Blending type Approximations by Kantorovich variant of α-Schurer operators

Nadeem Rao, Pradeep Malik and Mamta Rani

Abstract. In the present manuscript, we present a new sequence of operators, *i.e.*, α-Bernstein-Schurer-Kantorovich operators depending on two parameters $\alpha \in [0,1]$ and $\rho > 0$ foe one and two variables to approximate measurable functions on $[0.1+q], q > 0$. Next, we give basic results and discuss the rapidity of convergence and order of approximation for univariate and bivariate of these sequences in their respective sections . Further, Graphical and numerical analysis are presented. Moreover, local and global approximation properties are discussed in terms of first and second order modulus of smoothness, Peetre's K-functional and weight functions for these sequences in different spaces of functions.

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1. Introduction

Bernstein (1912) [1] proposed the Bernstein polynomials as:

$$\mathbb{B}_l(h;v) = \sum_{\nu=0}^{l} P_{l,\nu}(v) h\left(\frac{\nu}{l}\right), \; l \in \mathbb{N}, \tag{1.1}$$

where $P_{l,\nu}(v) = \binom{l}{\nu} v^\nu (1-v)^{\nu-l}$. For the operators given by (1.1), he showed that $\mathbb{B}(g;v)$ converges to g uniformly where $g \in C[0,1]$. In 1962, Schurer [2] modified operators given in (1.1) as: for $q > 0$, a real number

$$\mathbf{B}_{l,q}(g;v) = \sum_{\nu=0}^{l+q} \binom{l+q}{\nu} v^\nu (1-v)^{\nu+q-l} g\left(\frac{\nu}{l}\right), \; v \in [0,1+q], \tag{1.2}$$

where $g \in C[0,1+q]$. One can note that for $q = 0$, the polynomials presented in (1.2) reduces to polynomials given by (1.1). The operators are introduced in (1.1) and (1.2) are restricted for continuous functions only and are different in respect to the domain of function f. Several researchers, *e.g.*, Mursaleen et al. ([3], [4],) Acar et al. ([5], [6]), Mohiuddine et al. [7], Ana et al. [8], İçöz et al. ([9]), [10]), Kajla et al. ([11], [12]) constructed new sequences of linear positive operators to investigate the rapidity of convergence and order of approximation in different functional spaces in terms of several generating functions. In the recent past, for $g \in [0,1], m \in \mathbb{N}$ and $\lambda \in [-1,1]$, Chen et al. [13] constructed a sequence of new linear positive operators as:

$$T_{m,\lambda}(g;y) = \sum_{i=0}^{m} g\left(\frac{i}{m}\right) p_{m,i}^{\lambda}(y) \qquad (y \in [0,1]), \tag{1.3}$$

where $p_{1,0}^{(\lambda)} = 1-y, \quad p_{1,1}^{(\lambda)} = y$ and

$$\begin{aligned} p_{m,i}^{\lambda}(y) &= \left[(1-\lambda)y\binom{m-2}{i} + (1-\lambda)(1-y)\binom{m-2}{i-2} + \lambda y(1-y)\binom{m}{i}\right] \\ & \quad y^{i-1}(1-y)^{m-i-1} \quad (m \geq 2). \end{aligned} \tag{1.4}$$

The operators defined in (1.3) are named as $\lambda-$Berstein operator of order m.

Remark 1.1. One can not that for $\lambda = 1$, the relation (1.3) reduces to classical Bernstein operators [1].

Later, Aral and Erbay [14] introduced a parametric extension of Baskakov operators. Recently, Özger et al. [15] constructed a sequence λ-Bernstein-Schurer operators as: For every $g \in C_B[0,\infty)$ where $C_B[0,\infty)$ stands for the continuous and bounded function,

$$\Psi_{n,\nu,\lambda}(g;u) = \sum_{k=0}^{n+\nu} g_k q_{n,\nu,k}^{(\lambda)}(u) \tag{1.5}$$

$$g_k = g\left(\frac{k}{n}\right)$$

Now, we construct the $\lambda-$ Bernstein- Schurer kantorovich operators and their moments

$$\Psi_{n,\nu,\lambda}^{*}(g;u) = (n+1)\sum_{k=0}^{n+\nu} q_{n,\nu,k}^{(\lambda)}(u) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} g(s)ds \tag{1.6}$$

$$\begin{aligned} \mathcal{Q}_{n,k}^{(\lambda)}(u) &= \frac{u^{k-1}}{(1+u)^{n+k-1}}\left\{\frac{\lambda u}{1+u}\binom{n+k-1}{k} - (1-\lambda)(1+u)\binom{n+k-3}{k-2}\right. \\ &+ \left.(1-\lambda)u\binom{n+k-1}{k}\right\}, \end{aligned}$$

with $\binom{n-3}{-2} = \binom{n-2}{-1} = 0$. Motivating by the above development, we introduce positive linear operators as follows:

$$K^{\rho}_{n,\lambda}(f;y) = \sum_{i=0}^{m+q} p^{\lambda}_{m,i}(y) \int_0^1 g\left(\frac{i+t^{\rho}}{m+1}\right) ds, \tag{1.7}$$

$$\begin{aligned} p^{\lambda}_{m,i}(y) &= \Bigg[(1-\lambda)y\binom{m+q-2}{i} + (1-\lambda)(1-y)\binom{m+q-2}{i-2} \\ &+ \lambda y(1-y)\binom{m+q}{i}\Bigg] y^{i-1}(1-y)^{m+q-i-1} \quad (m \geq 2). \end{aligned}$$

where $\rho > o$ and $\mathcal{Q}^{(\lambda)}_{n,k}(u)$ is given by (1.7).
In the subsequent sections, we investigate basic Lemmas, rate of convergence, order of approximation results. Locally and globally approximation results in terms of modulus of continuity, Peetre's K-functional, second order modulus of smoothness, Lipschitz class and Lipschitz maximal function, weight functions. Further, λ-Bivariate Schurer Kantorovich operators are constructed and their pointwise and uniform approximation results are investigated.

2. Basic Estimates

Lemma 2.1. [15]*For the operator defined in* (1.5)*, one has*

$$\begin{aligned} \Psi_{n,\nu,\lambda}(e_0;u) &= 1, \\ \Psi_{n,\nu,\lambda}(e_1;u) &= u + \frac{\nu}{n}u, \\ \Psi_{n,\nu,\lambda}(e_2;u) &= u^2 + \frac{(n+\nu+2(1-\lambda))(u-u^2)}{n^2} + \frac{\nu(\nu+2n)u^2}{n^2}. \end{aligned}$$

Lemma 2.2. *Let $e_k(s) = s^k$, $k \in \{0,1,2\}$. Then, for the operators defined in* (1.6)*, we have*

$$\begin{aligned} \Psi^*_{n,\nu,\lambda}(e_0;u) &= 1, \\ \Psi^*_{n,\nu,\lambda}(e_1;u) &= \left(\frac{n+\nu}{n+1}\right)u + \frac{1}{2(n+1)} \\ \Psi^*_{n,\nu,\lambda}(e_2;u) &= \left[\frac{n^2}{(n+1)^2} - \frac{n+\nu+2(1-\lambda)}{(n+1)^2} + \frac{\nu(\nu+2n)}{(n+1)^2}\right]u^2 \\ &+ \left[\frac{n+\nu+2(1-\lambda)}{(n+1)^2} + \frac{n+\nu}{(n+1)^2}\right]u + \frac{1}{3(n+1)^2}. \end{aligned}$$

Lemma 2.3. *For the operator defined in (1.7), we have*

$$K^{\rho}_{n,\lambda}(e_0;u) = 1,$$

$$K^{\rho}_{n,\lambda}(e_1;u) = \frac{n+2(\lambda-1)}{n+1}u + \frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)},$$

$$\begin{aligned} K^{\rho}_{n,\lambda}(e_2;u) = &\left(1+\frac{4\lambda-3}{n}\right)\frac{n^2u^2}{(n+1)^2} \\ &+ \frac{[(\rho+1)(n(2\lambda+3)+(\lambda-1)(2\lambda+7))]+4(\lambda-1)}{(\rho+1)(n+1)^2}u \\ &+ \frac{2n(2\rho+1)+(\lambda+1)(2\rho+1)((\lambda+2)(\rho+1)+2)+\rho+1}{(2\rho+1)(\rho+1)(n+1)^2}. \end{aligned}$$

Proof. Using Lemma 2.3, one can easily prove Lemma 2.3. □

Lemma 2.4. *Let $e_k(s) = (e_1(s)-u)^k = \psi_u^k(s)$, $k \in \mathbb{N}$ be the central moments of $K^{\rho}_{n,\lambda}(.;.)$ constructed in (1.7). Then, we have*

$$K^{\rho}_{n,\lambda}((e_1(s)-u);u) = \frac{2\lambda-3}{n+1}u + \frac{(\lambda+1)(\rho+1)+1}{(\rho+1)(n+1)},$$

$$\begin{aligned} K^{\rho}_{n,\lambda}((e_1(s)-u)^2;u) = &\left[\left(1+\frac{4\lambda-3}{n}\right)\frac{n^2}{(n+1)^2} - \frac{2n+4\lambda-1}{n+1}+1\right]u^2 \\ &+ \frac{[(\rho+1)(n(2\lambda+3)+(\lambda-1)(2\lambda+7)-2(\lambda+1))]+\lambda-6}{(\rho+1)(n+1)^2}u \\ &+ \frac{2n(2\rho+1)+(\lambda+1)(2\rho+1)((\lambda+2)(\rho+1)+2)+\rho+1}{(2\rho+1)(\rho+1)(n+1)^2}. \end{aligned}$$

Proof. Using Lemma 2.2, Lemma 2.3 can easily be proved. □

3. Convergence behaviour of $K^{\rho}_{n,\lambda}(.;.)$

Definition 3.1. [7] For $g \in C[0,1+q], q>0$, the modulus of continuity for a uniformly continuous function g is defined as:

$$\omega(g;\delta) = \sup_{|r_1-r_2|\leq\delta} |g(r_1)-g(r_2)|, \qquad r_1, r_2 \in [0,1+p], q>0.$$

Let g be a uniformly continuous function in $C[0,1+q], q>0$ and $\delta>0$. Then, one get

$$|g(r_1)-g(r_2)| \leq \left(1+\frac{(r_1-r_2)^2}{\delta^2}\right)\omega(g;\delta). \tag{3.1}$$

Theorem 3.1. *Let $K^{\rho}_{n,\lambda}(.;.)$ be sequence of operators proposed by (1.7). Then, $K^{\rho}_{n,\lambda}$ converges uniformly to f on each bounded subset of $[0,1+q], q>0$ where $f \in C[0,1+q], q>0 \bigcap \left\{ f : u \geq 0, \frac{f(u)}{1+u^2} \text{ converges as } u \to \infty \right\}$.*

Proof. To prove this result, it is adequate to prove that

$$K^{\rho}_{n,\lambda}(e_i;u) \to e_i(u), \text{ for } i \in \{0,1,2\}.$$

Using Lemma 2.3, it is clear that $K^{\rho}_{n,\lambda}(e_i;u) \to e_i(u)$ for $i = 0,1,2$ as $n \to \infty$. Hence, Theorem 3.1 is proved. □

Example. One can note that, for the following set of parameters $q = 5$, $\rho = 0.1$ and $\lambda = 0.5$, the operator $K^{\rho}_{n,\lambda}(f;x)$ converges uniformly to the function $f(y) = y^3 - 5y^2 + 6y + 2$ as n increases which is illustrated in the figure1.

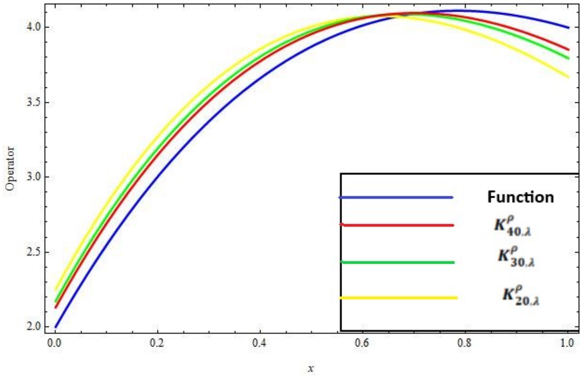

FIGURE 1. Approximation by operator $K^{\rho}_{n,\lambda}(;,;)$

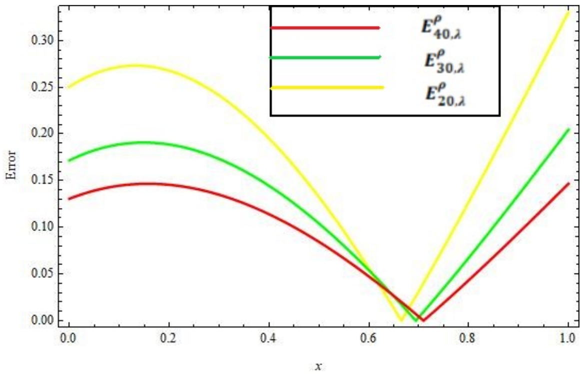

FIGURE 2. Error estimation of operators $K^{\rho}_{n,\lambda}(;,;)$ for the different values of n

The figure 2 and Table 1 are also demonstrated our analytical results.

Theorem 3.2. *(See [16]) Let $L : C([a,b]) \to B([a,b])$ be a linear and positive operator and let φ_x be the function defined by*

x	$E^{\rho}_{20,\lambda}(f;x)$	$E^{\rho}_{30,\lambda}(f;x)$	$E^{\rho}_{40,\lambda}(f;x)$
0.1	0.2717372121	0.1887446733	0.1445360958
0.2	0.2677254718	0.1886482134	0.1455017073
0.3	0.2412358918	0.1732429202	0.1348677403
0.4	0.1951644878	0.1444179547	0.1140349291
0.5	0.1324072752	0.1040624783	0.0844040078
0.6	0.0558602697	0.0540656519	0.0473757106
0.7	0.0315805132	0.0036833631	0.0043507716
0.8	0.1270190580	0.0672954058	0.0432700748
0.9	0.2275593491	0.134881315	0.0940860947
1	0.3303053711	0.2045519293	0.1466965539

TABLE 1. Error estimation table

$$\varphi_x(t) = |t-x|,\ \ (x,t) \in [a,b] \times [a,b].$$

If $f \in C_B([a,b])$ for any $x \in [a,b]$ and any $\delta > 0$, the operator L verifies:

$$\begin{aligned}|(Lf)(x) - f(x)| &\leq |f(x)||(Le_0)(x) - 1| \\ &\quad + \{(Le_0)(x) + \delta^{-1}\sqrt{(Le_0)(x)(L\varphi_x^2)(x)}\}\omega_f(\delta).\end{aligned}$$

Theorem 3.3. *Let the operators $K^{\rho}_{n,\lambda}(.;.)$ be introduced by (1.7) and $f \in C_B[0,1+p], q > 0$, we have*

$$|K^{\rho}_{n,\lambda}(f;u) - f(u)| \leq 2\omega(f;\delta),$$

where $\delta = \sqrt{K^{\rho}_{n,\lambda}(\psi_u^2;u)}$.

Proof. In view of Theorem 3.2, Lemma 2.3 and Lemma 2.4, one has

$$|K^{\rho}_{n,\lambda}(f;u) - f(u)| \leq \{1 + \delta^{-1}\sqrt{K^{\rho}_{n,\lambda}(f;u)(\psi_u^2;u)}\}\omega(f;\delta).$$

On choosing $\delta = \sqrt{K^{\rho}_{n,\lambda}(\psi_u^2;u)}$, we completes the proof of this result. □

4. Pointwise Approximation results

Here, we consider the Lipschitz type space [19] as

$$Lip_M^{k_1,k_2}(\gamma) := \left\{ f \in C_B[0,1+p], q>0 : |f(t) - f(u)| \leq M \frac{|t-u|^{\gamma}}{(t+k_1u+k_2u^2)^{\frac{\gamma}{2}}} : u,t \in (0,\infty) \right\},$$

where $M \geq 0$ is a real valued constant number, $k_1, k_2 > 0$, $\rho > 0$ and $\gamma \in (0,1]$.

Theorem 4.1. *For $f \in Lip_M^{k_1,k_2}(\gamma)$, one yield*

$$|K^{\rho}_{n,\lambda}(f;u) - f(u)| \leq M\left(\frac{\eta_n^*(u)}{k_1u + k_2u^2}\right)^{\frac{\gamma}{2}}, \tag{4.1}$$

where $u > 0$ *and* $\eta_n^*(u) = K_{n,\lambda}^{\rho}(\psi_u^2; u)$.

Proof. For $\gamma = 1$, we have

$$\begin{aligned}|K_{n,\lambda}^{\rho}(f;u) - f(u)| &\leq K_{n,\lambda}^{\gamma}(|f(t) - f(u)|)(u)\\ &\leq MK_{n,\lambda}^{\rho}\left(\frac{|t-u|}{(t+k_1u+k_2u^2)^{\frac{1}{2}}};u\right).\end{aligned}$$

Since $\frac{1}{t+k_1u+k_2u^2} < \frac{1}{k_1u+k_2u^2}$ for all $t, u \in (0, \infty)$, we get

$$\begin{aligned}|K_{n,\lambda}^{\rho}(f;u) - f(u)| &\leq \frac{M}{(k_1u+k_2u^2)^{\frac{1}{2}}}(K_{n,\lambda}^{\rho}((t-u)^2;u))^{\frac{1}{2}}\\ &\leq M\left(\frac{\eta_n^*(u)}{k_1u+k_2u^2}\right)^{\frac{1}{2}}.\end{aligned}$$

This implies that for $\gamma = 1$, this result stand good. Now, for $\gamma \in (0, 1)$ and using Hölder's inequality with $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, one obtain

$$\begin{aligned}|K_{n,\lambda}^{\rho}(f;u) - f(u)| &\leq \left(K_{n,\lambda}^{\rho}((|f(t) - f(u)|)^{\frac{2}{\gamma}};u)\right)^{\frac{\gamma}{2}}\\ &\leq M\left(K_{n,\lambda}^{\rho}\left(\frac{|t-u|^2}{(t+k_1u+k_2u^2)};u\right)\right)^{\frac{\gamma}{2}}.\end{aligned}$$

Since $\frac{1}{t+k_1u+k_2u^2} < \frac{1}{k_1u+k_2u^2}$ for all $t, u \in (0, \infty)$, we obtain

$$\begin{aligned}|K_{n,\lambda}^{\rho}(f;u) - f(u)| &\leq M\left(\frac{\mathcal{P}_n^{\mu,\nu}\left(|t-u|^2;u\right)}{k_1u+k_2u^2}\right)^{\frac{\gamma}{2}}\\ &\leq M\Big(\frac{\eta_n^*(u)}{k_1u+k_2u^2}\Big)^{\frac{\gamma}{2}}.\end{aligned}$$

Hence, we arrive at the desired result. □

5. Global Approximation

From [17], we recall some notation to prove the global approximation results. For the weight function $1 + u^2$ and $0 \leq u < \infty$, we have
$B_{1+u^2}[0, 1+p], q > 0 = \{f(u) : |f(u)| \leq M_f(1+u^2), M_f$ is constant depending on $f\}$.
$C_{1+u^2}[0, 1+p], q > 0 \subset B_{1+u^2}[0, 1+p], q > 0$ space of continuous functions endowed with the norm $\|f\|_{1+u^2} = \sup\limits_{u\in[0,1+p],q>0} \frac{|f|}{1+u^2}$.

and

$$C_{1+u^2}^k[0, 1+p], q > 0 = \{f \in C_{1+u^2} : \lim_{u\to\infty} \frac{f(u)}{1+u^2} = k, \text{ where } k \text{ is a constant}\}.$$

Theorem 5.1. *Let the* $K^{\rho}_{n,\lambda}(.;.)$ *be the operators given by* (1.7) *and* $K^{\rho}_{n,\lambda}(.;.):C^{k}_{1+u^2}[0,1+p],q>0\rightarrow B_{1+u^2}[0,1+p],q>0$. *Then, we have*

$$\lim_{n\to\infty}\|K^{\rho}_{n,\lambda}(f;u)-f\|_{1+u^2}=0,$$

where $f\in C^{k}_{1+u^2}[0,1+p],q>0$.

Proof. To prove this result, it is sufficient to show that

$$\lim_{n\to\infty}\|K^{\rho}_{n,\lambda}(e_i;u)-u^i\|_{1+u^2}=0,\ \ i=0,1,2.$$

From Lemma 2.3, we get

$$\|K^{\rho}_{n,\lambda}(e_0;u)-u^0\|_{1+u^2}=\sup_{u\in[0,1+p],q>0}\frac{|K^{\rho}_{n,\lambda}(1;u)-1|}{1+u^2}=0\text{ for }i=0.$$

For $i=1$

$$\begin{aligned}\|K^{\rho}_{n,\lambda}(e_1;u)-u^1\|_{1+u^2}&=\sup_{u\in[0,1+p],q>0}\frac{\frac{n+2(\lambda-1)}{n+1}u+\frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)}}{1+u^2}\\&=\left(\frac{n+2(\lambda-1)}{n+1}-1\right)\sup_{u\in[0,1+p],q>0}\frac{u}{1+u^2}\\&\quad+\frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)}\sup_{u\in[0,1+p],q>0}\frac{1}{1+u^2}.\end{aligned}$$

Which implies that $\|K^{\rho}_{n,\lambda}(e_1;u)-u^1\|_{1+u^2}\rightarrow 0$ an $n\rightarrow\infty$.
Similarly, we see that $\|K^{\rho}_{n,\lambda}(e_2;u)-u^2\|_{1+u^2}\rightarrow 0$ as $n\rightarrow\infty$. □

6. Construction of bivariate Szász-Durrmeyer-Operators $H^{*}_{n_1,n_2}(.;.)$ and their Basic Estimates

Take $\mathcal{I}^2=\{(y_1,y_2):0\leq y_1<1+q_1,\ 0\leq y_2<1+q_2\}$ and $C\left(\mathcal{I}^2\right)$ is the class of all continuous functions on $\mathcal{I}^2$ equipped with the norm $||g||_{C(\mathcal{I}^2)}=\sup_{(y_1,y_2)\in\mathcal{I}^2}|g(y_1,y_2)|$. Then for all $h\in C\left(\mathcal{I}^2\right)$ and $n_1,n_2\in\mathbb{N}$, we construct sequence of bivariate bivariate generalized baskakov operators as follows:

$$\begin{aligned}K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(f;y_1,y_2)&=\sum_{i_1=0}^{m_1+q_1}p^{\lambda_1}_{m_1,i_1}(y_1)\sum_{i_2=0}^{m_2+q_2}p^{\lambda_2}_{m_2,i_2}(y_2)\\&\quad\int_0^1\int_0^1 g\left(\frac{i_1+t_1^{\rho}}{m_1+1},\frac{i_2+t_2^{\rho}}{m_2+1}\right)dt_1dt_2,\end{aligned}\tag{6.1}$$

where

$$\begin{aligned}p^{\lambda_k}_{m_k,i_k}(y_k)&=\left[(1-\lambda)y_k\binom{m_k+q_k-2}{i_k}+(1-\lambda_k)(1-y_k)\binom{m_k+q_k-2}{i_k-2}\right.\\&+\left.\lambda y_k(1-y_k)\binom{m_k+q_k}{i_k}\right]y_k^{i_k-1}(1-y_k)^{m_k+q_k-i_k-1}\quad(m_1,m_2\geq 2).\end{aligned}$$

Lemma 6.1. *Let $e_{i,j} = y_1^i y_2^j$ be the central moments. Then, for the operators* (6.1)*, we have*

$$\begin{aligned}
K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_{0,0}; y_i, y_2) &= 1,\\
K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_{1,0}; y_i, y_2) &= \frac{m_1 + 2(\lambda_1 - 1)}{m_1 + 1} y_1 + \frac{(\lambda_1 + 1)(\rho + 1) + 1}{2(\rho + 1)(m_1 + 1)},\\
K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_{0,1}; y_i, y_2) &= \frac{m_1 + 2(\lambda_1 - 1)}{m_2 + 1} y_2 + \frac{(\lambda_2 + 1)(\rho + 1) + 1}{2(\rho + 1)(m_2 + 1)},\\
K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_{1,1}; y_i, y_2) &= \frac{m_1 + 2(\lambda_1 - 1)}{m_1 + 1} y_1 + \frac{(\lambda_1 + 1)(\rho + 1) + 1}{2(\rho + 1)(m_1 + 1)}\\
&\times \frac{m_2 + 2(\lambda_2 - 1)}{m_2 + 1} y_2 + \frac{(\lambda_2 + 1)(\rho + 1) + 1}{2(\rho + 1)(m_2 + 1)},
\end{aligned}$$

$$\begin{aligned}
&K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_{2,0}; y_i, y_2) = \left(1 + \frac{4\lambda_1 - 3}{m_1}\right) \frac{m_1^2 y_1^2}{(m_1 + 1)^2}\\
&+ \frac{[(\rho + 1)(m_1(2\lambda_1 + 3) + (\lambda_1 - 1)(2\lambda_1 + 7))] + 4(\lambda_1 - 1)}{(\rho + 1)(m_1 + 1)^2} y_1\\
&+ \frac{2m_1(2\rho + 1) + (\lambda_1 + 1)(2\rho + 1)((\lambda_1 + 2)(\rho + 1) + 2) + \rho + 1}{(2\rho + 1)(\rho + 1)(m_1 + 1)^2},\\
&K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_{0,2}; y_i, y_2) = \left(1 + \frac{4\lambda_2 - 3}{m_2}\right) \frac{m_2^2 y_2^2}{(m_2 + 1)^2}\\
&+ \frac{[(\rho + 1)(m_2(2\lambda_2 + 3) + (\lambda_2 - 1)(2\lambda_2 + 7))] + 4(\lambda_2 - 1)}{(\rho + 1)(m_2 + 1)^2} y_2\\
&+ \frac{2m_2(2\rho + 1) + (\lambda_2 + 1)(2\rho + 1)((\lambda_2 + 2)(\rho + 1) + 2) + \rho + 1}{(2\rho + 1)(\rho + 1)(m_2 + 1)^2}.
\end{aligned}$$

Proof. In the light of Lemma (2.3) and linearity property, we have

$$\begin{aligned}
K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_{0,0}; y_i, y_2) &= K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_0; y_i, y_2) K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_0; y_i, y_2),\\
K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_{1,0}; y_i, y_2) &= K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_1; y_i, y_2) K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_0; y_i, y_2),\\
K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_{0,1}; y_i, y_2) &= K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_0; y_i, y_2) K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_1; y_i, y_2),\\
K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_{1,1}; y_i, y_2) &= K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_1; y_i, y_2) K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_1; y_i, y_2),\\
K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_{2,0}; y_i, y_2) &= K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_2; y_i, y_2) K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_0; y_i, y_2),\\
K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_{0,2}; y_i, y_2) &= K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_0; y_i, y_2) K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(e_2; y_i, y_2),
\end{aligned}$$

which proves Lemma (6.1). ☐

Lemma 6.2. *Let $\Psi_{y_1,y_2}^{i,j}(t,s) = \eta_{i,j}(t,s) = (t - y_1)^i (s - y_2)^j, \quad i, j \in \{0, 1, 2\}$ be the central moments functions. Then from the operators $K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(.;.)$ defined* *by* (6.1) *satisfies the following identities*

$$\begin{aligned}
K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_{0,0};y_i,y_2) &= 1\\
K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_{1,0};y_i,y_2) &= \frac{2\lambda_1-3}{m_1+1}y_1+\frac{(\lambda_1+1)(\rho+1)+1}{(\rho+1)(m_1+1)},\\
K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_{0,1};y_i,y_2) &= \frac{2\lambda_1-3}{m_2+1}y_2+\frac{(\lambda_2+1)(\rho+1)+1}{(\rho+1)(m_2+1)}\\
K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_{1,1};y_i,y_2) &= \frac{2\lambda_1-3}{m_1+1}y_1+\frac{(\lambda_1+1)(\rho+1)+1}{(\rho+1)(m_1+1)}\\
&\times \frac{2\lambda_1-3}{m_2+1}y_2+\frac{(\lambda_2+1)(\rho+1)+1}{(\rho+1)(m_2+1)}
\end{aligned}$$

$$\begin{aligned}
&K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_{2,0};y_i,y_2) = \left[\left(1+\frac{4\lambda_1-3}{m_1}\right)\frac{m_1^2}{(m_1+1)^2}-\frac{2m_1+4\lambda_1-1}{m_1+1}+1\right]y_1^2\\
+&\quad \frac{[(\rho+1)(m_1(2\lambda_1+3)+(\lambda_1-1)(2\lambda_1+7)-2(\lambda_1+1))]+\lambda_1-6}{(\rho+1)(m_1+1)^2}y_1\\
+&\quad \frac{2m_1(2\rho+1)+(\lambda_1+1)(2\rho+1)((\lambda_1+2)(\rho+1)+2)+\rho+1}{(2\rho+1)(\rho+1)(m_1+1)^2},\\
&K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_{0,2};y_i,y_2) = \left[\left(1+\frac{4\lambda_2-3}{m_2}\right)\frac{m_2^2}{(m_2+1)^2}-\frac{2m_2+4\lambda_2-1}{m_2+1}+1\right]y_2^2\\
+&\quad \frac{[(\rho+1)(m_1(2\lambda_2+3)+(\lambda_2-1)(2\lambda_2+7)-2(\lambda_2+1))]+\lambda_2-6}{(\rho+1)(m_2+1)^2}y_2\\
+&\quad \frac{2m_2(2\rho+1)+(\lambda_2+1)(2\rho+1)((\lambda_2+2)(\rho+1)+2)+\rho+1}{(2\rho+1)(\rho+1)(m_2+1)^2}.
\end{aligned}$$

Proof. In the light of Lemma (6.1) and linearity property, we have

$$\begin{aligned}
K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_{0,0};y_i,y_2) &= K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_0;y_i,y_2)K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_0;y_i,y_2),\\
K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_{1,0};y_i,y_2) &= K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_1;y_i,y_2)K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_0;y_i,y_2),\\
K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_{0,1};y_i,y_2) &= K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_0;y_i,y_2)K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_1;y_i,y_2),\\
K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_{1,1};y_i,y_2) &= K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_1;y_i,y_2)K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_1;y_i,y_2),\\
K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_{2,0};y_i,y_2) &= K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_2;y_i,y_2)K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_0;y_i,y_2),\\
K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_{0,2};y_i,y_2) &= K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_0;y_i,y_2)K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\eta_2;y_i,y_2),
\end{aligned}$$

which proves Lemma (6.2). □

7. Degree of Convergence

For any $g \in C(\mathcal{I}^2)$ and $\delta > 0$ modulus of continuity of order second is given by

$$\omega\,(g;\delta_{n_1},\delta_{n_2}) = \sup\{\mid g(t,s)-g(y_1,y_2)\mid : (t,s),(y_1,y_2)\in\mathcal{I}^2\}$$

with $\mid t-y_1 \mid\leq \delta_{n_1},\ \mid s-y_2 \mid\leq \delta_{n_2}$ with the partial modulus of continuity defined as:

$$\omega_1\,(g;\delta)=\sup_{0\leq y_2\leq\infty}\ \sup_{|x_1-x_2|\leq\delta}\{\mid g(x_1,y_2)-g(x_2,y_2)\mid\},$$

$$\omega_2\,(g;\delta)=\sup_{0\leq y_1\leq\infty}\ \sup_{|y_1-y_2|\leq\delta}\{\mid g(y_1,y_1)-g(y_1,y_2)\mid\}.$$

Theorem 7.1. *For any $g\in C(\mathcal{I}^2)$ we have*

$$\mid K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(g;y_1,y_2)-g(y_1,y_2)\mid\leq 2\bigg(\omega_1(g;\delta_{y_1,n_1})+\omega_2(g;\delta_{n_2,y_2})\bigg).$$

Proof. In order to give the prove of Theorem 7.1, in general we use well-known Cauchy-Schwarz inequality. Thus we see that

$$\begin{aligned}
\mid K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(g;y_1,y_2) &- g(y_1,y_2)\mid\leq K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}\,(\mid g(t,s)-g(y_1,y_2)\mid;y_1,y_2)\\
&\leq K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}\,(\mid g(t,s)-g(y_1,s)\mid;y_1,y_2)\\
&+ K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}\,(\mid g(y_1,s)-g(y_1,y_2)\mid;y_1,y_2)\\
&\leq K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}\,(\omega_1(g;\mid t-y_1\mid);y_1,y_2)\\
&+ K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}\,(\omega_2(g;\mid s-y_2\mid);y_1,y_2)\\
&\leq \omega_1(g;\delta_{n_1})\left(1+\delta_{n_1}^{-1}K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\mid t-y_1\mid;y_1,y_2)\right)\\
&+ \omega_2(g;\delta_{n_2})\left(1+\delta_{n_2}^{-1}K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\mid s-y_2\mid;y_1,y_2)\right)\\
&\leq \omega_1(g;\delta_{n_1})\left(1+\frac{1}{\delta_{n_1}}\sqrt{K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}((t-y_1)^2;y_1,y_2)}\right)\\
&+ \omega_2(g;\delta_{n_2})\left(1+\frac{1}{\delta_{n_2}}\sqrt{K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}((s-y_2)^2;y_1,y_2)}\right).
\end{aligned}$$

If we choose $\delta^2_{n_1}=\delta^2_{n_1,y_1}=K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}((t-y_1)^2;y_1,y_2)$ and $\delta^2_{n_2}=\delta^2_{n_2,y_2}=K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}((s-y_2)^2;y_1,y_2)$, then we easily to reach our desired results. ☐

Here, we find convergence in terms of the Lipschitz class for bivariate function. For $M>0$ and $\tau,\tau\in[0,1+p],q>0$, Lipschitz maximal function space on $E\times E\subset\mathcal{I}^2$ defined by

$$\begin{aligned}
\mathcal{L}_{\tau,\tau}(E\times E) &= \bigg\{g:\sup(1+t)^\tau(1+s)^\tau\,(g_{\tau,\tau}(t,s)-g_{\tau,\tau}(y_1,y_2))\\
&\leq M\frac{1}{(1+y_1)^\tau}\frac{1}{(1+y_2)^\tau}\bigg\},
\end{aligned}$$

where g is continuous and bounded on $\mathcal{I}^2$, and

$$g_{\tau,\tau}(t,s)-g_{\tau,\tau}(y_1,y_2)=\frac{\mid g(t,s)-g(y_1,y_2)\mid}{\mid t-y_1\mid^\tau\mid s-y_2\mid^\tau};\quad (t,s),(y_1,y_2)\in\mathcal{I}^2. \tag{7.1}$$

Theorem 7.2. *Let $g \in \mathcal{L}_{\tau,\tau}(E \times E)$, then for any $\tau, \tau \in [0, 1+p], q > 0$, there exists $M > 0$ such that*

$$\begin{aligned} \mid K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(g; y_1, y_2) - g(y_1, y_2) \mid \ &\leq \ M\Bigg\{\Bigg((d(y_1, E))^{\tau} + \left(\delta^2_{n_1,y_1}\right)^{\frac{\tau}{2}} \Bigg) \\ &\times \ \Bigg((d(y_2, E))^{\tau} + \left(\delta^2_{n_2,y_2}\right)^{\frac{\tau}{2}} \Bigg) \\ &+ \ (d(y_1, E))^{\tau} \, (d(y_2, E))^{\tau} \Bigg\}, \end{aligned}$$

where δ_{n_1,y_1} and δ_{n_2,y_2} defined by Theorem 7.1.

Proof. Take $\mid y_1 - x_0 \mid = d(y_1, E)$ and $\mid y_2 - y_0 \mid = d(y_2, E)$. For any $(y_1, y_2) \in \mathcal{I}^2$, and $(x_0, y_0) \in E \times E$ we let $d(y_1, E) = \inf\{\mid y_1 - y_2 \mid : y_2 \in E\}$. Thus we can write here

$$\mid g(t,s) - g(y_1, y_2) \mid \leq M \mid g(t,s) - g(x_0, y_0) \mid + \mid g(x_0, y_0) - g(y_1, y_2) \mid . \quad (7.2)$$

Apply $K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}$, we obtain

$$\begin{aligned} \mid K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(g; y_1, y_2) \ &- \ g(y_1, y_2) \mid \\ &\leq \ K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2} \left(\mid g(y_1, y_2) - g(x_0, y_0) \mid + \mid g(x_0, y_0) - g(y_1, y_2) \mid\right) \\ &\leq \ M K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2} \left(\mid t - x_0 \mid^{\tau} \mid s - y_0 \mid^{\tau}; y_1, y_2\right) \\ &+ \ M \mid y_1 - x_0 \mid^{\tau} \mid y_2 - y_0 \mid^{\tau} . \end{aligned}$$

For all $A, B \geq 0$ and $\tau \in [0, 1+p], q > 0$ we know inequality $(A+B)^{\tau} \leq A^{\tau} + B^{\tau}$, thus

$$\mid t - x_0 \mid^{\tau} \leq \mid t - y_1 \mid^{\tau} + \mid y_1 - x_0 \mid^{\tau},$$

$$\mid s - y_0 \mid^{\tau} \leq \mid s - y_2 \mid^{\tau} + \mid y_2 - y_0 \mid^{\tau} .$$

Therefore

$$\begin{aligned} \mid K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}(g; y_1, y_2) - g(y_1, y_2) \mid \ &\leq \ M K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2} \left(\mid t - y_1 \mid^{\tau} \mid s - y_2 \mid^{\tau}; y_1, y_2\right) \\ &+ \ M \mid y_1 - x_0 \mid^{\tau} K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2} \left(\mid s - y_2 \mid^{\tau}; y_1, y_2\right) \\ &+ \ M \mid y_2 - y_0 \mid^{\tau} K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2} \left(\mid t - y_1 \mid^{\tau}; y_1, y_2\right) \\ &+ \ 2M \mid y_1 - x_0 \mid^{\tau} \mid y_2 - y_0 \mid^{\tau} K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2} \left(\mu_{0,0}; y_1, y_2\right). \end{aligned}$$

On apply Hölder inequality on $K_{m_1,m_2}^{\rho,\lambda_1,\lambda_2}$, we get

$$
\begin{aligned}
K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\mid t-y_1\mid^{\tau}\mid s-y_2\mid^{\tau};y_1,y_2) &= \mathcal{U}^{\lambda_1}_{n_1,k}(\mid t-y_1\mid^{\tau};y_1,y_2)\\
&\times \mathcal{V}^{\lambda_2}_{n_2,l}(\mid s-y_2\mid^{\tau};y_1,y_2)\\
&\leq \left(K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\mid t-y_1\mid^{2};y_1,y_2)\right)^{\frac{\tau}{2}}\\
&\times \left(K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\mu_{0,0};y_1,y_2)\right)^{\frac{2-\tau}{2}}\\
&\times \left(K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\mid s-y_2\mid^{2};y_1,y_2)\right)^{\frac{\tau}{2}}\\
&\times \left(K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(\mu_{0,0};y_1,y_2)\right)^{\frac{2-\tau}{2}}.
\end{aligned}
$$

Thus, we can obtain

$$
\begin{aligned}
\mid K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(g;y_1,y_2)-g(y_1,y_2)\mid &\leq M\left(\delta^2_{n_1,y_1}\right)^{\frac{\tau}{2}}\left(\delta^2_{n_2,y_2}\right)^{\frac{\tau}{2}}\\
&+ 2M\left(d(y_1,E)\right)^{\tau}\left(d(y_2,E)\right)^{\tau}\\
&+ M\left(d(y_1,E)\right)^{\tau}\left(\delta^2_{n_2,y_2}\right)^{\frac{\tau}{2}}+L\left(d(y_2,E)\right)^{\tau}\left(\delta^2_{n_1,y_1}\right)^{\frac{\tau}{2}}.
\end{aligned}
$$

We have complete the proof. □

Example. It is observed that in this example that for the following set of parameters $q=5$, $\rho=0.9$ and $\lambda=0.5$, the operator $K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(,;,)$ converges uniformly to the function $f(y)=y_1^3y_2^2$ (Blue) as $m_1=m_2=10$ (Green) and $m_1=m_2=20$ (Red) increases which is shown in the Figure 3.

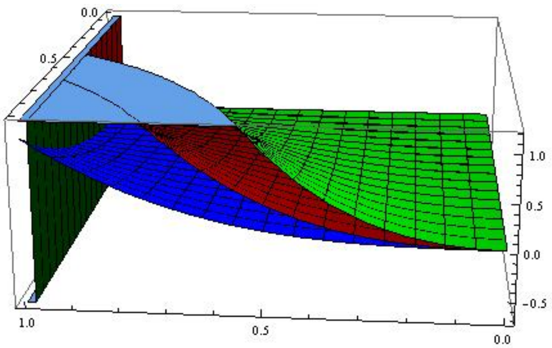

FIGURE 3. $K^{\rho,\lambda_1,\lambda_2}_{m_1,m_2}(,;,)$ converges to $f(x)=y_1^3y_2^2$

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Nadeem Rao
Department of Mathematics
Shree Guru Gobind Singh Tercentenary University
Gurugram-122505, Haryan
India
e-mail: nadeemrao1990@gmail.com

Pradeep Malik
Department of Mathematics
Shree Guru Gobind Singh Tercentenary University
Gurugram-122505, Haryan
India
e-mail: pradeepmaths@gmail.com

Mamta Rani
Shree Guru Gobind Singh Tercentenary University
Gurugram-122505, Haryan
India
e-mail: mamtakarla84@gmail.com