(|u|^2)$ and $W(x)$,

$$\|e^{it\Delta}u(t)-u_+\|_{L^2}\to 0\quad\text{as}\quad t\to+\infty.$$

Especially, if $V(x)\equiv 0$ and $W(x)\equiv 0$, we can obtain the scattering result on (1.3) with general $F(|u|^2)$,

$$\|e^{it\Delta}u(t)-u_+\|_{\Sigma}\to 0\quad\text{as}\quad t\to+\infty.$$

To establish the scattering results, in the course of proof, we factitiously let a continuous function be the sum of two piecewise functions and chose different admissible pairs in Strichartz estimates for the two terms. For example, let $\frac{1}{|x|^m}=V_1(x)+V_2(x)$, where

$$V_1(x)=\begin{cases}\frac{1}{|x|^m}, & 0<|x|\le 1,\\ 0, & |x|>1,\end{cases}\qquad\text{and}\quad V_2(x)=\begin{cases}0, & 0<|x|\le 1,\\ \frac{1}{|x|^m}, & |x|>1.\end{cases}$$

Then we have

$$\begin{aligned}&\|\int_t^\tau e^{is\Delta}[\frac{1}{|x|^m}u(s)]ds\|_{L^2}=\|\int_t^\tau e^{is\Delta}[V_1(x)u(s)+V_2(x)u(s)]ds\|_{L^2}\\&\le\|\int_t^\tau e^{is\Delta}V_1(x)u(s)ds\|_{L^2}+\|\int_t^\tau e^{is\Delta}V_2(x)u(s)ds\|_{L^2}\\&\le C\sum_{j=1}^2\left(\int_t^\tau\left(\int_{\mathbb{R}^N}|V_j(x)u|^{r_j'}dx\right)^{\frac{q_j'}{r_j'}}dt\right)^{\frac{1}{q_j'}}.\end{aligned}\tag{1.5}$$

Here (q_j, r_j), $j = 1, 2$, are admissible pairs, q_j' and r_j' are the conjugated exponents of q_j and r_j respectively.

The organization of this paper is as follows. In Section 2, we will give the local well-posedness result, prove mass and energy conservation laws, obtain some sufficient conditions on the global existence of the solution to (1.1) and establish pseudoconformal conservation law. In Section 3, we will give Morawetz type estimates based on pseudoconformal conservation law. In Section 4, we consider spacetime bound estimates for the solution. In Section 5, we will establish scattering theory for (1.3) as the applications of these estimates.

2 Local well-posedness, global existence and pseudoconformal conservation law for the solution of (1.1)

In convenience, we will use C, C', and so on, to denote some constants in the sequels, the values of it may vary line to line.

First, we state the local well-posedness result below.

Theorem A. *1. Assume that $h(s) = 0$, $V(x)$ and $W(x)$ satisfy (WV1), $u_0 \in H^1(\mathbb{R}^N)$, $F(s) = F_1(s) + ... + F_m(s)$ when $N \geq 1$, or $F(s) = F_1(s) + ... + F_m(s) - As^{\frac{2^*}{2}-1}$ with $A > 0$ when $N \geq 3$ where each $F_j(s)$ is continuous in s and for every $K_j > 0$, there exists $L(K_j) < +\infty$ such that*

$$|F_j(|u|^2)u - F_j(|v|^2)v| \leq L(K_j)|u - v|$$

for all $|u|, |v| \leq K_j$, $j = 1, 2..., m$. Furthermore,

$$\begin{cases} L(K_j) \in C([0, +\infty)) & \text{if} \quad N = 1 \\ \\ L(K_j) \leq C(1 + K_j^{\alpha_j}) \quad \text{with } 0 \leq \alpha_j < \frac{4}{N-2} & \text{if} \quad N \geq 2. \end{cases} \tag{2.1}$$

Then there exist a unique, strong H^1-solution u of (1.1) defined in a maximal interval $(0, T_{\max})$.

2. Suppose that $V(x)$ and $W(x)$ satisfy (WV2). For any $K \in Z^+$, $\partial^K W(x) \in L^1(\mathbb{R}^N)$, $F(s), h(s) \in \mathfrak{B}^\infty([0, M], \mathbb{R})$ for any $M > 0$ and $u_0 \in H^\infty(\mathbb{R}^N)$, $N \geq 1$.

Then there exist $a > 0$ and a unique solution $u \in C^1([0, a], H^\infty(\mathbb{R}^N)$ of (1.1).

Proof: 1. By the classic results on semilinear Schrödinger equation(see Theorem 3.3.1 in [12]), the conclusion in Case (WV1) is true.

2. Case (WV2). By the results of Theorem 1.1 in [13], Theorem A in [27]) and Theorem 6.4 in [40]), we only need to verify that for any $J \in Z^+$ and $M > 0$, $(W * |u|^2) \in C_b^J(\mathbb{R}^N \times \overline{B_M(0)})$, where $\overline{B_M(0)} = \{z \in C : |z| \leq M\}$, while $C_b^J(\mathbb{R}^N \times \overline{B_M(0)})$ is the space of all functions $a(x, u)$ such that all of the k-order($k = 0, 1, ..., J$) partial derivatives are bounded on $\mathbb{R}^N \times \overline{B_M(0)}$. In fact, since $\partial^K W(x) \in L^1(\mathbb{R}^N)$, denoting $(W * |u|^2) = a(x, u)$, it is easy to verify that

$$(W * |u|^2) = \int_{\mathbb{R}^N} W(x - y)|u(y)|^2 dy \in C_b^J(\mathbb{R}^N \times \overline{B_M(0)}),$$
$$(\partial^K W * |u|^2) = \int_{\mathbb{R}^N} \partial^K W(x - y)|u(y)|^2 dy \in C_b^J(\mathbb{R}^N \times \overline{B_M(0)})$$

for any $J \in Z^+$ and $M > 0$. Therefore there exist $a > 0$ and a unique solution $u \in C^1([0, a], H^\infty)$ of (1.1). $\square$

We prove a lemma as follows.

Lemma 2.1. *Assume that u is the solution of (1.1). Then in the time interval* $[0,t]$ *when it exists, u satisfies*

(i) Mass conservation:

$$m(u)=\left(\int_{\mathbb{R}^N}|u(x,t)|^2dx\right)^{\frac{1}{2}}=\left(\int_{\mathbb{R}^N}|u_0(x)|^2dx\right)^{\frac{1}{2}}=m(u_0);$$

(ii) Energy conservation:

$$\begin{aligned}E(u)=&\frac{1}{2}\int_{\mathbb{R}^N}[|\nabla u|^2+|\nabla h(|u|^2)|^2]dx-\frac{1}{2}\int_{\mathbb{R}^N}V(x)|u|^2dx\\&-\frac{1}{2}\int_{\mathbb{R}^N}G(|u|^2)dx-\frac{1}{4}\int_{\mathbb{R}^N}(W*|u|^2)|u|^2dx=E(u_0);\end{aligned}\tag{2.2}$$

(iii)

$$\frac{d}{dt}\int_{\mathbb{R}^N}|x|^2|u|^2dx=-4\Im\int_{\mathbb{R}^N}\bar{u}(x\cdot\nabla u)dx;$$

(iv)

$$\begin{aligned}&\frac{d}{dt}\Im\int_{\mathbb{R}^N}\bar{u}(x\cdot\nabla u)dx\\=&-2\int_{\mathbb{R}^N}|\nabla u|^2dx-(N+2)\int_{\mathbb{R}^N}|\nabla h(|u|^2)|^2dx\\&-8N\int_{\mathbb{R}^N}h''(|u|^2)h'(|u|^2)|u|^4|\nabla u|^2dx-\int_{\mathbb{R}^N}(x\cdot\nabla V)|u|^2dx\\&+N\int_{\mathbb{R}^N}[|u|^2F(|u|^2)-G(|u|^2)]dx-\int_{\mathbb{R}^N}[\frac{(x\cdot\nabla W)}{2}*|u|^2]|u|^2dx.\end{aligned}\tag{2.3}$$

Proof: (i) Multiplying (1.1) by $2\bar{u}$, taking the imaginary part of the result, we get

$$\frac{\partial}{\partial t}|u|^2=\Im(2\bar{u}\Delta u)=\nabla\cdot(2\Im\bar{u}\nabla u).\tag{2.4}$$

Integrating (2.4) over $\mathbb{R}^N\times[0,t]$, we have

$$\int_{\mathbb{R}^N}|u|^2dx=\int_{\mathbb{R}^N}|u_0|^2dx,$$

which implies mass conservation law.

(ii) Multiplying (1.1) by $2\bar{u}_t$, taking the real part of the result, then integrating it over $\mathbb{R}^N\times[0,t]$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}[|\nabla u|^2+|\nabla h(|u|^2)|^2]dx-\int_{\mathbb{R}^N}V(x)|u|^2dx\\&-\int_{\mathbb{R}^N}G(|u|^2)dx-\frac{1}{2}\int_{\mathbb{R}^N}(W*|u|^2)|u|^2dx\\=&\int_{\mathbb{R}^N}[|\nabla u_0|^2+|\nabla h(|u_0|^2)|^2]dx-\int_{\mathbb{R}^N}V(x)|u_0|^2dx\\&-\int_{\mathbb{R}^N}G(|u_0|^2)dx-\frac{1}{2}\int_{\mathbb{R}^N}(W*|u_0|^2)|u|^2dx,\end{aligned}$$

which implies energy conservation law.

(iii) Multiplying (2.4) by $|x|^2$ and integrating it over $\mathbb{R}^N$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^N}|x|^2|u|^2dx=\int_{\mathbb{R}^N}|x|^2\nabla\cdot(2\Im(\bar{u}\nabla u))dx=-4\Im\int_{\mathbb{R}^N}\bar{u}(x\cdot\nabla u)dx.$$

(iv) Let $a(x,t)=\Re u(x,t)$ and $b(x,t)=\Im u(x,t)$. Then

$$\begin{aligned}\frac{d}{dt}\Im\int_{\mathbb{R}^N}\bar{u}(x\cdot\nabla u)dx &= \int_{\mathbb{R}^N}\sum_{k=1}^{N}[x_k(b_t)_{x_k}a-x_k(a_t)_{x_k}b]dx+\int_{\mathbb{R}^N}\sum_{k=1}^{N}(x_kb_{x_k}a_t-x_ka_{x_k}b_t)dx\\
&=-2\int_{\mathbb{R}^N}|\nabla u|^2dx-(N+2)\int_{\mathbb{R}^N}|\nabla h(|u|^2)|^2dx\\
&\quad-8N\int_{\mathbb{R}^N}h'(|u|^2)h''(|u|^2)|u|^4|\nabla u|^2dx-\int_{\mathbb{R}^N}(x\cdot\nabla V)|u|^2dx\\
&\quad+N\int_{\mathbb{R}^N}[|u|^2F(|u|^2)-G(|u|^2)]dx-\int_{\mathbb{R}^N}[\frac{(x\cdot\nabla W)}{2}*|u|^2]|u|^2dx.\end{aligned}$$

Lemma 2.1 is proved. $\square$

Next, we establish some sufficient conditions on the global existence of the solution.

Theorem 1. *Let $u(x,t)$ be the solution of (1.1) with $u_0\in X$. Assume that $V(x)\le 0$, $W(x)\le 0$ for $x\in\mathbb{R}^N$, and satisfy (WV1) or (WV2), $F(s)$ and $h(s)$ satisfy the assumptions of Theorem A. Then $u(x,t)$ is a global solution in one of the following cases:*

Case 1. Defocusing case, i.e., $F(s)=F_2(s)\le 0$ for $s\ge 0$, $N\ge 1$, and the initial data u_0 satisfies $0<M(u_0)<+\infty$ and $0\le E(u_0)<+\infty$;

Case 2. $h(s)\ne 0$, $F(s)=F_1(s)-F_2(s)$ in the combined defocusing and focusing case, $F_1(s)\ge 0$ and $F_2(s)\ge 0$ for $s\ge 0$, $N\ge 3$, and there exist $c_1,c_1',c_2,c_2'>0$, $0<\gamma_1,\tilde{\gamma}_1<1$ and $\gamma_2,\tilde{\gamma}_2>1$ such that

$$\frac{2^*(1-\gamma_1)}{2(\gamma_2-\gamma_1)}\le 1,\quad \frac{2^*(1-\tilde{\gamma}_1)}{2(\tilde{\gamma}_2-\tilde{\gamma}_1)}\le 1, \tag{2.5}$$

$$[|G_1(s)|]^{\gamma_1}\le c_1s,\quad [|G_1(s)|]^{\gamma_2}\le c_1'[s^{\frac{1}{2}}+h(s)]^{2^*}\ \text{ for } 0\le s\le 1, \tag{2.6}$$

$$[|G_1(s)|]^{\tilde{\gamma}_1}\le c_2s,\quad [|G_1(s)|]^{\tilde{\gamma}_2}\le c_2'[s^{\frac{1}{2}}+h(s)]^{2^*}\ \text{ for } s>1, \tag{2.7}$$

besides $0\le E(u_0)<+\infty$, the initial data u_0 satisfies

$$\frac{2^*(1-\gamma_1)}{2(\gamma_2-\gamma_1)}=1,\quad \frac{2^*(1-\tilde{\gamma}_1)}{2(\tilde{\gamma}_2-\tilde{\gamma}_1)}<1,\quad (c_1\|u_0\|_{L^2}^2)^{\frac{2}{N}}(2^{2^*-1}c_1'C_s)^{\frac{N-2}{N}}<\frac{1}{4},$$

or

$$\frac{2^*(1-\gamma_1)}{2(\gamma_2-\gamma_1)}<1,\quad \frac{2^*(1-\tilde{\gamma}_1)}{2(\tilde{\gamma}_2-\tilde{\gamma}_1)}=1,\quad (c_2\|u_0\|_{L^2}^2)^{\frac{2}{N}}(2^{2^*-1}c_2'C_s)^{\frac{N-2}{N}}<\frac{1}{4},$$

or

$$\frac{2^*(1-\gamma_1)}{2(\gamma_2-\gamma_1)}=1,\quad \frac{2^*(1-\tilde{\gamma}_1)}{2(\tilde{\gamma}_2-\tilde{\gamma}_1)}=1,\quad \sum_{j=1}^{2}(c_j\|u_0\|_{L^2}^2)^{\frac{2}{N}}(2^{2^*-1}c_j'C_s)^{\frac{N-2}{N}}<\frac{1}{4}.$$

Here $G_i(s)=\int_0^sF_i(\eta)d\eta(i=1,2)$ and $G(s)=G_1(s)-G_2(s)$, C_s denotes the best constant in the Sobolev's inequality

$$\int_{\mathbb{R}^N}w^{2^*}dx\le C_s\left(\int_{\mathbb{R}^N}|\nabla w|^2dx\right)^{\frac{2^*}{2}}\quad \text{for any } w\in H^1(\mathbb{R}^{\mathbb{N}}). \tag{2.8}$$

Remark 2.1. Before we give the proof of this theorem, we would like to point out that there exist functions satisfy the assumptions of this theorem. For example, if $h(s) = s^p$, $f_1(s) = s^p$ and $p > \frac{1}{2^*-1}$, then we can take $\gamma_1 = \frac{1}{p+1}$, $\gamma_2 = \frac{2^*p}{p+1}$. If $h(s) = s^p$, $f_1(s) = s^p + s^q$ and $p > \frac{1}{2^*-1}$, $p < q < 2^*p - 1$, then we can take γ_1, γ_2 as above, while $\tilde{\gamma}_1 = \frac{1}{q+1}$ and $\tilde{\gamma}_2 = \frac{2^*p}{q+1}$.

The proof of Theorem 1:

Case 1. By energy conservation law and the assumptions on $V(x)$, $W(x)$, we have

$$\begin{aligned} E(u) = &\frac{1}{2}\int_{\mathbb{R}^N}[|\nabla u|^2 + |\nabla h(|u|^2)|^2]dx + \frac{1}{2}\int_{\mathbb{R}^N}|V(x)||u|^2dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^N}|G(|u|^2)|dx + \frac{1}{4}\int_{\mathbb{R}^N}(|W| * |u|^2)|u|^2dx = E(u_0) < +\infty. \end{aligned} \tag{2.9}$$

Case 2. Note the fact

$$\int_{\mathbb{R}^N}|G_1(|u|^2)|dx \le \sum_{j=1}^{2}(c_j\|u_0\|_{L^2}^2)^{\frac{1}{\tilde{\tau}'_j}}(2^{2^*-1}c'_jC_s)^{\frac{1}{\tilde{\tau}_j}}\left(\int_{\mathbb{R}^N}[|\nabla u|^2 + |\nabla h(|u|^2)|^2dx\right)^{\frac{2^*}{2\tilde{\tau}_j}}. \tag{2.10}$$

Here

$$\frac{1}{\tilde{\tau}_1} = \frac{1-\gamma_1}{\gamma_2-\gamma_1}, \quad \frac{1}{\tilde{\tau}'_1} = \frac{\gamma_2-1}{\gamma_2-\gamma_1}, \qquad \frac{1}{\tilde{\tau}_2} = \frac{1-\tilde{\gamma}_1}{\tilde{\gamma}_2-\tilde{\gamma}_1}, \quad \frac{1}{\tilde{\tau}'_2} = \frac{\tilde{\gamma}_2-1}{\tilde{\gamma}_2-\tilde{\gamma}_1}. \tag{2.11}$$

If

$$\frac{2^*(1-\gamma_1)}{2(\gamma_2-\gamma_1)} = 1, \quad \frac{2^*(1-\tilde{\gamma}_1)}{2(\tilde{\gamma}_2-\tilde{\gamma}_1)} < 1, \quad (c_1\|u_0\|_{L^2}^2)^{\frac{2}{N}}(2^{2^*-1}c'_1C_s)^{\frac{N-2}{N}} < \frac{1}{4},$$

applying Young inequality to (2.10), we obtain

$$\int_{\mathbb{R}^N}|G_1(|u|^2)|dx \le C + \frac{1}{4}\int_{\mathbb{R}^N}[|\nabla u|^2 + |\nabla h(|u|^2)|^2]dx. \tag{2.12}$$

Similarly, if

$$\frac{2^*(1-\gamma_1)}{2(\gamma_2-\gamma_1)} < 1, \quad \frac{2^*(1-\tilde{\gamma}_1)}{2(\tilde{\gamma}_2-\tilde{\gamma}_1)} = 1, \quad (c_2\|u_0\|_{L^2}^2)^{\frac{2}{N}}(2^{2^*-1}c'_2C_s)^{\frac{N-2}{N}} < \frac{1}{4}$$

applying Young inequality to (2.10), we get

$$\int_{\mathbb{R}^N}|G(|u|^2)|dx \le C + \frac{1}{4}\int_{\mathbb{R}^N}[|\nabla u|^2 + |\nabla h(|u|^2)|^2]dx. \tag{2.13}$$

If

$$\frac{2^*(1-\gamma_1)}{2(\gamma_2-\gamma_1)} = 1, \quad \frac{2^*(1-\tilde{\gamma}_1)}{2(\tilde{\gamma}_2-\tilde{\gamma}_1)} = 1, \quad \sum_{j=1}^{2}(c_j\|u_0\|_{L^2}^2)^{\frac{2}{N}}(2^{2^*-1}c'_jC_s)^{\frac{N-2}{N}} < \frac{1}{4},$$

(2.10) becomes

$$\begin{aligned} \int_{\mathbb{R}^N}|G_1(|u|^2)|dx &\le \sum_{j=1}^{2}(c_j\|u_0\|_{L^2}^2)^{\frac{1}{\tilde{\tau}'_j}}(c'_jC_s)^{\frac{1}{\tilde{\tau}_j}}\int_{\mathbb{R}^N}[\|\nabla u|^2 + |\nabla h(|u|^2)|^2]dx \\ &< \frac{1}{4}\int_{\mathbb{R}^N}[|\nabla u|^2 + |\nabla h(|u|^2)|^2dx. \end{aligned} \tag{2.14}$$

Noticing (2.12)–(2.14), in any case, we have

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^N}[|\nabla u|^2+|\nabla h(|u|^2)|^2+|V(x)||u|^2+|G_2(|u|^2)|]dx+\frac{1}{4}\int_{\mathbb{R}^N}(|W|*|u|^2)|u|^2dx\\ &=E(u_0)+\frac{1}{2}\int_{\mathbb{R}^N}|G_1(|u|^2)|dx\le C+\frac{1}{4}\int_{\mathbb{R}^N}[|\nabla u|^2+|\nabla h(|u|^2)|^2]dx, \qquad (2.15)\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^N}|\nabla u|^2dx+\int_{\mathbb{R}^N}|\nabla h(|u|^2)|^2dx\le C.$$

Theorem 1 is proved. □

Now we state pseudo-conformal conservation law as follows.

Theorem 2. (**Pseudoconformal conservation law**) *Let $u(x,t)$ be the global solution of (1.1), $V(x)\le 0$, $W(x)\le 0$ for $x\in\mathbb{R}^N$, and satisfy (WV1) or (WV2), $F(s)$ and $h(s)$ satisfy the assumptions of Theorem A, $u_0\in X$ and $xu_0\in L^2(\mathbb{R}^N)$. Then*

$$\begin{aligned}P(t)&=\int_{\mathbb{R}^N}|(x-2it\nabla)u|^2dx+4t^2\int_{\mathbb{R}^N}|\nabla h(|u|^2)|^2dx-4t^2\int_{\mathbb{R}^N}G(|u|^2)dx\\ &\quad-4t^2\int_{\mathbb{R}^N}V(x)|u|^2dx-2t^2\int_{\mathbb{R}^N}(W*|u|^2)|u|^2dx\\ &=\int_{\mathbb{R}^N}|xu_0|^2dx+4\int_0^t\tau\theta(\tau)d\tau. \qquad (2.16)\end{aligned}$$

Here

$$\begin{aligned}\theta(t)=&\int_{\mathbb{R}^N}-4N[2h''(|u|^2)h'(|u|^2)|u|^2+(h'(|u|^2))^2]|u|^2|\nabla u|^2dx\\ &-\int_{\mathbb{R}^N}[(N+2)G(|u|^2)-NF(|u|^2)|u|^2]dx\\ &-\int_{\mathbb{R}^N}[2V+(x\cdot\nabla V)]|u|^2dx-\int_{\mathbb{R}^N}\left([W+\frac{(x\cdot\nabla W)}{2}]*|u|^2\right)|u|^2dx. \qquad (2.17)\end{aligned}$$

Proof of Theorem 2: Assume that u is the solution of (1.1), $u_0\in X$ and $xu_0\in L^2(\mathbb{R}^N)$. Using energy conservation law, we get

$$P(t)=\int_{\mathbb{R}^N}|xu|^2dx+4t\Im\int_{\mathbb{R}^N}\bar{u}(x\cdot\nabla u)dx+8t^2E(u_0). \qquad (2.18)$$

Recalling that

$$\frac{d}{dt}\int_{\mathbb{R}^N}|x|^2|u|^2dx=-4\Im\int_{\mathbb{R}^N}\bar{u}(x\cdot\nabla u)dx,$$

using (2.18), we get

$$
\begin{aligned}
P'(t) &= \frac{d}{dt}\int_{\mathbb{R}^N} |xu|^2 dx + 4\Im\int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx + 4t\frac{d}{dt}\Im\int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx + 16tE(u_0)\\
&= 4t\frac{d}{dt}\Im\int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx + 16tE(u_0)\\
&= 4t\int_{\mathbb{R}^N} -4N[2h''(|u|^2)h'(|u|^2)|u|^2 + (h'(|u|^2))^2]|u|^2|\nabla u|^2 dx\\
&\quad + 4t\int_{\mathbb{R}^N} [N|u|^2F(|u|^2) - (N+2)G(|u|^2)]dx - 4t\int_{\mathbb{R}^N} [2V + (x\cdot\nabla V)]|u|^2 dx\\
&\quad - 4t\int_{\mathbb{R}^N} \left([W + \frac{(x\cdot\nabla W)}{2}] * |u|^2\right)|u|^2 dx. \qquad (2.19)
\end{aligned}
$$

Integrating (2.19) from 0 to t, we obtain (2.16). $\square$

3 Morawetz type estimates based on pseudoconformal conservation law

In this section, we will establish Morawetz type estimates based on pseudoconformal conservation law.

Theorem 3. *Let $u(x,t)$ be the global solution of (1.1) in energy space X, $u_0 \in X$ and $xu_0 \in L^2(\mathbb{R}^N)$. Assume that $V(x) \le 0$, $W(x) \le 0$ for $x \in \mathbb{R}^N$, and satisfy (WV1) or (WV2), $F(s)$ and $h(s)$ satisfy the assumptions of Theorem A, the space dimension $N \ge 1$ in defocusing case, $N \ge 3$ in combined defocusing and focusing case, $0 < M(u_0) < +\infty$ and $0 \le E(u_0) < +\infty$. In addition, suppose that $h(s) \ge 0, \not\equiv 0$, $F(s) = F_1(s) - F_2(s)$ in the combined defocusing and focusing case, $F_1(s) \ge 0$ and $F_2(s) \ge 0$ for $s \ge 0$, and there exist $c_3, c_3', c_4, c_4' > 0$, $0 < \gamma_3, \tilde{\gamma}_3 < 1$ and $\gamma_4, \tilde{\gamma}_4 > 1$ such that*

$$
\frac{2^*(1-\gamma_3)}{2(\gamma_4-\gamma_3)} = 1, \quad \frac{2^*(1-\tilde{\gamma}_3)}{2(\tilde{\gamma}_4-\tilde{\gamma}_3)} = 1, \qquad (3.1)
$$

$$
C_r(u_0) := \sum_{j=3}^{4}(c_j\|u_0\|_{L^2}^2)^{\frac{2}{N}}(2^{2^*-1}c_j'C_s)^{\frac{N-2}{N}} < 1 \qquad (3.2)
$$

$$
[|G_1(s)|]^{\gamma_3} \le c_3 s, \quad [|G_1(s)|]^{\gamma_4} \le c_3'[h(s)]^{2^*} \text{ for } 0 \le s \le 1, \qquad (3.3)
$$

$$
[|G(s)|]^{\tilde{\gamma}_3} \le c_4 s, \quad [|G_2(s)|]^{\tilde{\gamma}_4} \le c_4'[h(s)]^{2^*} \text{ for } s > 1. \qquad (3.4)
$$

1. *Assume that $[2h''(s)h'(s)s + (h'(s))^2] \ge 0$, $[(N+2)G_1(s) - NF_1(s)s] \ge 0$ and $[NF_2(s)s - (N+2)G_2(s)] \ge 0$ for $s \ge 0$, $[2V + (x\cdot\nabla V)] \ge 0$ and $[2W + (x\cdot\nabla W)] \ge 0$ for $x \in \mathbb{R}^N$. Then*

Estimate (A):

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^N} \frac{\left[|\nabla h(|u|^2)|^2 + |V(x)||u|^2 + |G_1(|u|^2)| + |G_2(|u|^2)| + \frac{1}{2}(|W| * |u|^2)|u|^2\right]^\theta}{a_1(x,t)} dxdt\\
&\le M_1(u_0,\theta), \qquad (3.5)
\end{aligned}
$$

where the function $a_1(x,t)$ satisfies $a_1(x,t) \ge a(x) \ge 0$ for $x \in \mathbb{R}^N$ and $t \ge 0$, and the function $a(x)$ satisfies $\frac{1}{a(x)} \in L^{\frac{1}{1-\theta}}(\mathbb{R}^N)$, $\frac{1}{2} < \theta < 1$, $M_1(u_0,\theta)$ is a positive constant depending on u_0 and θ.

Estimate (B):

$$\int_0^\infty \int_{\mathbb{R}^N} \frac{t^2\left[|\nabla h(|u|^2)|^2 + |V(x)||u|^2 + |G_1(|u|^2)| + |G_2(|u|^2)| + \frac{1}{2}(|W| * |u|^2)|u|^2\right]}{a_2(x,t)} dxdt$$
$$\leq M_2(u_0, k), \tag{3.6}$$

where the function $a_2(x,t)$ satisfies $a_2(x,t) \geq b(x) + t^k$ for $x \in \mathbb{R}^N$ and $t \geq 0$, $1 < k < 3$ if the function $b(x)$ satisfies $b(x) \geq 0$, or $1 < k$ if the function $b(x)$ satisfies $b(x) \geq b > 0$; $M_2(u_0, k)$ is a positive constant depending on u_0 and k.

Especially, let $b(x) \equiv 0$, $k = 2$, then

Estimate (C):

$$\int_0^\infty \int_{\mathbb{R}^N} \left[|\nabla h(|u|^2)|^2 + |V(x)||u|^2 + |G_1(|u|^2)| + |G_2(|u|^2)| + \frac{1}{2}(|W| * |u|^2)|u|^2\right] dxdt$$
$$\leq M_3(u_0). \tag{3.7}$$

Here $M_3(u_0)$ is a positive constant depending on u_0.

2. *Assume that*

(i) $-k_1(h'(s))^2 \leq [2h''(s)h'(s)s + (h'(s))^2] \leq 0$ for some $k_1 > 0$;

(ii) $-k_2|G_1(s)| \leq (N+2)G_1(s) - NF_1(s)s \leq 0$ for some $k_2 > 0$;

(iii) $-k_3|G_2(s)| \leq NF_2(s)s - (N+2)G_2(s) \leq 0$ for some $k_3 > 0$;

(iv) $-k_4|V| \leq 2V + (x \cdot \nabla V) \leq 0$ for some $k_4 > 0$;

(v) $-k_5|W| \leq 2W + (x \cdot \nabla W) \leq 0$ for some $k_5 > 0$.

Let

$$l = \max(Nk_1, k_2, k_3, k_4, k_5). \tag{3.8}$$

Then

Estimate (D):

$$\int_0^\infty \int_{\mathbb{R}^N} \frac{t^2\left[|\nabla h(|u|^2)|^2 + |G_1(|u|^2)| + |G_2(|u|^2)| + |V(x)||u|^2 + \frac{1}{2}(|W| * |u|^2)|u|^2\right]}{a_3(x,t)} dxdt$$
$$\leq M_4(u_0, k, l) \tag{3.9}$$

Here the function $a_3(x,t)$ satisfies $a_3(x,t) \geq (c(x)+t)^k$ for $x \in \mathbb{R}^N$ and $t \geq 0$, $l+1 < k < 3$ if $l < 2$ in defocusing case, $k > 1 + \frac{l[1+C_r(u_0)]}{1-C_r(u_0)}$ in combined defocusing and focusing case, if the function $c(x) \geq 0$. While $l+1 < k$ in defocusing case, $k > 1 + \frac{l[1+C_r(u_0)]}{1-C_r(u_0)}$ in combinied defocusing and focusing case, if the function $c(x) \geq c > 0$. $M_4(u_0, k, l)$ is a positive constant depending on u_0, k and l.

Especially, if $c(x) \equiv 0$, $l < 1$ and $k = 2$, then

Estimate (E):

$$\int_0^\infty \int_{\mathbb{R}^N} \left[|\nabla h(|u|^2)|^2 + |G_1(|u|^2)| + |G_2(|u|^2)| + |V(x)||u|^2 + \frac{1}{2}(|W| * |u|^2)|u|^2\right] dxdt$$
$$\leq M_5(u_0, l). \tag{3.10}$$

Here $M_5(u_0, l)$ is a positive constant depending on u_0 and l.

We divide this section into two subsection according to Case 1 and Case 2.

3.1 The proof of Theorem 3 in Case 1

In this subsection, we prove Theorem 3 in Case 1.

The proof of Theorem 3 in Case 1: First, we give estimates for

$$\begin{aligned}\int_{\mathbb{R}^N}\Phi(V,u,W)dx :=& \int_{\mathbb{R}^N}[|\nabla h(|u|^2)|^2+|G_1(|u|^2)|+|G_2(|u|^2)|]dx\\ &+\int_{\mathbb{R}^N}[|V(x)||u|^2+\frac{1}{2}(|W|*|u|^2)|u|^2]dx \qquad (3.11)\end{aligned}$$

in two subcases.

Subcase (1). Defocusing case, $N\geq 1$. By energy conservation law, we get

$$\int_{\mathbb{R}^N}\Phi(V,u,W)dx\leq 2E(u_0)\text{ for }t\geq 0(\text{especially for }0\leq t\leq 1). \qquad (3.12)$$

Using (2.16) and (2.17), we have

$$4t^2\int_{\mathbb{R}^N}\Phi(V,u,W)dx\leq\int_{\mathbb{R}^N}|xu_0|^2dx,\ \int_{\mathbb{R}^N}\Phi(V,u,W)dx\leq\frac{C(u_0)}{4t^2}\quad\text{for }t\geq 1. \qquad (3.13)$$

Here

$$C(u_0)=\int_{\mathbb{R}^N}|xu_0|^2dx. \qquad (3.14)$$

Subcase (2). Combined defocusing and focusing case, $N\geq 3$.

By energy conservation law, we get

$$\begin{aligned}&[1-C_r(u_0)]\int_{\mathbb{R}^N}[|\nabla h(|u|^2)|^2+|V(x)||u|^2+|G_2(|u|^2)|+\frac{1}{2}(|W|*|u|^2)|u|^2]dx\\ &\leq\int_{\mathbb{R}^N}[|\nabla h(|u|^2)|^2-|G_1(|u|^2)|+|G_2(|u|^2)+|V(x)||u|^2+\frac{1}{2}(|W|*|u|^2)|u|^2]dx\\ &=2E(u_0),\end{aligned}$$

and

$$\int_{\mathbb{R}^N}[|\nabla h(|u|^2)|^2+|V(x)||u|^2+|G_2(|u|^2)|+\frac{1}{2}(|W|*|u|^2)|u|^2]dx\leq\frac{2E(u_0)}{[1-C_r(u_0)]}, \qquad (3.15)$$

consequently,

$$\int_{\mathbb{R}^N}\Phi(V,u,W)dx\leq\frac{2E(u_0)[1+C_r(u_0)]}{[1-C_r(u_0)]}\quad\text{for }t\geq 0\text{ (especially for }0\leq t\leq 1). \qquad (3.16)$$

Using (2.16) and (2.17), we obtain

$$\begin{aligned}&[1-C_r(u_0)]\left(4t^2\int_{\mathbb{R}^N}[|\nabla h(|u|^2)|^2+|V(x)||u|^2+|G_2(|u|^2)|]dx+2t^2\int_{\mathbb{R}^N}(|W|*|u|^2)|u|^2dx\right)\\ &\leq[1-C_r(u_0)]4t^2\int_{\mathbb{R}^N}|\nabla h(|u|^2)|^2dx+4t^2\int_{\mathbb{R}^N}[|V(x)||u|^2+|G_2(|u|^2)|]dx\\ &\quad+2t^2\int_{\mathbb{R}^N}(|W|*|u|^2)|u|^2dx\leq\int_{\mathbb{R}^N}|xu_0|^2dx,\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^N}[|\nabla h(|u|^2)|^2+|V(x)||u|^2+|G_2(|u|^2)|+\frac{1}{2}(|W|*|u|^2)|u|^2]dx$$
$$\leq\frac{C(u_0)}{4[1-C_r(u_0)]t^2}, \tag{3.17}$$

and consequently

$$\int_{\mathbb{R}^N}\Phi(V,u,W)dx\leq\frac{C(u_0)[1+C_r(u_0)]}{4[1-C_r(u_)]t^2}\quad\text{for}\quad t\geq 1. \tag{3.18}$$

Now Morawetz estimates can be proved below.

Estimate (A):

For any $\frac{1}{2}<\theta<1$, $a_1(x,t)\geq a(x)$ and $\frac{1}{a(x)}\in L^{\frac{1}{1-\theta}}(\mathbb{R}^N)$, using (3.12)–(3.18), we have

$$\int_0^\infty\int_{\mathbb{R}^N}\frac{[\Phi(V,u,W)]^\theta}{a_1(x,t)}dxdt\leq\int_0^1\int_{\mathbb{R}^N}\frac{[\Phi(V,u,W)]^\theta}{a(x)}dxdt$$
$$\leq\left[\int_0^1 Cdt+\int_1^\infty\frac{C'}{t^{2\theta}}dt\right]\left(\int_{\mathbb{R}^N}\frac{1}{[a(x)]^{\frac{1}{1-\theta}}}dx\right)^{1-\theta}\leq M_1(u_0,\theta). \tag{3.19}$$

Estimate (B):

If $b(x)\geq 0$, $1<k<3$, we get

$$\int_0^\infty\int_{\mathbb{R}^N}\frac{t^2\Phi(V,u,W)}{a_2(x,t)}dxdt\leq\int_0^\infty\int_{\mathbb{R}^N}\frac{t^2\Phi(V,u,W)}{b(x)+t^k}dxdt$$
$$\leq\int_0^1 t^{2-k}\int_{\mathbb{R}^N}\Phi(V,u,W)dxdt+\int_1^\infty\frac{1}{t^k}\int_{\mathbb{R}^N}t^2\Phi(V,u,W)dxdt\leq M_2(u_0,k). \tag{3.20}$$

If $b(x)\geq b>0$, $1<k$, we obtain

$$\int_0^\infty\int_{\mathbb{R}^N}\frac{t^2\Phi(V,u,W)}{a_2(x,t)}dxdt\leq\int_0^\infty\int_{\mathbb{R}^N}\frac{t^2\Phi(V,u,W)}{b(x)+t^k}dxdt$$
$$\leq\int_0^1\frac{Ct^2}{b}dt+\int_1^\infty\frac{C'}{t^k}dt\leq M_2'(u_0,k). \tag{3.21}$$

Especially, if $b(x)\equiv 0$ and $k=2$, we have

Estimate (C):

$$\int_0^\infty\int_{\mathbb{R}^N}\left[|\nabla h(|u|^2)|^2+|G(|u|^2)|+|V(x)||u|^2+\frac{1}{2}(|W|*|u|^2)|u|^2\right]dxdt$$
$$\leq M_3(u_0). \tag{3.22}$$

3.2 The proof of Theorem 3 in Case 2

In this subsection, we prove Theorem 3 in Case 2.

The proof of Theorem 3 in Case 2:

Estimate (D): We prove it in two subcases.

Subcase (i). Defocusing case, $N\geq 1$. By energy conservation law, we also have

$$\int_{\mathbb{R}^N}\Phi(V,u,W)dx\leq 2E(u_0)\quad\text{for } t\geq 0(\text{especially for } 0\leq t\leq 1).$$

Letting

$$A(t) = 4\int_0^t \tau \int_{\mathbb{R}^N} \Phi(V,u,W)dxd\tau, \tag{3.23}$$

using (2.16) and (2.17), we have

$$tA'(t) \le \int_{\mathbb{R}^N} |xu_0|^2 dx + lA(t) = C(u_0) + lA(t),$$

i.e.,

$$A'(t) \le \frac{l}{t}A(t) + \frac{C(u_0)}{t}. \tag{3.24}$$

Applying Gronwall inequality to (3.24), we get

$$A(t) \le e^{\int_1^t \frac{l}{\eta}d\eta}[A(1) + \int_1^t \frac{C(u_0)}{\eta} e^{-\int_1^\eta \frac{l}{\xi}d\xi} d\eta] \le [4E(u_0) + \frac{C(u_0)}{l}]t^l \tag{3.25}$$

for $t \ge 1$. (3.24) and (3.25) mean that

$$\int_{\mathbb{R}^N} \Phi(V,u,W)dx \le \frac{C(u_0)}{4t^2} + \frac{[4lE(u_0) + C(u_0)]}{4t^{2-l}} \quad \text{for } t \ge 1. \tag{3.26}$$

In defocusing case, for $c(x) \ge 0$, $l+1 < k < 3$ if $l < 2$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^N} \frac{t^2\Phi(V,u,W)}{a_3(x,t)}dxdt \le \int_0^\infty \int_{\mathbb{R}^N} \frac{t^2\Phi(V,u,W)}{(c(x)+t)^k}dxdt \le C. \tag{3.27}$$

Similarly, for $c(x) \ge c > 0$, $l+1 < k$, we have

$$\int_0^\infty \int_{\mathbb{R}^N} \frac{t^2\Phi(V,u,W)}{a_3(x,t)}dxdt \le \int_0^\infty \int_{\mathbb{R}^N} \frac{t^2\Phi(V,u,W)}{(c(x)+t)^k}dxdt \le C. \tag{3.28}$$

Subcase (ii). Combined defocusing and focusing case, $N \ge 3$. Recall that (3.16)

$$\int_{\mathbb{R}^N} \Phi(V,u,W)dx \le \frac{2E(u_0)[1 + C_r(u_0)]}{[1 - C_r(u_0)]}$$

for $t \ge 0$(especially for $0 < t \le 1$).

Using (2.16) and (2.17), we get

$$\begin{aligned}
&[1 - C_r(u_0)]4t^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx + 4t^2 \int_{\mathbb{R}^N} [|V(x)||u|^2 + |G_2(|u|^2)|]dx \\
&\quad + 2t^2 \int_{\mathbb{R}^N} (|W| * |u|^2)|u|^2 dx \\
&\le C(u_0) + 4l[1 + C_r(u_0)] \int_0^t \tau \int_{\mathbb{R}^N} [|\nabla h(|u|^2)|^2 dx + |V(x)||u|^2 + |G_2(|u|^2)| \\
&\quad + \frac{1}{2}(|W| * |u|^2)|u|^2] dxd\tau.
\end{aligned} \tag{3.29}$$

Letting

$$B(t) = 4\int_0^t \tau \int_{\mathbb{R}^N} [|\nabla h(|u|^2)|^2 dx + |V(x)||u|^2 + |G_2(|u|^2)| + \frac{1}{2}(|W| * |u|^2)|u|^2] dxd\tau,$$

we have from (3.29)

$$B'(t) \leq \frac{C(u_0)}{[1-C_r(u_0)]t} + \frac{l[1+C_r(u_0)]}{[1-C_r(u_0)]t}B(t). \tag{3.30}$$

Applying Gronwall inequality to (3.30), and using (3.15), we obtain

$$B(t) \leq [\frac{4lE(u_0)[1+C_r(u_0)] + C(u_0)[1-C_r(u_0)]}{l[1-C_r^2(u_0)]}]t^{\frac{l[1+C_r(u_0)]}{1-C_r(u_0)}},$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 + |V(x)||u|^2 + \frac{1}{2}(|W| * |u|^2)|u|^2]dx \\ &\leq \frac{C(u_0)}{4[1-C_r(u_0)]t^2} + \frac{4lE(u_0)[1+C_r(u_0)] + C(u_0)[1-C_r(u_0)]}{4[1-C_r(u_0)]^2 t^{2-\frac{l[1+C_r(u_0)]}{1-C_r(u_0)}}}\end{aligned} \tag{3.31}$$

for $t \geq 1$. Consequently,

$$\int_{\mathbb{R}^N} \Phi(V,u,W)dx \leq C\left(\frac{1}{t^2} + \frac{1}{t^{2-\frac{l[1+C_r(u_0)]}{1-C_r(u_0)}}}\right) \tag{3.32}$$

for $t \geq 1$.

Similar to (3.27), in combined defocusing and focusing case,

$$\int_0^\infty \int_{\mathbb{R}^N} \frac{t^2\Phi(V,u,W)}{a_3(x,t)}dxdt \leq \int_0^\infty \int_{\mathbb{R}^N} \frac{t^2\Phi(V,u,W)}{(c(x)+t)^k}dxdt \leq C \tag{3.33}$$

for $c(x) \geq 0$, $k > 1 + \frac{l[1+C_r(u_0)]}{1-C_r(u_0)}$. Combining (3.27) and (3.33), we have

$$\int_0^\infty \int_{\mathbb{R}^N} \frac{t^2\Phi(V,u,W)}{a_3(x,t)}dxdt \leq \int_0^\infty \int_{\mathbb{R}^N} \frac{t^2\Phi(V,u,W)}{(c(x)+t)^k}dxdt \leq C. \tag{3.34}$$

Similarly to (3.28), (3.33) in combined defocusing and focusing case,

$$\int_0^\infty \int_{\mathbb{R}^N} \frac{t^2\Phi(V,u,W)}{a_3(x,t)}dxdt \leq \int_0^\infty \int_{\mathbb{R}^N} \frac{t^2\Phi(V,u,W)}{(c(x)+t)^k}dxdt \leq C \tag{3.35}$$

for $c(x) \geq c > 0$, $k > 1 + \frac{l[1+C_r(u_0)]}{1-C_r(u_0)}$. Combining (3.28) and (3.35), we get

$$\int_0^\infty \int_{\mathbb{R}^N} \frac{t^2\Phi(V,u,W)}{a_3(x,t)}dxdt \leq \int_0^\infty \int_{\mathbb{R}^N} \frac{t^2\Phi(V,u,W)}{(c(x)+t)^k}dxdt \leq M_4(u_0,k,l). \tag{3.36}$$

Estimate (E):
Especially, if $c(x) \equiv 0$, $k = 2$, $l < \frac{1-C_r(u_0)}{1+C_r(u_0)}$, by the discussions above, we have

$$\int_0^\infty \int_{\mathbb{R}^N} \Phi(V,u,W)dxdt \leq M_5(u_0,l). \tag{3.37}$$

Remark 3.1. 1. The assumptions of Case 2 can be weaken as: Assume that at least one of (i)–(iv) holds. And the corresponding value of l can be take one of Nk_1, k_2, k_3 and $2k_4$.

2. By the proof of Theorem 3, in defocusing case, we obtain

$$\int_{\mathbb{R}^N} |(x-2it\nabla)u|^2dx \leq C \quad \text{in Case 1,} \qquad \int_{\mathbb{R}^N} |(x-2it\nabla)u|^2dx \leq Ct^l \quad \text{in Case 2.} \tag{3.38}$$

4 Spacetime bound estimates based on pseudoconformal conservation law

In this section, we will establish spacetime bound estimates based on pseudoconformal conservation law.

Theorem 4. *Let $u(x,t)$ be the solution of (1.1) in energy space X, $u_0 \in X$ and $xu_0 \in L^2(\mathbb{R}^N)$. Assume that $V(x) \le 0$, $W(x) \le 0$ for $x \in \mathbb{R}^N$, and satisfy (WV1) or (WV2), $F(s)$ and $h(s)$ satisfy the assumptions of Theorem A and Theorem 3, the space dimension $N \ge 1$ in defocusing case, $N \ge 3$ in combined defocusing and focusing case, $0 < M(u_0) < +\infty$ and $0 \le E(u_0) < +\infty$. Then*

Bound (F): *Weighted spacetime bound*

$$\left(\int_0^{+\infty} \left(\int_{\mathbb{R}^N} w(x,t)\,[\Phi(V,u,W)]^\theta\,dx\right)^p dt\right)^{\frac{1}{p}} \le C(u_0,p,\theta). \tag{4.1}$$

Here

$$\Phi(V,u,W) = |\nabla h(|u|^2)|^2 + |G_1(|u|^2)| + |G_2(|u|^2)| + |V(x)||u|^2 + \frac{1}{2}(|W| * |u|^2)|u|^2. \tag{4.2}$$

$0 < \theta \le 1$, $w(x,t)$ satisfies (w1) $0 \le w(x,t) \le c_w$ for all $x \in \mathbb{R}^N$ and $t \ge 0$ if $\theta = 1$, or (w2) $0 \le w(x,t)$ for all $x \in \mathbb{R}^N$ and $t \ge 0$, $\int_{\mathbb{R}^N} |w(x,t)|^{\frac{1}{1-\theta}}dx \le c_w'$ if $0 < \theta < 1$, $p > \frac{1}{2\theta}$ in defocusing case, and

$$p > \max\left(\frac{1}{2\theta}, \frac{[1 - C_r(u_0)]}{\theta[2(1 - C_r(u_0)) - l(1 + C_r(u_0))]}\right), \quad 0 < l < \frac{2[1 - C_r(u_0)]}{[1 + C_r(u_0)]}$$

in combined defocusing and focusing case.

Moreover, if $N \ge 3$, then

Bound (G): *Weighted spacetime norm*

$$\begin{aligned}\|G_1(|u|^2)\|_{L^q_w(\mathbb{R}^+)L^r_w(\mathbb{R}^N)} &= \left(\int_0^{+\infty} \left(\int_{\mathbb{R}^N} w(x,t)[|G_1(|u|^2)|]^r dx\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}} \\ &\le C(u_0,r,q,\gamma_1,\gamma_2,\tilde{\gamma}_1,\tilde{\gamma}_2).\end{aligned} \tag{4.3}$$

Here $1 \le r < \gamma_2$, $1 \le r < \tilde{\gamma}_2$, $w(x,t)$ satisfies (w1) $0 \le w(x,t) \le c_v$ for all $x \in \mathbb{R}^N$ and $t \ge 0$ if $1 \le r < \gamma_2$, $1 \le r < \tilde{\gamma}_2$, or (w2) $0 \le w(x,t)$ for all $x \in \mathbb{R}^N$ and $t \ge 0$, $\int_{\mathbb{R}^N} |w(x,t)|^{\frac{\delta}{\delta-1}}dx \le c_w'$ for some $1 < \delta < \frac{\gamma_2}{r} \le \gamma_2$ and $1 < \delta < \frac{\tilde{\gamma}_2}{r} \le \tilde{\gamma}_2$.

$$q > \frac{r\sigma(\gamma_2 - \gamma_1)}{2^*(r\sigma - \gamma_1)}, \quad q > \frac{r\sigma(\tilde{\gamma}_2 - \tilde{\gamma}_1)}{2^*(r\sigma - \tilde{\gamma}_1)}$$

for combined defocusing and focusing subcase of Case 1 in Theorem 3,

$$q > \frac{2r\sigma(\gamma_2 - \gamma_1)[1 - C_r(u_0)]}{2^*(r\sigma - \gamma_1)[2(1 - C_r(u_0)) - l(1 + C_r(u_0))]},$$

$$q > \frac{2r\sigma(\tilde{\gamma}_2 - \tilde{\gamma}_1)[1 - C_r(u_0)]}{2^*(r\sigma - \tilde{\gamma}_1)[2(1 - C_r(u_0)) - l(1 + C_r(u_0))]},$$

$0 < l < \frac{2[1-C_r(u_0)]}{[1+C_r(u_0)]}$ for combined defocusing and focusing subcase of Case 2 in Theorem 3, where $\sigma = 1$ if (w1) holds, while $\sigma = \delta$ if (w2) holds.

Proof of Theorem 4: Similar to (2.10), we get

$$\int_{\mathbb{R}^N} |G_1(|u|^2)|dx \le \sum_{j=1}^{2} (c_j\|u_0\|_{L^2}^2)^{\frac{1}{\tilde{\tau}_j'}} (2^{2^*-1}c_j'C_s)^{\frac{1}{\tilde{\tau}_j}} \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx$$

$$:= C_r(u_0) \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx. \tag{4.4}$$

if $N \ge 3$. $\tilde{\tau}_1$, $\tilde{\tau}_1'$, $\tilde{\tau}_2$ and $\tilde{\tau}_2'$ are the same as those in (2.11).

Bound (F): We will prove (4.1) in three cases. We only give the details in Case (I), the proofs in Case (II) and Case(III) are similar to that in Case (I).

Case (I). Defocusing subcase in Case 2 of Theorem 3. In this case,

$$\int_{\mathbb{R}^N} \Phi(V,u,W)dx \le 2E(u_0) \quad \text{for} \quad 0 \le t \le 1,$$
$$\int_{\mathbb{R}^N} \Phi(V,u,W)dx \le \frac{C(u_0)}{4t^2} + \frac{[4lE(u_0)+C(u_0)]}{4t^{2-l}} \quad \text{for} \quad t > 1.$$

We discuss it in two subcases.

Subcase (i). $0 \le w(x,t) \le c_w$ for all $x \in \mathbb{R}^N$ and $t \ge 0$ if $\theta = 1$. By (3.12) and (3.26), we obtain

$$\left(\int_0^{+\infty} \left(\int_{\mathbb{R}^N} w(x,t)\Phi(V,u,W)dx\right)^p dt\right)^{\frac{1}{p}}$$
$$\le c_w \left(\int_0^1 [2E(u_0)]^p dt + \int_1^{+\infty} \left(\frac{C(u_0)}{4t^2} + \frac{4lE(u_0)+C(u_0)}{4t^{(2-l)}}\right)^p dt\right)^{\frac{1}{p}} \le C. \tag{4.5}$$

Subcase (ii). $0 \le w(x,t)$ for all $x \in \mathbb{R}^N$ and $t \ge 0$, $\int_{\mathbb{R}^N} |w(x,t)^{\frac{1}{1-\theta}} dx < c_w'$ if $0 < \theta < 1$, we get

$$\left(\int_0^{+\infty} \left(\int_{\mathbb{R}^N} w(x,t)[\Phi(V,u,W)]^\theta dx\right)^p dt\right)^{\frac{1}{p}}$$
$$\le \left(\int_0^{+\infty} \left\{\left(\int_{\mathbb{R}^N} |w(x,t)^{\frac{1}{1-\theta}} dx\right)^{1-\vartheta} \left(\int_{\mathbb{R}^N} \Phi(V,u,W)dx\right)^\theta\right\}^p dt\right)^{\frac{1}{p}} \le C. \tag{4.6}$$

Case (II). Combined defocusing and focusing subcase in Case 2 of Theorem 3. In this case,

$$\int_{\mathbb{R}^N} \Phi(V,u,W)dx \le C, \quad 0 \le t \le 1, \tag{4.7}$$

$$\int_{\mathbb{R}^N} \Phi(V,u,W)dx \le C\left(\frac{1}{t^2} + \frac{1}{t^{2-\frac{l[1+C_r(u_0)]}{1-C_r(u_0)}}}\right), \quad t \ge 1. \tag{4.8}$$

Similarly, we get

$$\left(\int_0^{+\infty} \left(\int_{\mathbb{R}^N} w(x,t)[\Phi(V,u,W)]^\theta dx\right)^p dt\right)^{\frac{1}{p}} \le C. \tag{4.9}$$

Case (III). Case 1 of Theorem 3. In this case,

$$\int_{\mathbb{R}^N} \Phi(V,u,W)dx \le C \quad \text{for} \quad 0 \le t \le 1, \qquad \int_{\mathbb{R}^N} \Phi(V,u,W)dx \le \frac{C'}{t^2} \quad \text{for} \quad t \ge 1.$$

Similarly, we have

$$\left(\int_0^{+\infty}\left(\int_{\mathbb{R}^N} w(x,t)[\Phi(V,u,W)]^{\theta}dx\right)^p dt\right)^{\frac{1}{p}} \leq C. \tag{4.10}$$

Bound (G): We only give the details in Case 2 in combined defocusing and focusing case of Theorem 3, the proof in Case 1 in combined defocusing and focusing case of Theorem 3 is similar.

We also discuss it in two subcases.

Subcase (i). $0 \leq w(x,t) \leq c_w$ for any $x \in \mathbb{R}^N$ and $t \geq 0$, $1 \leq r < \min(\gamma_2, \tilde{\gamma}_2)$.

$$\begin{aligned}
&\left(\int_0^{+\infty}\left(\int_{\mathbb{R}^N} w(x,t)|G_1(|u|^2)|^r dx\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}} \\
&\leq (c_w)^{\frac{1}{r}} C\left\{\left(\int_0^1 [2E(u_0)]^{\frac{2^*q}{2r\tau_4}} dt\right)^{\frac{1}{q}} + \left(\int_1^{+\infty}\left(\frac{C}{4t^2} + \frac{C}{t^{2-\frac{l[1+C_r(u_0)]}{1-C_r(u_0)}}}\right)^{\frac{2^*q}{2r\tau_4}} dt\right)^{\frac{1}{q}}\right\} \\
&\quad + (c_w)^{\frac{1}{r}} C\left\{\left(\int_0^1 [2E(u_0)]^{\frac{2^*q}{2r\tilde{\tau}_4}} dt\right)^{\frac{1}{q}} + \left(\int_1^{+\infty}\left(\frac{C}{4t^2} + \frac{C}{t^{2-\frac{l[1+C_r(u_0)]}{1-C_r(u_0)}}}\right)^{\frac{2^*q}{2r\tilde{\tau}_4}} dt\right)^{\frac{1}{q}}\right\} \\
&\leq C.
\end{aligned} \tag{4.11}$$

Subcase (ii). $0 \leq w(x,t)$ for any $x \in \mathbb{R}^N$ and $t \geq 0$, $\int_{\mathbb{R}^N} |w(x,t)|^{\frac{\delta}{\delta-1}} dx \leq c'_w$ for some $1 < \delta < \frac{\gamma_2}{r} \leq \gamma_2$ and $1 < \delta < \frac{\tilde{\gamma}_2}{r} \leq \tilde{\gamma}_2$. We have

$$\begin{aligned}
&\left(\int_0^{+\infty}\left(\int_{\mathbb{R}^N} w(x,t)|G_1(|u|^2)|^r dx\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^{+\infty}\left\{\left(\int_{\mathbb{R}^N} |w(x,t)|^{\frac{\delta}{\delta-1}} dx\right)^{\frac{\delta-1}{\delta}}\left(\int_{\mathbb{R}^N} |G_1(|u|^2)|^{r\delta} dx\right)^{\frac{1}{\delta}}\right\}^{\frac{q}{r}} dt\right)^{\frac{1}{q}} \\
&\leq (c'_v)^{\frac{\delta-1}{r\delta}} C\left\{\left(\int_0^1 [2E(u_0)]^{\frac{2^*q}{2r\delta\tau_4}} dt\right)^{\frac{1}{q}} + \left(\int_1^{+\infty}\left(\frac{C}{t^2} + \frac{C}{t^{2-\frac{l[1+C_r(u_0)]}{1-C_r(u_0)}}}\right)^{\frac{2^*q}{2r\delta\tau_4}} dt\right)^{\frac{1}{q}}\right\} \\
&\quad + (c'_v)^{\frac{\delta-1}{r\delta}} C\left\{\left(\int_0^1 [2E(u_0)]^{\frac{2^*q}{2r\delta\tilde{\tau}_4}} dt\right)^{\frac{1}{q}} + \left(\int_1^{+\infty}\left(\frac{C}{t^2} + \frac{C}{t^{2-\frac{l[1+C_r(u_0)]}{1-C_r(u_0)}}}\right)^{\frac{2^*q}{2r\delta\tilde{\tau}_4}} dt\right)^{\frac{1}{q}}\right\} \\
&\leq C.
\end{aligned} \tag{4.12}$$

Theorem 4 is proved. $\square$

As a corollary of Theorem 3 and Theorem 4, we can obtain the decay rate and asymptotic behavior for the solution as $t \to +\infty$.

Corollary 4.1. *Let $u(x,t)$ be the global solution of (1.1). Under the assumptions of Theorem 3 and Theorem 4,*

$$\int_{\mathbb{R}^N}[|\nabla h(|u|^2)|^2+|V(x)||u|^2+|G_1(|u|^2)|+|G_2(|u|^2)|+\frac{1}{2}(W*|u|^2)|u|^2]dx\leq\frac{C}{t^2} \tag{4.13}$$

in Case 1,

$$\int_{\mathbb{R}^N}[|\nabla h(|u|^2)|^2+|V(x)||u|^2+|G_2(|u|^2)|+\frac{1}{2}(W*|u|^2)|u|^2]dx\leq\frac{C}{t^{2-l}} \tag{4.14}$$

in defocusing subcase of Case 2,

$$\begin{aligned}&\int_{\mathbb{R}^N}[|\nabla h(|u|^2)|^2+|V(x)||u|^2+|G_1(|u|^2)|+|G_2(|u|^2)|+\frac{1}{2}(W*|u|^2)|u|^2]dx\\&\leq\frac{C}{t^{2-\frac{l(1+C_r(u_0))}{1-C_r(u_0)}}}\end{aligned} \tag{4.15}$$

in combined defocusing and focusing subcase of Case 2, and

$$\lim_{t\to+\infty}\int_{\mathbb{R}^N}[|\nabla h(|u|^2)|^2+|V(x)||u|^2+|G(|u|^2)|+\frac{1}{2}(W*|u|^2)|u|^2]dx=0, \tag{4.16}$$

$$\lim_{t\to+\infty}\int_{\mathbb{R}^N}|\nabla u|^2dx=2E(u_0),\quad \lim_{t\to+\infty}\int_{\mathbb{R}^N}[|u|^2+|\nabla u|^2]dx=M(u_0)+2E(u_0). \tag{4.17}$$

Consequently, for any $2\leq r<2^*$, $2^*=\frac{2N}{N-2}$ if $N\geq 3$, $2^*=+\infty$ if $N=1,2$,

$$\int_{\mathbb{R}^N}|u|^rdx\leq C. \tag{4.18}$$

Proof of Corollary 4.1: (4.13),(4.14), (4.15) and (4.16) are the direct results of (3.13), (3.18), (3.26) and (3.32).

By mass and energy conservation laws, we have

$$\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx=E(u_0)-\frac{1}{2}\int_{\mathbb{R}^N}[|\nabla h(|u|^2)|^2+|V(x)||u|^2+|G(|u|^2)|+\frac{1}{2}(W*|u|^2)|u|^2]dx,$$

which means (4.17). (4.16) and (4.17) imply that

$$\int_{\mathbb{R}^N}|u|^2dx\leq C,\quad \int_{\mathbb{R}^N}|\nabla u|^2dx\leq C,$$

by embedding theorem, we get (4.18). □

We give two examples to show the results on Theorem 3 and Theorem 4.

Remark 4.1. 1. If $h(s)\equiv 0$, $F(|u|^2)=-|u|^{2\beta}$, $V(x)=-\frac{1}{|x|^m}$ and $W(x)=-\frac{1}{|x|^n}$, $x\neq\mathbf{0}$, $\beta,m,n>0$, then we can verify the assumptions of Theorem 3 and 4 and especially have

$$\int_0^\infty\int_{\mathbb{R}^N}\left[\frac{|u|^{2\beta+2}}{\beta+1}+\frac{1}{|x|^m}|u|^2+\frac{1}{2}(\frac{1}{|x|^n}*|u|^2)|u|^2\right]dxdt\leq C, \tag{4.19}$$

$$\int_0^\infty\left(\int_{\mathbb{R}^N}\left[\frac{|u|^{2\beta+2}}{\beta+1}+\frac{1}{|x|^m}|u|^2+\frac{1}{2}(\frac{1}{|x|^n}*|u|^2)|u|^2\right]dx\right)^p dt\leq C, \tag{4.20}$$

$$\|u\|_{L^q(\mathbb{R}^+)L^{\tilde{r}}(\mathbb{R}^N)}=\left(\int_0^\infty\left(\int_{\mathbb{R}^N}|u|^{\tilde{r}}dx\right)^{\frac{q}{\tilde{r}}}dt\right)^{\frac{1}{q}}\leq C. \tag{4.21}$$

2. Consider the following Cauchy problem:

$$\begin{cases} iu_t = \Delta u + 2\alpha |u|^{2\alpha-2}u\Delta(|u|^{2\alpha}) - \frac{|x|^2 u}{|x|^2+1} \mp |u|^{2\beta}u - (\frac{|x|^2}{(a|x|^2+1)^m} * |u|^2)u, x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^N. \end{cases} \tag{4.22}$$

Here $\alpha, \beta \in Z^+$. Then

$$2h''(s)h'(s)s + (h'(s))^2 = (2\alpha-1)\alpha^2 s^{2\alpha-2},$$
$$NF(s)s - (N+2)G(s) = \mp[N - \frac{N+2}{\beta+1}]s^{\beta+1},$$
$$2V + (x\cdot\nabla V) \le 0, \quad 2W + (x\cdot\nabla W) \le 0 \quad \text{for suitable} \quad a,\ m.$$

We can verify the assumptions of Theorem 3 and 4, and get

$$k_1 = 2\alpha - 1, \quad k_2 = \frac{|N\beta - 2|}{\beta+1}, \quad l = \max(k_1, k_2),$$

$$\int_0^\infty \int_{\mathbb{R}^N} \left[|\nabla h(|u|^2)|^2 + \frac{|u|^{2\beta+2}}{\beta+1} + |V(x)||u|^2 + \frac{1}{2}(|W(x)| * |u|^2)|u|^2 \right] dxdt \le C,$$
$$\int_0^\infty \left(\int_{\mathbb{R}^N} \left[|\nabla h(|u|^2)|^2 + \frac{|u|^{2\beta+2}}{\beta+1} + |V(x)||u|^2 + \frac{1}{2}(|W(x)| * |u|^2)|u|^2 \right] dx \right)^p dt \le C,$$
$$\|u\|_{L^q(\mathbb{R}^+)L^{\tilde r}(\mathbb{R}^N)} = \left(\int_0^\infty \left(\int_{\mathbb{R}^N} |u|^{\tilde r} dx \right)^{\frac{q}{\tilde r}} dt \right)^{\frac{1}{q}} \le C.$$

5 Scattering theory for (1.3) in defocusing case and arbitrary space dimension

In this section, applying the results of Theorem 3 and Theorem 4, we will establish scattering theory in $L^2(\mathbb{R}^N)$ and Σ $(N \ge 1)$ under certain assumptions. Here

$$\Sigma = \{f \in H^1(\mathbb{R}^N), \quad |x|f \in L^2(\mathbb{R}^N)\}. \tag{5.1}$$

5.1 Scattering theory in $L^2(\mathbb{R}^N)$ for (1.3) in defocusing case and arbitrary space dimension

In this subsection, we will establish scattering theory in $L^2(\mathbb{R}^N)$ for (1.3) in defocusing case and arbitrary space dimension.

Theorem 5. *Let $u \in C(\mathbb{R}, \Sigma)$ be the solution of (1.3) in defocusing case, i.e., $h(s) \equiv 0$, $F(s) \le 0$ for $s \ge 0$, $V(x) \le 0$ and $W(x) \le 0$, $W(x)$ is even for $x \in \mathbb{R}^N$, $N \ge 1$, and $u_0 \in \Sigma$.*

Assume that there exist $C > 0$, θ_1, p_1, θ_2 and p_2 such that

$$[|F(s)|s^{\frac{1}{2}}]^{\theta_1} \le Cs, \quad [|F(s)|s^{\frac{1}{2}}]^{p_1} \le C|G(s)|, \quad 0 < s < 1, \tag{5.2}$$
$$[|F(s)|s^{\frac{1}{2}}]^{\theta_2} \le Cs, \quad [|F(s)|s^{\frac{1}{2}}]^{p_2} \le C|G(s)|, \quad s > 1, \tag{5.3}$$

and there exist c_1, c_2, $V_1(x)$, $V_2(x)$, $W_1(x)$ and $W_2(x)$ such that

$$V(x) = V_1(x) + V_2(x), \quad c_1(|V_1(x)| + |V_2(x)|) \le |V(x)|, \tag{5.4}$$
$$W(x) = W_1(x) + W_2(x), \quad c_2(|W_1(x)| + |W_2(x)|) \le |W(x)|. \tag{5.5}$$

In addition, suppose that there exist admissible pairs (q_1,r_1), (q_2,r_2), (q_3,r_3), $(\tilde q_1,\tilde r_1)$ and $(\tilde q_2,\tilde r_2)$ such that

$$V_1(x)\in L^{\frac{r_1}{r_1-2}}(\mathbb{R}^N),\quad V_2(x)\in L^{\frac{r_2}{r_2-2}}(\mathbb{R}^N), \tag{5.6}$$

$$W_1(x)\in L^{\frac{\tilde r_1}{2(\tilde r_1-2)}}(\mathbb{R}^N),\quad W_2(x)\in L^{\frac{\tilde r_2}{2(\tilde r_2-2)}}(\mathbb{R}^N), \tag{5.7}$$

and

$$q_1'>1,\quad q_2'>1,\quad \frac{2q_3'(r_3'-\theta_1)}{r_3'(p_1-\theta_1)}>1,\quad \frac{2q_3'(r_3'-\theta_2)}{r_3'(p_2-\theta_2)}>1,\quad \tilde q_1'>1,\quad \tilde q_2'>1 \tag{5.8}$$

if $[(N+2)G(s)-NF(s)s]\geq 0$ for $s\geq 0$, $[2V+(x\cdot\nabla V)]\geq 0$ and $[2W+(x\cdot\nabla W)]\geq 0$ for $x\in\mathbb{R}^N$, while

$$\frac{(2-l)q_1'}{2}>1,\quad \frac{(2-l)q_2'}{2}>1,\quad \frac{(2-l)q_3'(r_3'-\theta_1)}{r_3'(p_1-\theta_1)}>1, \tag{5.9}$$

$$\frac{(2-l)q_3'(r_3'-\theta_2)}{r_3'(p_2-\theta_2)}>1,\quad \frac{(2-l)\tilde q_1'}{2}>1,\quad \frac{(2-l)\tilde q_2'}{2}>1 \tag{5.10}$$

if at least one of the following cases holds:

(i) $-k_1|G(s)|\leq (N+2)G(s)-NF(s)s\leq 0$ for some $k_1>0$;

(iv) $-k_2|V|\leq 2V+(x\cdot\nabla V)\leq 0$ for some $k_2>0$;

(v) $-k_3|W|\leq 2W+(x\cdot\nabla W)\leq 0$ for some $k_3>0$.

Here

$$l=\max(k_1,k_2,k_3), \tag{5.11}$$

q_j', r_j', $\tilde q_m'$, $\tilde r_m'$ are the conjugated exponents of q_j, r_j, $\tilde q_m$, $\tilde r_m$ respectively.

Then there exists $u_+\in L^2(\mathbb{R}^N)$ such that

$$e^{it\Delta}u(t)\longrightarrow u_+\quad\text{in}\quad L^2(\mathbb{R}^N)\quad\text{as}\quad t\to+\infty.$$

Proof: Duhamel's principle implies that

$$u(t)=e^{-it\Delta}u_0-i\int_0^t e^{-i(t-s)\Delta}\big(V(x)u(s)+F(|u|^2)u(s)+(W*|u|^2)u(s)\big)\,ds.$$

By Strichartz estimates, for any $0<t<\tau$, we obtain

$$\begin{aligned}
&\|e^{it\Delta}u(t)-e^{i\tau\Delta}u(\tau)\|_{L^2}\\
&\leq \|\int_t^\tau e^{is\Delta}V(x)u(s)ds\|_{L^2}+\|\int_t^\tau e^{is\Delta}F(|u|^2)u(s)ds\|_{L^2}+\|\int_t^\tau e^{is\Delta}(W*|u|^2)u(s)ds\|_{L^2}\\
&\leq \sum_{j=1}^2\|\int_t^\tau e^{is\Delta}V_j(x)u(s)ds\|_{L^2}+\|\int_t^\tau e^{is\Delta}F(|u|^2)u(s)ds\|_{L^2}\\
&\quad+\sum_{m=1}^2\|\int_t^\tau e^{is\Delta}(W_m*|u|^2)u(s)ds\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{j=1}^{2}\left(\int_t^\tau\left(\int_{\mathbb{R}^N}|V_j(x)u|^{r_j'}dx\right)^{\frac{q_j'}{r_j'}}dt\right)^{\frac{1}{q_j'}}+C\left(\int_t^\tau\left(\int_{\mathbb{R}^N}[|F(|u|^2)||u|]^{r_3'}dx\right)^{\frac{q_3'}{r_3'}}dt\right)^{\frac{1}{q_3'}}\\
&\quad+C\sum_{m=1}^{2}\left(\int_t^\tau\left(\int_{\mathbb{R}^N}[(|W_m|*|u|^2)|u|]^{\tilde{r}_m'}dx\right)^{\frac{\tilde{q}_m'}{\tilde{r}_m'}}dt\right)^{\frac{1}{\tilde{q}_m'}}\\
&:=(I)+(II)+(III).
\end{aligned}\tag{5.12}
$$

Using Hölder inequality, it is easy to get

$$
\begin{aligned}
(I)&\le C\left(\int_t^\tau\left(\int_{\mathbb{R}^N}|V_1(x)||u|^2dx\right)^{\frac{q_1'}{2}}\left(\int_{\mathbb{R}^N}|V_1(x)|^{\frac{r_1'}{2-r_1'}}dx\right)^{\frac{q_1'(2-r_1')}{2r_1'}}dt\right)^{\frac{1}{q_1'}}\\
&\quad+C\left(\int_t^\tau\left(\int_{\mathbb{R}^N}|V_2(x)||u|^2dx\right)^{\frac{q_2'}{2}}\left(\int_{\mathbb{R}^N}|V_2(x)|^{\frac{r_2'}{2-r_2'}}dx\right)^{\frac{q_2'(2-r_2')}{2r_2'}}dt\right)^{\frac{1}{q_2'}}\\
&\quad\longrightarrow 0\quad\text{as}\quad t,\ \tau\to+\infty,
\end{aligned}\tag{5.13}
$$

$$
\begin{aligned}
(II)&\le C\left(\int_t^\tau\left(\int_{\{|u|\le1\}}[|F(|u|^2)||u|]^{\theta_1}dx\right)^{\frac{q_3'}{\tau_1'r_3'}}\left(\int_{\{|u|\le1\}}[|F(|u|^2)||u|]^{p_1}dx\right)^{\frac{q_3'}{\tau_1r_3'}}dt\right)^{\frac{1}{q_3'}}\\
&\quad+C\left(\int_t^\tau\left(\int_{\{|u|>1\}}[|F(|u|^2)||u|]^{\theta_2}dx\right)^{\frac{q_3'}{\tau_2'r_3'}}\left(\int_{\{|u|>1\}}[|F(|u|^2)||u|]^{p_2}dx\right)^{\frac{q_3'}{\tau_2r_3'}}dt\right)^{\frac{1}{q_3'}}\\
&\le C\left(\int_t^\tau\left(\int_{\mathbb{R}^N}G(|u|^2)dx\right)^{\frac{q_3'}{\tau_1r_3'}}dt\right)^{\frac{1}{q_3'}}+C\left(\int_t^\tau\left(\int_{\mathbb{R}^N}|G(|u|^2)|dx\right)^{\frac{q_3'}{\tau_2r_3'}}dt\right)^{\frac{1}{q_3'}}\\
&\quad\longrightarrow 0\quad\text{as}\quad t,\ \tau\to+\infty,
\end{aligned}\tag{5.14}
$$

$$
\begin{aligned}
(III)&\le\sum_{m=1}^{2}C\left\{\int_t^\tau\left(\int_{\mathbb{R}^N}(|W_m|*|u|^2)|u|^2dx\right)^{\frac{\tilde{q}_m'}{2}}\left(\int_{\mathbb{R}^N}|u(x)|^{\frac{r_m'}{r_m'-1}}dx\right)^{\frac{\tilde{q}_m'(\tilde{r}_m'-1)}{\tilde{r}_m'}}\right.\\
&\qquad\left.\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|W_m(x-y)|^{\frac{\tilde{r}_m'}{2-\tilde{r}_m'}}dydx\right)^{\frac{\tilde{q}_m'(2-\tilde{r}_m')}{2\tilde{r}_m'}}dt\right\}^{\frac{1}{\tilde{q}_m'}}\\
&\quad\longrightarrow 0\quad\text{as}\quad t,\ \tau\to+\infty,
\end{aligned}\tag{5.15}
$$

because

$$
\int_{\mathbb{R}^N}|u|^2dx=\int_{\mathbb{R}^N}|u_0|^2dx,\quad\int_{\mathbb{R}^N}|\nabla u|^2dx\le C,\quad\int_{\mathbb{R}^N}|u(x)|^{\frac{\tilde{r}_m'}{\tilde{r}_m'-1}}dx\le C
$$

by the results of Section 3 and Section 4, moreover,

$$
\int_{\mathbb{R}^N}|V(x)||u|^2dx\le\frac{C}{t^2},\quad\int_{\mathbb{R}^N}|G(|u|^2)|dx\le\frac{C}{t^2},\quad\int_{\mathbb{R}^N}(|W|*|u|^2)|u|^2dx\le\frac{C}{t^2}
$$

and (5.8) in Case 1, while

$$\int_{\mathbb{R}^N} |V(x)||u|^2 dx \le \frac{C}{t^{2-l}}, \quad \int_{\mathbb{R}^N} |G(|u|^2)| dx \le \frac{C}{t^{2-l}}, \quad \int_{\mathbb{R}^N} (|W| * |u|^2)|u|^2 dx \le \frac{C}{t^{2-l}}$$

and (5.9), (5.10) in Case 2. Here

$$\frac{1}{\tau_1} = \frac{r_3' - \theta_1}{p_1 - \theta_1}, \quad \frac{1}{\tau_1'} = \frac{p_1 - r_3'}{p_1 - \theta_1}, \quad \frac{1}{\tau_2} = \frac{r_3' - \theta_2}{p_2 - \theta_2}, \quad \frac{1}{\tau_2'} = \frac{p_2 - r_3'}{p_2 - \theta_2}.$$

Consequently, there exists $u_+ \in L^2(\mathbb{R}^N)$ such that

$$\|e^{it\Delta}u(t) - u_+\|_{L^2} \to 0 \quad \text{as} \quad t \to +\infty.$$

That is, every solution in Σ of (1.3) has scattering state in $L^2(\mathbb{R}^N)$. □

Remark 5.1. 1. A special case in the assumptions of Theorem 5 is $\theta_1 = \theta_2 = \theta$, $p_1 = p_2 = p$. For example, if $F(|u|^2)u = b|u|^{2\beta}u$, then $\theta_1 = \theta_2 = \frac{2}{2\beta+1}$, $p_1 = p_2 = \frac{2\beta+2}{2\beta+1}$, and the assumptions of Theorem 5 can be satisfied.

2. In the proof of Theorem 5, we take different admissible pairs in Strichartz estimates for different terms on the right side of Duhamel's formula in order to keep the terms containing $V(x)u$, $F(|u|^2)u$ and $(W * |u|^2)u$ independent each other. Consequently, Theorem 5 can directly deduce scattering theory in $L^2(\mathbb{R}^N)$ for Cauchy problem of the equation contains one of $V(x)u$, $F(|u|^2)u$ and $(W * |u|^2)u$.

Corollary 5.1. *Let u be the solution of the following problem*

$$\begin{cases} iu_t = \Delta u + V(x)u, \ x \in \mathbb{R}^N, \ t > 0 \\ u(x,0) = u_0(x) \in \Sigma, \quad x \in \mathbb{R}^N. \end{cases} \tag{5.16}$$

Assume that $V(x) \le 0$ for $x \in \mathbb{R}^N$, $N \ge 1$, and (5.4), (5.6), (5.8) and (5.9) hold. Then there exists $u_+ \in L^2(\mathbb{R}^N)$ such that

$$\|e^{it\Delta}u(t) - u_+\|_{L^2} \to 0 \quad \text{as} \quad t \to +\infty.$$

Corollary 5.2. *Let u be the solution of the following problem*

$$\begin{cases} iu_t = \Delta u + F(|u|^2)u, \ x \in \mathbb{R}^N, \ t > 0 \\ u(x,0) = u_0(x) \in \Sigma, \quad x \in \mathbb{R}^N. \end{cases} \tag{5.17}$$

Assume that $F(s)$ satisfies (G):

(G) $\frac{|G(s)|}{s^{\frac{2^*}{2}}} \to 0$ *as $s \to +\infty$, where $G(s) = \int_0^s F(\eta)d\eta$,*

$F(s) \le 0$ for $s \ge 0$, $N \ge 1$, and (5.2), (5.3), (5.8), (5.9) and (5.10) hold. Then there exists $u_+ \in L^2(\mathbb{R}^N)$ such that

$$\|e^{it\Delta}u(t) - u_+\|_{L^2} \to 0 \quad \text{as} \quad t \to +\infty.$$

Corollary 5.3. *Let u be the solution of the following problem*

$$\begin{cases} iu_t = \Delta u + (W * |u|^2)u, \ x \in \mathbb{R}^N, \ t > 0 \\ u(x,0) = u_0(x) \in \Sigma, \quad x \in \mathbb{R}^N. \end{cases} \tag{5.18}$$

Assume that $W(x)$ is even and $W(x) \leq 0$ for $x \in \mathbb{R}^N$, $N \geq 1$, and (5.5), (5.7), (5.8) and (5.10) hold. Then there exists $u_+ \in L^2(\mathbb{R}^N)$ such that

$$\|e^{it\Delta}u(t) - u_+\|_{L^2} \to 0 \quad \text{as} \quad t \to +\infty.$$

2. If the nonlinearities of a semilinear Schödinger equation are combined by any two terms of $V(x)u$, $F(|u|^2)u$ and $(W * |u|^2)u$, then we also can establish the scattering theory in $L^2(\mathbb{R}^N)$ directly. For example, we have

Corollary 5.4. *Let u be the solution of*

$$\begin{cases} iu_t = \Delta u + V(x)u + (W * |u|^2)u, \ x \in \mathbb{R}^N, \ t > 0 \\ u(x,0) = u_0(x) \in \Sigma, \quad x \in \mathbb{R}^N. \end{cases} \tag{5.19}$$

Assume that $V(x) \leq 0$ and $W(x) \leq 0$ for $x \in \mathbb{R}^N$, $N \geq 1$, $W(x)$ is even, and (5.4)–(5.10) hold. Then there exists $u_+ \in L^2(\mathbb{R}^N)$ such that

$$\|e^{it\Delta}u(t) - u_+\|_{L^2} \to 0 \quad \text{as} \quad t \to +\infty.$$

As a corollary of Theorem 5, we give the scattering theory in $L^2(\mathbb{R}^N)$ of (1.4) below.

Corollary 5.5. *Assume that $u(x,t)$ is the solution of (1.4) and $u_0 \in \Sigma$. Then there exists $u_+ \in L^2(\mathbb{R}^N)$ such that*

$$\|e^{it\Delta}u(t) - u_+\|_{L^2} \to 0 \quad \text{as} \quad t \to +\infty$$

if one of the following cases holds:
(I). $N \geq 2$, $\frac{4}{3} < m < 2$, $m < n < 4$, $\frac{4}{3} < m < N\beta < 2^*$, $8 < 4m + n$;
(II). $N \geq 2$, $\beta_0 < N\beta < m < 2$, $\beta_0 < N\beta < n < 4$, $4 < 2N\beta + m$, $8 < 4N\beta + n$;
(III). $N \geq 2$, $\frac{8}{5} < n < m < 2$, $n < N\beta < 2^*$.
Here

$$\beta_0 = \frac{4 - 3N + \sqrt{9N^2 + 40N + 16}}{8N},$$

$2^* = \frac{2N}{N-2}$ *if $N \geq 3$ and $2^* = +\infty$ if $N = 2$.*

Proof: Let

$$V_1(x) = \begin{cases} -\frac{1}{|x|^m}, & 0 < |x| \leq 1, \\ 0, & |x| > 1, \end{cases} \quad \text{and} \quad V_2(x) = \begin{cases} 0, & 0 < |x| \leq 1, \\ -\frac{1}{|x|^m}, & |x| > 1, \end{cases}$$

$$F(|u|^2)u = |u|^{2\beta}u, \quad \theta_1 = \theta_2 = \frac{2}{2\beta + 1}, \quad p_1 = p_2 = \frac{2\beta + 2}{2\beta + 1},$$

$$W_1(x) = \begin{cases} -\frac{1}{|x|^n}, & 0 < |x| \leq 1, \\ 0, & |x| > 1, \end{cases} \quad \text{and} \quad W_2(x) = \begin{cases} 0, & 0 < |x| \leq 1, \\ -\frac{1}{|x|^n}, & |x| > 1. \end{cases}$$

Since

$$\begin{aligned} &-(2-m)|V(x)| = 2V(x) + x \cdot \nabla V(x) = -\frac{(2-m)}{|x|^m} < 0, \\ &2W(x) + x \cdot \nabla W(x) = \frac{(n-2)}{|x|^n} = (n-2)|W(x)|, \\ &NF(s)s - (N+2)G(s) = -\frac{(2 - N\beta)}{\beta + 1}|u|^{2\beta+2} = -(2 - N\beta)|G(s)|, \end{aligned}$$

it belongs to Case 2 of Theorem 6.

We can take r_1', r_2', r_3', $\tilde{r}_1'$ and $\tilde{r}_2'$ respectively as follows:

(I). $\min(m, N\beta, n) = m$.

$$\begin{aligned}
&\frac{2N}{N+2} < r_1' < \frac{2N}{N+m}, \quad \frac{2N}{N+2} < \tilde{r}_1' < \frac{4N}{2N+n},\\
&\frac{2N}{N+m} < r_2' < \frac{2N}{N+4-2m}, \quad \frac{4N}{2N+n} < \tilde{r}_2' < \frac{2N}{N+4-2m},\\
&\max(\frac{2N}{N+2}, \frac{4m-2N\beta}{2m+4m\beta-4\beta-N\beta}) < r_3' < \frac{2\beta+2}{2\beta+1} \quad \text{if} \quad 2m > N\beta,\\
&\frac{2N}{N+2} < r_3' < \frac{2\beta+2}{2\beta+1} \quad \text{if} \quad 2m \le N\beta, \quad 2m+4m\beta-4\beta-N\beta \ge 0,\\
&\frac{2N}{N+2} < r_3' < \min(\frac{2\beta+2}{2\beta+1}, \frac{2N\beta-4m}{4\beta+N\beta-2m-4m\beta})\\
&\qquad \text{if } 2m < N\beta,\ 2m+4m\beta-4\beta-N\beta < 0;
\end{aligned}$$

(II). $\min(m, N\beta, n) = N\beta$.

$$\begin{aligned}
&\frac{2N}{N+2} < r_1' < \frac{2N}{N+m}, \quad \frac{2N}{N+2} < \tilde{r}_1' < \frac{4N}{2N+n},\\
&\frac{2N}{N+m} < r_2' < \frac{2N}{N+4-2N\beta}, \quad \frac{4N}{2N+n} < \tilde{r}_2' < \frac{2N}{N+4-2N\beta},\\
&\max(\frac{2N}{N+2}, \frac{2}{2\beta+1}, \frac{2N}{N+4N\beta-4}) < r_3' < \frac{2\beta+2}{2\beta+1};
\end{aligned}$$

(III). $\min(m, N\beta, n) = n$.

$$\begin{aligned}
&\frac{2N}{N+2} < r_1' < \frac{2N}{N+m}, \quad \frac{2N}{N+2} < \tilde{r}_1' < \frac{4N}{2N+n},\\
&\frac{2N}{N+m} < r_2' < \frac{2N}{N+4-2n}, \quad \frac{4N}{2N+n} < \tilde{r}_2' < \frac{2N}{N+4-2n},\\
&\max(\frac{2N}{N+2}, \frac{4n-2N\beta}{2n+4n\beta-4\beta-N\beta}) < r_3' < \frac{2\beta+2}{2\beta+1} \quad \text{if} \quad 2n > N\beta,\\
&\frac{2N}{N+2} < r_3' < \frac{2\beta+2}{2\beta+1} \quad \text{if} \quad 2n \le N\beta, \quad 2n+4n\beta-4\beta-N\beta \ge 0,\\
&\frac{2N}{N+2} < r_3' < \min(\frac{2\beta+2}{2\beta+1}, \frac{2N\beta-4n}{4\beta+N\beta-2n-4n\beta})\\
&\qquad \text{if } 2n < N\beta,\ 2n+4n\beta-4\beta-N\beta < 0;
\end{aligned}$$

It is easy to verify the assumptions of Theorem 5 and establish scattering theory in $L^2(\mathbb{R}^N)$ for (1.4). □

Remark 5.2. Our idea can be applied to deal with the following problem:

$$\begin{cases} iu_t = \Delta u + \sum_{m=1}^{M} V_m(x)u + \sum_{k=1}^{K} F_k(|u|^2)u + \sum_{l=1}^{L}(W_l * |u|^2)u, \ x \in \mathbb{R}^N, \ t > 0 \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^N. \end{cases}$$

And we can obtain the general scattering results similar to Theorem 5.

5.2 Scattering theory in Σ for (1.3) in defocusing case and arbitrary space dimension

In this subsection, we will establish classic scattering theory in Σ for the solution of (1.3) in defocusing case and arbitrary space dimension.

Theorem 6. *Let $u \in C(\mathbb{R}, \Sigma)$ be the solution of (1.3) in defocusing case with $u_0 \in \Sigma$. Assume that $V(x) \equiv 0$ and $W(x) \equiv 0$ for $x \in \mathbb{R}^N$, $F(s)$ satisfies $(\mathbf{G})$ and there exist $C > 0$, θ_1, p_1, θ_2, p_2, $2 < r < \frac{2N}{N-2}$ if $N \geq 3$, $2 < r < +\infty$ if $N = 1, 2$, $0 < l < 2$, such that*

$$\theta_1 < \frac{r}{r-2} < p_1, \quad \theta_2 < \frac{r}{r-2} < p_2, \tag{5.20}$$

$$[|F(s)| + |F'(s)|s^{\frac{1}{2}}]^{\theta_1} \leq Cs, \ [|F(s)| + |F'(s)|s^{\frac{1}{2}}]^{p_1} \leq C|G(s)|, \quad 0 < s < 1, \tag{5.21}$$

$$[|F(s)| + |F'(s)|s^{\frac{1}{2}}]^{\theta_2} \leq Cs, \ [|F(s)| + |F'(s)|s^{\frac{1}{2}}]^{p_2} \leq C|G(s)|, \ s > 1. \tag{5.22}$$

Moreover,

$$\frac{4[r(1-\theta_j) + 2\theta_j]}{[2N - (N-2)r](p_j - \theta_j)} > 1, \quad j = 1, 2, \tag{5.23}$$

in Case 1: $[NF(s)s - (N+2)G(s)] \leq 0$ for $s \geq 0$,

$$\frac{2(2-l)[r(1-\theta_j) + 2\theta_j]}{[2N - (N-2)r](p_j - \theta_j)} > 1, \quad j = 1, 2, \tag{5.24}$$

in Case 2: $0 \leq (N+2)G(s) - NF(s)s \leq l|G(s)|$.

Then there exists $u_+ \in \Sigma$ such that

$$e^{it\Delta}u(t) \to u_+ \quad \text{in} \quad \Sigma \quad \text{as} \quad t \to +\infty.$$

Proof: We only prove it in Case (B). The proof in Case (A) can be obtained similarly.

Let (q, r) be the admissible pair satisfying

$$\frac{2}{q} = N(\frac{1}{2} - \frac{1}{r}),$$

where $2 < r < \frac{2N}{N-2}$ if $N \geq 3$, $2 < r < +\infty$ if $N = 1, 2$.

First, we prove that

$$\|u\|_{L^q((0,t), W^{1,r})} \leq C \quad \text{for} \quad t > 0. \tag{5.25}$$

Duhamel's principle implies that

$$u(t) = e^{-it\Delta}u_0 - i\int_0^t e^{-i(t-s)\Delta}F(|u|^2)u(s)ds.$$

By Strichartz estimates, using Hölder's inequality, we have

$$
\begin{aligned}
&\|u\|_{L^q((0,t),W^{1,r})} \le C\|u_0\|_{H^1} + C\|F(|u|^2)u\|_{L^{q'}((0,t),W^{1,r'})} \\
&\le C + C\left(\int_0^T \left(\int_{\mathbb{R}^N} [|F(|u|^2)| + |F'(|u|^2)||u|^2]^{\frac{r}{r-2}} dx\right)^{\frac{q(r-2)}{r(q-2)}} dt\right)^{\frac{q-2}{q}} \|u\|_{L^q((0,T),W^{1,r})} \\
&\quad + C\left(\int_T^t \left(\int_{\mathbb{R}^N} [|F(|u|^2)| + |F'(|u|^2)||u|^2]^{\frac{r}{r-2}} dx\right)^{\frac{q(r-2)}{r(q-2)}} dt\right)^{\frac{q-2}{q}} \|u\|_{L^q((T,t),W^{1,r})} \\
&\le C' + C\sum_{j=1}^2 \left(\int_T^t \left\{\left(\int_{\mathbb{R}^N} |u|^2 dx\right)^{\frac{1}{\tau_j'}} \left(\int_{\mathbb{R}^N} |G(|u|^2)| dx\right)^{\frac{1}{\tau_j}}\right\}^{\frac{q(r-2)}{r(q-2)}} dt\right)^{\frac{q-2}{q}} \|u\|_{L^q((T,t),W^{1,r})} \\
&\le C' + C\sum_{j=1}^2 \left(\int_T^t \left(\int_{\mathbb{R}^N} |G(|u(x)|^2)| dx\right)^{\frac{q[r(1-\theta_j)+2\theta_j]}{r(q-2)(p_j-\theta_j)}} dt\right)^{\frac{q-2}{q}} \|u\|_{L^q((T,t),W^{1,r})} \\
&\le C + \frac{1}{2}\|u\|_{L^q((0,t),W^{1,r})}
\end{aligned} \tag{5.26}
$$

if T is large enough because

$$\int_{\mathbb{R}^N} |G(|u|^2)| dx \le \frac{C}{t^2}, \quad \frac{4[r(1-\theta_j)+2\theta_j]}{[2N-(N-2)r](p_j-\theta_j)} > 1, \quad j = 1, 2,$$

in Case 1, while

$$\int_{\mathbb{R}^N} |G(|u|^2)| dx \le \frac{C}{t^{2-l}}, \quad \frac{2(2-l)[r(1-\theta_j)+2\theta_j]}{[2N-(N-2)r](p_j-\theta_j)} > 1, \quad j = 1, 2,$$

in Case 2. Here

$$\frac{1}{\tau_j} = \frac{r(1-\theta_j)+2\theta_j}{(r-2)(p_j-\theta_j)}, \quad \frac{1}{\tau_j'} = \frac{(r-2)(p_j-\theta_j) - [r(1-\theta_j)+2\theta_j]}{(r-2)(p_j-\theta_j)}, \quad j = 1, 2.$$

(5.26) implies (5.25).

As a byproduct of (5.26), we get

$$\|F(|u|^2)u\|_{L^{q'}((t,\tau),W^{1,r'})} \longrightarrow 0 \quad \text{as} \quad t,\ \tau \to +\infty. \tag{5.27}$$

Consequently, we obtain

$$
\begin{aligned}
&\|e^{it\Delta}u(t) - e^{i\tau\Delta}u(\tau)\|_{H^1} \le \|\int_t^\tau e^{is\Delta}F(|u|^2)u(s)ds\|_{H^1} \\
&\le C\|F(|u|^2)u\|_{L^{q'}((t,\tau),W^{1,r'})} \longrightarrow 0 \quad \text{as} \quad t,\ \tau \to +\infty
\end{aligned} \tag{5.28}
$$

by the result of (5.27).

Therefore, there exists $u_+ \in H^1(\mathbb{R}^N)$ such that

$$e^{it\Delta}u(t) \to u_+ \quad \text{in} \quad H^1(\mathbb{R}^N) \quad \text{as} \quad t \to +\infty.$$

Now we will prove that

$$\|(x - 2it\nabla)u\|_{L^q((0,t),W^{1,r})} \le C \quad \text{for} \quad t > 0. \tag{5.29}$$

Since

$$(x-2it\nabla)u(t)=e^{-it\Delta}xu_0-i\int_0^t e^{-i(t-s)\Delta}(x-2is\nabla)[F(|u|^2)u(s)]ds,$$

by Strichartz estimates, we obtain

$$\|(x-2it\nabla)u\|_{L^q((0,t),L^r)}\le C\|xu_0\|_{L^2}+C\|(x-2it\nabla)[F(|u|^2)u]\|_{L^{q'}((0,t),L^{r'})}. \tag{5.30}$$

Letting $H(t):=(x-2it\nabla)$, it is easy to verify that

$$H(t)[F(|u|^2)u]=\partial_u[F(|u|^2)u]H(t)u-\partial_{\bar u}[F(|u|^2)u]\overline{H(t)u}$$

and

$$\begin{aligned}&\|(x-2it\nabla)[F(|u|^2)u]\|_{L^{q'}((0,t),L^{r'})}\\&\le\|\partial_u[F(|u|^2)u]H(t)u\|_{L^{q'}((0,t),L^{r'})}+\|\partial_{\bar u}[F(|u|^2)u]\overline{H(t)u}\|_{L^{q'}((0,t),L^{r'})}\\&\le C\|[|F(|u|^2)+|F'(|u|^2)|u|^2](x-2it\nabla)u\|_{L^{q'}((0,t),L^{r'})}.\end{aligned} \tag{5.31}$$

By (5.30) and (5.31), we get

$$\begin{aligned}&\|(x-2it\nabla)u\|_{L^q((0,t),L^r)}\\&\le C\|xu_0\|_{L^2}+C\|[|F(|u|^2)|+|F'(|u|^2)||u|^2]|(x-2is\nabla)u|\|_{L^{q'}((0,t),L^{r'})}\\&\le C'+C\|[|F(|u|^2)|+|F'(|u|^2)||u|^2]|(x-2is\nabla)u|\|_{L^{q'}((0,t),L^{r'})}.\end{aligned} \tag{5.32}$$

Similar to the discussion of (5.26) and (5.27), we have (5.29) and

$$\|[F(|u|^2)+|F'(|u|^2)||u|^2](x-2is\nabla)u\|_{L^{q'}((t,\tau),L^{r'})}\longrightarrow 0\quad\text{as}\quad t,\ \tau\to+\infty. \tag{5.33}$$

Consequently,

$$\begin{aligned}\|xe^{it\Delta}u(t)-xe^{i\tau\Delta}u(\tau)\|_{L^2}&=\|\int_t^\tau e^{is\Delta}(x-2is\nabla)[F(|u|^2)u(s)\\ \le C\|[|F(|u|^2)+|F'(|u|^2)|u|^2]|(x-2is\nabla)u|\|_{L^{q'}((t,\tau),L^{r'})}&\longrightarrow 0\end{aligned} \tag{5.34}$$

as $t,\ \tau\to+\infty$ by the result of (5.33).

Hence, there exists $u_+\in\Sigma$ such that

$$e^{it\Delta}u(t)\to u_+\quad\text{in}\quad\Sigma\quad\text{as}\quad t\to+\infty.$$

That is, if $V(x)\equiv 0$ and $W(x)\equiv 0$, under the assumptions on $F(s)$, every solution with initial data $u_0\in\Sigma$ of (1.3) has scattering state in Σ. □

Remark 5.3. The typical example of $F(|u|^2)u$ satisfying the assumptions of Theorem 6 is

$$\begin{aligned}&F(|u|^2)u=a_1|u|^{2\beta_1}u+...+a_m|u|^{2\beta_m}u,\quad a_j<0,\quad j=1,2,...,m,\\&\frac{2-N+\sqrt{N^2+12N+4}}{4N}<\beta_1<...<\beta_m<\frac{2^*}{N}.\end{aligned}$$

In the last part of this subsection, we considered the following Cauchy problem

$$\begin{cases}iu_t=\Delta u+\tilde F(|u|^2)u-A|u|^{2^*-2}u,\ x\in\mathbb{R}^N,\ t>0,\\ u(x,0)=u_0(x)\in\Sigma,\quad x\in\mathbb{R}^N,\end{cases} \tag{5.35}$$

where $\tilde{F}(s)$ and $\tilde{G}(s) = \int_0^s \tilde{F}(\eta)d\eta$ satisfies $(\mathbf{G})$.

We would like to say something about the nonlinearities in (5.35) below.

1. Let $F(|u|^2)u = \tilde{F}(|u|^2)u - A|u|^{2^*-2}u$. Then $G(s)$ satisfies

$(\tilde{G})$ $\quad \frac{|G(s)|}{s^{\frac{2^*}{2}}} \geq M$ for some $M > 0$ as $s \to +\infty$,

which is a complementary condition of $(\mathbf{G})$ in Theorem 6.

2. If $N\tilde{F}(s)s - (N+2)\tilde{G}(s) \leq 0$, then

$$\int_{\mathbb{R}^N} \left[|\tilde{G}(|u|^2)| + \frac{2A}{2^*}|u|^{2^*} \right] dx \leq \frac{C}{t^2}$$

by the result of Corollary 4.1 because

$$NF(s)s - (N+2)G(s) = N\tilde{F}(s)s - (N+2)\tilde{G}(s) - \frac{4A}{N}|u|^{2^*} \leq 0,$$

which satisfies the assumptions of Case 1 in Theorem 3 if $h(s) \equiv 0$ and

$$F(s) = -F_2(s) = \tilde{F}(|u|^2)u - A|u|^{2^*-2}u.$$

Now we will give the following scattering result on (5.35).

Theorem 7(Scattering theory in Σ) *Let $u \in C(\mathbb{R}, \Sigma)$ be the global solution of (5.35), $N \geq 3$, $A > 0$ and $u_0 \in \Sigma$. Suppose that $N\tilde{F}(s)s - (N+2)\tilde{G}(s) \leq 0$ for $s \geq 0$, and there exist $C > 0$, $\theta_1 < \frac{N}{2} < p_1$ and $\theta_2 < \frac{N}{2} < p_2$ such that*

$$[|\tilde{F}(s)| + |\tilde{F}'(s)|s]^{\theta_1} \leq Cs, \ [|\tilde{F}(s)| + |\tilde{F}'(s)|s]^{p_1} \leq C|\tilde{G}(s)|, \quad 0 < s < 1, \tag{5.36}$$

$$[|\tilde{F}(s)| + |\tilde{F}'(s)|s]^{\theta_2} \leq Cs, \ [|\tilde{F}(s)| + |\tilde{F}'(s)|s]^{p_2} \leq C|\tilde{G}(s)|, \quad s > 1. \tag{5.37}$$

Then there exists $u_+ \in \Sigma$ such that

$$\|e^{it\Delta}u(t) - u_+\|_\Sigma \to 0 \quad \text{as} \quad t \to +\infty.$$

Proof: Note that $(2, 2^*)$ is an admissible pair (q, r) satisfying

$$\frac{2}{q} = N(\frac{1}{2} - \frac{1}{r}). \tag{5.38}$$

We first prove that for this admissible pair

$$\|u\|_{L^q((0,t),W^{1,r})} \leq C \quad \text{for} \quad t > 0. \tag{5.39}$$

Duhamel's principle implies that

$$u(t) = e^{-it\Delta}u_0 - i\int_0^t e^{-i(t-s)\Delta}\left(\tilde{F}(|u|^2)u(s) + A|u|^{2^*-2}u(s)\right)ds.$$

By Strichartz estimates, using Hölder's inequality, we have

$$
\begin{aligned}
&\|u\|_{L^q((0,t),W^{1,r})}\\
&\le C\|u_0\|_{H^1}+C\|\tilde{F}(|u|^2)u\|_{L^{q'}((0,t),W^{1,r'})}+C\||u|^{2^*-2}u\|_{L^{q'}((0,t),W^{1,r'})}\\
&\le C+C\left(\int_0^T\left(\int_{\mathbb{R}^N}[|\tilde{F}(|u|^2)|+|\tilde{F}'(|u|^2)||u|^2]^{\frac{N}{2}}dx\right)^{\frac{4}{N}}\left(\int_{\mathbb{R}^N}[|u|^r+|\nabla u|^r]dx\right)^{\frac{2}{r}}dt\right)^{\frac{1}{2}}\\
&\quad+C\left(\int_T^t\left(\int_{\mathbb{R}^N}[|\tilde{F}(|u|^2)|+|\tilde{F}'(|u|^2)||u|^2]^{\frac{N}{2}}dx\right)^{\frac{4}{N}}\left(\int_{\mathbb{R}^N}[|u|^r+|\nabla u|^r]dx\right)^{\frac{2}{r}}dt\right)^{\frac{1}{2}}\\
&\quad+C\left(\int_0^T\left(\int_{\mathbb{R}^N}[|u|^{2^*}dx\right)^{\frac{4}{N}}\left(\int_{\mathbb{R}^N}[|u|^r+|\nabla u|^r]dx\right)^{\frac{2}{r}}dt\right)^{\frac{1}{2}}\\
&\quad+C\left(\int_T^t\left(\int_{\mathbb{R}^N}[|u|^{2^*}dx\right)^{\frac{4}{N}}\left(\int_{\mathbb{R}^N}[|u|^r+|\nabla u|^r]dx\right)^{\frac{2}{r}}dt\right)^{\frac{1}{2}}\\
&\le C\sum_{j=1}^{2}\max_{[T,t]}\left\{\left(\int_{\mathbb{R}^N}|u|^2dx\right)^{\frac{1}{\tau_j}}\left(\int_{\mathbb{R}^N}|\tilde{G}(|u|^2)|dx\right)^{\frac{1}{\tau_j'}}\right\}^{\frac{4}{N}}\|u\|_{L^q((T,t),W^{1,r})}\\
&\quad+C'+\frac{1}{4}\|u\|_{L^q((T,t),W^{1,r})}\\
&\le C\sum_{j=1}^{2}\max_{[T,t]}\left(\int_{\mathbb{R}^N}|\tilde{G}(|u(x)|^2)|dx\right)^{\frac{2[N-2\theta_j]}{N(p_j-\theta_j)}}\|u\|_{L^q((T,t),W^{1,r})}\\
&\quad+C'+\frac{1}{4}\|u\|_{L^q((T,t),W^{1,r})}\\
&\le C+\frac{1}{2}\|u\|_{L^q((0,t),W^{1,r})}
\end{aligned}
\tag{5.40}
$$

if T is large enough because $\frac{2[N-2\theta_j]}{N(p_j-\theta_j)}>0$ and $\int_{\mathbb{R}^N}|\tilde{G}(|u(x)|^2)|dx\le\frac{c}{t^2}$, we have

$$C\sum_{j=1}^{2}\max_{[T,t]}\left(\int_{\mathbb{R}^N}|\tilde{G}(|u(x)|^2)|dx\right)^{\frac{2[N-2\theta_j]}{N(p_j-\theta_j)}}<\frac{1}{4}.$$

Here

$$\frac{1}{\tau}=\frac{2p_j-N}{2(p_j-\theta_j)},\quad \frac{1}{\tau_j'}=\frac{N-2\theta_j}{2(p_j-\theta_j)}.$$

(5.40) implies (5.39).

As a byproduct of (5.40), we get

$$\|\tilde{F}(|u|^2)u\|_{L^{q'}((0,t),W^{1,r'})}+\||u|^{2^*-2}u\|_{L^{q'}((0,t),W^{1,r'})}\le C,\quad t>0,$$

which implies that

$$\|\tilde{F}(|u|^2)u\|_{L^{q'}((t,\tau),W^{1,r'})}+\||u|^{2^*-2}u\|_{L^{q'}((t,\tau),W^{1,r'})}\longrightarrow 0\quad\text{as}\quad t,\ \tau\to+\infty. \tag{5.41}$$

Therefore, we obtain

$$
\begin{aligned}
&\|e^{it\Delta}u(t) - e^{i\tau\Delta}u(\tau)\|_{H^1} \\
\leq\ & \|\int_t^\tau e^{is\Delta}\tilde{F}(|u|^2)u(s)ds\|_{H^1} + \|\int_t^\tau e^{is\Delta}A|u|^{2^*-2}u(s)ds\|_{H^1} \\
\leq\ & C\|\tilde{F}(|u|^2)u\|_{L^{q'}((\tau,t),W^{1,r'})} + C\||u|^{2^*-2}u\|_{L^{q'}((\tau,t),W^{1,r'})} \\
&\longrightarrow 0 \quad \text{as} \quad t,\ \tau \to +\infty. \qquad (5.42)
\end{aligned}
$$

Next, we prove that for this admissible pair (q, r),

$$
\|(x - 2it\nabla)u\|_{L^q((0,t),W^{1,r})} \leq C \quad \text{for} \quad t > 0. \qquad (5.43)
$$

In fact, since

$$
(x - 2it\nabla)u(t) = e^{-it\Delta}xu_0 - i\int_0^t e^{-i(t-s)\Delta}(x - 2is\nabla)\left(\tilde{F}(|u|^2)u(s) + A|u|^{2^*-2}u(s)\right)ds,
$$

by Strichartz estimates, we have

$$
\begin{aligned}
\|(x - 2it\nabla)u\|_{L^q((0,t),L^r)} \leq\ & C\|xu_0\|_{L^2} + C\|(x - 2it\nabla)[\tilde{F}(|u|^2)u]\|_{L^{q'}((0,t),L^{r'})} \\
&+ C\|(x - 2it\nabla)[|u|^{2^*-2}u]\|_{L^{q'}((0,t),L^{r'})}. \qquad (5.44)
\end{aligned}
$$

Letting $H(t) := (x - 2it\nabla)$, it is easy to verify that

$$
H(t)[\tilde{F}(|u|^2)u] = \partial_u[\tilde{F}(|u|^2)u]H(t)u - \partial_{\bar{u}}[\tilde{F}(|u|^2)u]\overline{H(t)u}, \qquad (5.45)
$$

$$
H(t)[|u|^{2^*-2}u] = \partial_u[|u|^{2^*-2}u]H(t)u - \partial_{\bar{u}}[|u|^{2^*-2}u]\overline{H(t)u}. \qquad (5.46)
$$

By Strichartz estimates (5.44), we get

$$
\begin{aligned}
\|(x - 2it\nabla)u\|_{L^q((0,t),L^r)} \leq\ & C\|xu_0\|_{L^2} + C\|\tilde{F}(|u|^2)(x - 2is\nabla)u\|_{L^{q'}((0,t),L^{r'})} \\
&+ C\||u|^{2^*-2}(x - 2is\nabla)u\|_{L^{q'}((0,t),L^{r'})}. \qquad (5.47)
\end{aligned}
$$

Similar to the discussion of (5.40) and (5.41), we obtain (5.43) and

$$
\begin{aligned}
&\|F(|u|^2)(x - 2is\nabla)u\|_{L^{q'}((t,\tau),L^{r'})} + \||u|^{2^*-2}(x - 2is\nabla)u\|_{L^{q'}((t,\tau),L^{r'})} \\
&\longrightarrow 0 \quad \text{as} \quad t,\ \tau \to +\infty. \qquad (5.48)
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\|xe^{it\Delta}u(t) - xe^{i\tau\Delta}u(\tau)\|_{L^2} \\
=\ & \|\int_t^\tau e^{it\Delta}(x - 2is\nabla)\left(\tilde{F}(|u|^2)u(s) + A|u|^{2^*-2}u(s)\right)ds\|_{L^2} \\
\leq\ & C\|\tilde{F}(|u|^2)(x - 2is\nabla)u\|_{L^{q'}((t,\tau),L^{r'})} + C\||u|^{2^*-2}(x - 2is\nabla)u\|_{L^{q'}((t,\tau),L^{r'})} \\
&\longrightarrow 0 \quad \text{as} \quad t,\ \tau \to +\infty \qquad (5.49)
\end{aligned}
$$

by the result of (5.48).

Hence, the solution of (5.35) has scattering state in Σ. □

Remark 5.4 1. A typical example of $\tilde{F}(s)$ satisfying (5.36) and (5.37) is

$$
\tilde{F}(|u|^2)u = a_1|u|^{2\beta_1}u + ... + a_m|u|^{2\beta_m}u.
$$

If $\frac{2}{N} < \beta_1 \leq ... \leq \beta_m < \frac{2}{N-2}$ and $u_0 \in \Sigma$. Taking

$$\theta_1 = \frac{1}{\beta_1}, \quad p_1 = \frac{\beta_1 + 1}{\beta_1}, \quad \theta_2 = \frac{1}{\beta_m}, \quad p_2 = \frac{\beta_m + 1}{\beta_m},$$

we can verify the assumptions of Theorem 7 and obtain the corresponding scattering results.

2. Especially, if $\tilde{F}(|u|^2) \equiv 0$ and $\tilde{F}(|u|^2)u = a_1|u|^{2\beta_1}u$, then our results meet those of [50] and [47] respectively.

3. Obviously, Theorem 6 and Theorem 7 are complementary each other, the equation in (1.3) only contains nonlinearities with subcritical Sobolev exponent, while the equation in (5.35) contains nonlinearities with subcritical and critical Sobolev exponent. However, the constrictions on space dimensions and nonlinearities are different. For example, if $F(|u|^2)u = a_1|u|^{2\beta_1}u + ... + a_m|u|^{2\beta_m}u$, $a_j < 0$, $j = 1, 2, ..., m$, we can take

$$\frac{2 - N + \sqrt{N^2 + 12N + 4}}{4N} < \beta_1 < ... < \beta_m < \frac{2^*}{N}$$

and $N \geq 1$, that is, each β_j can be smaller than $\frac{2}{N}$ or larger than $\frac{2}{N}$ in Theorem 6. However, if $\tilde{F}(|u|^2)u = a_1|u|^{2\beta_1}u + ... + a_m|u|^{2\beta_m}u$, $a_j < 0$, $j = 1, 2, ..., m$, we have to require that $\frac{2}{N} < \beta_1 ... < \beta_m < \frac{2^*}{N}$ and $N \geq 3$, i.e., every β_j must be larger than $\frac{2}{N}$ in Theorem 7.

References

[1] V. Banica and N. Visciglia, Scattering for NLS with a delta potential, J. Differential Equations, 260(2016), 4410–4439, .

[2] V. Banica and L. Vega, Scattering for 1D cubic NLS and singular vortex dynamics, *J. Eur. Math. Soc.*, 14, 209–253.

[3] F. G. Bass and N. N. Nasanov, Nonlinear electromagnetic spin waves, *Phys. Rep.*, 189(1990), 165–223.

[4] N. Bez and M. Sugimoto, Optimal forward and reverse estimates of Morawetz and Kato–Yajima type with angular smoothing index, *J. Fourier Anal. Appl.*, 21(2015), 318–341.

[5] A. V. Borovskii and A. L. Galkin, Dynamical modulation of an ultrashort high-intensity laser pulse in matter, *JETP*, 77(1993), 562–573.

[6] A. de Bouard, N. Hayashi and J. C. Saut, Global existence of small solutions to a relativistic nonlinear Schrödinger equation, *Commun. Math. Phys.*, 189(1997), 73–105.

[7] J. Bourgain, Global Solutions of Nonlinear Schrödinger Equations, American Mathematical Society Colloquium Publications 46, American Mathematical Society, Providence, RI, 1999.

[8] R. Carles, Nonlinear Schrödinger equations with replusive harmonic potential and applications, *SIAM J. Math. Anal.*, 35(2003), 823–843.

[9] R. Carles, On semi-classical limit of nonlinear quantum scattering, *Ann. Sci. Éc. Norm. Supér.*, 49(2016):711–756.

[10] R. Carles, N. J. Mauser and H. P. Stimming, (Semi)classical limit of the Hartree equation with Harmonic potential, *SIAM J. Appl. Math.*, 66(2005), 29–56.

[11] B. Cassano and P. D'Ancona, Scattering in the energy space for the NLS with variable coeffcients, *Math. Ann.*, 366(2016), 479–543.

[12] T. Cazenave, Semilinear Schröodinger equations, Courant Lecture Notes in Mathematics 10, New York University, Courant Institute of Mathematical Sciences, AMS, Providence, RI, 2003.

[13] M. Colin, On the local well-posedness of quasilinear Schrödinger equations in arbitrary space dimension, *Comm. Partial Differential Equations*, 27(2002), 325–354.

[14] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in R^3, *Commun. Pure Appl. Math.*, 57(2004), 987–1014.

[15] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in R^3, *Ann. of Math.*, 167(2008), 767–865.

[16] O. Costin, R. Donninger, W. Schlag and S. Tanveer, Semiclassical low energy scattering for one-Dimensional Schrödinger operators with exponentially decaying potentials, *Ann. Henri Poincaré*, 13(2012), 1371–1426.

[17] B. Dodson, Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d = 2$, *Duke Math. J.*, 165(2016), 3435–3516.

[18] T. Duyckaerts, J. Holmer and S. Roudenko, Scattering for the non-radial 3D cubic nonlinear Schrödinger equation, *Math. Res. Lett.*, 15(2008), 1233–1250.

[19] P. Germain, F. Pusateri and F. Rousset, The nonlinear Schrödinger equation with a potential, *Ann. I. H. Poincaré-AN*, 35(2018), 1477–1530.

[20] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations, *J. Funct. Anal.*, 32(1979), 1–71.

[21] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations with nonlocal interaction, *Math. Z.*, 170(1980), 109–136.

[22] J. Ginibre and G. Velo, Scattering theory in the energy space for a class of nonlinear Schrödinger equations, *J. Math. Pures Appl.*, 64(1985), 363–401.

[23] J. Ginibre and G. Velo, Quadratic Morawetz inequalities and asymptotic completeness in the energy space for nonlinear Schrödinger and Hartree equations, *Quart. Appl. Math.*, 68(2010), 113–134.

[24] M. V. Goldman and M. Porkolab, Upper hybrid solitons ans oscillating two-stream instabilities, *Phys. Fluids*, 19(1976), 872–881.

[25] N. Hayashi and P. I. Naumkin, Asymptotics for large time of solutions to the nonlinear Schrodinger and Hartree equations, *American J. Math.*, 120(1998), 369–389.

[26] Y. H. Hong, Scattering for a nonlinear Schrödinger equation with a potential, *Commun. Pure Appl. Anal.*, 15(2016), 1571–1601.

[27] C. E. Kenig, G. Ponce and L. Vega, The Cauchy problem for quasi-linear Schrödinger equations, *Invent. Math.*, 158(2004), 343–388.

[28] R. Killip, J. Murphy, M. Visan, and J. Q. Zheng, The focusing cubic NLS with inverse-square potential in three space dimensions, Differential Integral Equations, 30(2017), 161–206.

[29] R. Killip, M. Visan and X. Y. Zhang, Energy-critical NLS with quadratic potentials, *Comm. Partial Differential Equations*, 34(2009), 1531–1565.

[30] R. Killip and M. Visan, Nonlinear Schrödinger equations at critical regularity. Evolution equations, 325–437, Clay Math. Proc., 17, Amer. Math. Soc., Providence, RI, 2013.

[31] S. Kurihura, Large-amplitude quasi-solitons in superfluid films, *J. Phys. Soc. Japan*, 50(1981), 3262–3267 .

[32] D. Lafontaine, Scattering for NLS with a potential on the line, *Asymptot. Anal.*, 100(2016), 21–39.

[33] E. W. Laedke, K. H. Spatschek, L. Stenflo, Evolution theorem for a class of perturbed envelope soliton solutions, *J. Math. Phys.*, 24(1983), 2764–2769.

[34] A. G. Litvak and A. M. Sergeev, One dimensional collapse of plasma waves, *JETP Lett.*, 27(1978), 517–520.

[35] J. Lu, C. X. Miao and J. Murphy, Scattering in H^1 for the intercritical NLS with an inverse-square potential, *J. Differential Equations*, 264(2018), 3174–3211.

[36] V. G. Makhankov and V. K. Fedynanin, Non-linear effects in quasi-one-dimensional models of condensed matter theory, *Phys. Rep.*, 104(1984), 1–86.

[37] C. Morawetz, Time decay for the nonlinear Klein-Gordon equations, *Proc. Roy. Soc. Ser. A*, 306(1968), 291–296.

[38] K. Nakanishi, Energy Scattering for Nonlinear Klein–Gordon and Schrödinger Equations in Spatial Dimensions 1 and 2, *J. Fuct. Anal.*, 169(1999), 201-225.

[39] T. Ozawa and K. M. Rogers, Sharp Morawetz estimates, *J. Anal. Math.*, 121(2013), 163–175.

[40] M. Poppenberg, On the local well posedness of quasi-linear Schrödinger equations in arbitrary space dimension, *J. Differential Equations*, 172(2001), 83–115.

[41] B. Ritchie, Relativistic self-focusing and channel formation in laser-plasma interactions, *Phys. Rev. E*, 50(1994), 687–689.

[42] B. Schlein, Quantum dynamics, coherent states and Bogoliubov transformations. XVIIth International Congress on Mathematical Physics, 197–216, World Sci. Publ., Hackensack, NJ, 2014.

[43] B. Schlein, Effective equations for quantum dynamics. European Congress of Mathematics, 515-529, Eur. Math. Soc., Zürich, 2013.

[44] B. Schlein, Derivation of effective evolution equations from microscopic quantum dynamics. Evolution equations, 511-572, Clay Math. Proc., 17, Amer. Math. Soc., Providence, RI, 2013.

[45] W. A. Strauss, Nonilinear scattering theory at low energy, *J. Funct. Anal.*, 41(1981), 110–133.

[46] T. Tao, Nonlinear dispersive equations. Local and global analysis. CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. xvi+373 pp. ISBN: 0-8218-4143-2.

[47] T. T. Tao, M. Visan and X. Y. Zhang, The nonlinear Schrödinger equation with combined power-type nonlinearities, *Comm. Partial Differential Equations*, 32 (2007), 1281–1343.

[48] Y. Tsutsumi, Scattering problem for nonlinear Schrödinger equation, *Ann. Inst. H. Poincaré Phys. Théor.*, 43(1985), 321-347.

[49] Y. Tsutsumi and K. Yajima, The asymptotic behavior of nonlinear Schrödinger equations, *Bull. Amer. Math. Soc.*, 11(1984), 186–188.

[50] M. Visan, The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions, *Duke Math. J.*, 138(2007), 281–374.

[51] J. Y. Zhang, J. Q. Zheng, Scattering theory for nonlinear Schrödinger equations with inverse-square potential, *J. Fuct. Anal.*, 267(2014), 2907–2932.