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Propagation of shock wave of the time fractional Gardner Burger equation in a multicomponent plasma using novel analytical method

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The time-fractional Gardner Burger (TFGB) equation is an efficient tool for studying nonlinear fluctuations of different types of the wave profiles, such as the gravity solitary waves in the ocean, the dust ion-acoustic wave (DIAW) in a plasma environment, etc. Here, to create an example for the existence of the classical Gardner Burger (GB) equation, a multi-component plasma environment has been considered and a classical GB equation has been derived using the basic governing equation by employing reductive perturbation technique (RPT). Further, the classical GB equation has been converted into the TFGB equation using the Agrawal's approach, where the Riesz fractional derivative has been adopted on the time-fractional term. A new approach, using the improved Bernoulli sub-equation function method (IBSEFM), has been employed to solve the TFGB equation. Finally, some $2D$ and $3D$ graphs have been plotted to explore the physical structures of the solution and the effect of the Burgers term and fractional order of the equation have been determined.

KEYWORDS:
time fractional Gardner Burger equation, dust ion acoustic wave, reductive perturbation technique, improved Bernoulli sub-equation function method

1 | INTRODUCTION

During last few decades, nonlinear evolution equations (NLEEs) have achieved a lot of attention of the authors, due to their vast applications in different branches of nonlinear sciences. For example, NLEEs have been utilized to formulate various problems associated with the protein chemistry, the chemical kinetics, the quantum mechanics, the plasma physics, the propagation of shallow water waves etc. The classical KdV equation is an example of NLEE which extensively utilized to model weakly nonlinear long waves. However, in many situations, the classical KdV equation becomes inappropriate when one encounters with a situations where the cubic nonlinear terms have to be included, for instance, the fluid in the neighbourhood of the critical velocity or at the critical level of density, and by considering the high-order terms, the modified KdV (mKdV) equation is constructed in those cases. Sometimes dual nonlinear terms are to be considered to generate combined KdV- mKdV equation which in turn reduces to the Gardner equation.
In situ measurements and remote sensing observations it is found that long solitary type waves are very familiar to density stratified shallow water[1]-[3]. Recent investigations show that although KdV framework is well approved for a wide range of parameters but there are still situations, where the KdV model is not adopted. For example, the critical values of the parameter for which the coefficients of the nonlinear term in the KdV equation are incorporated in modeling long internal solitary waves,

vanish if symmetrical stratification appears. Thus, the extension of small quadratic approximation of nonlinearity in the KdV model to higher-order nonlinearity by incorporating cubic nonlinear term becomes important in many applications. For the first time, Miura addressed the Gardner equation about a century ago by expanding the KdV equation[3,4]. Actually, the dual nonlinearity in the KdV model raises Gardner equation. Gardner equation adopts the same type of behavior as the standard KdV equation, however; the former claim the validity of the wider parametric domain for internal wave motion in a particular environment. The extension of the parametric domain for modeling of internal wave motion is found in[5]-[14].

Standard presentation of Gardner model $u = u(x,t)$, signifies the amplitude of wave mode and the nonlinear terms represent wave steepening and the third-order derivative terms denote dispersive effects. In water wave phenomena, the coefficients of dual nonlinear terms and the dispersive term are respectively determined by the density of the oceanic background and stratification of water flow through the linear eigen mode of the internal waves. Dual nonlinearity is frequently introduced in many situations, such as in the density stratified ocean in which internal gravity waves are found. The KdV type model, which contains only single nonlinearity, cannot define correctly the shallow water wave. It is observed that dual-power law nonlinearity may arrange a framework that fulfills the designing of the shallow water wave in the ocean engineering system. In the year 1995, an experiment was conducted in Oregon Bay, which was popularly known as the Coastal Ocean Probe Experiment (COPE). It was found that the internal waves were strong enough in COPE. The KdV model was again failed there to model the water wave. There are various situations in which it is necessary to consider dual nonlinear terms. For example, in a density stratified ocean, where the internal gravity waves are observed, the single nonlinear term cannot correctly model the shallow water waves. The Gardner equation is extensively used in diverse fields such as the quantum field theory, the plasma physics, the solid state physics and so on[15]-[17]. But some excellent observations noted in[18–21] reports that wave propagation significantly modified by viscosity effect. Being aware of the fact we introduce Burgers term in Gardner framework and the model becomes more relevant for dissipative plasma.

To solve different kinds of NLEEs, several researchers have proposed and applied various analytical as well as numerical techniques, such as the modified trial equation method (MTEM)[22], the first integral method[24]-[26], the sine-Gordon expansion method[27], the modified Kudryashov method[28,29], the exp-function method[30,31], (G'/G)-expansion method[32,33],the functional variable method[34], the Riccati sub-equation method[35], the undetermined coefficients method[37], the improved fractional sub-equation method[38] and many other symbolic techniques involving tedious computations[39]-[43]. Several authors observed the compound KdV-Burgers equation and derived traveling wave solutions utilizing various technique. For instance, Zheng et al.[44] employed the improved sine-cosine technique to obtain exact traveling wave solutions where as, Gong and Pan[45] used the new algebraic method to build exact solutions for the same. Zayed and Gepreel[46] find traveling wave solutions of this equation by applying (G'/G)-expansion method. Naher et al. found a class of traveling wave solutions for the same via the improved (G'/G)-expansion method[47].

We analyzed the propagation of the dust acoustic waves in plasma system consisting of cold inertial ions, immobile negatively charged dust grains, non-inertial electrons and positrons. Applying reduction perturbation theory, Gardner-Burgers equation is derived and for the first time, that classical equation is converted to time fractional Gardner-Burgers equation. Finally, using novel IBSEFM, different types of new exact analytical solution are obtained for this evolution equation. The effects of different parameters on the dust acoustic solutions are also discussed.

In this article, different types of solution of TFGB equation have been derived. Though we have used Gardener Burger model to study DIAW in multi-component plasma medium, this example has been set mainly to establish the existence of such models in several physical situations. In fact, in the light of above study, emphasis is given in determining the model solutions. Because, with the aid of this model, several systems of real situations like the water wave phenomena, the quantum mechanics, etc. can be studied. It is to be mentioned in this connection that the fractional models in non homogeneous medium will be more effective as expected. The entire investigation is presented as follows: Some basic definition are stated in Sec. 2. An outline for the proposed method has been adopted in Sec. 2. Sec. 4 is utilized for derivation of the classical GB equation. Sec. 5 is allotted for conversion of GB equation to TFGB equation. A set of new wave solution are deduced in Sec. 6. Sec. 7 introduces a brief discussion of the numerical structure of the solutions. Concluding remarks are given in Sec. 8.

2 | PRELIMINARIES OF FRACTIONAL CALCULUS

Nowadays fractional calculus takes an active role in modern research. In the year 1695, Leibnitz for the first time introduced fractional calculus as a generalization of the classical standard calculus. Further, the fractional derivative operators, as defined

by many mathematicians turn out to be a non-local operator and fractional differential equations have found an extensive growth due to their non-local property. Thus, the fractional differential equations extensively utilized to model various non-conservative processes in the real world problem. Some time it is incorporate better with the experimental data than the standard one. For example, It has been observed that theoretical analysis with the fractional derivative parameter $\alpha = 0.78$ provides a good agreement with the observation from the Viking satellite in the dayside auroral zone. Much interest has gained to apply this new approach in the theoretical model [17] where long term memory effects and asymptotic scaling appears. A numbers of definitions are available in literature. Some of the important definitions that are relevant in our study are presented bellow,
Definition 1. The left Riemann-Liouville fractional derivative (LRLFD) ${}_aD_t^\alpha$ is described as follows[48,49]

$$ {}_aD_t^\alpha = \frac{1}{\Gamma(M-\alpha)}\frac{d^M}{dt^M}\left(\int_a^t d\tau(t-\tau)^{M-\alpha-1}f(\tau)\right), \quad M-1 \leq \alpha \leq M, \quad t \in [a,b]. \tag{1} $$

Definition 2. The right Riemann-Liouville fractional derivative (RRLFD) ${}_tD_b^\alpha$ is defined as follows[48,49]

$$ {}_tD_b^\alpha = \frac{(-1)^M}{\Gamma(M-\alpha)}\frac{d^M}{dt^M}\left(\int_a^t d\tau(t-\tau)^{M-\alpha-1}f(\tau)\right), \quad M-1 \leq \alpha \leq M, \quad t \in [a,b]. \tag{2} $$

Definition 3. The Riesz fractional derivative operator ${}_0^RD_t^\alpha f(t)$ can be stated as follows[48,50]

$$ \begin{aligned} {}_0^RD_t^\alpha f(t) &= \frac{1}{2}[{}_aD_t^\alpha f(t) + (-1)^k\, {}_tD_b^\alpha f(t)] \\ &= \frac{(-1)^k}{2\Gamma(k-\alpha)}\frac{d^k}{dt^k}\left(\int_a^t |\tau - t|^{k-\alpha-1}f(\tau)d\tau\right), \end{aligned} \tag{3} $$

where $k-1 \leq \alpha \leq k, \quad t \in [a,b],$ a and b are real.

3 | DESCRIPTION OF THE PROPOSED METHOD

Now, we describe the general structure of the IBSEFM[51]. Let us consider the fractional differential equation as:

$$ u_\tau^\alpha = \mathcal{N}(u_\xi, u_{\xi\xi}, u_{\tau\xi}, u_{\xi\zeta\zeta}, u_{\xi\xi\xi}...) \tag{4} $$

where $u = u(\xi,\tau)$ and $\alpha \in (0,1]$ is the order of the conformable derivative. In the following, we give the main steps of this method:
Step 1. Introducing a complex z for substitution of different real variables such as ξ and τ, we define

$$ u(\xi,\tau) = V(\theta), \quad \theta = k\xi - \frac{\omega\tau^\alpha}{\alpha} \tag{5} $$

where the constants k and ω, will be determined later. Thus, Eq.(4) converts into a nonlinear ordinary differential equation as:

$$ \omega V' = Q(V, V', V'', V''', \cdots) \tag{6} $$

for $V = V(\theta)$ where Q is a polynomial of V and its derivatives and the superscripts signifies the differential w.r.t. θ.
Step 2. We consider the solution of Eq.(6) in general form as

$$ V(z) = \frac{\sum_{i=0}^p a_i Y^i(\theta)}{\sum_{j=0}^r a_j Y^j(\theta)} = \frac{a_0 + a_1Y(\theta) + a_2Y^2(\theta) + \cdots + a_pY^p(\theta)}{b_0 + b_1Y(\theta) + b_2Y^2(\theta) + \cdots + b_rY^r(\theta)}. \tag{7} $$

We can find the general form of Bernoulli differential equation for Y' according to Bernoulli theory as

$$ Y' = PY^M + QY, \quad P \neq 0, \quad Q \neq 0, \quad M \in \mathbb{R} - \{0,1,2\} \tag{8} $$

where $Y = Y(\theta)$ denotes Bernoulli differential polynomial function. Considering Eq.(7) and Eq.(6) the polynomial equation Y may be determined. Using homogeneous balance principle the relation between p, r and M will be determined.
Step 3. Equating all the coefficients of (Y) an algebraic equation system is generated. By solving the system, we can determine the values of $a_i, i = 0.\cdots, s$ and $b_j, j = 0, \cdots, r$.

Step 4. Substitute the parameter values which are obtained and the general solution of Eq.(8) in Eq.(7), then the solutions of Eq.(4) are obtained.

4 | PROBLEM FORMULATION AND DERIVATION OF NONLINEAR GARDNER BURGERS EQUATION

We consider an unmagnetized collisionless four-component plasma consisting of cold inertial ions, immobile negatively charged dust grains, non-inertial κ-distributed electrons and positrons. The nonlinear dynamic of the DIA waves, whose phase speed is much smaller(larger) than the electron (ion) thermal speed, is governed by the normalized equations of the form.

$$\frac{\partial n_j}{\partial t} + \frac{\partial (n_j v_j)}{\partial x} = 0, \tag{9a}$$

$$\frac{\partial v_j}{\partial t} + v_j \frac{\partial v_j}{\partial x} = -\frac{\partial \phi}{\partial x} + \delta_j \frac{\partial^2 v_j}{\partial x^2}, \tag{9b}$$

$$\frac{\partial^2 \phi}{\partial x^2} = -\rho = \mu_e n_e + \mu_n n_j - n_p + \mu_d \tag{9c}$$

Here n_l is the number density of lth species (where $l = j, e, p$ and d stands for ions, electrons, positrons and dust grains respectively) normalized by n_{l0}; v_j represents the normalized ion fluid velocity compared to wave speed $C_j = \sqrt{\frac{k_B T_e}{m_j}}$; ϕ is normalized electrostatic wave potential with respect to $\frac{k_B T_e}{e}$; ρ is the normalized surface charge density; k_B represents Boltzmann constant;e is the magnitude of the electron charge; T_e is the electron temperature, $\sigma = T_e/T_p$ (electron temperature to positron temperature ratio), $\mu_e = n_{e0}/n_{p0}$ (electron to positron number density ratio), $\mu_n = n_{j0}/n_{p0}$ (ion to positron number density ratio), $\mu_d = Z_d n_{d0}/n_{p0}$ (dust grains to positron number density ratio multiplied by Z_d); x, t are affirmed as space and time coordinates, respectively and they are normalized by the Debye length $\lambda_{Dj} = \sqrt{\frac{k_B T_e}{4\pi n_{j0} e^2}}$ and period of ion plasma $\omega_{pj}^{-1} = \sqrt{\frac{m_j}{4\pi n_{j0} e^2}}$, respectively. The quantities n_{j0}, n_{e0}, n_{p0} and n_{d0} are the equilibrium number densities of the species ions, electrons, positrons and dust grains respectively. In equilibrium, the charge neutrality condition is $n_{j0} + n_{p0} = n_{e0} + Z_d n_{d0}$.
The normalized number densities of electrons and positrons are given by

$$\begin{aligned} n_e &= \left(1 - \frac{\phi}{\kappa - \frac{3}{2}}\right)^{-\kappa + \frac{1}{2}} \\ &= 1 + L_1 \phi + L_2 \phi^2 + \cdots \end{aligned}$$

where,

$$L_1 = \frac{2\kappa - 1}{2\kappa - 3},\ L_2 = \frac{(2\kappa - 1)(2\kappa + 1)}{2!(2\kappa - 3)^2},\ L_3 = \frac{(2\kappa - 1)(2\kappa + 1)(2\kappa + 3)}{3!(2\kappa - 3)^3}$$

and

$$n_p = e^{-\sigma \phi}$$

In order to derive KdV equation from the basic governing equation, the depending variables n_j, v_j, ϕ, ρ and δ_j are expanded in power series of ϵ as follows:

$$\begin{aligned} n_j &= 1 + \epsilon n_1 + \epsilon^2 n_2 + \epsilon^3 n_3 + \cdots \\ v_j &= 0 + \epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 + \cdots \\ \phi &= 0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \epsilon^3 \phi_3 + \cdots \\ \rho &= 0 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \epsilon^3 \rho_3 + \cdots \\ \delta_j &= \epsilon^{\frac{1}{2}} \delta_{j0} \end{aligned} \tag{10}$$

Further, we introduce the new stretched coordinates as follows:

$$\xi = \epsilon^{\frac{1}{2}}(x - v_p t), \quad \tau = \epsilon^{\frac{3}{2}} t. \tag{11}$$

4.1 | Formation of KdV Burgers equation

Here, $v_p = \sqrt{\frac{\mu_n}{(\sigma+L_1\mu_e)}}$ denotes the phase speed of the perturbation mode and the small parameter ϵ helps to measure the weakness as well as the dispersion of the wave perturbation. Using standard perturbation technique, and comparing the coefficients of ϵ, we obtain following KdV Burgers equation as

$$\frac{\partial\phi}{\partial\tau} + A_1\phi\frac{\partial\phi}{\partial\xi} + C\frac{\partial^3\phi}{\partial\xi^3} + D\frac{\partial^2\phi}{\partial\xi^2} = 0, \tag{12}$$

where $A_1 = \frac{v_p^3}{2\mu_n}\left(\sigma^2 - 2\mu_e L_2 - \frac{3\mu_n}{v_p^4}\right),\ \ C = \frac{v_p^3}{2\mu_n},\ \ D = -\frac{\delta_{j0}}{2}.$

4.2 | Derivation of mKdV Burgers equation

Here, we consider the plasma environment which leads to adapt the basic governing equations as given above. In this case, we introduce the following stretching co-ordinates

$$\xi = \epsilon(x - v_p t),\ \tau = \epsilon^3 t. \tag{13}$$

δ_j is expanded in power series of ϵ as $\delta_j = \epsilon\,\delta_{j0}$. Considering the third order calculation for ϵ, we finally obtained following mKdV-Burgers equation as

$$\frac{\partial\phi}{\partial\tau} + B\phi^2\frac{\partial\phi}{\partial\xi} + C\frac{\partial^3\phi}{\partial\xi^3} + D\frac{\partial^2\phi}{\partial\xi^2} = 0, \tag{14}$$

where $B = \frac{v_p^3}{2\mu_n}\left(\frac{15\mu_n}{2v_p^6} - 3\mu_e L_3 v_p\right),\ \ C = \frac{v_p^3}{2\mu_n},\ \ D = -\frac{\delta_{j0}}{2}.$

4.3 | Derivation of Gardner Burgers Equation

As argued earlier, in a case where the coefficient of non-linearity tends to zero. There is a strong possibility that one may need to handle a situation of possible formation of infinite amplitude solitons both for KdV and mKdV equations. To restrict these infinite amplitude solitons, we formulate the Gardner equation in this section. To the next higher order in ϵ, we get the following equation:

$$\frac{\partial^2\phi_1}{\partial\xi^2} + \frac{1}{2}c_1 s\phi_1^2 = (\mu_e L_1 + \sigma)\phi_3 + (2L_2\mu_3 - \sigma^2)\phi_1\phi_2 + L_3\phi_1^3 + \mu_n n_3 \tag{15}$$

After simplification (15), and taking $\phi_1 = \Phi$, we have

$$\frac{\partial\Phi}{\partial\tau} + A\Phi\frac{\partial\Phi}{\partial\xi} + B\Phi^2\frac{\partial\Phi}{\partial\xi} + C\frac{\partial^3\Phi}{\partial\xi^3} + D\frac{\partial^2\Phi}{\partial\xi^2} = 0. \tag{16}$$

where $A = \frac{c_1 s v_p^3}{2\mu_n}$, $B = \frac{v_p^3}{2\mu_n}\left(\frac{15\mu_n}{2v_p^6} - 3\mu_e L_3 v_p\right)$, $C = \frac{u_p^3}{2\mu_n}$, $D = -\frac{\delta_{j0}}{2}$.
Eq.(16) is known as Gardner Burgers (GB) equation. In the absence of Burgers term, Eq.(16) converted to Gardner equation as

$$\frac{\partial\Phi}{\partial\tau} + A\Phi\frac{\partial\Phi}{\partial\xi} + B\Phi^2\frac{\partial\Phi}{\partial\xi} + C\frac{\partial^3\Phi}{\partial\xi^3} = 0. \tag{17}$$

5 | THE TIME FRACTIONAL GARDNER BURGERS EQUATION (TFGBE)

The fractional differential equations have been achieved immense interest for last few decades. Several researcher utilized various fractional order differential equations in the diverse field of science and engineering such as elctromagnetics, acoustics, cosmology, surface engineering [49,52,53] and references therein. For the numerical treatment of fractional differential equations, the reader is referred to [54,55]. It is observed from the historically data that as suitable framework for observing various nonlinear physical phenomenon a differential equation of fractional order is more suited than that of integer order. By applying the potential function $\Phi(\xi,\tau)$ where $\Phi(\xi,\tau) = U_\xi(\xi,\tau)$, the GB equation (16) is converted into,

$$U_{\xi\tau} + AU_\xi U_{\xi\xi} + BU_\xi^2 U_{\xi\xi} + CU_{\xi\xi\xi\xi} + D\Phi_{\xi\xi} = 0. \tag{18}$$

Here the subscripts designates partial differentiation of the function with respect to the parameter. The functional of the potential equation can be represented by

$$J(U)=\int_R d\xi\int_T d\tau U(p_1U_{\xi\tau}+p_2AU_\xi U_{\xi\xi}+p_3BU_\xi^2U_{\xi\xi}+p_4CU_{\xi\xi\xi\xi}+Dp_5\Phi_{\xi\xi}). \tag{19}$$

where p_1, p_2, p_3, p_4 and p_5 are constant Lagrangeian multipliers. Here R presents the boundaries of the space domain and T denotes to the initial and final values of the time. Integrating (19) by parts where $U_\xi|_R = U_\tau|_T = U_{\xi\xi}|_R = 0$, and employing the variation of this functional w. r. t. $U(\xi,\tau)$ leads to

$$J(U)=\int_R d\xi\int_T d\tau(-p_1U_\xi U_\tau-\frac{1}{2}p_2AU_\xi^3-\frac{1}{3}p_3BU_\xi^4+p_4CU_{\xi\xi}^2+Dp_5U\Phi_{\xi\xi}). \tag{20}$$

The unknown constants $(p_i, i = 1,\cdots,5)$ can be found by considering the variation of the functional. By applying the variation of this functional and then integrating by parts using $\delta U|_T = \delta U|_R = \delta U_\xi|_R = 0$ and optimizing i.e., $\delta J(U) = 0$ we find,

$$2p_1U_{\xi\tau}+3p_2AU_\xi U_{\xi\xi}+4p_3BU_\xi^2U_{\xi\xi}+2p_4CU_{\xi\xi\xi\xi}+Dp_5\Phi_{\xi\xi}=0 \tag{21}$$

Using Eq.(18)-Eq.(21), we obtain

$$p_1=\frac{1}{2},\ \ p_2=\frac{1}{3},\ \ p_3=\frac{1}{4},\ \ p_4=\frac{1}{2},\ \ p_5=1. \tag{22}$$

The functional relation yields directly the Lagrangian of the potential equation as

$$L=-\frac{1}{2}U_\tau U_\xi-\frac{1}{6}AU_\xi^3-\frac{1}{12}BU_\xi^4+\frac{1}{2}CU_{\xi\xi}+UD\Phi_{\xi\xi}. \tag{23}$$

The time fractional Lagrangian equation for the Gardner Burgers equation can be written as

$$F({}_0D_\tau^\alpha U,U_\xi,U_{\xi\xi})=-\frac{1}{2}\,{}_0D_\tau^\alpha U_\tau U_\xi-\frac{1}{6}AU_\xi^3-\frac{1}{12}BU_\xi^4+\frac{1}{2}CU_{\xi\xi}+UD\Phi_{\xi\xi},\ \ 0\le\alpha<1, \tag{24}$$

where ${}_0D_\tau^\alpha$ is left Riemann-Liouville fractional derivative defined as follows[48,49]

$${}_aD_\tau^\alpha=\frac{1}{\Gamma(M-\alpha)}\frac{d^M}{dt^M}\left(\int_a^t d\tau(t-\tau)^{M-\alpha-1}f(\tau)\right),\ \ M-1\le\alpha\le M,\ \ t\in[a,b]. \tag{25}$$

Then, the functional of the TFGBE can be written as

$$J(U)=\int_R d\xi\int_T d\tau F({}_0D_\tau^\alpha U,U_\xi,U_{\xi\xi}). \tag{26}$$

By considering the variational functional Eq.(26), and imposing the optimization constraints, i.e., $\delta U|_T = \delta U|_R = \delta U_\xi|_R = 0$ w. r. t. $U(\xi,\tau)$ yields the following Euler-Lagrange equation

$${}_\tau D_{T_0}^\alpha\left(\frac{\partial F}{\partial_0 D_\tau^\alpha U}\right)-\frac{\partial}{\partial\xi}\left(\frac{\partial F}{\partial U_\xi}\right)+\frac{\partial^2}{\partial\xi^2}\left(\frac{\partial F}{\partial U_{\xi\xi}}\right)=0. \tag{27}$$

Employing the Lagrangian (24) of the TFGBE in the Euler-Lagrange formula (27), we get

$$-\frac{1}{2}\,{}_\tau D_{T_0}^\alpha U_\xi(\xi,\tau)+\frac{1}{2}\,{}_0D_\tau^\alpha U_\xi(\xi,\tau)+(AU_\xi(\xi,\tau)+BU_\xi^2(\xi,\tau))U_{\xi\xi}(\xi,\tau)+CU_{\xi\xi\xi\xi}(\xi,\tau)+D\Phi_{\xi\xi}(\xi,\tau)=0. \tag{28}$$

Switching for the potential function $U_\xi(\xi,\tau) = \Phi(\xi,\tau)$ yields the TFGE for the state function $\Phi(\xi,\tau)$ in the following form

$$-\frac{1}{2}\,{}_\tau D_{T_0}^\alpha\Phi(\xi,\tau)+\frac{1}{2}\,{}_0D_\tau^\alpha\Phi(\xi,\tau)+(A\Phi(\xi,\tau)+B\Phi^2(\xi,\tau))\Phi_\xi(\xi,\tau)+C\Phi_{\xi\xi\xi}(\xi,\tau)+D\Phi_{\xi\xi}(\xi,\tau)=0. \tag{29}$$

The TFGBE can be rewritten as follows

$${}_0^RD_\tau^\alpha\Phi(\xi,\tau)+(A\Phi(\xi,\tau)+B\Phi^2(\xi,\tau))\Phi_\xi(\xi,\tau)+C\Phi_{\xi\xi\xi}(\xi,\tau)+D\Phi_{\xi\xi}(\xi,\tau)=0, \tag{30}$$

where the fractional operator ${}_0^RD_\tau^\alpha$ is Riesz fractional derivative operator and can be represented as follows[48,50]

$$\begin{aligned}{}_0^RD_t^\alpha f(t)&=\frac{1}{2}[{}_aD_t^\alpha f(t)+(-1)^k\,{}_tD_b^\alpha f(t)]\\&=\frac{(-1)^k}{2\Gamma(k-\alpha)}\frac{d^k}{dt^k}\left(\int_a^t|\tau-t|^{k-\alpha-1}f(\tau)d\tau\right),\end{aligned} \tag{31}$$

where $k-1 \leq \alpha \leq k,\ \ t \in [a,b]$, a and b are real. In the absence of a Burgers term in Eq.(30), the equation will be known as the time fractional Gardner equation is given by

$$ {}_{0}^{R}D_{\tau}^{\alpha}\Phi(\xi,\tau) + (A\Phi(\xi,\tau) + B\Phi^2(\xi,\tau))\Phi_{\xi}(\xi,\tau) + C\Phi_{\xi\xi\xi}(\xi,\tau) = 0. \tag{32} $$

6 | SOLUTION OF TIME FRACTIONAL GARDNER BURGERS EQUATION

This section presents a new class of solution for the present system. We consider the TFGB equation as

$$ D_{\tau}^{\alpha}\Phi(\xi,\tau) + (A\Phi(\xi,\tau) + B\Phi^2(\xi,\tau))\Phi_{\xi}(\xi,\tau) + C\Phi_{\xi\xi\xi}(\xi,\tau) + D\Phi_{\xi\xi} = 0. \tag{33} $$

here $\alpha \in (0,1]$, α is chosen as the order of time fractional derivative and A, B, C and D are the arbitrary constants. Now, we apply the wave transformation

$$ \Phi(\xi,\tau) = V(\theta),\ \ \theta = k\xi - \frac{\omega\tau^{\alpha}}{\alpha} \tag{34} $$

Substituting Eq.(34) into Eq.(33), we have obtained the following nonlinear differential equation:

$$ -\omega V' + AkVV' + BkV^2V' + Ck^3V''' + Dk^2V'' = 0 \tag{35} $$

Integarting, the Eq.(35) is converted to the equation of the form

$$ -\omega V + \frac{AkV^2}{2} + \frac{BkV^3}{3} + Ck^3V'' + Dk^2V' = 0 \tag{36} $$

Appluing homogeneous balance principle between V'' and V^3, we find a relationship for r, p and M as,

$$ M + r = p + 1 \tag{37} $$

Case 1. Taking $M = 3$ and $r = 1$, gives $p = 3$. We choose a trial solution of Eq.(36) as

$$ V = \frac{a_0 + a_1Y + a_2Y^2 + a_3Y^3}{b_0 + b_1Y} = \frac{f}{g} \tag{38} $$

$$ V' = \frac{f'g - g'g}{g^2} \tag{39} $$

$$ V'' = \frac{f''g - g'f}{g^2} - \frac{(fg')'g^2 - 2f(g')^2g}{g^4}. \tag{40} $$

where $Y' = PY^3 + QY$, $a_3 \neq 0$, $b_1 \neq 0$. Putting Eq.(38) into Eq.(36) a system of algebraic equations is obtained and by solving this system with the help of symbolic computation software Maple, we find the values of the involved coefficients. These are determined as,

Case 1 (a): When $P = Q$

Set 1.

$$ \begin{aligned} B &= -\frac{3}{2}\frac{CA^2}{(6CPk + D)^2},\ \omega = 4CP^2k^3 + 2PDk^2,\ a_0 = 0,\ a_1 = 0, \\ a_2 &= \frac{4Pkb_0(6CPk + D)}{A},\ a_3 = 0,\ b_0 = b_0,\ b_1 = 0. \end{aligned} \tag{41} $$

By using Eq.(41), the soliton solution can be written as

$$ V_1 = \frac{4Pk(6CPk + D)(1 + \tanh(Q\theta))}{A((N-1) - (N+1)\tanh(Q\theta))} \tag{42} $$

where $\theta = k\xi - (4CP^2k^3 + 2PDk^2)\frac{\tau^{\alpha}}{\alpha}$.

Set 2.

$$ \begin{aligned} B &= -\frac{3}{8}\frac{CA^2}{D^2},\ V = -\frac{1}{2}\frac{k(4C^2P^2k^2 - D^2)}{C},\ a_0 = \frac{2b_0D(2CPk + D)}{AC},\ a_1 = 0, \\ a_2 &= \frac{8b_0DkP}{A},\ a_3 = 0,\ b_0 = b_0,\ b_1 = 0 \end{aligned} \tag{43} $$

By using Eq.(43), the solution can be written as

$$ V_2 = \frac{1}{b_0}\left(\frac{2b_0D(2CPk + D)}{AC} + \frac{8b_0DkP(1 + \tanh(Q\theta))}{A((N-1) - (N+1)\tanh(Q\theta))}\right) \tag{44} $$

where $\theta = k(\xi + \frac{4C^2P^2k^2-D^2}{2C})\frac{\tau^\alpha}{\alpha}$.

Set 3.

$$B = -\frac{3}{2}\frac{CA^2}{(6CPk+D)^2},\ V = 4CP^2k^3 + 2PDk^2,\ a_0 = 0,\ a_1 = 0,$$
$$a_2 = \frac{4kb_0(6CPk+D)}{A},\ a_3 = \frac{4Pkb_1(6CPk+D)}{A},\ b_0 = b_0,\ b_1 = b_1 \tag{45}$$

By using Eq.(45), the soliton solution can be termed as

$$V_3 = \frac{\frac{4kb_0(6CPk+D)(1+\tanh(Q\theta))}{A((N-1)-(N+1)\tanh(Q\theta))} + \frac{4Pkb_1(6CPk+D)}{A}\left(\frac{1+\tanh(Q\theta)}{(N-1)-(N+1)\tanh(Q\theta)}\right)^{3/2}}{b_0 + b_1\sqrt{\frac{1+\tanh(Q\theta)}{(N-1)-(N+1)\tanh(Q\theta)}}} \tag{46}$$

where $\theta = k\xi - (4CP^2k^3 + 2PDk^2)\frac{\tau^\alpha}{\alpha}$.

Set 4.

$$B = -\frac{3}{8}\frac{CA^2}{D^2},\ V = -\frac{1}{2}\frac{k(4C^2P^2k^2 - D^2)}{C},\ a_0 = \frac{2b_0D(2CPk+D)}{AC},$$
$$a_1 = \frac{2b_1D(2CPk+D)}{AC},\ a_2 = \frac{8b_0DkP}{A},\ a_3 = \frac{8b_1DkP}{A},\ b_0 = b_0,\ b_1 = b_1 \tag{47}$$

By using Eq.(47), the soliton solution becomes

$$V_4 = \frac{2D(2CPk+D)}{AC} + \frac{8DkP(1+\tanh(Q\theta))}{A[(N-1)-(N+1)\tanh(Q\theta)]} \tag{48}$$

where $\theta = k(\xi + \frac{4C^2P^2k^2-D^2}{2C})\frac{\tau^\alpha}{\alpha}$.

Set 5.

$$B = -\frac{3}{2}\frac{CA^2}{(6CPk-D)^2},\ V = 4CP^2k^3 - 2PDk^2,\ a_0 = \frac{4Pkb_0(6CPk-D)}{A},\ a_1 = \frac{4Pkb_1(6CPk-D)}{A},$$
$$a_2 = \frac{4Pkb_0(6CPk-D)}{A},\ a_3 = \frac{4Pkb_1(6CPk-D)}{A},\ b_0 = b_0,\ b_1 = b_1 \tag{49}$$

By using Eq.(49), the soliton solution can be formed as

$$V_5 = \frac{4Pk(6CPk-D)(1+\tanh(Q\theta))}{A[(N-1)-(N+1)\tanh(Q\theta)]} + \frac{4Pk(6CPk-D)}{A} \tag{50}$$

where $\theta = k\xi - (4CP^2k^3 - 2PDk^2)\frac{\tau^\alpha}{\alpha}$.

Case 1 (b): When $P \neq Q$

Set 1.

$$B = -\frac{3}{2}\frac{CA^2}{(6CQk+D)^2},\ V = 4CQ^2k^3 + 2QDk^2,\ a_0 = 0,\ a_1 = 0,$$
$$a_2 = \frac{4Pkb_0(6CQk+D)}{A},\ a_3 = 0,\ b_0 = b_0,\ b_1 = 0 \tag{51}$$

By using Eq.(51), the solution becomes

$$V_6 = \frac{4Pkb_0(6CQk+D)Qe^{2Q\theta}}{Ab_0(N - Pe^{2Q\theta})} \tag{52}$$

where $\theta = k\xi - (4CQ^2k^3 + 2QDk^2)\frac{\tau^\alpha}{\alpha}$.

Set 2.

$$B = -\frac{3}{8}\frac{CA^2}{D^2},\ V = -\frac{1}{2}\frac{k(4C^2Q^2k^2 - D^2)}{C},\ a_0 = \frac{1}{4}\frac{a_2(2CQk+D)}{PkC},\ a_1 = 0,$$
$$a_2 = a_2,\ b_0 = \frac{a_2A}{PDk},\ a_3 = 0,\ b_1 = 0 \tag{53}$$

By using Eq.(55), the soliton solution can be formed as

$$V_7 = \frac{D(2CQk+D)(N - Pe^{2Q\theta}) + 4DPkCQe^{2Q\theta}}{4CA(N - Pe^{2Q\theta})} \tag{54}$$

where $\theta = k(\xi + \frac{4C^2Q^2k^2-D^2}{2C})\frac{\tau^\alpha}{\alpha}$.

Set 3.

$$B = -\frac{3}{2}\frac{CA^2}{(6CQk-D)^2},\ V = 4CQ^2k^3 - 2QDk^2,\ a_0 = \frac{4Qkb_0(6CQk-D)}{A},$$
$$a_1 = 0,\ a_2 = \frac{4Pkb_0(6CQk-D)}{A},\ a_3 = 0,\ b_0 = b_0,\ b_1 = 0 \tag{55}$$

By using Eq.(55), we can write the solution as

$$V_8 = \frac{4Qk(6CQk-D)}{A}\left(1 + \frac{Pe^{2Q\theta}}{N - Pe^{2Q\theta}}\right) \tag{56}$$

where $\theta = k\xi - (4CQ^2k^3 - 2QDk^2)\frac{\tau^\alpha}{\alpha}$.

Set 4.

$$B = -\frac{3}{2}\frac{CA^2}{(6CQk+D)^2},\ V = 4CQ^2k^3 + 2QDk^2,\ k = k,\ a_0 = 0,\ a_1 = 0,$$
$$a_2 = \frac{4kb_0(6CQk+D)}{A},\ a_3 = \frac{4Pkb_1(6CQk+D)}{A},\ b_0 = b_0,\ b_1 = b_1 \tag{57}$$

By using Eq.(57), the soliton solution can be formed as

$$V_9 = \frac{4k(6CQk+D)Qe^{2Q\theta}\left(b_0 + Pb_1\sqrt{\frac{Qe^{2Q\theta}}{N-Pe^{2Q\theta}}}\right)}{A(N - Pe^{2Q\theta})\left(b_0 + b_1\sqrt{\frac{Qe^{2Q\theta}}{N-Pe^{2Q\theta}}}\right)} \tag{58}$$

where $\theta = k\xi - (4CQ^2k^3 + 2QDk^2)\frac{\tau^\alpha}{\alpha}$.

Set 5.

$$B = -\frac{3}{8}\frac{CA^2}{D^2},\ V = -\frac{1}{2}\frac{k(4C^2Q^2k^2 - D^2)}{C},\ a_0 = \frac{2b_0D(2CQk+D)}{AC},\ a_1 = \frac{2b_1D(2CQk+D)}{AC},$$
$$a_2 = \frac{8b_0DkP}{A},\ a_3 = \frac{8b_1DkP}{A},\ b_0 = b_0,\ b_1 = b_1. \tag{59}$$

By using Eq.(59), the solution becomes

$$V_{10} = \frac{2D(2CQk+D)}{AC} + \frac{8DkP}{A}\frac{Qe^{2Q\theta}}{N - Pe^{2Q\theta}} \tag{60}$$

where $\theta = k(\xi + \frac{4C^2Q^2k^2-D^2}{2C})\frac{\tau^\alpha}{\alpha}$.

Set 6.

$$B = -\frac{3}{2}\frac{CA^2}{(6CQk-D)^2},\ V = 4CQ^2k^3 - 2QDk^2,\ a_0 = \frac{4Qkb_0(6CQk-D)}{A},\ a_1 = \frac{4Qkb_1(6CQk-D)}{A},$$
$$a_2 = \frac{4Pkb_0(6CQk-D)}{A},\ a_3 = \frac{4Pkb_1(6CQk-D)}{A},\ b_0 = b_0,\ b_1 = b_1 \tag{61}$$

By using Eq.(61), the soliton solution can be formed as

$$V_{11} = \frac{4Qk(6CQk-D)}{A}\left(1 + \frac{Pe^{2Q\theta}}{N - Pe^{2Q\theta}}\right) \tag{62}$$

where $\theta = k\xi - (4CQ^2k^3 - 2QDk^2)\frac{\tau^\alpha}{\alpha}$.

Case 2. Taking $M = 3$ and $r = 2$, gives $p = 2$. Thus one may write the trial solution of Eq.(36) as

$$V = \frac{a_0 + a_1Y(\theta) + a_2Y^2(\theta)}{b_0 + b_1Y(\theta) + b_2Y^2(\theta)} \tag{63}$$

where $Y' = QY + PY^3$, $a_2 \neq 0$, $b_2 \neq 0$. Putting Eq.(63) into Eq.(36), we obtain a system of algebraic equations and solve this system of equations with the help of symbolic computation package Maple, we get distinct solution set as,

Case 2 (a): When $P = Q$

Set 1.

$$B = -\frac{3}{2}\frac{CA^2}{(6CPk + D)^2},\ V = 4CP^2k^3 + 2PDk^2,\ a_0 = 0,\ a_1 = 0,$$
$$a_2 = \frac{4Pk(6CPkb_0 - 6CPkb_2 + Db_0 - Db_2)}{A},\ b_0 = b_0,\ b_1 = 0 \tag{64}$$

By using Eq.(64), the soliton solution can be formed as

$$V_{12} = \frac{4Pk(6CPkb_0 - 6CPkb_2 + db_0 - db_2)(1 + \tanh(Q\theta))}{Ab_0((N-1) - (N+1)\tanh(Q\theta)) + Ab_2(1 + \tanh(Q\theta))} \tag{65}$$

where $\theta = k\xi - (4CP^2k^3 + 2PDk^2)\frac{\tau^\alpha}{\alpha}$.

Set 2.

$$B = -\frac{3}{8}\frac{CA^2}{D^2},\ V = -\frac{1}{2}\frac{k(4C^2P^2k^2 - D^2)}{C},\ a_0 = \frac{1}{2}\frac{b_2D(4C^2P^2k^2 - D^2)}{AC^2Pk},$$
$$a_1 = 0,\ a_2 = 0,\ b_0 = \frac{1}{4}\frac{b_2(2CPk - D)}{CPk},\ b_1 = 0 \tag{66}$$

By using Eq.(66), the soliton solution can be formed as

$$V_{13} = \frac{2D(4C^2P^2k^2 - D^2)((N-1) - (N+1)\tanh(Q\theta))}{AC(2CPk - D)((N-1) - (N+1)\tanh(Q\theta)) + ACPk(1 + \tanh(Q\theta))} \tag{67}$$

where $\theta = k(\xi + \frac{4C^2P^2k^2 - D^2}{2C})\frac{\tau^\alpha}{\alpha}$.

Set 3.

$$B = -\frac{3}{8}\frac{CA^2}{D^2},\ V = -\frac{1}{2}\frac{(2CPk + D)k(2CPk - D)}{C},\ a_0 = \frac{2b_0D(2CPk + D)}{AC},\ a_1 = 0,$$
$$a_2 = \frac{2D(4CPkb_0 - 2CPkb_2 + Db_2}{AC},\ b_0 = b_0,\ b_1 = 0 \tag{68}$$

By using Eq.(68), the soliton solution can be formed as

$$V_{14} = \frac{2b_0D(2CPk + D)((N-1) - (N+1)\tanh(Q\theta))}{AC(b_0[(N-1) - (N+1)\tanh(Q\theta)] + b_2[1 + \tanh(Q\theta)])} + \frac{2D(4CPkb_0 - 2CPkb_2 + Db_2)[1 + \tanh(Q\theta)]}{AC(b_0[(N-1) - (N+1)\tanh(Q\theta)] + b_2[1 + \tanh(Q\theta)])} \tag{69}$$

where $\theta = k\xi + \frac{(2CPk+D)k(2CPk-D)}{2C}\frac{\tau^\alpha}{\alpha}$.

Case 2 (b): When $P \neq Q$

Set 1.

$$B = -\frac{3}{2}\frac{CA^2}{(6CQk + D)^2},\ V = 4CQ^2k^3 + 2QDk^2,\ a_0 = 0,\ a_1 = 0,$$
$$a_2 = \frac{4k(6CPkb_0 - 6CQ^2kb_2 + PDb_0 - QDb_2)}{A},\ b_0 = b_0,\ b_1 = 0 \tag{70}$$

By using Eq.(70), the soliton solution can be formed as

$$V_{15} = \frac{4k(6CPkb_0 - 6CQ^2kb_2 + PDb_0 - QDb_2)Qe^{2Q\theta}}{Ab_0(N - Pe^{2Q\theta}) + Ab_2Qe^{2Q\theta}} \tag{71}$$

where $\theta = k\xi - (4CQ^2k^3 + 2QDk^2)\frac{\tau^\alpha}{\alpha}$.

Set 2.

$$B = -\frac{3}{8}\frac{CA^2}{D^2},\ V = -\frac{1}{2}\frac{(2CQk + D)k(2CQk - D)}{C},\ a_0 = \frac{2Db_0(2CQk + D)}{AC},$$
$$a_1 = 0,\ a_2 = \frac{2D(4CPkb_0 2CQkb_2 + Db_2)}{AC},\ \ b_0 = b_0,\ b_1 = 0 \tag{72}$$

By using Eq.(72), the soliton solution can be formed as

$$V_{16} = \frac{2Db_0(2CQk + D)[N - Pe^{2Q\theta}] + 2D(4CPkb_0 2CQkb_2 + Db_2)Qe^{2Q\theta}}{AC[b_0(N - Pe^{2Q\theta}) + b_2Qe^{2Q\theta}]} \tag{73}$$

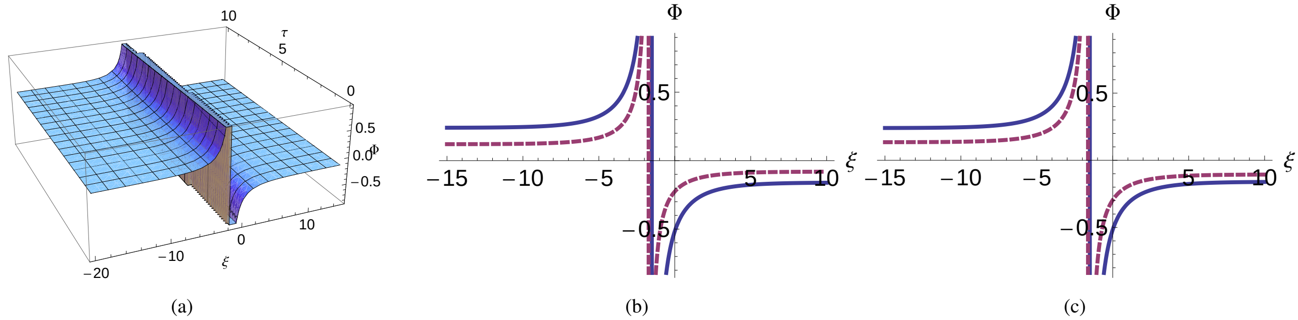


FIGURE 1 (a) The 3D Profiles of Solution (44) for $b_0 = 0.05,\ k = 0.25,\ D = 0.03,\ A = .25,\ B = 0.4,\ C = 0.5,\ Q = 1,\ P = 1,\ N = 0.5,\ \alpha = 1$. (b) The 2D Profiles of Solution (44) for $\alpha = 1$ and $\alpha = 0.25$ while the other parameters are same as (a). (c) The 2D Profiles of Solution (44) for $D = 0.03$ and $D = 0.05$ while the other parameters are same as (a).

where $\theta = k\xi + \frac{(2CQk+D)k(2CQk-D)}{2C}\frac{\tau^\alpha}{\alpha}$.

Set 3.

$$\begin{aligned} &B = -\frac{3}{2}\frac{CA^2}{(6CQk-D)^2},\ V = 4CQ^2k^3 - 2QDk^2,\ D = D,\ a_0 = \frac{4Qkb_0(6CQk-D)}{A}, \\ &a_1 = 0,\ a_2 = \frac{4Pkb_0(6CQk-D)}{A},\ b_0 = b_0,\ b_1 = 0 \end{aligned} \tag{74}$$

By using Eq.(74), the soliton solution can be formed as

$$V_{17} = \frac{4Qkb_0(6CQk-D)(N - Pe^{2Q\theta}) + 4Pkb_0(6CQk-D)Qe^{2Q\theta}}{Ab_0[N - Pe^{2Q\theta}] + Ab_2Qe^{2Q\theta}} \tag{75}$$

where $\theta = k\xi - (4CQ^2k^3 - 2QDk^2)\frac{\tau^\alpha}{\alpha}$.

Set 4.

$$\begin{aligned} &B = -\frac{3}{8}\frac{CA^2}{D^2},\ P = \frac{1}{4}\frac{b_2(2CQk-D)}{Ckb_0},\ V = -\frac{1}{2}\frac{(2CQk+D)(2CQk-D)k}{C}, \\ &a_0 = \frac{2b_0D(2CQk+D)}{AC},\ a_1 = 0,\ a_2 = 0,\ b_0 = b_0,\ b_1 = 0 \end{aligned} \tag{76}$$

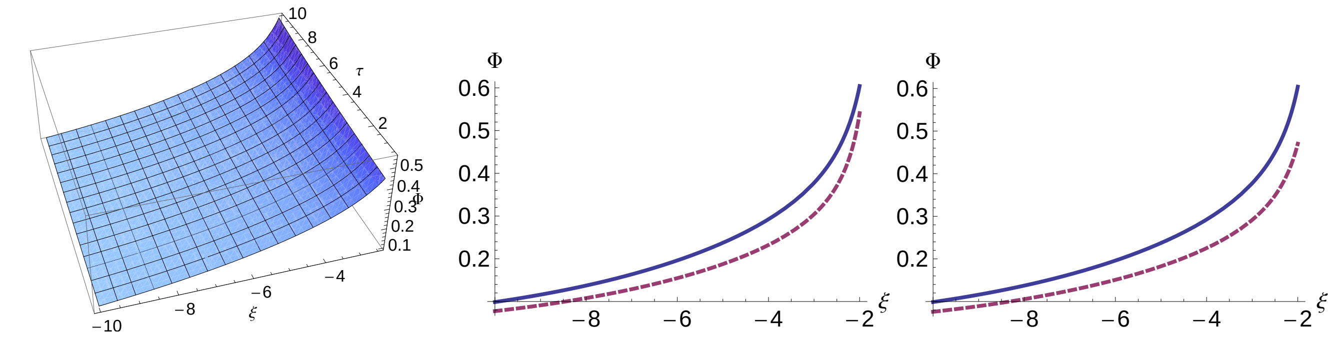
By using Eq.(76), the soliton solution can be formed as

$$V_{18} = \frac{2b_0D(2CQk+D)(N - Pe^{2Q\theta})}{AC\left(b_0(N - Pe^{2Q\theta}) + b_2Qe^{2Q\theta}\right)} \tag{77}$$

where $\theta = k\xi + \frac{(2CQk+D)(2CQk-D)k}{2C}\frac{\tau^\alpha}{\alpha}$.

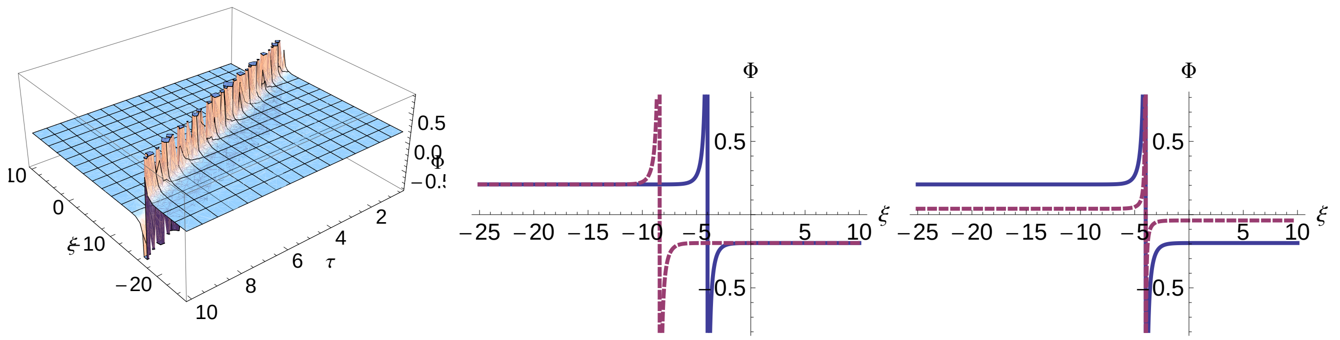
7 | RESULTS AND DISCUSSION

In this work we have acquire new class of general solutions for TFGB model and and the most important thing is that the solutions are in different form such as exponential type, rational type, hyperbolic etc. Moreover, the obtained solutions able to present different types of nonlinear structure such as kink shaped solitons, kink with periodic hump soliton, anti-kink type solitons, half dark-bright solitons, etc. Fig. 1 (a) presents $3D$ plots of the half dark-bright soliton solutions for Eq.(44) when $b_0 = 0.5,\ k = 0.5,\ D = 0.01,\ A = 1,\ C = 0.8,\ Q = 2,\ P = 0.5,\ N = 0.05,\ \alpha = 1,\ b_1 = 0.5$ and for a particular range of ξ and τ. For the values of $b_0 = 0.5,\ k = 0.5,\ D = 0.01,\ A = 1,\ C = 0.8,\ Q = 2,\ P = 0.5,\ N = 0.05,\ \alpha = 1,\ b_1 = 0.5$, in a particular range of ξ and τ, Fig. 2 (a) shows the simplest types of flow structure for Eq.(50). The most important wave feature is shown in Fig. 3 (a) of Eq.(60) in a particular range of ξ and τ, for $k = 0.5,\ D = 0.05,\ A = 2,\ C = 0.4,\ Q = 1,\ P = 1,\ N = 0.1,\ \alpha = 1,\ b_2 = 0.25$, in which the periodic hump is included in a kink shaped wave. Fig. 4 (a) exhibits the purely kink shaped $3D$ plots of Eq.(73) for a particular range of ξ and τ. This type of complex nonlinear structure is important in different nonlinear media. The nature of the solutions due to the variation of the fractional derivative parameter α is clearly described in figures Figs. 1 (b), 2 (b), 3 (b)



(a) (b) (c)

FIGURE 2 The 3D Profiles of Solution (50) is plotted against $b_0 = 0.5$, $k = 0.5$, $D = 0.03$, $A = 1$, $C = 0.8$, $Q = 2$, $P = 0.5$, $N = 0.05$, $\alpha = 1$, $b_1 = 0.5$. (b) The 2D Profiles of Solution (50) for $\alpha = 1$ and $\alpha = 0.25$ while the other parameters are same as (a). (c) The 2D Profiles of Solution (50) for $D = 0.03$ and $D = 0.05$ while the other parameters are same as (a).



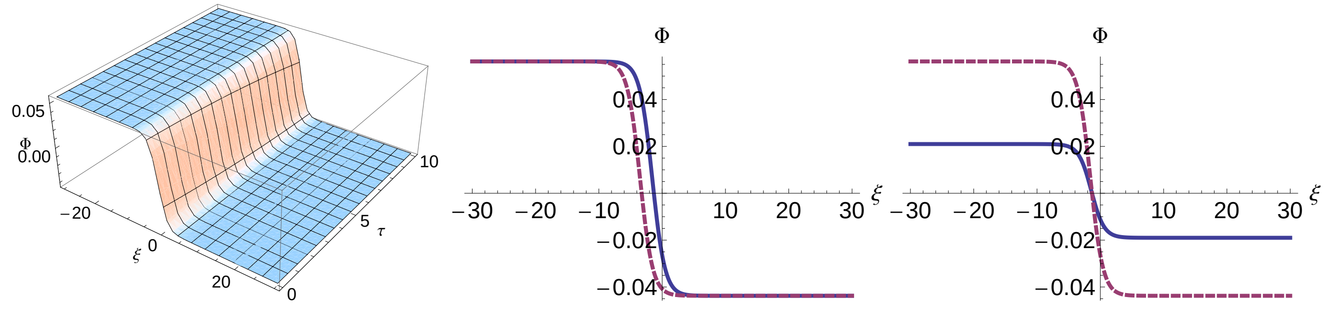
(a) (b) (c)

FIGURE 3 The 3D Profiles of Solution (60) is plotted against $b_0 = 0.5$, $k = 0.5$, $D = 0.03$, $A = 1$, $C = 0.8$, $Q = 2$, $P = 0.5$, $N = 0.05, \alpha = 1$, $b_1 = 0.5$. (b) The 2D Profiles of Solution (60) for $\alpha = 1$ and $\alpha = 0.25$ while the other parameters are same as (a). (c) The 2D Profiles of Solution (60) for $D = 0.03$ and $D = 0.05$ while the other parameters are same as (a).

and 4 (b). The remaining figures Figs. 1 (c), 2 (c), 3 (c) and 4 (c) presents the significant effect of Burgers term on solution structure.

8 | CONCLUSION

In this investigation, TFGB model is adopted to study DIAW in multicomponent plasma. We have implemented a novel analytical treatment on TFGB model and a class of new solutions is found, which may be useful in the development of the dynamics of solitons, the quantum plasma, the dynamics of adiabatic parameters, the dynamics in Fluid flow, problems on the industrial phenomena etc. Some of the specified solutions are obtained by employing symbolic software package Mathematica and have been presented graphically to analyze the physical structure of the solutions. Moreover, the effect of the Burgers term and the fractional derivative parameter α are clearly observed in $2D$ graphs. It is also concluded from the $2D$ figures Figs. 1 (b), 2 (b), 3 (b) and 4 (b) that nonlinearity and dispersive effects are significantly modified due to the variation in α. In figures Figs. 1 (c), 2 (c), 3 (c) and 4 (c), it is found that the amplitudes of kink and shock wave enhance due to the increase in Burgers term. This type of nonlinear phenomena appears as enhancement in viscosity (due to enhance in Burgers term) leads to increase in dispersion and as a result shock rises.



(a) (b) (c)

FIGURE 4 The 3D Profiles of Solution (73) is plotted against $k = 0.5$, $D = 0.03$, $A = 2$, $C = 0.4$, $Q = 1$, $P = 1$, $N = 0.1$, $\alpha = 1$, $b_2 = 0.25$. (b) The 2D Profiles of Solution (73) for $\alpha = 1$ and $\alpha = 0.25$ while the other parameters are same as (a). (c) The 2D Profiles of Solution (73) for $D = 0.03$ and $D = 0.05$ while the other parameters are same as (a).

References

[1] Helfrich KR, Melville WK. Long nonlinear internal waves. Annu. Rev. Fluid Mech 2006;38:395-425.

[2] Lamb KG, Yan L. The evolution of internal wave undular bores: comparisons of a fully nonlinear numerical model with weakly nonlinear theory. Journal of physical oceanography. 1996;**26**(12):2712-34.

[3] Lee CY, Beardsley RC. The generation of long nonlinear internal waves in a weakly stratified shear flow. Journal of Geophysical Research 1974;**79**(3):453-62.

[4] Miura RM. Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation. Journal of Mathematical Physics 1968;**9**(8):1202-4.

[5] Grimshaw R, editor. Environmental stratified flows. Springer Science & Business Media 2002.

[6] Grimshaw R, Pelinovsky E, Tian X. Interaction of a solitary wave with an external force. Physica D: Nonlinear Phenomena 1994;**77**(4):405-33.

[7] Grimshaw R, Pelinovsky E, Talipova T. Solitary wave transformation due to a change in polarity. Studies in Applied Mathematics 1998;**101**(4):357-88.

[8] Grimshaw R, Pelinovsky D, Pelinovsky E, Slunyaev A. Generation of large-amplitude solitons in the extended Korteweg-de Vries equation. Chaos: An Interdisciplinary Journal of Nonlinear Science 2002;**12**(4):1070-6.

[9] Grimshaw R, Pelinovsky E, Talipova T. Damping of large-amplitude solitary waves. Wave motion. 2003;**37**(4):351-64.

[10] GGrimshaw R, Pelinovsky E, Stepanyants Y, Talipova T. Modelling internal solitary waves on the Australian North West Shelf. Marine and Freshwater Research. 2006;**57**(3):265-72.

[11] Holloway PE, Pelinovsky E, Talipova T, Barnes B. A nonlinear model of internal tide transformation on the Australian North West Shelf. Journal of Physical Oceanography 1997;**27**(6):871-96.

[12] Holloway PE, Pelinovsky E, Talipova T. A generalized Korteweg-de Vries model of internal tide transformation in the coastal zone. Journal of Geophysical Research: Oceans 1999;**104**(C8):18333-50.

[13] Holloway P, Pelinovsky E, Talipova T. Internal tide transformation and oceanic internal solitary waves. In Environmental stratified flows 2003 (pp. 29-60).

[14] Daoui AK, Triki H, Mirzazadeh M, Biswas A. Solitary waves, shock waves and singular solitons of GardnerâĂŹs equation for shallow water dynamics. Acta Physica Polonica, B 2014;**45**(6):1135-45.

[15] Vassilev VM, Djondjorov PA, Hadzhilazova MT, Mladenov IM. Traveling wave solutions of the Gardner equation and motion of plane curves governed by the mKdV flow. In AIP Conference Proceedings 2011 (Vol. 1404, No. 1, pp. 86-93).

[16] Betchewe G, Victor KK, Thomas BB, Crepin KT. New solutions of the Gardner equation: Analytical and numerical analysis of its dynamical understanding. Applied Mathematics and Computation. 2013;223:377-88.

[17] Xu XG, Meng XH, Gao YT, Wen XY. Analytic N-solitary-wave solution of a variable-coefficient Gardner equation from fluid dynamics and plasma physics. Applied mathematics and computation 2009;**210**(2):313-20..

[18] Nakamura Y, Sarma A. Observation of ion-acoustic solitary waves in a dusty plasma. Physics of Plasmas 2001; **8**(9): 3921-6.

[19] Popel SI, Yu MY. Ion Acoustic Solitons in Impurity-Containg Plasmas. Contributions to Plasma Physics 1995;**35**(2):103-8.

[20] Mamun AA, Shukla PK. Electrostatic solitary and shock structures in dusty plasmas. Physica Scripta 2002;2002(T98):107.

[21] Raut S, Mondal KK, Chatterjee P., & Roy A. Propagation of dust-ion-acoustic solitary waves for damped modified Kadomtsev-Petviashvili equation in dusty plasma with a q-nonextensive nonthermal electron velocity distribution, SeMA Journal, 2021; pp.1-23.

[22] Yel G, Sulaiman TA, Baskonus HM. On the complex solutions to the (3+ 1)-dimensional conformable fractional modified KdV-Zakharov-Kuznetsov equation. Modern Physics Letters B 2020;**34**(05):2050069.

[23] Kocak ZF, Yel G. Trigonometric Function Solutions of Fractional Drinfeld's Sokolov-Wilson System. InITM Web of Conferences 2017 (Vol. 13, p. 01006).

[24] Ilie M, Biazar J, Ayati Z. The first integral method for solving some conformable fractional differential equations. Optical and Quantum Electronics 2018;**50**(2):1-1.

[25] Eslami M, Rezazadeh H. The first integral method for Wu-Zhang system with conformable time-fractional derivative. Calcolo 2016;**53**(3):475-85.

[26] Eslami M, Rezazadeh H, Rezazadeh M, Mosavi SS. Exact solutions to the spaceâĂŞtime fractional Schrodinger-Hirota equation and the spaceâĂŞtime modified KDV-Zakharov-Kuznetsov equation. Optical and Quantum Electronics 2017;**49**(8):1-5.

[27] Korkmaz A, Hepson OE, Hosseini K, Rezazadeh H, Eslami M. Sine-Gordon expansion method for exact solutions to conformable time fractional equations in RLW-class. Journal of King Saud University-Science 2020;**32**(1):567-74.

[28] Kumar D, Seadawy AR, Joardar AK. Modified Kudryashov method via new exact solutions for some conformable fractional differential equations arising in mathematical biology. Chinese journal of physics 2018;**56**(1):75-85.

[29] Hosseini K, Ansari R. New exact solutions of nonlinear conformable time-fractional Boussinesq equations using the modified Kudryashov method. Waves in Random and Complex Media 2017;**27**(4):628-36.

[30] He JH. Exp-function method for fractional differential equations. International Journal of Nonlinear Sciences and Numerical Simulation 2013;**14**(6):363-6.

[31] Tasbozan O, Cenesiz Y, Kurt A, Baleanu D. New analytical solutions for conformable fractional PDEs arising in mathematical physics by exp-function method. Open Physics 2017;**15**(1):647-51.

[32] Al-Shawba AA, Gepreel KA, Abdullah FA, Azmi A. Abundant closed form solutions of the conformable time fractional Sawada-Kotera-Ito equation using (G'/G)-expansion method. Results in Physics 2018;9:337-43.

[33] Wang M, Li X, Zhang J. The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Physics Letters A 2008;**372**(4):417-23.

[34] Zerarka A, Ouamane S, Attaf A. On the functional variable method for finding exact solutions to a class of wave equations. Applied Mathematics and Computation 2010;**217**(7):2897-904.

[35] Khodadad FS, Nazari F, Eslami M, Rezazadeh H. Soliton solutions of the conformable fractional Zakharov-Kuznetsov equation with dual-power law nonlinearity. Optical and Quantum Electronics 2017;**49**(11):1-2.

[36] El-Ajou A, Al-Zhour Z, Momani S, Hayat T. Series solutions of nonlinear conformable fractional KdV-Burgers equation with some applications. The European Physical Journal Plus 2019;**134**(8):1-6.

[37] Jin Q, Xia T, Wang J. The exact solution of the space-time fractional modified Kdv-Zakharov-Kuznetsov equation. Journal of Applied Mathematics and Physics 2017;**5**(4):844-52.

[38] Sahoo S, Ray SS. Improved fractional sub-equation method for (3+ 1)-dimensional generalized fractional KdV-Zakharov-Kuznetsov equations. Computers & Mathematics with Applications 2015;**70**(2):158-66.

[39] Rezazadeh H, Korkmaz A, Eslami M, Vahidi J, Asghari R. Traveling wave solution of conformable fractional generalized reaction Duffing model by generalized projective Riccati equation method. Optical and Quantum Electronics 2018;**50**(3):1-3.

[40] Eslami M. Exact traveling wave solutions to the fractional coupled nonlinear Schrodinger equations. Applied Mathematics and Computation 2016;285:141-8.

[41] Ferdous F, Hafez MG, Biswas A, Ekici M, Zhou Q, Alfiras M, Moshokoa SP, Belic M. Oblique resonant optical solitons with Kerr and parabolic law nonlinearities and fractional temporal evolution by generalized exp (-$\phi(\xi)$)-expansion. Optik 2019;178:439-48.

[42] Eslami M. Trial solution technique to chiral nonlinear SchrodingerâĂŹs equation in (1+2)-dimensions. Nonlinear Dynamics 2016;**85**(2):813-6.

[43] Nazarzadeh A, Eslami M, Mirzazadeh M. Exact solutions of some nonlinear partial differential equations using functional variable method. Pramana 2013;**81**(2):225-36.

[44] Zheng XD, Xia TC, Zhang HQ. New exact traveling wave solutions for compound KdV-Burgers equations in mathematical physics. Appl. Math. E-Notes 2002;2:45-50.

[45] Lun-Xun G, Jun-Ting P. Some new solitary wave solutions to a compound KdV-Burgers equation. Communications in Theoretical Physics 2008;**50**(1):51.

[46] Zayed EM, Gepreel KA. The (G'/G)-expansion method for finding traveling wave solutions of nonlinear partial differential equations in mathematical physics. Journal of Mathematical Physics 2009;**50**(1):013502.

[47] Naher H, Abdullah FA, Bekir A. Abundant traveling wave solutions of the compound KdV-Burgers equation via the improved (GâĂš/G)-expansion method. AIP Advances 2012;**2**(4):042163.

[48] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. elsevier 2006.

[49] Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. Wiley 1993.

[50] Agrawal OP. Fractional variational calculus in terms of Riesz fractional derivatives. Journal of Physics A: Mathematical and Theoretical 2007;**40**(24):6287.

[51] Yel G, Sulaiman TA, Baskonus HM. On the complex solutions to the (3+ 1)-dimensional conformable fractional modified KdV-Zakharov-Kuznetsov equation. Modern Physics Letters B 2020;**34**(05):2050069.

[52] West BJ, Bologna M, Grigolini P. Fractional Laplace Transforms. InPhysics of Fractal Operators. Springer, New York, NY 2003 (pp. 157-183).

[53] Podlubny I. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Elsevier 1998.

[54] Burrage K, Hale N, Kay D. An efficient implicit FEM scheme for fractional-in-space reaction-diffusion equations. SIAM Journal on Scientific Computing 2012;**34**(4):A2145-72.

[55] Diethelm K. An investigation of some nonclassical methods for the numerical approximation of Caputo-type fractional derivatives. Numerical Algorithms 2008;**47**(4):361-90.

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