Explicit Commutativity and Stability for the Heun's Linear Time-Varying Differential Systems

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Abstract

This paper studies the commutativity and stability for the Heun's linear time-varying system (LTVS) with both zero and non-zero initial conditions(ICs). Given a LTVS A of order 2, we find it's commutative pair, that is a new LTVS B of order $m \leq n$. Explicit commutative theories and conditions for second-order LTVSs are derived and solved to simplify and guarantee the equivalency between the connected input-output of systems AB and BA. The explicit results obtained are juxtaposed by simulation in order to investigate the commutativity of Heun's differential system, sensitivity of Heun's system, effects due to disturbance on Heun's system, robustness on Heun's system and problems regarding the stability of Heun's system. This findings will help to fill the gap on stability problem, system behaviors, commutativity theory, and general theory for solutions of differential equations, which has significant contribution to science and unlimited application in engineering, our results are verify using Heun's differential system as well as authenticated by Wolfrom Mathematica 11 and Matlab.

Keywords: Commutativity, Heun's Differential Equation, Linear Time-Varying Systems, Asymptotic Stability, and Analogue Control.

1 Introduction

The most intuitive concepts and theory of differential equations are used in a wide variety of disciplines, which involve studying and predicting the changes that occur all around us, at all time, that is continuous time-varying systems (CTVSs). Some is rapid, some change is gradual, some is predictable, some change is random. The theory of differential equations have unlimited application in engineering field, such as heat transfer, theory of electric circuits, mechanical vibration, system identification in structural dynamics [1, 2].

For example, the application of differential equations has great significant in the area of control system, modern design, digital technology, modeling of physical systems, such as resistor-capacitor-inductor, circuit, voltage, current, temperature, pressure, displacement, and population models. The electric equipment we use today is an outcome of converting physical systems into mathematical model and mathematical equations that involves the use of differential equations [3, 4].

Cascade connection of subsystems is one of the most basic series connections, where the output of one component is fed into the input of another. The cascade connection methods have been developed to assist engineers for connection of components or subsystems to form a network. Cascade connection is used in control systems, electric and electronic to measure the open-circuit voltage and short-circuit current [5, 6].

Commutativity is a process that involves a cascade (or series) connection between two dynamical systems A and B, the relation between the input-output of the combined systems is base on the parameters of each system. Whenever the two connections AB and BA produce similar input-output pairs irrespective of the applied input, then we called it commutative systems; that is AB and BA are equivalent, i.e., AB=BA.

The explicit commutativity conditions provide significant important toward development of commutativity theory, it ease the use of commutativity conditions. Marshall introduced the concept of explicit commutativity conditions for first-order systems in [7]. The author in [8, 9] studied the explicit commutativity conditions for second-order systems. And that for third-order systems was presented in [10]. Commutativity conditions for fourth-order systems and that of fifth-order CTVLs was derived and summarised in [11] and [12] respectively. Recently, commutativity for sixth-order CTVLs was studied in [13]. Regarding the application of commutativity, decomposition of LTVSs play a vital role, the authors in [14, 15, 16] studied and presented the decomposition of fourth-order LTVSs. The relationship between feedback conjugates with time-varying forward and feedback path gains have been validated in [17]. Explicit commutative pairs of some well-known second-order LTVSs and explicit commutativity conditions for commutativity of second-order CTVLs was investigated in [18] and [19] respectively. Moreover, transitivity property of commutativity for second-order linear time-varying analogue systems has been studied in [20]. Commutativity theories has been extended to discrete LTVSs and the authors in [21, 22] studied the commutativity of discrete LTVSs and first order discrete LTVSs respectively.

Explicit methods for the stability of linear time-varying differential state apace systems was explored in [23]. Stability and robustness for input and output feedback systems was verified in [24].

A reason why we developed more interested in the commutativity and stability of Heun's differential systems might be due to the fact that the spheroidal wave functions, Lame function, Mathieu function and hypergeometric function are all special case of Heun's functions. Because of this, their applications to science and engineering is significant. Mathematician, engineer and scientist have tackle more difficult problems such as Heun's differential system, which lead to solution, no solution or singularities, more especially at the complex plane see [25].

This paper derived and proof the simplex explicit commutativity theory and condition for second-order LTVSs with non-zero ICs. We consider Heun's differential system as a case study in order to verify our explicit results, which was supported by simulation. Furthermore, stability for Heun's differential system was investigated. However, the explicit commutativity and stability for Heun's differential system have not been present in the literature yet; and this paper fills in the gab. This paper is outlined as follows: Some preliminary results for second-order LTVSs are given in Section 2. Section 3 present the explicit commutativity conditions for non-relaxed second-order LTVSs systems. Application of second-order LTVSs on Heun's differential system are given in order to demonstrate the effectiveness of our results in section 4. Section 5 consider the stability of Heun's differential system. Finally, we conclude in Section 6.

2 Preliminary Results on Second-Order LTVSs

Given two analog second-order LTVSs as;

$$A : a_2(t)y_A''(t) + a_1(t)y_A'(t) + a_0(t)y_A(t) = x_A(t), \tag{1a}$$
$$B : b_2(t)y_B''(t) + b_1(t)y_B'(t) + b_0(t)y_B(t) = x_B(t); \tag{1b}$$

where $x_A(t)$, $y_A(t)$ and $x_B(t)$, $y_B(t)$ are the input and output of system A and B respectively; $y_A(t)$, $y_A'(t)$, $y_B(t)$, $y_B'(t)$ are the ICs at initial time (IT) t_0.

Definition 1 Two LTVSs A described by Eq. (1a) is said to be commutative with another LTVSs B of the same type as expressed in Eq. (1b) if they have the same input-output relation irrespective of the applied input.

Suppose the connection between two LTVSs A and B take place in cascade as shown in Fig. 1, the input-output relation of the system depends on the parameters of each system and on the fact that which system appears first. If both the connections in Fig. 1(a) and Fig. 1(b) have the same input-output relation irrespective of the applied input, then the systems are said to be commutative. The inputs and output are expressed as x and y, AB and BA represent the cascade connection in Fig. 1(a) and Fig. 1(b) below.

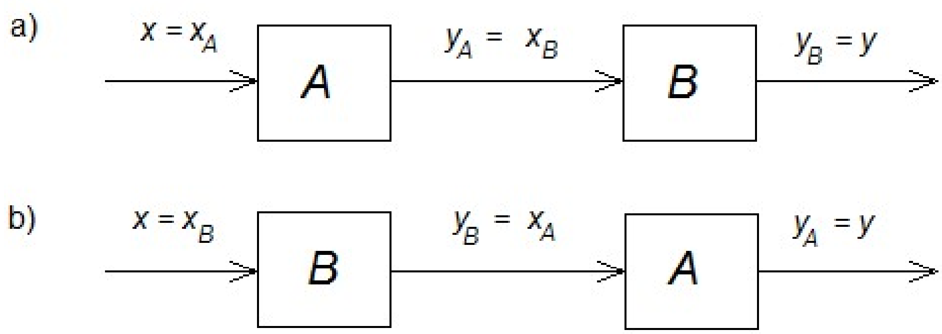

Figure 1: Cascade connection of differential systems.

The explicit commutative relation between the input and output of LTVS A described by Eq. (1a) with that of LTVS B of the same type as expressed in Eq. (1b) is that Eq. (2) and Eq. (3) must be equivalent.

$$
\begin{aligned}
\mathrm{AB} = \left(a_2 b_2 y^{(4)}\right) + \left(a_2 b_1 + a_1 b_2 + 2a_2 b_2'\right) y^{(3)} + \\
\left(a_2 b_0 + a_1 b_1 + a_0 b_2 + 2a_2 b_1' + a_1 b_2' + a_2 b_2''\right) y'' + \\
\left(a_1 b_0 + a_0 b_1 + 2a_2 b_0' + a_1 b_1' + a_2 b_1''\right) y' + \left(a_0 b_0 + a_1 b_0' + a_2 b_0''\right) y.
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
\mathrm{BA} = \left(a_2 b_2 y^{(4)}\right) + \left(a_1 b_2 + a_2 b_1 + 2b_2 a_2'\right) y^{(3)} + \\
\left(b_2 a_0 + a_1 b_1 + b_0 a_2 + 2b_2 a_1' + b_1 a_2' + b_2 a_2''\right) y'' + \\
\left(b_1 a_0 + b_0 a_1 + 2b_2 a_0' + b_1 a_1' + b_2 a_1''\right) y' + \left(a_0 b_0 + b_1 a_0' + b_2 a_0''\right) y.
\end{aligned} \tag{3}
$$

Theorem 1 *(See [8]) The necessary and sufficient conditions for a second-order LTVS A to be commutative with another LTVS B under zero initial is that the coefficients of B are expressed in terms of the coefficients of A as*

$$
\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_2 & 0 & 0 \\ a_1 & a_2^{0.5} & 0 \\ a_0 & f_{32} & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix}, \quad f_{32} = \frac{1}{4}[a_2^{-0.5}(2a_1 - a_2')]; \tag{4a}
$$

$$
-a_2^{0.5} \frac{d}{dt}[a_0 - f_{32}^2 - a_2^{0.5} f_{32}'] c_1 \ = 0, \tag{4b}
$$

where c_2, c_1, c_0 are constants and it must satisfy ($4b$). Eq. (4a) and Eq. (4b) are commutativity formulas of relaxed second-order LTVSs in matrix and differential form respectively.

3 Commutativity for Non-Relaxed Second-Order Systems

Regarding the case of commutative of second-order LTVSs under non-zero ICs, Eq. (4a) and Eq. (4b) are not sufficient, and this necessitate the realization of another set of commutativity conditions:
Theorem 2 *The commutativity for second-order LTVS A with non-zero initial conditions with another second or lower-order LTVS B are that:*

i) Explicit formulas for 2^{nd} order LTVSs in Eq.(4a) and Eq. (4b) must be satisfied.

ii) The ICs at the initial time (IT) $t_0 \leq t$ must hold:

$$\left\{ \binom{2}{m} \begin{bmatrix} 1 & 0 \\ -A_2^{-1}A_1 & A_2^{-1} \end{bmatrix} - \binom{m}{2} \begin{bmatrix} 0 & 1 \\ B_2^{-1} & -B_2^{-1}B_1 \end{bmatrix} \right\} \begin{bmatrix} Y_A \\ Y_B \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}; \tag{5}$$

where
$Y_A = [y_A(t),\ y'_A(t)]^T$,
$Y_B = [y_B(t),\ y'_B(t)]^T$ and the matrix A_1 (A_2, B_1, B_2) are described by there entries a'_{ij} (a''_{ij}, b'_{ij}, b''_{ij}) respectively:

$$\begin{aligned}
a'_{ij} &= \sum_{s=max(0,i-j)}^{i-1} \frac{(i-1)!}{s!(i-1-s)!} a^s_{j-i+s}; \quad i = 1, m, \quad j = 1, 2, \\
a''_{ij} &= \sum_{s=0}^{i-j} \frac{(i-1)!}{s!(i-1-s)!} a^s_{j-i+n+s}; \quad i = 1, m, \quad j = 1, m; \\
&= 0 \ \ for \ \ i = 1, \ldots, m-1, \quad j = i+1, \ldots, m, \\
b'_{ij} &= \sum_{s=max(0,i-j)}^{i-1} \frac{(i-1)!}{s!(i-1-s)!} b^s_{j-i+s}; \quad i = 1, 2, \quad j = 1, m, \\
b''_{ij} &= \sum_{s=max(0,i-j-m)}^{i-j} \frac{(i-1)!}{s!(i-1-s)!} b^s_{j-i+m+s}; \quad i = 1, 2, \quad j = 1, \ldots, i; \\
&= 0 \ \ for \ \ i = 1, \quad j = i+1, \ldots, 2,
\end{aligned} \tag{6}$$

. ***Proof:***
Part i) The author in [8] proof the explicit formula for the commutativity of second-order LTVSs with zero ICs, we analogously make used of it.

Part ii) While the second part of theorem 2 is the special case of [Theorem: Koksal] in [12] at $n = 2$ and $m \leq 2$, hence the proof follows from the general case by considering $n = 2$ and $m \leq 2$.

3.1 Explicit commutativity conditions for non-relaxed second-order systems

Commutativity conditions obtained from the previous section are presented in explicit form. Simplifying Eqs. (5) and (6) for $n = m = 2$ gives:

$$Y_B = \begin{bmatrix} y_B(t) \\ y'_B(t) \end{bmatrix} = \begin{bmatrix} y_A(t) \\ y'_A(t) \end{bmatrix} = Y_A, \tag{7a}$$

$$A_1 Y_A + A_2 \begin{bmatrix} y''_A \\ y^{(3)}_A \end{bmatrix} = Y_B \tag{7b}$$

$$B_1 Y_B + B_2 \begin{bmatrix} y''_B \\ y^{(3)}_B \end{bmatrix} = Y_A \tag{7c}$$

$$\begin{bmatrix} A_2^{-1}(I - A_1) - B_2^{-1}(I - B_1) \end{bmatrix} \begin{bmatrix} y_A(t) \\ y'_A(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{7d}$$

Eq. (7a) indicate that the ICs for second-order LTVSs A and B must be the same, while Eq. (7b)) and Eq. (7c) are the ICs for systems AB and BA respectively, and Eq. (7d) indicates that, the vector $Y_A = Y_B$ is in the null space of $\begin{bmatrix} A_2^{-1}(I - A_1) - B_2^{-1}(I - B_1) \end{bmatrix}$ at IT t_0. Computing A_1, A_2, B_1, B_2 by using Eq. (6) generates:

$$A_1 = \begin{bmatrix} a_0 & a_1 \\ a'_0 & a'_1 + a_0 \end{bmatrix}, \; A_2 = \begin{bmatrix} a_2 & 0 \\ a'_2 + a_1 & a_2 \end{bmatrix}; \tag{8a}$$

$$B_1 = \begin{bmatrix} b_0 & b_1 \\ b'_0 & b'_1 + b_0 \end{bmatrix}, \; B_2 = \begin{bmatrix} b_2 & 0 \\ b'_2 + b_1 & b_2 \end{bmatrix}. \tag{8b}$$

Theorem 3 *The simplex form of the explicit necessary and sufficient conditions for the commutativity of a second-order LTVS A with non-zero ICs with another second or lower-order LTVS B are that:*

i) *The conditions of Theorem 2 are satisfied.*

ii) *In addition, $c'_i s$ should satisfy.*

$$c_2 + c_0 - 1 = \mp \quad c_1 \sqrt{1 - a_0 + \frac{a_1^2}{4a_2} + \frac{a'_1}{2} - \frac{a_1 a'_2}{2a_2} + \frac{3(a'_2)^2}{16a_2} - \frac{a''_2}{4}}. \tag{9}$$

iii) *Further more, the initial conditions must satisfy*

$$c_1 y'_A = - \quad \frac{1}{\sqrt{a_2}} \left(c_2 + c_0 - 1 + \frac{a_1 c_1}{2a_2^{1/2}} - \frac{c_1 a'_2}{4a_2^{1/2}} \right) y_A. \tag{10}$$

Proof:
Part i) It is obvious that Theorem 2 has been addressed.
Part ii) In order to proof the relation between the $c'_i s$ in Eq.(9), the following steps will lead to the result.
Inserting Eq. (8a) into Eq.(7b), organizing and simplifying the terms, and also multiplying the

inverse of A_2 with the simplified terms, we obtain the following matrix after a rigorous work:

$$\begin{bmatrix} \frac{(1-a_0)}{a_2} & -\frac{a_1}{a_2} \\ -\frac{a_0'}{a_2}+\frac{-a_1-a_2'}{a_2^2}-\frac{a_0\left(-a_1-a_2'\right)}{a_2^2} & \frac{1}{a_2}-\frac{a_0+a_1'}{a_2}-\frac{a_1\left(-a_1-a_2'\right)}{a_2^2} \end{bmatrix}\begin{bmatrix} y_A(t) \\ y_A'(t) \end{bmatrix}=\begin{bmatrix} y_A'' \\ y_A^{(3)} \end{bmatrix}. \tag{11}$$

Repeating the same procedure of Eq. (11) by inserting Eq. (8b) into Eq.(7c), organizing and simplifying the terms, and also multiplying the inverse of B_2 with the simplified terms, we obtain:

$$\begin{bmatrix} \frac{(1-b_0)}{b_2} & -\frac{b_1}{b_2} \\ -\frac{b_0'}{b_2}+\frac{-b_1-b_2'}{b_2^2}-\frac{b_0\left(-b_1-b_2'\right)}{b_2^2} & \frac{1}{b_2}-\frac{b_0+b_1'}{b_2}-\frac{b_1\left(-b_1-b_2'\right)}{b_2^2} \end{bmatrix}\begin{bmatrix} y_A(t) \\ y_A'(t) \end{bmatrix}=\begin{bmatrix} y_B'' \\ y_B^{(3)} \end{bmatrix}. \tag{12}$$

Substituting Eq. (8a) and Eq. (8b) into Eq. (7d) gives:

$$\begin{bmatrix} \frac{1}{a_2}-\frac{a_0}{a_2}-\frac{1}{b_2}+\frac{b_0}{b_2} & -\frac{a_1}{a_2}+\frac{b_1}{b_2} \\ k_1 & k_2 \end{bmatrix}\begin{bmatrix} y_A(t) \\ y_A'(t) \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{13}$$

where

$$k_1=-\frac{a_0'}{a_2}+\frac{-a_1-a_2'}{a_2^2}-\frac{a_0\left(-a_1-a_2'\right)}{a_2^2}+\frac{b_0'}{b_2}-\frac{-b_1-b_2'}{b_2^2}+\frac{b_0\left(-b_1-b_2'\right)}{b_2^2}$$

$$k_2=\frac{1}{a_2}-\frac{1}{b_2}-\frac{a_0+a_1'}{a_2}+\frac{a_1^2+a_1a_2'}{a_2^2}+\frac{b_0+b_1'}{b_2}-\frac{b_1^2+b_1b_2'}{b_2^2}.$$

Substituting b_2, b_1 and b_0 of Eq. (4a) in Eq. (13), after some mathematical computations, we obtain the following matrix

$$\begin{bmatrix} \frac{1}{a_2}-\frac{1}{a_2c_2}+\frac{c_0}{a_2c_2}+\frac{a_1c_1}{2a_2^{3/2}c_2}-\frac{c_1a_2'}{4a_2^{3/2}c_2} & \frac{c_1}{\sqrt{a_2}c_2} \\ k_3 & k_4 \end{bmatrix}\begin{bmatrix} y_A(t) \\ y_A'(t) \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{14}$$

where

$$\begin{aligned} k_3=&-\frac{a_1}{a_2^2}+\frac{c_1}{a_2^{3/2}c_2^2}+\frac{a_1}{a_2^2c_2}-\frac{c_0c_1}{a_2^{3/2}c_2^2}-\frac{a_1c_1^2}{2a_2^2c_2^2}-\frac{a_1c_0}{a_2^2c_2}-\frac{a_1^2c_1}{2a_2^{5/2}c_2}-\frac{a_0c_1}{a_2^{3/2}c_2}+\\ &\frac{c_1a_1'}{2a_2^{3/2}c_2}-\frac{a_2'}{a_2^2}+\frac{a_2'}{a_2^2c_2}-\frac{a_1c_1a_2'}{2a_2^{5/2}c_2}+\frac{c_1^2a_2'}{4a_2^2c_2^2}-\frac{c_0a_2'}{a_2^2c_2}+\frac{3c_1\left(a_2'\right)^2}{8a_2^{5/2}c_2}-\frac{c_1a_2''}{4a_2^{3/2}c_2},\end{aligned}$$

$$k_4=\frac{1}{a_2}-\frac{c_1^2}{a_2c_2^2}-\frac{1}{a_2c_2}-\frac{3a_1c_1}{2a_2^{3/2}c_2}+\frac{c_0}{a_2c_2}-\frac{3c_1a_2'}{4a_2^{3/2}c_2}.$$

For the existence of commutative with non-zero ICs, the coefficient matrix in Eq. (14) must be singular, so that its determinant must be zero at time $t=t_0$. All the result obtained from the evaluation conducted at time $t=t_0$ lead to Eq. (9), hence the result of the second part of Theorem 3.

Part iii) It can be easily be verify that Eq. (10) is obtained as a result of the matrix in Eq. (14), this proof the third part of Theorem 3

Corollary 1 *Supposed* $c_1=0$*, then the conditions in Theorem* 3

i) *Regarding the constants in ii) must be*

$$c_2 + c_0 - 1 = 0. \tag{15}$$

ii) *Pertaining the initial condition in iii) should be*

$$y'_A = y_A. \tag{16}$$

Proof:
Part i) It is obvious that Corollary 1 is a special case of Theorem 3 at $c_1 = 0$.
Part ii) By considering $c_1 = 0$ and Part i) of Corollary 1, the initial conditions can be arbitrary selected. Hence the proof of Corollary 1.

4 Application of Commutativity on Heun's Differential system

In this section, we want to validate our work by applying the method and explicit results obtained from the previous section on Heun's differential system.

Considering the Heun's differential system given by

$$y''_A(t) + \left(\frac{\gamma}{t} + \frac{\delta}{t-1} + \frac{\epsilon}{t-a}\right) y'_A(t) + \left[\frac{\alpha\beta\mathrm{t} - q}{t(t-1)(t-a)}\right] y_A(t) = x_A(t), \tag{17}$$

where

$\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0.$
To find the unknown constant in Eq.(17), we substitute the coefficients of Eq. (17) into Eq. (4b)

$$\begin{aligned} k = \frac{-q + t\alpha\beta}{(-1+t)t(-a+t)} - \frac{1}{4}\Bigg[2\left(-\frac{\gamma}{t^2} - \frac{\delta}{(-1+t)^2} - \frac{\epsilon}{(-a+t)^2}\right) + \\ \left(\frac{\gamma}{t} + \frac{\delta}{-1+t} + \frac{\epsilon}{-a+t}\right)^2\Bigg] c_1. \end{aligned} \tag{18}$$

Simplifying Eq. (18) and collecting terms together lead to

$$\begin{aligned} 4a^2kt^2 - 8a(1+a)kt^3 + 4\left(1+4a+a^2\right)kt^4 - 8(1+a)kt^5 + 4kt^6 = \\ -a^2(-2+\gamma)\gamma c_1 + 2at\{-2q + \gamma[-2+\gamma + a(-2+\gamma+\delta) + \epsilon]c_1\} + \\ 2t^3\left[-2(q + \alpha\beta + a\alpha\beta) + (-2+\gamma+\delta+\epsilon)(\gamma + a\gamma + a\delta + \epsilon)c_1\right] + \\ t^4\{4\alpha\beta - \left[\gamma^2 + \delta^2 + 2\delta(-1+\epsilon) + (-2+\epsilon)\epsilon + 2\gamma(-1+\delta+\epsilon)\right] c_1\} + \\ t^2\{4(q + aq + a\alpha\beta) - \left[\left(1+4a+a^2\right)\gamma^2 + a^2(-2+\delta)\delta + 2a\delta\epsilon\right] c_1\} + \\ t^2\{(-2+\epsilon)\epsilon + 2\gamma\left[-1 + a^2(-1+\delta) + \epsilon + 2a(-2+\delta+\epsilon)\right] c_1\}. \end{aligned} \tag{19}$$

By equating the coefficients of Eq. (19) and solving the equations for the unknown constants with the help of Wolfrom Mathematica 11, we categorically classified the results into three cases as follows

Case 1: With constant feedback conjugate (CFC) ($c_1 = 0$), with

$$q = 0,\ a = -1,\ \gamma = 0,\ \delta = 1,\ \epsilon = 1,\ \alpha = 1,\ \beta = 0,\ k = 0,\ c_1 = 0. \tag{20}$$

Case 2: Without CFC ($c_1 \neq 0$), with

$$q = 0,\ a = -1,\ \gamma = 0,\ \delta = 2,\ \epsilon = 2,\ \alpha = 2,\ \beta = 1,\ k = 0,\ c_1 \neq 0. \tag{21}$$

Case 3: With CFC ($c_1 \neq 0$), with

$$q = 0,\ a = 1,\ \gamma = 0,\ \delta = 1,\ \epsilon = 0,\ \alpha = \frac{1}{2},\ \beta = -\frac{1}{2},\ k = -\frac{1}{4},\ c_1 \neq 0. \tag{22}$$

Regarding the first case, substituting the constants from Eq. (20) into Eq. (17), we obtain a new Heun's differential system as

$$A : y_A''(t) + \left(\frac{1}{-1+t} + \frac{1}{1+t} \right) y_A'(t) = x_A(t). \tag{23}$$

Substituting the coefficients of Eq. (23) into Eq. (4a) gives

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{-1+t} + \frac{1}{1+t} & 1 & 0 \\ 0 & \frac{1}{2}\left(\frac{1}{-1+t} + \frac{1}{1+t} \right) & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix}. \tag{24}$$

From Eq. (24), the commutative pairs of A is given as

$$B : c_2 y_B''(t) + \left[c_2 \left(\frac{1}{-1+t} + \frac{1}{1+t} \right) + c_1 \right] y_B'(t) + \frac{1}{2} \left[c_1 \left(\frac{1}{-1+t} + \frac{1}{1+t} \right) + c_0 \right] y_B(t) = x_B(t). \tag{25}$$

We consider the commutativity of relaxed second-order LTVSs with zero ICs in differential form in Eq. (4b), inserting coefficients of Eq. (17) into Eq. (4b), one can obtain

$$\frac{1}{2} \left[\frac{1}{(-1+t)^3} + \frac{1}{(1+t)^3} - \frac{1}{(-1+t)(1+t)^2} - \frac{1}{(-1+t)^2(1+t)} \right] c_1 \quad = \quad 0. \tag{26}$$

Solving for the unknowns c_1 in Eq. (26) lead to $c_1 = 0$, which indicates that the only commutative pairs of A are its CFC. Base on this fact, the commutative pairs of A becomes

$$B : c_2 y_B''(t) + c_2 \left(\frac{1}{-1+t} + \frac{1}{1+t} \right) y_B'(t) + c_0 y_B(t) = x_B(t). \tag{27}$$

applying the coefficients of Eq. (23) into Eq. (11) at $t_0 = 0$, we acquired

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_A(0) \\ y_A'(0) \end{bmatrix} = \begin{bmatrix} y_A'' \\ y_A^{(3)} \end{bmatrix}. \tag{28}$$

Further substitution of the coefficients of Eq. (27) into Eq. (12) at $t_0 = 0$ result to

$$\begin{bmatrix} \frac{(1-c_0)}{c_2} & 0 \\ 0 & \frac{(1-c_0+2c_2)}{c_2} \end{bmatrix} \begin{bmatrix} y_A(0) \\ y_A'(0) \end{bmatrix} = \begin{bmatrix} y_B'' \\ y_B^{(3)} \end{bmatrix}. \tag{29}$$

We again substitute the coefficients of Eq. (23) and Eq. (27) into Eq. (13) at t_0, we obtain

$$\begin{bmatrix} 1 - \frac{1}{c_2} + \frac{c_0}{c_2} & 0 \\ -\frac{1}{-1+t} - \frac{1}{1+t} + \frac{\frac{2t}{t^2-1}}{c_2} + \frac{\left(-\frac{1}{-1+t} - \frac{1}{1+t}\right)c_0}{c_2} & k_5 \end{bmatrix} \begin{bmatrix} y_A(t) \\ y_A'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{30}$$

where

$$k_5 = 1 + \frac{1}{(-1+t)^2} + \frac{1}{(1+t)^2} - \frac{1}{c_2} + \frac{c_0 + \left(-\frac{1}{(-1+t)^2} - \frac{1}{(1+t)^2} \right) c_2}{c_2}.$$

For the commutativity with non-zero ICs to take place, the coefficient matrix in Eq. (30) must be singular, that is, determinant must be zero and this can only be guarantee if and only if

$$c_0 \to 1 - c_2. \tag{31}$$

The result obtained in Eq. (31) is the same with first part of Corollary 3.1 in Eq. (15), that is "$c_0 + c_2 - 1 = 0.\ iff\ c_1 = 0$", moreover, base on the fact that $c_1 = 0$, second part of Corollary 1 in Eq. (16) that says "$y_A' = y_A$, which means the initial conditions can be selected arbitrarily".

The results obtained from case 1 are supported using Simulink and illustrated in Figs. 2-4.

Considering a sinusoid of amplitude 5, bias -3 and frequency 7, under automatic variable step length with ODE 23 [Bogacki - Shampine] as the solver, the results are illustrated in **Fig. 2**. For $c_2 = c_0 = 1$ and $c_1 = 0$, AB and BA (solid blue curve) gives the same output response for zero ICs, by arbitrary choosing IT t_0 to be 0 and the initial states as $y_A(0) = y_B(0) = y_A'(0) = y_B'(0) = 1$, this lead to $AB1$ (doted-dash red curve) and $BA1$ (dashed-green curve) deviating from each other, this is because of the parameters c_6 and c_0 violated Eq. (31), commutativity of A and B is invalid.

Considering a sinusoid of amplitude 5, bias -3 and frequency 7, under automatic variable step length with ODE 23 [Bogacki - Shampine] as the solver, the results are illustrated in **Fig. 3**. For $c_2 = c_0 = 0.5$ and $c_1 = 0$, with IT t_0 to be 0 and the initial states as $y_A(0) = y_B(0) = y'_A(0) = y'_B(0) = -1$, AB and BA (solid blue curve) gives the same output response for nonzero ICs, and commutativity hold; for the sensitivity of AB and BA to ICs, (switching $y_A(0) = -1$ to 3), $AB1$ (doted dash-Red curve) and $BA1$ (dashed-green curve) are no longer commutative, these give different response as a result of violating the ICs in Eq. (16).

For $c_1 = 0$, $c_2 = 0.75$ and $c_0 = 0.25$ with unit step length as the input, a white noise with a noise power of 15 was inserted between A and B with the same ICs in **Fig. 4**, using the same ICs, a saw-tooth with amplitude 5 and frequency 5 rad/sec is use as the 2nd disturbance, see **Fig. 4** for both ABI, BAI (dot-red) and $ABII$ (dashed-green), $BAII$ (long dashed-black) respectively; hence commutativity is not satisfied because of the effect of external noise on the connections.

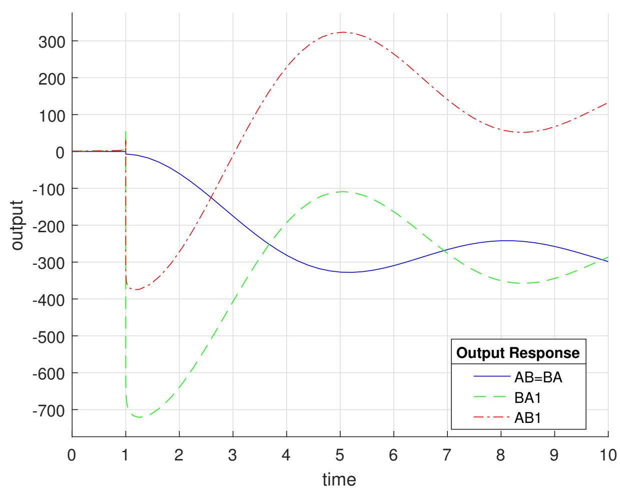

Figure 2: Simulation results of case 1 for $c_6 = 1$ and $c_0 = 1$

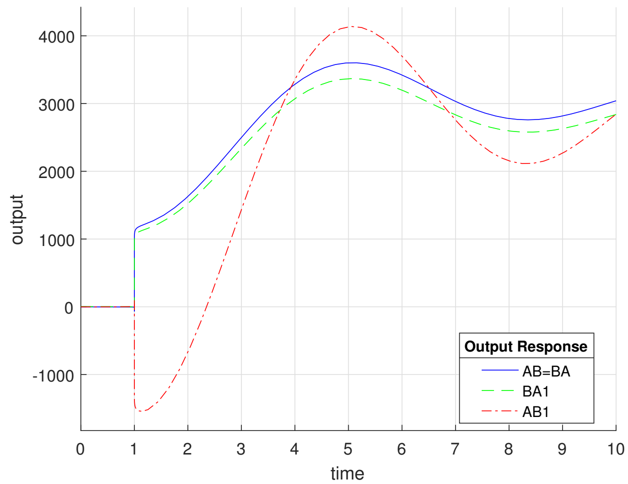

Figure 3: Simulation results of case 1 for $c_6 = 0.5$ and $c_0 = 0.5$

Considering the second case, we analogously substitute the constants in Eq. (21) into Eq. (17) in order to obtain a new Heun's differential system as

$$A : y_A''(t) + \left(\frac{2}{-1+t} + \frac{2}{1+t}\right) y_A'(t) + \left[\frac{2}{(-1+t)(1+t)}\right] y_A(t) = x_A(t). \tag{32}$$

Replacing the coefficients of Eq. (32) with that of Eq. (4a) gives the following matrix equation for second-order commutative system

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{-1+t} + \frac{2}{1+t} & 1 & 0 \\ \left(\frac{2t}{t^2-1}\right) & \frac{1}{2}\left(\frac{2}{-1+t} + \frac{2}{1+t}\right) & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix}. \tag{33}$$

From Eq. (33), the commutative pairs of A is given as

$$\begin{gathered} B : c_2 y_B''(t) + \left(c_1 + \left(\frac{2}{-1+t} + \frac{2}{1+t}\right) c_2\right) y_B'(t) + \\ \left(c_0 + \frac{1}{2}\left(\frac{2}{-1+t} + \frac{2}{1+t}\right) c_1 + \frac{2tc_2}{(t^2-1)}\right) y_B(t) = x_B(t). \end{gathered} \tag{34}$$

We consider the commutativity of relaxed second-order LTVSs with zero initial condition in differential form in Eq. (4b), inserting coefficients of Eq. (32) into Eq. (4b), one can obtain

$$[0]c_1 \quad = \quad 0. \tag{35}$$

Solving Eq. (35) lead to $c_1 \neq 0$, this indicates that the commutative pairs of A are not constant feedback conjugates.

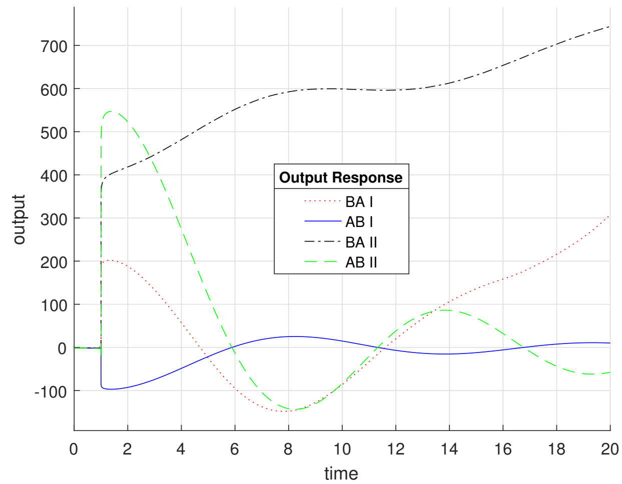

Figure 4: Simulation results of case 1 for $c_6 = 0.75$ and $c_0 = 0.25$

Substituting the coefficients of Eq. (32) with that of Eq. (11) at $t_0 = 0$ gives

$$\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} y_A(0) \\ y'_A(0) \end{bmatrix} = \begin{bmatrix} y''_A \\ y_A^{(3)} \end{bmatrix}. \tag{36}$$

Further substitution with the coefficients of Eq. (34) and that of Eq. (12) at $t_0 = 0$ lead to

$$\begin{bmatrix} \frac{1-(c_0-2c_2)}{c_2} & -\frac{c_1}{c_2} \\ \frac{2c_1}{c_2} - \frac{c_1}{c_2^2} + \frac{(c_0-2c_2)c_1}{c_2^2} & \frac{c_1^2}{c_2^2} + \frac{6c_2-c_0}{c_2} + \frac{1}{c_2} \end{bmatrix} \begin{bmatrix} y_A(0) \\ y'_A(0) \end{bmatrix} = \begin{bmatrix} y''_B \\ y_B^{(3)} \end{bmatrix}. \tag{37}$$

We again substitute the coefficients of Eq. (32) and Eq. (34) into Eq. (13) at $t = t_0$ result to

$$\begin{bmatrix} \frac{c_1}{c_2(t-1)} + \frac{c_1}{c_2(t+1)} + \frac{c_0}{c_2} - \frac{1}{c_2} + 1 & \frac{c_1}{c_2} \\ k_6 & -\frac{3c_1}{c_2(t-1)} - \frac{3c_1}{c_2(t+1)} - \frac{c_1^2}{c_2^2} + \frac{c_0}{c_2} - \frac{1}{c_2} + 1 \end{bmatrix} \begin{bmatrix} y_A(t) \\ y'_A(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{38}$$

where

$$k_6 = -\frac{c_1^2}{c_2^2(t-1)} - \frac{c_1^2}{c_2^2(t+1)} - \frac{3c_1}{c_2(t-1)^2} - \frac{6c_1}{c_2(t-1)(t+1)} - \frac{3c_1}{c_2(t+1)^2} - \frac{2c_0}{c_2(t-1)} + \frac{2}{c_2(t-1)} - \frac{2c_0}{c_2(t+1)} + \frac{2}{c_2(t+1)} - \frac{c_0c_1}{c_2^2} + \frac{c_1}{c_2^2} - \frac{2}{t-1} - \frac{2}{t+1}.$$

Solving the coefficient matrix in Eq. (38) to be singular $t_0 = 0$ give

$$c_1 = \mp c_2 + c_0 - 1. \tag{39}$$

Moreover, Eq. (39) at t_0 requires

$$(c_2 + c_0 - 1)y_A + c_1 y'_A = 0. \tag{40}$$

Inserting Eq. (39) into Eq. (40), the commutativity of A and B exist only if

$$y'_A = \pm\, y_A. \tag{41}$$

In order to verify the results obtained in Eq. (39) and Eq. (40), we consider Eq. (9) and Eq. (10) of Theorem 3. Applying the coefficients of Eq.(32) into Eq.(9) and Eq.(10), one can easily obtain the relations in Eq.(39) and Eq.(40) respectively.
The obtained results from case 2 are supported using Simulink and illustrated in Figs. 5-7. With a sinusoid of amplitude 150, bias -30 and frequency 100, under automatic variable step length with ODE 23 [Bogacki - Shampine] as the solver, Simulink results are illustrated in **Fig. 5**. For $c_2 = c_1 = c_0 = 1$, with initial time t_0 to be 0 and the initial states as $y_A(0) = y_B(0) = -2$ and $y'_A(0) = y'_B(0) = 2$, AB and BA (solid blue curve) gives the same output response, commutativity hold; for the sensitivity of AB and BA toward parameters, (changing $c_0 = 1$ to 100), $AB1$ (doted dash-Red curve) and $BA1$ (dashed-green curve) gives different response as a result of tempering with Eq. (39).

With a sinusoid of amplitude 15, bias -3 and frequency 1, under automatic variable step length with ODE 23 [Bogacki - Shampine] as the solver, Simulink results are depicted in **Fig. 6**. For $c_2 = c_1 = c_0 = 1$, with IT t_0 to be 0 and the initial states as $y_A(0) = y_B(0) = 1$ and $y'_A(0) = y'_B(0) = -1$, AB and BA (solid blue curve) gives the same output response, commutativity hold; for the sensitivity of AB and BA to ICs, with little change from ($y_A(0) = 1$ to 0.9), the commutativity for AB1 (doted dash-Red curve) and BA1 (dashed-green curve) is unequal, these give different response as a result of disrupting the initial conditions in Eq. (41).

For $c_1 = 1$, $c_2 = 1$ and $c_0 = 1$ with amplitude 150, bias -30 and frequency 100 applied on the connection as input, with zero initial time and initial states. The response for AB and BA (solid blue curve) is the same, hence commutativity hold; using the same conditions, a saw-tooth with amplitude 7 and frequency 15 rad/sec is use as the disturbance, see ABI (dashed-green) and BAI (doted-red) in **Fig. 7**;hence commutativity is not satisfied because of the effect of external noise on the connections. BAI is more affected than BAI.

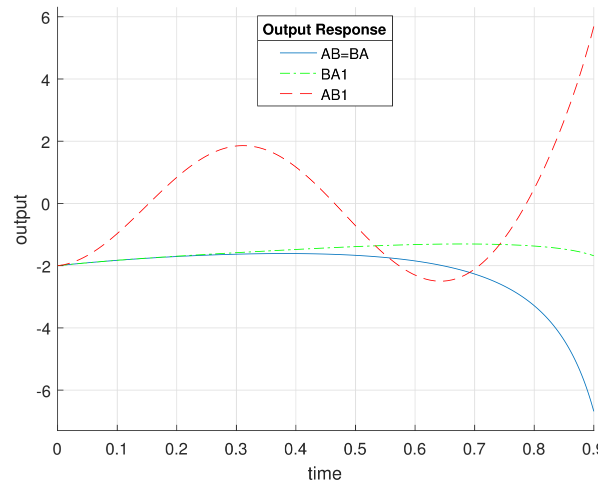

Figure 5: Simulation results of case 2 for $c_6 = c_1 = 1$ and $c_0 = 100$.

Pertaining the third case, substituting the constants in Eq. (22) into Eq. (17), we obtain a new Heun's differential system as

$$A = y_A''(t) + \left(\frac{1}{(-1+t)}\right) y_A'(t) - \frac{1}{4}\left(\frac{1}{(-1+t)^2} - 1\right) y_A(t) = x_A(t). \tag{42}$$

Applying the coefficient of Eq. (42) into Eq. (4a) gives the following matrix equation for second-order commutative system

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{(-1+t)} & 1 & 0 \\ \frac{1}{4} - \frac{1}{4(-1+t)^2} & \frac{1}{2(-1+t)} & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix}. \tag{43}$$

From Eq. (43), the commutative pairs of A is given as

$$\begin{aligned} B : c_2 y_B''(t) + \left(\frac{c_2}{t-1} + c_1\right) y_B'(t) + \\ \left(\frac{c_1}{2(t-1)} - \frac{c_2}{4(t-1)^2} + c_0 + \frac{c_2}{4}\right) y_B(t) = x_B(t). \end{aligned} \tag{44}$$

We consider the commutativity of relaxed second-order LTVSs with zero initial condition in differential form in Eq. (4b), inserting coefficients of Eq. (44) into Eq. (4b), one can obtain

$$[0]c_1 \quad = \quad 0. \tag{45}$$

Solving c_1 in Eq. (45) lead to $c_1 \neq 0$, which indicates the commutative pairs of A are not CFC.

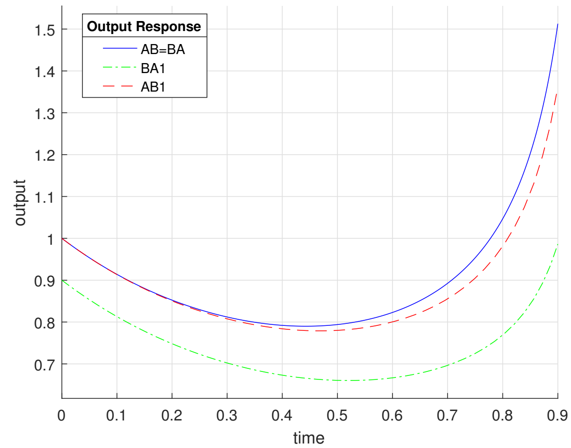

Figure 6: Simulation results of case 2 for $c_6 = c_1 = c_0 = 1$.

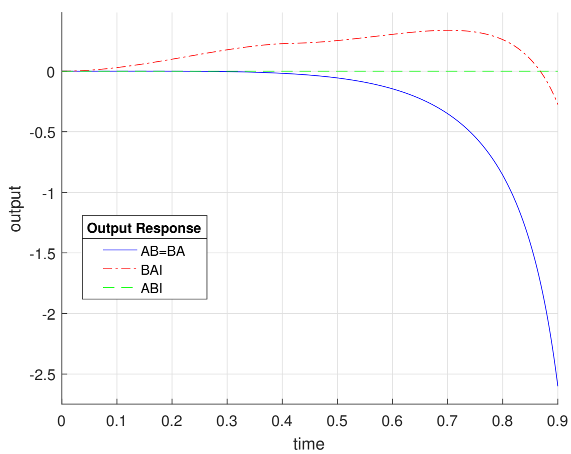

Figure 7: Simulation results of case 2 for $y_A(0) = y_B(0) = y'_A(0) = y'_B(0) = 0$.

Applying the coefficients of Eq. (42) into Eq. (11) at $t_0 = 0$, we acquired

$$\begin{bmatrix} 1 & 1 \\ \frac{3}{2} & 3 \end{bmatrix} \begin{bmatrix} y_A(0) \\ y'_A(0) \end{bmatrix} = \begin{bmatrix} y''_A \\ y^{(3)}_A \end{bmatrix}. \tag{46}$$

Further substitution with the coefficient of Eq. (44) and that of Eq. (12) at $t_0 = 0$ lead to

$$\begin{bmatrix} -\frac{c_0}{c_2} + \frac{1}{c_2} + \frac{c_1}{2c_2} & 1 - \frac{c_1}{c_2} \\ k_7 & \frac{c_1^2}{c_2^2} - \frac{3c_1}{2c_2} - \frac{c_0}{c_2} + \frac{1}{c_2} + 2 \end{bmatrix} \begin{bmatrix} y_A(0) \\ y'_A(0) \end{bmatrix} = \begin{bmatrix} y''_B \\ y^{(3)}_B \end{bmatrix}. \tag{47}$$

where

$$k_7 = -\frac{c_1^2}{2c_2^2} + \frac{c_1}{c_2} + \frac{c_0c_1}{c_2^2} - \frac{c_1}{c_2^2} - \frac{c_0}{c_2} + \frac{1}{c_2} + \frac{1}{2}.$$

We again substitute the coefficients of Eq. (42) and Eq. (44) into Eq. (13) at $t = t_0$, we obtain

$$\begin{bmatrix} \frac{c_1}{2c_2(t-1)} + \frac{c_0}{c_2} - \frac{1}{c_2} + 1 & \frac{c_1}{c_2} \\ k_8 & -\frac{c_1^2}{c_2^2} + \frac{3c_1}{2c_2} + \frac{c_0}{c_2} - \frac{1}{c_2} + 1 \end{bmatrix} \begin{bmatrix} y_A(t) \\ y_A'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{48}$$

where

$$k_8 = -\frac{c_1^2}{2c_2^2(t-1)} - \frac{3c_1}{4c_2(t-1)^2} - \frac{c_0}{c_2(t-1)} + \frac{1}{c_2(t-1)} - \frac{c_1}{4c_2} - \frac{c_0c_1}{c_2^2} + \frac{c_1}{c_2^2} + \frac{1}{t-1}.$$

Solving the coefficient matrix in Eq. (48) to be singular at $t_0 = 0$ give

$$c_1 = \mp \frac{2\,(c_0 + c_2 - 1)}{\sqrt{3}}. \tag{49}$$

Moreover, Eq. (49) at t_0 requires

$$(c_0 + c_2 - 1 + \frac{c_1}{2(t-1)})y_A + c_1 y_A' = 0. \tag{50}$$

Inserting Eq. (49) into Eq. (50), the commutativity of A and B exist only if

$$y_A' = \pm\,(\frac{\sqrt{3}-1}{2})y_A. \tag{51}$$

In order to verify the results obtained in Eq. (49) and Eq. (50), we consider Eq. (9) and Eq. (10) of Theorem 3. Applying the coefficients of Eq.(42) into Eq.(9) and Eq.(10), one can easily obtain the relation in Eq. (49) and Eq. (50) respectively.

The obtained results from case 3 are supported using Simulink and illustrated in Figs. 8-10. With a sinusoid of amplitude 2, bias -10 and frequency 5, under automatic variable step length with ODE 23 [Bogacki - Shampine] as the solver, Simulink results are illustrated in **Fig. 8**. For $c_2 = c_0 = 1$ and $c_1 = \frac{2}{\sqrt{3}}$, with IT t_0 to be 0 and the initial states as $y_A(0) = y_B(0) = \frac{1}{10}$ and $y_A'(0) = y_B'(0) = -\frac{\sqrt{3}-1}{20}$, AB and BA (solid blue curve) gives the same response. With the same ICs, but zero input, ABI and BAI (doted dash-Red curve) gives the same zero-input response, and with zero ICs and non-zero input, $ABII$ and $BAII$ (dashed-green curve) gives the same zero state response. Commutativity is satisfied for both relaxed case and unrelaxed case.

With a sinusoid of amplitude 2, bias -3 and frequency 5, under automatic variable step length with ODE 23 [Bogacki - Shampine] as the solver, Simulink results are illustrated in **Fig. 9**. For $c_2 = c_0 = 1$ and $c_1 = \frac{2}{\sqrt{3}}$, with initial time t_0 to be 0 and the initial states as $y_A(0) = y_B(0) = \frac{21}{100}$, $y_A'(0) = y_B'(0) = \frac{21(1-\sqrt{3})}{200}$, AB and BA (solid blue curve) gives the same response, hence commutativity hold; (reducing $y_A = \frac{21}{100}$ to $\frac{20}{100}$), the commutativity for

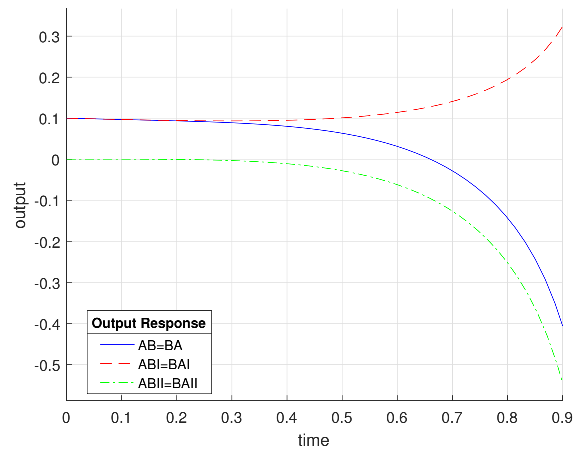

Figure 8: Simulation results of case 3

$AB1$ (doted dash-Red curve) and $BA1$ (dashed-green curve) is not the same, these give different response as a result of altering with the conditions in Eq.(51).

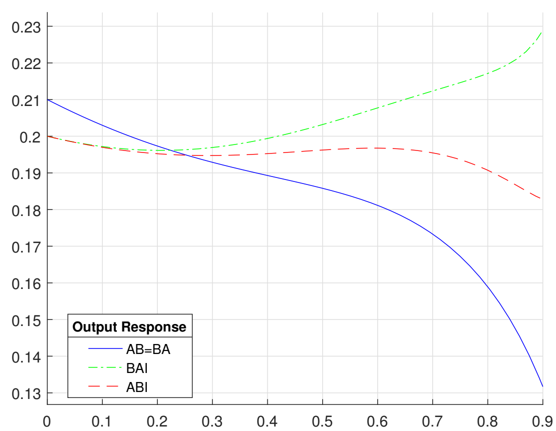

Figure 9: Simulation results of case 3 for $c_6 = c_0 = 1$ and $c_1 = \frac{2}{\sqrt{3}}$.

With a sinusoid of amplitude 7, bias -7 and frequency 7, under automatic variable step length with ODE 23 [Bogacki - Shampine] as the solver, Simulink results are illustrated in **Fig. 10**. For $c_2 = c_1 = c_0 = 1$, with initial time t_0 to be 0 and the initial states as $y_A(0) = y_B(0) = 1$, $y'_A(0) = y'_B(0) = \frac{1-\sqrt{3}}{2}$, AB and BA (solid blue curve) give the same response, hence commutativity hold; for sensitivity of AB and BA toward parameter, (changing $c_1 = \frac{2}{\sqrt{3}}$ to 1), the commutativity for $AB1$ (dashed-green curve) and $BA1$ (doted dash-Red curve) is not the same, these give different response as a result of tempering with the conditions in Eq.(50).

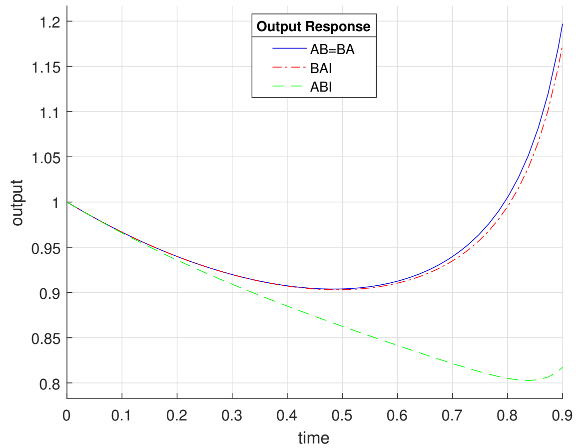

Figure 10: Simulation results of case 3 for $c_6 = c_0 = 1$ and $c_1 = 1$.

5 Stability of Heun's Differential system

Stability is a quantitative property of science and engineering systems, it plays a vital role in systems theory of science and engineering. Almost every workable system is designed to be stable, physically, an unstable system whose natural response grows without bound can damage the system. Based on this fact, the stability of Heun's LTVSs systems will be verified in this section. **Definition 2** The LTV system is said to be asymptotically stable if and only if x (t) starting from any finite initial state x_0 is bounded and $x(t) \to 0$ as $t \to \infty$. The asymptotically stable is frequently used in stability analysis.

The author in [23] proved that

i) The spectral decomposition of state transition matrix (STM) for LTV systems is explicitly written as

$$\Phi(t,0) = \sum_{i=1}^{n} exp\left[\int_{t_0}^{t} \lambda_i(\tau)d\tau\right] \alpha_i(t)\tau_i^T(t_0), \quad \forall t, \quad \forall t_0, \tag{52}$$

where $\{\lambda_i(t), \alpha_i(t)\}$ are the extended eigenpair of a LTV system, and $\tau_i^T(t)$ is the reciprocal bases of $\alpha_i(t)$.

ii) The LTV systems is asymptotically stable if and only if

$$\|exp\left[\int_{t_0}^{t} \lambda_i(\tau)d\tau\right] \alpha_i(t)\| \to 0, \quad as\ t \to \infty. \tag{53}$$

Let the extended eigenvectors be represented by

$$T(t) \quad = [\alpha_1(t), \alpha_2(t), ..., \alpha_n(t)]. \tag{54}$$

Let $T^{-1}(t)$ be the inverse matrix corresponding to the rows $\tau_i^T(t)$

$$T^{-1}(t) \quad = \begin{bmatrix} \tau_1^T(t) \\ . \\ . \\ . \\ \tau_n^T(t) \end{bmatrix}. \tag{55}$$

Also, the LTV systems can also be written inform of Riccati equation as

$$\lambda'(t) + \lambda^2(t) + a_1(t)\lambda(t) + a_0(t) = 0. \tag{56}$$

where $a_1(t)$, $a_0(t)$ are the coefficient of LTV systems.

We analogously make use of Eqs.(52-56) in order to verify the stability of Heun's differential system. Let

$$A = y''(t) + \left(\frac{1}{-1+t} + \frac{1}{1+t}\right) y'(t) = x_A(t). \tag{57}$$

Inserting the $a_1(t)$, $a_0(t)$ of Eq. (57) into Eq. (56), we now obtain the Riccati equation as

$$\lambda'(t) + \lambda^2(t) + \left(\frac{1}{-1+t} + \frac{1}{1+t}\right) \lambda(t) = 0. \tag{58}$$

Solving the non linear Riccati equation in Eq. (58) result to the following eigenvalues

$$\begin{aligned} \lambda_1(t) &= \frac{2}{(-1+t^2)\left(2 - \mathrm{Log}[-1-t] + \mathrm{Log}[-1+t]\right)}, \\ \lambda_2(t) &= -\frac{2}{(-1+t^2)\left(2 + \mathrm{Log}[-1-t] - \mathrm{Log}[-1+t]\right)}, \end{aligned} \tag{59}$$

while the corresponding eigenvectors are

$$\alpha_1(t) = \begin{bmatrix} 1 \\ \lambda_1(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{(-1+t^2)(2-\mathrm{Log}[-1-t]+\mathrm{Log}[-1+t])} \end{bmatrix}, \tag{60}$$

and

$$\alpha_2(t) = \begin{bmatrix} 1 \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{(-1+t^2)(2+\mathrm{Log}[-1-t]-\mathrm{Log}[-1+t])} \end{bmatrix}. \tag{61}$$

from Eq. (55), the reciprocal basis vectors are given as

$$\tau_1(t) = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}, \qquad \tau_2(t) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}. \tag{62}$$

Finally, we apply the explicit results obtained from Eqs. (59-62) into Eq. (54). After solving, we obtain

$$\Phi(t,0) = \begin{bmatrix} 1 & \frac{1}{2}(\mathrm{Log}[-1-t] - \mathrm{Log}[-1+t]) \\ 0 & \frac{1}{1-t^2} \end{bmatrix}. \tag{63}$$

Base on the result obtained from Eq. (63), the Heun's differential system in Eq. (57) proof to be unstable. Furthermore, the simulation from **Fig. 2**, **Fig. 3** and **Fig. 4** shows that, commutativity of system A and system B are unstable. Moreover, one can verify that, at $t_0 = 1$, there is an overlap signal response that lead to unstable.

6 Conclusion

This paper thoroughly investigates the commutativity and stability of Heun's LTVSs. Explicit commutativity conditions for second-order LTVSs with non-zero ICs are derived after vigorous mathematical computation and manipulation. The explicit formulas for second-order LTVSs with non-zero ICs in Eq.(9), Eq.(10), Eq.(15) and Eq.(16) are expressed in terms of c_2, c_1, c_0, a_2, a_1 and a_0, this findings provide us with a simple and easier way to find and verify the Commutativity of any second-order LTVSs with non-zero ICs. Base on our findings, we discovered that Heun's LTVSs systems posses both constant feedback conjugates and non CFC as its commutative pairs. The system is sensible toward changes in parameters and ICs. moreover, the system shows great level of commutativity imbalance toward noise disturbance. Explicit commutativity method was used in order to reduced the singularity of Heun's LTVSs, the stability issue of Heun's linear time-varying differential systems was also address, which proof to be unstable both explicitly and numerically. Commutativity play a vital role in solving problems regarding the stability of systems, singularity of systems, effects due to disturbance on systems, robustness on systems, sensitivity of systems and show up possible applications in science and engineering. All the explicitly and numerical results are well verified by examples treated with Wolfram Mathematica 11 and Matlab. Finally, our findings can only be applicable to second-order LTVSs, so it will be an open problem and a great idea to examine higher-order linear time-varying systems, complex systems, non-linear systems, discrete time systems, partial order systems and fractional order systems.

Conflict of interest Disclosure: The authors declare that there is no conflict of interest regarding the publication of this paper.

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