SINGULAR LIMIT OF LAMÉ EQUATIONS

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Abstract. In this paper, we study the asymptotic behavior of the solution to the Lamé equations with a parameter ε. We prove that the solution will converge to the solution of a Maxwell type equations as $\varepsilon \to 0$; Meanwhile we will show that the solution converges to the solution of a Stokes type equations as $\varepsilon \to \infty$.

1. Introduction

Consider the following Lamé equations

$$\begin{cases} \operatorname{curl}^2 \mathbf{u}_\varepsilon - \varepsilon^2 \nabla \operatorname{div} \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon = \mathbf{f} & \text{in } \Omega, \\ \mathbf{u}_\varepsilon = \mathbf{u}^0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded, connected and C^2 domain in $\mathbb{R}^3$, $\varepsilon > 0$ is a parameter. We are interested in the limit of the solution $\mathbf{u}_\varepsilon$ as $\varepsilon \to 0$ or $\varepsilon \to \infty$.

Note that

$$\operatorname{curl}^2 = -\Delta + \nabla \operatorname{div}.$$

The first equation of (1.1) can be written as

$$-\mu \Delta \mathbf{u}_\varepsilon - (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon = \mathbf{f}, \tag{1.2}$$

where $\mu = 1$ and $\lambda = \varepsilon^2 - 2$ are Lamé constants. Lamé equations are proposed in elasticity. We refer to [12] for the physical background and applications. Lamé equations are also closely related to the Maxwell equations, see for instance [5].

Equation (1.2) is a non-degenerately elliptic equation. When $\varepsilon \to 0$, the equation becomes degenerate. We are interested in the connection between Lamé equations and degenerately elliptic equations. Equations involving operator curl with a small parameter have been extensively studied by many mathematicians. They dealt with eigenvalue problems, Landau-de Gennes model, Meissner solution of Ginzburg-Landau model and so on in this topic, see for instance [4, 10, 13].

When $\varepsilon \to \infty$, it can be considered as a penalization parameter for the vanishing divergence condition in the Lamé equations. For this kind of problem, M. Costabel and M. Dauge [4] studied the following Lamé eigenproblems

$$\operatorname{curl}^2 \mathbf{u} - s \nabla \operatorname{div} \mathbf{u} = \sigma \mathbf{u},$$

and obtained an interesting result: the eigenvalues converge to the Stokes eigenvalues as the parameter s tends to infinity.

Key words and phrases. Elliptic equations, Lamé equations, asymptotic behavior, Maxwell equations, Stokes equations, convergence.

In this paper, we make a comparison between the two cases where $\varepsilon \to 0$ and $\varepsilon \to \infty$, and we show completely different asymptotic behaviors of the solution $\mathbf{u}_\varepsilon$ to the Lamé equations (1.1).

• **Case 1:** When $\varepsilon \to 0$, we prove that the term $\varepsilon^2 \nabla \operatorname{div} \mathbf{u}_\varepsilon$ is negligible and $\mathbf{u}_\varepsilon$ weakly converges to $\mathbf{u}$ in $\mathcal{H}(\Omega, \operatorname{curl})$, the solution of the Maxwell type equations

$$\begin{cases} \operatorname{curl}^2 \mathbf{u} + \mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{u} \times \nu = \mathbf{u}^0 \times \nu & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where ν is the unit outer normal on $\partial\Omega$ and $\mathcal{H}(\Omega, \operatorname{curl})$ is defined as

$$\mathcal{H}(\Omega, \operatorname{curl}) = \{\mathbf{u} \in L^2(\Omega, \mathbb{R}^3) : \operatorname{curl} \mathbf{u} \in L^2(\Omega, \mathbb{R}^3)\}.$$

Note that curl^2 is degenerately elliptic and $\operatorname{div} \mathbf{u} \neq 0$ in Ω. So equations (1.3) is degenerately elliptic equations. Thus as ε tends to 0, the equations (1.1) links the non-degenerately elliptic equations and degenerately elliptic equations. As we all know, it is difficult and interesting to study degenerate equations. One can see [9] for the second order degenerate equations and [6, 7, 8, 11, 14] for the Maxwell equations.

An observation is that the Lamé equations and the Maxwell type equations admit different kinds of boundary conditions. As $\varepsilon \to 0$, the solution sequence $\mathbf{u}_\varepsilon$ may change dramatically in a thin layer near the boundary $\partial\Omega$.

• **Case 2:** When $\varepsilon \to \infty$, we prove that the solution $\mathbf{u}_\varepsilon$ strongly converges to $\mathbf{u}$ in $H^1(\Omega, \mathbb{R}^3)$, where $\mathbf{u}$ solves the following Stokes type equations

$$\begin{cases} -\Delta \mathbf{u} + \nabla p + \mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}^0 & \text{on } \partial\Omega. \end{cases}$$

In the equations above, $\operatorname{div} \mathbf{u} = 0$ in Ω. Thus the equations can be written as

$$\begin{cases} \operatorname{curl}^2 \mathbf{u} + \nabla p + \mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}^0 & \text{on } \partial\Omega. \end{cases}$$

As $\varepsilon \to \infty$, the Lamé equations are closely related to the Stokes equations. We find that the term $\varepsilon^2 \nabla \operatorname{div} \mathbf{u}_\varepsilon$ converges to a gradient term ∇p where p represents the pressure term in the Stokes equations. Moreover, the limiting equations admit the same kinds of boundary condition with the Lamé equations which is different from case 1.

The paper is organized as follows. In section 2, we list several known results that will be used in this paper. In section 3, we show existence of H^1 weak solution to (1.1). Then we prove that the solution $\mathbf{u}_\varepsilon$ converges to the solution of a Maxwell type equations in $\mathcal{H}(\Omega, \operatorname{curl})$ as $\varepsilon \to 0$. In section 4, we prove that the solution $\mathbf{u}_\varepsilon$ to (1.1) converges to the solution of a Stokes type equations in $H^1(\Omega, \mathbb{R}^3)$ as $\varepsilon \to \infty$ and moreover, we obtain the convergence rate.

2. Notions and Preliminaries

In this section, we will introduce our notations and several known results that will be used in this paper.

Throughout this paper we assume that $\Omega \subset \mathbb{R}^3$ is a bounded and connected domain with C^2 boundary. We use C to denote a generic constant independent of ε, which may vary from line to line. Let ν is the unit outer normal on $\partial\Omega$. Then we introduce some spaces:

$$\begin{aligned}
&L_0^2(\Omega) = L^2(\Omega)/\mathbb{R},\\
&\mathcal{H}(\Omega, \operatorname{div}) = \{\mathbf{u} \in L^2(\Omega, \mathbb{R}^3) : \operatorname{div}\mathbf{u} \in L^2(\Omega)\},\\
&H^1_{t_0}(\Omega, \operatorname{div} 0) = \{\mathbf{u} \in H^1(\Omega, \mathbb{R}^3) : \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u} \times \nu = \mathbf{0} \text{ on } \partial\Omega\}.
\end{aligned}$$

Remark 2.1. *The space $L_0^2(\Omega)$ is isomorphic with the closed subspace of $L^2(\Omega)$ made up of functions with a zero mean, see* [2, Lemma IV.1.9.].

The following lemmas are needed in this paper.

Lemma 2.2. (see [6, p.212, Corollary 1]) Let Ω be a bounded domain in $\mathbb{R}^3$ with a C^2 boundary $\partial\Omega$. Then there exists a constant $C = C(\Omega)$ such that for any $u \in H^1(\Omega, \mathbb{R}^3)$,

$$\|\mathbf{u}\|_{H^1(\Omega)} \le C\big(\|\mathbf{u}\|_{L^2(\Omega)} + \|\operatorname{curl}\mathbf{u}\|_{L^2(\Omega)} + \|\operatorname{div}\mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{u}\times\nu\|_{H^{1/2}(\partial\Omega)}\big).$$

Lemma 2.3. (Nečas inequality [2]) Let Ω be a Lipschitz domain in $\mathbb{R}^3$ with compact boundary. Define the space

$$\chi(\Omega) = \{p \in H^{-1}(\Omega), \nabla p \in H^{-1}(\Omega, \mathbb{R}^3)\},$$

endowed with the norm

$$\|p\|_{\chi(\Omega)} = \|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)}.$$

Then we have $\chi(\Omega) = L^2(\Omega)$ and, moreover, there is a $C > 0$ such that

$$\|p\|_{L^2(\Omega)} \le C\|p\|_{\chi(\Omega)},\ \forall\, p \in L^2(\Omega).$$

A new Poincaré type equality follows from the Nečas inequality, see [2, Proposition IV.1.7.].

Lemma 2.4. *Let Ω be a connected, bounded, Lipschitz domain in $\mathbb{R}^3$. There exists a $C > 0$ such that for all $p \in L^2(\Omega)$, we have*

$$\|p\|_{H^{-1}(\Omega)} \le C\Big(\frac{1}{|\Omega|}\Big|\int_\Omega p\,dx\Big| + \|\nabla p\|_{H^{-1}(\Omega)}\Big).$$

3. Asymptotic behavior of weak solution for small parameter

In this section, we study the asymptotic behavior of the solution $\mathbf{u}_\varepsilon$ to (1.1) as $\varepsilon \to 0$.

First, we prove the existence of the weak solution $\mathbf{u}_\varepsilon$ to problem (1.1) for given data $(\mathbf{f}, \mathbf{u}^0)$.

Lemma 3.1. *Let $\mathbf{f} \in H^{-1}(\Omega, \mathbb{R}^3)$ and $\mathbf{u}^0 \in H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^3)$. For any $\varepsilon > 0$, there exists a unique weak solution $\mathbf{u}_\varepsilon \in H^1(\Omega, \mathbb{R}^3)$ to* (1.1).

Proof. The natural quadratic form associated with equations (1.1) is

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \big(\operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\mathbf{v} + \varepsilon^2 \operatorname{div}\mathbf{u}\,\operatorname{div}\mathbf{v} + \mathbf{u}\cdot\mathbf{v}\big)\,dx, \quad \forall\, \mathbf{u}, \mathbf{v} \in H^1_0(\Omega, \mathbb{R}^3). \tag{3.1}$$

Since $\mathbf{u}^0 \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$, there exists a vector field $\tilde{\mathbf{u}}^0 \in H^1(\Omega, \mathbb{R}^3)$ such that

$$\tilde{\mathbf{u}}^0 = \mathbf{u}^0 \text{ on } \partial\Omega. \tag{3.2}$$

Thus we turn to the following problem:
Find $\mathbf{u}_\varepsilon \in H^1(\Omega, \mathbb{R}^3)$ such that

$$\begin{cases} \mathbf{u}_\varepsilon - \tilde{\mathbf{u}}^0 \in H_0^1(\Omega, \mathbb{R}^3), \\ a(\mathbf{u}_\varepsilon - \tilde{\mathbf{u}}^0, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}, H_0^1} - a(\tilde{\mathbf{u}}^0, \mathbf{v}) \quad \forall\, \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^3), \end{cases} \tag{3.3}$$

where $a(\cdot,\cdot)$ is the quadratic form in (3.1). Fix ε,

$$\exists\, c(\varepsilon) > 0, \quad \forall\, \mathbf{u}_\varepsilon \in H_0^1(\Omega), \quad a(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \geq c(\varepsilon) \|\mathbf{u}_\varepsilon\|^2_{H^1(\Omega)}.$$

Thus $a(\cdot,\cdot)$ is coercive on $H_0^1(\Omega, \mathbb{R}^3)$. By Lax-Milgram theorem, there exists a unique solution $\mathbf{u}_\varepsilon$ to (1.1). □

It is well known that every vector field $\mathbf{u} \in L^2(\Omega, \mathbb{R}^3)$ can be decomposed as

$$\mathbf{u} = \mathbf{v} + \nabla p,$$

where

$$\operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \quad p \in H_0^1(\Omega).$$

This is the classical Hodge decomposition of L^2-vector fields in bounded domains Ω in $\mathbb{R}^3$ with smooth boundary, see [6], [8].

With the help of the Hodge decomposition, for $\mathbf{f} \in \mathcal{H}(\Omega, \operatorname{div})$, we have the following decomposition:

$$\mathbf{f} = \mathbf{f}_0 + \nabla\phi_f, \quad \operatorname{div} \mathbf{f}_0 = 0, \quad \phi_f \in H_0^1(\Omega), \quad \Delta\phi_f \in L^2(\Omega). \tag{3.4}$$

For the boundary value $\mathbf{u}^0 \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$, we have the following decomposition:

Lemma 3.2. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a C^2 boundary $\partial\Omega$. Let $\mathbf{u}^0 \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$, then we have a unique decomposition*

$$\mathbf{u}^0 = \mathbf{v}^0 + \nabla\xi^0|_{\partial\Omega}, \quad \mathbf{v}^0 \cdot \nu = 0, \quad \xi^0 = 0 \textit{ on } \partial\Omega. \tag{3.5}$$

Proof. Using H^2 trace theorem([7, Theorem 1.6]), there exists a function $\xi^0 \in H^2(\Omega)$ such that

$$\xi^0 = 0, \quad \frac{\partial \xi^0}{\partial \nu} = \mathbf{u}^0 \cdot \nu \text{ on } \partial\Omega.$$

Since $\nabla\xi^0 \in H^1(\Omega, \mathbb{R}^3)$, the trace of $\nabla\xi^0$ is well defined and equals $(\mathbf{u}^0 \cdot \nu) \cdot \nu$. Let

$$\mathbf{v}^0 = \mathbf{u}^0 - (\mathbf{u}^0 \cdot \nu) \cdot \nu.$$

Then $\mathbf{v}^0$ is also uniquely determined and satisfies $\mathbf{v}^0 \cdot \nu = 0$ on $\partial\Omega$. □

For the solution $\mathbf{u}_\varepsilon \in H^1(\Omega, \mathbb{R}^3)$ to (1.1), we decompose it as follows:

$$\mathbf{u}_\varepsilon = \mathbf{v} + \nabla\xi_\varepsilon + \mathbf{w}_\varepsilon,$$

where $\mathbf{v}$ and ξ_ε satisfy the following equations:

$$\begin{cases} \operatorname{curl}^2 \mathbf{v} + \mathbf{v} = \mathbf{f}_0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v}_T = \mathbf{v}^0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

and

$$\begin{cases} -\varepsilon^2\Delta\xi_\varepsilon + \xi_\varepsilon = \phi_f & \text{in } \Omega, \\ \xi_\varepsilon = \xi^0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

where $\mathbf{f}_0$, ϕ_f and ξ^0 are functions in decompositions (3.4) and (3.5), and $\mathbf{v}_T = (\nu \times \mathbf{v}) \times \nu$ denotes the tangential component of $\mathbf{v}$. Then $\mathbf{w}_\varepsilon$ solves the following equations

$$\begin{cases} \operatorname{curl}^2\mathbf{w}_\varepsilon - \varepsilon^2\nabla\operatorname{div}\mathbf{w}_\varepsilon + \mathbf{w}_\varepsilon = \mathbf{0} & \text{in } \Omega, \\ \mathbf{w}_\varepsilon \times \nu = \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{w}_\varepsilon \cdot \nu = \mathbf{u}^0 \cdot \nu - \mathbf{v} \cdot \nu - \dfrac{\partial \xi_\varepsilon}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

3.1. **Analysis of** (3.6) **and** (3.7)**.**

Lemma 3.3. *Let $\mathbf{f}_0$ and $\mathbf{v}^0$ be the vector fields in decompositions* (3.4) *and* (3.5)*. Then there exists a unique weak solution $\mathbf{v} \in H^1(\Omega, \mathbb{R}^3)$ to* (3.6)*.*

Proof. Since

$$\exists\, c > 0, \quad \forall\, \mathbf{v} \in H^1_{t0}(\Omega, \operatorname{div} 0), \quad \int_\Omega \big(\operatorname{curl}\mathbf{v}\cdot\operatorname{curl}\mathbf{v} + \mathbf{v}\cdot\mathbf{v}\big)dx \geq c\|\mathbf{v}\|^2_{H^1_{t0}(\Omega,\operatorname{div} 0)},$$

thus the quadratic form is coercive on $H^1_{t0}(\Omega, \operatorname{div} 0)$. Also since $\operatorname{div}\mathbf{f}_0 = 0$ in Ω, by Lax-Milgram theorem, we can have the unique solution $\mathbf{v} \in H^1(\Omega, \mathbb{R}^3)$ to (3.6). □

For equations (3.7), we have the following lemma:

Lemma 3.4. *Let ϕ_f be the function in decomposition* (3.4) *and ξ_ε be the solution to* (3.7)*. Then we have*

(a)

$$\xi_\varepsilon \to \phi_f \text{ in } H^1(\Omega), \qquad \text{as } \varepsilon \to 0.$$

(b) *There exists a constant C independent of ε such that*

$$\Big\|\frac{\partial \xi_\varepsilon}{\partial \nu}\Big\|_{H^{1/2}(\partial\Omega)} \leq C\{\|\operatorname{div}\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{f}\|_{L^2(\Omega)}\}.$$

Proof. (a) Let $\zeta_\varepsilon = \xi_\varepsilon - \phi_f$. Then

$$\begin{cases} -\varepsilon^2\Delta\zeta_\varepsilon + \zeta_\varepsilon = \varepsilon^2\Delta\phi_f & \text{in } \Omega, \\ \zeta_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

We have

$$\varepsilon^2\int_\Omega |\nabla\zeta_\varepsilon|^2 dx + \int_\Omega |\zeta_\varepsilon|^2 dx = \varepsilon^2\int_\Omega \Delta\phi_f\zeta_\varepsilon\, dx.$$

So

$$\begin{aligned} \int_\Omega |\zeta_\varepsilon|^2 dx &\leq \varepsilon^2\int_\Omega \Delta\phi_f\zeta_\varepsilon\, dx \\ &\leq \varepsilon^2\Big(\int_\Omega |\Delta\phi_f|^2 dx\Big)^{\frac12}\Big(\int_\Omega \zeta_\varepsilon^2 dx\Big)^{\frac12}. \end{aligned}$$

Thus

$$\|\zeta_\varepsilon\|_{L^2(\Omega)} \leq \varepsilon^2\|\Delta\phi_f\|_{L^2(\Omega)}.$$

We have

$$
\begin{aligned}
\varepsilon^2 \int_\Omega |\nabla \zeta_\varepsilon|^2 dx &\le \varepsilon^2 \int_\Omega \Delta\phi_f \zeta_\varepsilon dx \\
&\le \varepsilon^2 \Big(\int_\Omega |\Delta\phi_f|^2 dx\Big)^{\frac{1}{2}} \Big(\int_\Omega \zeta_\varepsilon^2 dx\Big)^{\frac{1}{2}} \\
&\le \varepsilon^4 \int_\Omega |\Delta\phi_f|^2 dx.
\end{aligned}
$$

Thus

$$
\|\nabla\zeta_\varepsilon\|_{L^2(\Omega)} \le \varepsilon \|\Delta\phi_f\|_{L^2(\Omega)}.
$$

So

$$
\begin{aligned}
&\zeta_\varepsilon \to 0 \text{ in } H^1(\Omega), \qquad \text{as } \varepsilon \to 0. \\
&\xi_\varepsilon \to \phi_f \text{ in } H^1(\Omega), \qquad \text{as } \varepsilon \to 0.
\end{aligned}
$$

(b) Since $\xi_\varepsilon = 0$ on $\partial\Omega$, using the div-curl-gradient inequality (Lemma 2.2), we have

$$
\|\nabla\xi_\varepsilon\|_{H^1(\Omega)} \le C(\Omega)\{\|\Delta\xi_\varepsilon\|_{L^2(\Omega)} + \|\nabla\xi_\varepsilon\|_{L^2(\Omega)}\},
$$

where $C(\Omega)$ is a constant depending on Ω. Hence

$$
\begin{aligned}
\|\frac{\partial\xi_\varepsilon}{\partial\nu}\|_{H^{1/2}(\partial\Omega)} \le C(\Omega)\|\nabla\xi_\varepsilon\|_{H^1(\Omega)} &\le C\{\|\Delta\xi_\varepsilon\|_{L^2(\Omega)} + \|\nabla\xi_\varepsilon\|_{L^2(\Omega)}\} \\
&= C\{\frac{1}{\varepsilon^2}\|\xi_\varepsilon - \phi_f\|_{L^2(\Omega)} + \|\nabla\xi_\varepsilon\|_{L^2(\Omega)}\} \\
&= C\{\frac{1}{\varepsilon^2}\|\zeta_\varepsilon\|_{L^2(\Omega)} + \|\nabla\xi_\varepsilon\|_{L^2(\Omega)}\} \\
&\le C\{\|\Delta\phi_f\|_{L^2(\Omega)} + \|\nabla\phi_f\|_{L^2(\Omega)}\} \\
&= C\{\|\mathrm{div}\,\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{f}\|_{L^2(\Omega)}\}.
\end{aligned}
$$

□

3.2. **The limit of solution $\mathbf{u}_\varepsilon$ to** (1.1) **as** $\varepsilon \to 0$**.**

Theorem 3.5. *Assume* $\mathbf{f} \in \mathcal{H}(\Omega, \mathrm{div}\,)$ *and* $\mathbf{u}^0 \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$*. Let* $\mathbf{u}_\varepsilon \in H^1(\Omega, \mathbb{R}^3)$ *be the solution to* (1.1)*. Then we have*

$$
\mathbf{u}_\varepsilon \rightharpoonup \mathbf{v} + \nabla\phi_f \text{ weakly in } \mathcal{H}(\Omega, \mathrm{curl}\,), \qquad \text{as } \varepsilon \to 0,
$$

where $\mathbf{v}$ *is the unique solution to* (3.6)*,* ϕ_f *is the function in the decomposition* (3.4) *and* $\mathbf{v} + \nabla\phi_f$ *satisfies the Maxwell type equations* (1.3)*.*

Proof. Step 1: Reduce the problem (3.8) to a new problem with homogeneous boundary data.

Since Ω is C^2, by [7, Proposition 1.3], there exists a function $\psi_\varepsilon \in H^2(\Omega)$ such that

$$
\begin{cases}
\Delta^2\psi_\varepsilon = 0 & \text{in } \Omega, \\
\psi_\varepsilon = 0 & \text{on } \partial\Omega, \\
\dfrac{\partial\psi_\varepsilon}{\partial\nu} = \dfrac{\partial\xi^0}{\partial\nu} - \mathbf{v}\cdot\nu - \dfrac{\partial\xi_\varepsilon}{\partial\nu} & \text{on } \partial\Omega,
\end{cases}
$$

and

$$\|\psi_\varepsilon\|_{H^2(\Omega)} \le C\{\|\frac{\partial \xi^0}{\partial \nu}\|_{H^{1/2}(\partial\Omega)} + \|\mathbf{v}\cdot\nu\|_{H^{1/2}(\partial\Omega)} + \|\frac{\partial \xi_\varepsilon}{\partial \nu}\|_{H^{1/2}(\partial\Omega)}\}.$$

Lemma 3.4 leads to the estimate

$$\|\psi_\varepsilon\|_{H^2(\Omega)} \le C\{\|\frac{\partial \xi^0}{\partial \nu}\|_{H^{1/2}(\partial\Omega)} + \|\mathbf{v}\cdot\nu\|_{H^{1/2}(\partial\Omega)} + \|\mathrm{div}\,\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{f}\|_{L^2(\Omega)}\}.$$

Therefore we have

$$\psi_\varepsilon \rightharpoonup \psi \quad \text{weakly in } H^2(\Omega), \quad \text{as } \varepsilon \to 0.$$

Recall the decomposition:

$$\mathbf{u}_\varepsilon = \mathbf{v} + \nabla\xi_\varepsilon + \mathbf{w}_\varepsilon,$$

we set

$$\mathbf{z}_\varepsilon = \mathbf{w}_\varepsilon - \nabla\psi_\varepsilon.$$

Note that

$$\begin{aligned} \mathbf{z}_{\varepsilon T} &= \mathbf{w}^0_{\varepsilon T} - (\nabla\psi_\varepsilon)_T = \mathbf{0} \text{ on } \partial\Omega, \\ \mathbf{z}_\varepsilon\cdot\nu &= \mathbf{w}^0_\varepsilon\cdot\nu - \frac{\partial\psi_\varepsilon}{\partial\nu} = 0 \text{ on } \partial\Omega. \end{aligned}$$

Then we have

$$\begin{cases} \mathrm{curl}^2\mathbf{z}_\varepsilon - \varepsilon^2\nabla\mathrm{div}\,\mathbf{z}_\varepsilon + \mathbf{z}_\varepsilon = \varepsilon^2\nabla\Delta\psi_\varepsilon - \nabla\psi_\varepsilon & \text{in } \Omega, \\ \mathbf{z}_\varepsilon = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

Step 2: Analyze the vector field $\mathbf{z}_\varepsilon$.

Taking $\mathbf{z}_\varepsilon$ as a test vector field in the equations above, we have

$$\begin{aligned} &\int_\Omega (|\mathrm{curl}\,\mathbf{z}_\varepsilon|^2 + \varepsilon^2(\mathrm{div}\,\mathbf{z}_\varepsilon)^2 + |\mathbf{z}_\varepsilon|^2)dx = -\int_\Omega (\nabla\psi_\varepsilon\cdot\mathbf{z}_\varepsilon + \varepsilon^2\triangle\psi_\varepsilon\ \mathrm{div}\,\mathbf{z}_\varepsilon\ )dx \\ &\le \frac{1}{2}\int_\Omega |\nabla\psi_\varepsilon|^2dx + \frac{1}{2}\int_\Omega |\mathbf{z}_\varepsilon|^2dx + \frac{\varepsilon^2}{2}\int_\Omega |\triangle\psi_\varepsilon|^2dx + \frac{\varepsilon^2}{2}\int_\Omega |\mathrm{div}\,\mathbf{z}_\varepsilon|^2dx. \end{aligned}$$

That means

$$\begin{aligned} &\|\mathrm{curl}\,\mathbf{z}_\varepsilon\|_{L^2(\Omega)} + \|\varepsilon\mathrm{div}\,\mathbf{z}_\varepsilon\|_{L^2(\Omega)} + \|\mathbf{z}_\varepsilon\|_{L^2(\Omega)} \le C\big(\|\nabla\psi_\varepsilon\|_{L^2(\Omega)} + \|\triangle\psi_\varepsilon\|_{L^2(\Omega)}\big) \\ &\le C\{\|\mathbf{v}\cdot\nu\|_{H^{1/2}(\partial\Omega)} + \|\mathbf{u}^0\cdot\nu\|_{H^{1/2}(\partial\Omega)} + \|\Delta\phi_f\|_{L^2(\Omega)} + \|\nabla\phi_f\|_{L^2(\Omega)}\}. \end{aligned}$$

Thus there exists a $\mathbf{z}\in\mathcal{H}(\Omega,\mathrm{curl}\,)$ and a $q\in L^2(\Omega)$ such that, up to a subsequence,

$$\begin{aligned} &\mathbf{z}_\varepsilon \rightharpoonup \mathbf{z} \quad \text{weakly in } L^2(\Omega,\mathbb{R}^3), \\ &\mathrm{curl}\,\mathbf{z}_\varepsilon \rightharpoonup \mathrm{curl}\,\mathbf{z} \quad \text{weakly in } L^2(\Omega,\mathbb{R}^3), \\ &\varepsilon\mathrm{div}\,\mathbf{z}_\varepsilon \rightharpoonup q \quad \text{weakly in } L^2(\Omega), \text{ as } \ \varepsilon\to 0. \end{aligned} \tag{3.11}$$

Since $\mathbf{z}_\varepsilon$ is the solution of (3.10), we have

$$\int_\Omega (\mathrm{curl}\,\mathbf{z}_\varepsilon\cdot\mathrm{curl}\,\phi + \varepsilon^2\mathrm{div}\,\mathbf{z}_\varepsilon\ \mathrm{div}\ \phi + \mathbf{z}_\varepsilon\cdot\phi)\ dx = -\int_\Omega (\nabla\psi_\varepsilon\cdot\phi + \varepsilon^2\triangle\psi_\varepsilon\ \mathrm{div}\,\phi)\ dx,\ \forall\ \phi\in C_0^\infty(\Omega,\mathbb{R}^3).$$

Let $\varepsilon\to 0$, we have

$$\int_\Omega (\mathrm{curl}\,\mathbf{z}\cdot\mathrm{curl}\,\phi + \mathbf{z}\cdot\phi)\ dx = -\int_\Omega \nabla\psi\cdot\phi\ dx,\ \forall\ \phi\in C_0^\infty(\Omega,\mathbb{R}^3).$$

So $\mathbf{z}$ satisfies

$$\begin{cases} \operatorname{curl}^2\mathbf{z} + \mathbf{z} = -\nabla\psi & \text{in } \Omega, \\ \mathbf{z} \times \nu = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Step 3: Analyze the vector field $\mathbf{w}_\varepsilon$.

Since $\mathbf{w}_\varepsilon = \mathbf{z}_\varepsilon + \nabla\psi_\varepsilon$, using the results in step 1 and step 2, there exists a $\mathbf{w} \in \mathcal{H}(\Omega, \operatorname{curl})$ such that, up to a subsequence,

$$\mathbf{w}_\varepsilon \rightharpoonup \mathbf{w} \text{ weakly in } \mathcal{H}(\Omega, \operatorname{curl}), \quad \text{as } \varepsilon \to 0,$$

where

$$\mathbf{w} = \mathbf{z} + \nabla\psi, \tag{3.13}$$

and

$$\begin{cases} \operatorname{curl}^2\mathbf{w} + \mathbf{w} = \mathbf{0} & \text{in } \Omega, \\ \mathbf{w} \times \nu = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

From the equations above, we have $\operatorname{div}\mathbf{w} = 0$ in Ω. We claim that $\mathbf{w} = \mathbf{0}$ in Ω. Since $\mathbf{w} \in L^2(\Omega, \mathbb{R}^3)$, $\operatorname{curl}\mathbf{w} \in L^2(\Omega, \mathbb{R}^3)$, $\operatorname{div}\mathbf{w} = 0$ and $\mathbf{w} \times \nu = \mathbf{0}$, we have $\mathbf{w} \in H^1(\Omega, \mathbb{R}^3)$. Set $\theta \in H^2(\Omega)$, such that

$$\theta = 0, \quad \frac{\partial\theta}{\partial\nu} = \mathbf{w} \cdot \nu \text{ on } \partial\Omega.$$

Taking $\mathbf{w} - \nabla\theta \in H^1_0(\Omega, \mathbb{R}^3)$ as a test function in the weak formulation of (3.14), we have

$$\int_\Omega \big(|\operatorname{curl}\mathbf{w}|^2 + |\mathbf{w}|^2 - \mathbf{w} \cdot \nabla\theta\big)dx = 0.$$

Since $\operatorname{div}\mathbf{w} = 0$, we obtain

$$\int_\Omega \big(|\operatorname{curl}\mathbf{w}|^2 + |\mathbf{w}|^2\big)dx = 0.$$

Thus $\mathbf{w} = \mathbf{0}$ in Ω.

Step 4: Obtain the limit of $\mathbf{u}_\varepsilon$. Back to the decomposition:

$$\mathbf{u}_\varepsilon = \mathbf{v} + \nabla\xi_\varepsilon + \mathbf{w}_\varepsilon.$$

Using the result in step 3, we get

$$\mathbf{u}_\varepsilon \rightharpoonup \mathbf{v} + \nabla\phi_f \quad \text{weakly in } \mathcal{H}(\Omega, \operatorname{curl}), \quad \text{as } \varepsilon \to 0.$$

Combining with the definitions of $\mathbf{v}$ and ϕ_f, we prove that $\mathbf{v} + \nabla\phi_f$ solves the equations (1.3). □

Remark 3.6. *If we take divergence on both sides of* (3.10)*, then* $\operatorname{div}\mathbf{z}_\varepsilon$ *will satisfy the following equation in a very weak sense*

$$\begin{cases} -\varepsilon^2\Delta\operatorname{div}\mathbf{z}_\varepsilon + \operatorname{div}\mathbf{z}_\varepsilon = -\Delta\psi_\varepsilon & \text{in } \Omega, \\ \operatorname{div}\mathbf{z}_\varepsilon = \operatorname{div}_{\partial\Omega}(\pi\mathbf{z}_\varepsilon) + 2(\nu\cdot\mathbf{z}_\varepsilon)H + \dfrac{\partial}{\partial\nu}(\nu\cdot\mathbf{z}_\varepsilon) & \text{on } \partial\Omega, \end{cases} \tag{3.15}$$

where $\pi\mathbf{z}_\varepsilon(x)$*, for* $x \in \partial\Omega$*, denotes the projection of* $\mathbf{z}_\varepsilon$ *onto the tangent plane to* x *at* $\partial\Omega$*;* $\operatorname{div}_{\partial\Omega}$ *is the surface divergence of a tangent field to* $\partial\Omega$*;* $H(x)$ *is the mean curvature at* x*, see* [6] *for the expression of the divergence. The first and second terms of the divergence* $\operatorname{div}\mathbf{z}_\varepsilon$ *are well handled, but the estimate of the third term* $\frac{\partial}{\partial\nu}(\nu\cdot\mathbf{z}_\varepsilon)$ *will be a big challenge.*

As $\varepsilon \to 0$, the solution sequence $\mathbf{z}_\varepsilon$ may change dramatically in a thin layer near the boundary $\partial\Omega$.

4. Asymptotic behavior of weak solution for large parameter

Compared to the case where $\varepsilon \to 0$, we examine the asymptotic behavior of the solution $\mathbf{u}_\varepsilon$ to (1.1) as $\varepsilon \to \infty$. In this case, the parameter ε can be considered as a penalization parameter for the vanishing divergence condition in the Lamé equations. In this section, we rewrite the equations (1.1) as a penalty approximation of a Stokes type equations and then we use the Nečas inequality to obtain the limit of $\mathbf{u}_\varepsilon$ as $\varepsilon \to \infty$, see [2].

Given $\mathbf{f} \in H^{-1}(\Omega, \mathbb{R}^3)$ and $\mathbf{u}^0 \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$, it follows from the Lax-Milgram theorem that Lamé equations (1.1) has a unique weak solution in $H^1(\Omega, \mathbb{R}^3)$.

If $\mathbf{u}^0$ satisfies the compatibility condition, we will have the following main result:

Theorem 4.1. *Assume $\mathbf{f} \in H^{-1}(\Omega, \mathbb{R}^3)$ and $\mathbf{u}^0 \in H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^3)$ satisfying the compatibility condition*

$$\int_{\partial\Omega} \mathbf{u}^0 \cdot \nu d\sigma = 0.$$

Let $\mathbf{u}_\varepsilon$ be the unique solution of equations (1.1), then we have

$$\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} \text{ weakly in } H^1(\Omega, \mathbb{R}^3), \text{ as } \varepsilon \to \infty.$$

Moreover, $\mathbf{u}$ satisfies the following Stokes type equations

$$\begin{cases} -\Delta \mathbf{u} + \nabla p + \mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}^0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $p \in L^2_0(\Omega)$ is the pressure term.

Proof. Step 1: Reduce the problem (1.1) to a new problem with homogeneous boundary data.

Since $\mathbf{u}^0 \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$, using trace lifting operator theorem([2, Theorem III.2.22]), there exists a function $\tilde{\mathbf{u}}^0 \in H^1(\Omega, \mathbb{R}^3)$ such that $\tilde{\mathbf{u}}^0|_{\partial\Omega} = \mathbf{u}^0$. Since

$$\int_\Omega \operatorname{div} \tilde{\mathbf{u}}^0 dx = \int_{\partial\Omega} \mathbf{u}^0 \cdot \nu d\sigma = 0,$$

then from [2, Theorem IV.3.1], there exists a function $\bar{\mathbf{u}}^0 \in H^1_0(\Omega, \mathbb{R}^3)$ such that

$$\operatorname{div} \bar{\mathbf{u}}^0 = -\operatorname{div} \tilde{\mathbf{u}}^0.$$

Hence $\hat{\mathbf{u}}^0 = \tilde{\mathbf{u}}^0 + \bar{\mathbf{u}}^0$ satisfies the conditions

$$\operatorname{div} \hat{\mathbf{u}}^0 = 0 \text{ in } \Omega, \ \hat{\mathbf{u}}^0 = \mathbf{u}^0 \text{ on } \partial\Omega.$$

Let $\hat{\mathbf{u}}_\varepsilon = \mathbf{u}_\varepsilon - \hat{\mathbf{u}}^0$, then it satisfies the following equations

$$\begin{cases} \operatorname{curl}^2 \hat{\mathbf{u}}_\varepsilon - \varepsilon^2 \nabla \operatorname{div} \hat{\mathbf{u}}_\varepsilon + \hat{\mathbf{u}}_\varepsilon = \mathbf{f} - \operatorname{curl}^2 \hat{\mathbf{u}}^0 - \hat{\mathbf{u}}^0 & \text{in } \Omega, \\ \hat{\mathbf{u}}_\varepsilon = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Step 2: Estimate the vector field $\hat{\mathbf{u}}_\varepsilon$.

Taking $\hat{\mathbf{u}}_\varepsilon$ as a test function in the weak formulation of the equations above, we get

$$\|\operatorname{curl}\hat{\mathbf{u}}_\varepsilon\|^2_{L^2(\Omega)} + \varepsilon^2\|\operatorname{div}\hat{\mathbf{u}}_\varepsilon\|^2_{L^2(\Omega)} + \|\hat{\mathbf{u}}_\varepsilon\|^2_{L^2(\Omega)} \le \|\mathbf{f} - \operatorname{curl}^2\hat{\mathbf{u}}^0 - \hat{\mathbf{u}}^0\|_{H^{-1}(\Omega)}\|\hat{\mathbf{u}}_\varepsilon\|_{H^1_0(\Omega)}.$$

By using the Poincaré inequality, and the formular

$$\|\nabla\hat{\mathbf{u}}_\varepsilon\|^2_{L^2(\Omega)} = \|\operatorname{curl}\hat{\mathbf{u}}_\varepsilon\|^2_{L^2(\Omega)} + \|\operatorname{div}\hat{\mathbf{u}}_\varepsilon\|^2_{L^2(\Omega)},\ \forall\ \hat{\mathbf{u}}_\varepsilon \in H^1_0(\Omega,\mathbb{R}^3),$$

we deduce

$$\|\operatorname{curl}\hat{\mathbf{u}}_\varepsilon\|^2_{L^2(\Omega)} + (2\varepsilon^2 - 1)\|\operatorname{div}\hat{\mathbf{u}}_\varepsilon\|^2_{L^2(\Omega)} + 2\|\hat{\mathbf{u}}_\varepsilon\|^2_{L^2(\Omega)} \le C\|\mathbf{f} - \operatorname{curl}^2\hat{\mathbf{u}}^0 - \hat{\mathbf{u}}^0\|^2_{H^{-1}(\Omega)}.$$

According to the Nečas inequality(Lemma 2.3) and the trace lifting operator theorem([2, Theorem III.2.22]), we have

$$\|\mathbf{f} - \operatorname{curl}^2\hat{\mathbf{u}}^0 - \hat{\mathbf{u}}^0\|_{H^{-1}(\Omega)} \le C\big(\|\mathbf{f}\|_{H^{-1}(\Omega)} + \|\mathbf{u}^0\|_{H^{1/2}(\partial\Omega)}\big).$$

By div-curl-gradient inequality(Lemma 2.2), we obtain

$$\|\hat{\mathbf{u}}_\varepsilon\|_{H^1(\Omega)} \le C\big(\|\mathbf{f}\|_{H^{-1}(\Omega)} + \|\mathbf{u}^0\|_{H^{1/2}(\partial\Omega)}\big).$$

Step 3: Examine the limits of $\hat{\mathbf{u}}_\varepsilon$ and $\mathbf{u}_\varepsilon$.

We rewrite the equations (4.2) as follows

$$\begin{cases} -\Delta\hat{\mathbf{u}}_\varepsilon + \nabla p_\varepsilon + \hat{\mathbf{u}}_\varepsilon = \mathbf{f} - \operatorname{curl}^2\hat{\mathbf{u}}^0 - \hat{\mathbf{u}}^0 & \text{in } \Omega,\\ \operatorname{div}\hat{\mathbf{u}}_\varepsilon + \dfrac{1}{\varepsilon^2 - 1}p_\varepsilon = 0 & \text{in } \Omega,\\ \hat{\mathbf{u}}_\varepsilon = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

First, we use the Nečas inequality(Lemma 2.3) and the Poincaré inequality(Lemma 2.4) to estimate the pressure term p_ε,

$$\begin{aligned}\|p_\varepsilon\|_{L^2(\Omega)} \le C\|\nabla p_\varepsilon\|_{H^{-1}(\Omega)} &= \|\mathbf{f} - \operatorname{curl}^2\hat{\mathbf{u}}^0 - \hat{\mathbf{u}}^0 + \Delta\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}_\varepsilon\|_{H^{-1}(\Omega)}\\ &\le C\big(\|\mathbf{f}\|_{H^{-1}(\Omega)} + \|\mathbf{u}^0\|_{H^{1/2}(\partial\Omega)}\big) + C\|\nabla\hat{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)}.\end{aligned}$$

Combining this with the estimate above, we obtain

$$\|\hat{\mathbf{u}}_\varepsilon\|_{H^1(\Omega)} + \|p_\varepsilon\|_{L^2(\Omega)} \le C\big(\|\mathbf{f}\|_{H^{-1}(\Omega)} + \|\mathbf{u}^0\|_{H^{1/2}(\partial\Omega)}\big).$$

It follows from this result that there exists a vector field $\hat{\mathbf{u}} \in H^1_0(\Omega,\mathbb{R}^3)$ and a function $p \in L^2_0(\Omega)$ such that, up to a subsequence,

$$\begin{aligned}&\hat{\mathbf{u}}_\varepsilon \rightharpoonup \hat{\mathbf{u}} \text{ weakly in } H^1_0(\Omega,\mathbb{R}^3),\\ &p_\varepsilon \rightharpoonup p \text{ weakly in } L^2_0(\Omega),\ \text{ as } \varepsilon \to \infty.\end{aligned}$$

These weak convergences let us pass to the limit in the equations (4.3) and prove that the limits $(\hat{\mathbf{u}}, p)$ actually solve the following Stokes type equations

$$\begin{cases} -\Delta\hat{\mathbf{u}} + \nabla p + \hat{\mathbf{u}} = \mathbf{f} - \operatorname{curl}^2\hat{\mathbf{u}}^0 - \hat{\mathbf{u}}^0 & \text{in } \Omega,\\ \operatorname{div}\hat{\mathbf{u}} = 0 & \text{in } \Omega,\\ \hat{\mathbf{u}} = \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

Recall the definitions of $\hat{\mathbf{u}}_\varepsilon$ and $\hat{\mathbf{u}}^0$, we obtain

$$\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} = \hat{\mathbf{u}} + \hat{\mathbf{u}}^0 \text{ weakly in } H^1(\Omega,\mathbb{R}^3),\ \text{ as } \varepsilon \to \infty.$$

Note that

$$\operatorname{div}\hat{\mathbf{u}}^0 = 0 \text{ in } \Omega,$$

Thus

$$-\Delta\hat{\mathbf{u}}^0 = \operatorname{curl}\hat{\mathbf{u}}^0.$$

In conclusion, $\mathbf{u}$ satisfies equations (4.1). □

Finally, we claim that $\mathbf{u}_\varepsilon$ strongly converges to $\mathbf{u}$ in $H^1(\Omega, \mathbb{R}^3)$ as $\varepsilon \to \infty$ and we also obtain the convergence rate.

Theorem 4.2. *With the same assumption as in Theorem 4.1, there exists a constant $C > 0$ such that*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}\|_{H^1(\Omega)} \le \frac{C}{\varepsilon^2}.$$

Proof. We set $\mathbf{v}_\varepsilon = \mathbf{u}_\varepsilon - \mathbf{u}$, $q_\varepsilon = p_\varepsilon - p$, then we have

$$\begin{cases} -\Delta\mathbf{v}_\varepsilon + \nabla q_\varepsilon + \mathbf{v}_\varepsilon = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\mathbf{v}_\varepsilon + \dfrac{p_\varepsilon}{\varepsilon^2 - 1} = 0 & \text{in } \Omega, \\ \mathbf{v}_\varepsilon = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Taking $\mathbf{v}_\varepsilon$ as a test function in the first equation of (4.4), we get

$$\begin{aligned} \int_\Omega (|\nabla\mathbf{v}_\varepsilon|^2 + |\mathbf{v}_\varepsilon|^2)dx = \int_\Omega (\operatorname{div}\mathbf{v}_\varepsilon) q_\varepsilon dx &\le \frac{C}{\varepsilon^2}\|p_\varepsilon\|_{L^2(\Omega)}\|q_\varepsilon\|_{L^2(\Omega)} \le \frac{C}{\varepsilon^2}\|\nabla q_\varepsilon\|_{H^{-1}(\Omega)} \\ &\le \frac{C}{\varepsilon^2}\|\Delta\mathbf{v}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^{-1}(\Omega)} \le \frac{C}{\varepsilon^2}\|\nabla\mathbf{v}_\varepsilon\|_{L^2(\Omega)}. \end{aligned}$$

Therefore we obtain

$$\|\mathbf{v}_\varepsilon\|_{H^1(\Omega)} \le \frac{C}{\varepsilon^2},$$

and the theorem is proved. □

Acknowledgements. The author is grateful to Professor Xing-Bin Pan, for guidance and constant encouragement. This work was partially supported by the National Natural Science Foundation of China grant no. 11671143.

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