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Global solution to Cauchy problem of fractional drift diffusion system with power-law nonlinearity†

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In this paper we consider the global existence, regularizing-decay rate and asymptotic behavior of mild solutions to the Cauchy problem of fractional drift diffusion system with power-law nonlinearity. Using the properties of fractional heat semigroup and the classical estimates of fractional heat kernel, we first prove the global-in-time existence and uniqueness of the mild solutions in the frame of mixed time-space Besov space with multi-linear continuous mappings. Then we show the asymptotic behavior and regularizing-decay rate estimates of the solution to equations with power-law nonlinearity by the method of multi-linear operator and the classical Hardy-Littlewood-Sobolev inequality. Copyright © 2021 John Wiley & Sons, Ltd.

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1. Introduction

In this paper we consider the global solution to the Cauchy problem of fractional drift diffusion system with power-law nonlinearity

$$\begin{cases} \partial_t v + \Lambda^\alpha v = -\nabla\cdot(v^m \nabla\phi), & t>0, x\in\mathbb{R}^N,\\ \partial_t w + \Lambda^\alpha w = \nabla\cdot(w^m \nabla\phi), & t>0, x\in\mathbb{R}^N,\\ \Delta\phi = v-w, & t>0, x\in\mathbb{R}^N,\\ v(x,0)=v_0(x),\ \ w(x,0)=w_0(x), & x\in\mathbb{R}^N, \end{cases} \tag{1.1}$$

where $m\geq 1$ is an integer, $v(x,t), w(x,t)$ are the densities of negatively and positively charged particles, $\phi(x,t)$ is the electric potential determined by the Poisson equation $\Delta\phi = v-w$. The difficulties mainly come from higher order nonlinear couplings.

By the fundamental solution of Laplacian

$$\Phi_N(x) = \begin{cases} -\frac{1}{2}|x|, & N=1,\\ -\frac{1}{2\pi}\ln|x|, & N=2,\\ \frac{1}{N(N-2)\omega(N)|x|^{N-2}}, & N\geq 3, \end{cases} \tag{1.2}$$

where $\omega(N)$ denotes the volume of the unit ball in $\mathbb{R}^N$, the electric potential ϕ can be expressed by the convolution

$$\phi = (-\Delta)^{-1}(w-v) = \Phi_N*(w-v) = \int_{\mathbb{R}^N}\Phi_N(x-y)(w-v)(y)dy. \tag{1.3}$$

$\Lambda=\sqrt{-\Delta}$ is the Calderón-Zygmund operator, and the fractional Laplacian $\Lambda^\alpha=(-\Delta)^{\frac{\alpha}{2}}$ with $1<\alpha<2N$ is a non-local fractional differential operator defined as

$$\Lambda^\alpha v(x) = \mathcal{F}^{-1}|\xi|^\alpha\mathcal{F}v(\xi), \tag{1.4}$$

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where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse [1].

In probabilistic terms, replacing the Laplacian Δ by its fractional power $-\Lambda^\alpha = -(-\Delta)^{\frac{\alpha}{2}}$, it leads to interesting and largely open questions of extensions of results for Brownian motion driven stochastic equations to those driven by Lévy $\alpha-$stable flights.

In the physical literature, such fractal anomalous diffusions have been recently enthusiastically embraced by a slew of investigators in the context of hydrodynamics, acoustics, trapping effects in surface diffusion, statistical mechanics, relaxation phenomena, and biology [2].

An important technical difficulty is that the densities of the semigroups generated by $-\Lambda^\alpha = -(-\Delta)^{\frac{\alpha}{2}}$ do not decay rapidly in $x \in \mathbb{R}^N$ as is the case of the heat semigroup $S(t) = e^{t\Delta}$ $(\alpha = 2)$, the Gauss-Weierstrass kernel $K_t(x) = \mathcal{F}^{-1}(e^{-t|\xi|^2})$ decays exponentially while the densities $\mathcal{F}^{-1}(e^{-t|\xi|^\alpha})(0 < \alpha < 2)$ of non-Gaussian Lévy $\alpha-$stable semigroups $S_\alpha(t) = e^{-t(-\Delta)^{\frac{\alpha}{2}}}$ have only an algebraic decay rate $|x|^{-N-\alpha}$.

For a more general nonlinear term in (1.1), the motivation is the Keller-Segel model [3, 4], a prototype of cross-diffusion models related to pattern formation, it describes the time and space dynamics of the density of cells (or organisms) $n(t,x)$ interacting with a chemoattractant $S(t,x)$ according to the following system

$$\begin{cases} \partial_t n = \nabla_x \cdot (D_n(n,s)\nabla_x n - \chi(n,s) n \nabla_x s) + F(n,s), \\ \partial_t s = D_s(n,s)\Delta s + G(n,s), \end{cases} \tag{1.5}$$

where F and G are the source terms related to interactions [5]. The positive definite nonlinear terms $D_n(n,s)$ and $D_s(n,s)$ are the diffusivity of the chemoattractant and of the cells, respectively. In many applications the cross-diffusion function $\chi(n,s)$ has a complicated structure, even if it has a very simple structure, for example, a polynomial $\chi(n,s) = n^m$, but fails to satisfy a global Lipschitz condition.

For $m = 1$, (1.1) becomes a fractional drift-diffusion system

$$\begin{cases} \partial_t v + \Lambda^\alpha v = -\nabla \cdot (v \nabla \phi), & t > 0, x \in \mathbb{R}^N, \\ \partial_t w + \Lambda^\alpha w = \nabla \cdot (w \nabla \phi), & t > 0, x \in \mathbb{R}^N, \\ \Delta \phi = v - w, & t > 0, x \in \mathbb{R}^N, \\ v(x,0) = v_0(x), \ w(x,0) = w_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$

Zhao-Liu [6] established global well-posedness and asymptotic stability of mild solutions for the Cauchy problem (1.5) with small initial data in critical Besov spaces, and proved the regularizing-decay rate estimates which imply that mild solutions are analytic in space variables. Ogawa-Yamamoto [7] considered the global existence and asymptotic behavior of solutions for the Cauchy problem (1.5), they showed that the time global existence of the solutions with large initial data in Lebesgue space $L^p(\mathbb{R}^N)$ and Sobolev space $W^{\alpha,p}(\mathbb{R}^N)$ and obtained the asymptotic expansion of the solution up to the second terms as $t \to +\infty$.

For $\alpha = 2$, (1.6) corresponds to the usual drift-diffusion system

$$\begin{cases} \partial_t v - \Delta v = -\nabla \cdot (v \nabla \phi), & t > 0, x \in \mathbb{R}^N, \\ \partial_t w - \Delta w = \nabla \cdot (w \nabla \phi), & t > 0, x \in \mathbb{R}^N, \\ \Delta \phi = v - w, & t > 0, x \in \mathbb{R}^N, \\ v(x,0) = v_0(x), \ w(x,0) = w_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.7}$$

it has been studied widely [8, 9, 10, 11, 12, 13, 14].

For $w = 0$, (1.6) corresponds to the generalized Keller-Segel model of chemotaxis

$$\begin{cases} \partial_t v + \Lambda^\alpha v = -\nabla \cdot (v \nabla \phi), & t > 0, x \in \mathbb{R}^N, \\ \Delta \phi = v, & t > 0, x \in \mathbb{R}^N, \\ v(x,0) = v_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.8}$$

For $1 < \alpha < 2$, Escudero [20] proved that (1.8) admits a one-dimensional global solution (the same result also holds for $\alpha = 2$), Biler-Karch [21] studied the Blowup of solutions to generalized Keller-Segel model, and Biler-Wu [22] considered two-dimensional chemotaxis models with fractional diffusion. For $\alpha = 2$, Biler-Boritchev-Karch et al. considered the concentration phenomena [23] and gave sharp Sobolev estimates for concentration of solution [24] to the diffusive aggregation model

$$\partial_t v - \varepsilon \Delta v = -\nabla \cdot (v \nabla K * v)$$

with the Poisson kernel function K from the equation $\Delta \phi = v$.

Wu-Zheng [25] considered the parabolic-parabolic system corresponding to the parabolic-elliptic system (1.8), the Keller-Segel system with fractional diffusion generalizing the Keller-Segel model of chemotaxis

$$\begin{cases} \partial_t u + \Lambda^\alpha u = \pm \nabla \cdot (u \nabla \phi), & t > 0, x \in \mathbb{R}^N, \\ \varepsilon \partial_t \phi + \Lambda^\alpha \phi = u, & t > 0, x \in \mathbb{R}^N, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.9}$$

for the initial data (u_0, v_0) in the critical Fourier-Herz spaces $\dot{B}_q^{2-2\alpha}(\mathbb{R}^N) \times \dot{B}_q^{2-\alpha}(\mathbb{R}^N)$ with $2 \le q \le \infty$ for $\varepsilon > 0$ and $1 < \alpha \le 2$.

For the fractional evolution equations with higher order nonlinearity, Miao-Yuan-Zhang [15] studied the Cauchy problem for the semilinear fractional power dissipative equation

$$\begin{cases} \partial_t u + \Lambda^\alpha u = F(u), & t > 0, x \in \mathbb{R}^N, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.10}$$

with the nonlinear term $F(u) = f(u)$ or $F(u) = Q(D)f(u)$, where $Q(D)$ is a homogeneous pseudo differential operator and $f(u) = |u|^b u$ or $|u|^{b_1} u + |u|^{b_2} u$ with $b > 0, b_1 > 0$ and $b_2 > 0$. For example, the equation in (1.10) contains the semilinear fractional power dissipative equation $\partial_t u + \Lambda^\alpha u = \pm |u|^b u$, the generalized convection-diffusion equation $\partial_t u + \Lambda^\alpha u = a \cdot \nabla(|u|^b u)$, and so on.

By the fractional heat semigroup $S_\alpha(t) = e^{-t\Lambda^\alpha}$ and the well-known Duhamel principle, we rewrite the system (1.1) as a system of integral equations

$$\begin{cases} v(t) = S_\alpha(t) v_0 + B(v, \cdots, v, w), \\ w(t) = S_\alpha(t) w_0 + B(w, \cdots, w, v), \end{cases} \tag{1.11}$$

where

$$B(\underbrace{v, \cdots, v}_{m}, w) = \int_0^t S_\alpha(t-\tau) \nabla \cdot (v^m \nabla \phi)(\tau) d\tau, \ \phi = (-\Delta)^{-1}(w - v). \tag{1.12}$$

A solution of (1.11)-(1.12) is called a mild solution of (1.1).

Inspired by the contributions summarized in the above items, we aim to extend the results to the system (1.1) with higher order nonlinear terms $\nabla \cdot (v^m \nabla \phi)$ and $\nabla \cdot (w^m \nabla \phi)$. The goal of this article is to prove the global well-posedness of mild solutions to the Cauchy problem (1.1) with small initial data in critical Besov spaces. When $m = 1$ in the higher order nonlinear term $\nabla \cdot (v^m \nabla \phi)$, we recover the result proved in [6]. The outline of the rest of the article is as follows. In Section 2 we give the definition of homogeneous Besov space by the fractional heat semigroup operator and present some useful estimates. In Section 3 we establish the global existence and uniqueness of mild solution. In Section 4 we discuss the asymptotic stability of the mild solution. In Section 5 we give the regularizing-decay rate estimates of the mild solution. In Section 6 we consider a fractional drift diffusion system with generalized electric potential equation and we also give the global existence and asymptotic stability of the mild solution.

2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^N)$ be the Schwartz space and $\mathcal{S}'(\mathbb{R}^N)$ be its dual. Now, we introduce a definition of the homogeneous Besov space by the semigroup operator $S_\alpha(t) = e^{-t\Lambda^\alpha}$.

Definition 2.1 *[6] Let $l > 0$ and $1 \le p \le \infty$. Define*

$$\dot{B}^{-l}_{p,\infty}(\mathbb{R}^N) = \left\{ f \in \mathcal{S}'(\mathbb{R}^N) : S_\alpha f \in C((0, +\infty), L^p), \sup_{t>0} t^{\frac{l}{\alpha}} ||S_\alpha f||_{L^p} < \infty \right\} \tag{2.1}$$

with the norm

$$||f||_{\dot{B}^{-l}_{p,\infty}(\mathbb{R}^N)} = \sup_{t>0} t^{\frac{l}{\alpha}} ||S_\alpha(t) f||_{L^p}. \tag{2.2}$$

$(\dot{B}^{-l}_{p,\infty}(\mathbb{R}^N), ||\cdot||_{\dot{B}^{-l}_{p,\infty}})$ is a Banach space.

If $(v(x,t), w(x,t))$ is a solution of the Cauchy problem (1.1), for any $\lambda > 0$, denote

$$v_\lambda(x,t) = \lambda^{\frac{\alpha}{m}} v(\lambda x, \lambda^\alpha t), \ w_\lambda(x,t) = \lambda^{\frac{\alpha}{m}} w(\lambda x, \lambda^\alpha t), \tag{2.3}$$

then $(v_\lambda(x,t), w_\lambda(x,t))$ is also a solution of the Cauchy problem (1.1) with the initial data $(v_\lambda(x,0), w_\lambda(x,0)) = (\lambda^{\frac{\alpha}{m}} v_0(\lambda x), \lambda^{\frac{\alpha}{m}} w_0(\lambda x))$.

We can verify that $\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)$ is a critical space, which defined in [6], for initial data $(v_0(x), w_0(x))$ of the system (1.1). In order to find a critical space for the solutions of the Cauchy problem (1.1), we define some time-weighted space-time space.

Let X be a Banach space and I be a finite or infinite interval. We define the time-weighted space-time Banach space

$$C_\sigma(I; X) = \left\{ f \in C(I; X) : \sup_{t>0} t^{\frac{1}{\sigma}} ||f||_X < \infty \right\} \tag{2.4}$$

with the norm $||f||_{C_\sigma(I;X)} = \sup_{t>0} t^{\frac{1}{\sigma}}||f||_X$. The corresponding homogeneous time-weighted space-time Banach space

$$\dot{C}_\sigma(I;X) = \Big\{f \in C_\sigma(I;X) : \lim_{t\downarrow 0} t^{\frac{1}{\sigma}}||f||_X = 0\Big\}. \tag{2.5}$$

We use $C_*([0,\infty);X)$ denotes the set of bounded maps from $[0,\infty)$ to X which are continuous for $t>0$ and weakly continuously for $t=0$.

For the initial data $(v_0(x), w_0(x))$ in a critical homogeneous Besov space $\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)$ with minimal regularity, we want to find a mild solution of the Cauchy problem (1.1) and discuss the global existence of mild solution in the following critical space

$$\mathcal{X} = C_*([0,\infty), \dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)) \cap C_{\frac{m\alpha p}{\alpha p - mN}}([0,\infty), L^p(\mathbb{R}^N)) \tag{2.6}$$

with the norm

$$||u||_{\mathcal{X}} = \sup_{t>0}||u(t)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} + \sup_{t>0} t^{\frac{1}{m}-\frac{N}{\alpha p}}||u(t)||_{L^p(\mathbb{R}^N)}. \tag{2.7}$$

For the Laplacian operator Δ and the Calderón-Zygmund operator $\Lambda = \sqrt{-\Delta}$, we have the following classical Hardy-Littlewood-Sobolev inequality.

Lemma 2.1 *[18, 26] Let $1<p<N$, the nonlocal operator $\sqrt{-\Delta}$ is bounded from $L^p(\mathbb{R}^N)$ to $L^{\frac{Np}{N-p}}(\mathbb{R}^N)$, i.e., $\forall f \in L^p(\mathbb{R}^N)$,*

$$||\sqrt{-\Delta}f||_{L^{\frac{Np}{N-p}}(\mathbb{R}^N)} \le C(N,p)||f||_{L^p(\mathbb{R}^N)}, \tag{2.8}$$

$$||\nabla(-\Delta)^{-1}f||_{L^{\frac{Np}{N-p}}(\mathbb{R}^N)} \le C(N,p)||f||_{L^p(\mathbb{R}^N)}. \tag{2.9}$$

For the fractional power operator $\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ and the semigroup operator $S_\alpha(t) = e^{-t\Lambda^\alpha}$, we first consider the Cauchy problem for the homogeneous linear fractional heat equation

$$\begin{cases} \partial_t u + \Lambda^\alpha u = 0, & t>0, x\in\mathbb{R}^N, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{2.10}$$

By the Fourier transform the solution can be written as

$$u(t,x) = \mathcal{F}^{-1}(e^{-t|\xi|^\alpha}\mathcal{F}u_0(\xi)) = \mathcal{F}^{-1}(e^{-t|\xi|^\alpha}) * u_0(x) = K_t(x) * u_0(x) = S_\alpha(t)u_0(x), \tag{2.11}$$

where the fractional heat kernel

$$K_t(x) = (2\pi)^{-\frac{N}{2}}\int_{\mathbb{R}^N} e^{ix\xi}e^{-t|\xi|^\alpha}d\xi = t^{-\frac{N}{\alpha}}K(xt^{-\frac{1}{\alpha}}), \tag{2.12}$$

the function $K(x) \in L^\infty(\mathbb{R}^N)\cap C_0(\mathbb{R}^N)$, where $C_0(\mathbb{R}^{\mathbb{N}})$ denotes the space of functions $f \in C(\mathbb{R}^N)$ satisfying that $\lim_{|x|\to\infty} f(x) = 0$.

For the semigroup operator $S_\alpha(t)$ we have $L^p - L^q$ estimates

Lemma 2.2 *[9] Let $1 \le p \le q \le \infty$. Then, $\forall f \in L^p(\mathbb{R}^N)$,*

$$||S_\alpha(t)f||_{L^q} \le C(N,\alpha)t^{-\frac{N}{\alpha}(\frac{1}{p}-\frac{1}{q})}||f||_{L^p}, \tag{2.13}$$

$$||\Lambda^\gamma S_\alpha(t)f||_{L^q} \le C(N,\alpha)t^{-\frac{\gamma}{\alpha}-\frac{N}{\alpha}(\frac{1}{p}-\frac{1}{q})}||f||_{L^p}, \tag{2.14}$$

for $\alpha>0$ and $\gamma>0$.

Following the work of Kato [16, 17] and Lemarie-Rieusset [18] for the Navier-Stokes problem. Miao-Yuan [19] gave a general existence and uniqueness result for an abstract operator equation via a contraction argument.

Lemma 2.3 *[19] Let X be a Banach space and $B: X\times X\times\cdots\times X \to X$ be a $(m+1)-$linear continuous operator satisfying*

$$||B(u_1,u_2,\cdots,u_{m+1})||_X \le K||u_1||_X||u_2||_X\cdots||u_{m+1}||_X, \tag{2.15}$$

$\forall u_1,u_2,\cdots,u_{m+1}\in X$ for some constant $K>0$. Let $\varepsilon>0$ be such that $(m+1)(2\varepsilon)^m K < 1$. Then for every $y \in X$ with $||y||_X \le \varepsilon$ the equation

$$u = y + B(u,u,\cdots,u) \tag{2.16}$$

has a unique solution $u\in X$ satisfying that $||u||_X \le 2\varepsilon$. Moreover, the solution u depends continuously on y in the sense that, if $||y||_X\le\varepsilon$ and $v = y_1 + B(v,v,\cdots,v)$, $||v||_X \le 2\varepsilon$, then

$$||u-v||_X \le \frac{1}{1-(m+1)(2\varepsilon)^m K}||y-y_1||_X. \tag{2.17}$$

We will use the Lemma to prove the global-in-time existence and uniqueness of the mild solutions to the Cauchy problem (1.1) in the mixed time-space Besov space.

3. Global existence and uniqueness in Besov space

In this section we give the global existence and uniqueness of mild solution to the Cauchy problem (1.1).

Theorem 3.1 *Let $N \geq 2$ be a positive integer, $1 < \alpha \leq 2N$ and*

$$\max\{1, \frac{mN}{\alpha}\} < p < \min\{N, \frac{m(m+1)N}{\alpha}\}. \tag{3.1}$$

If $(v_0, w_0) \in \dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)$, then there exists $\varepsilon > 0$ such that if $||(v_0, w_0)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}} \leq \varepsilon$, the Cauchy problem (1.1) has a unique global mild solution $(v, w) \in \mathcal{X}$ such that $||(v, w)||_{\mathcal{X}} \leq 2\varepsilon$. Moreover, the solution depends continuously on initial data in the following sense: let $(\tilde{v}, \tilde{w}) \in \mathcal{X}$ be the solution of (1.1) with initial data $(\tilde{v}_0, \tilde{w}_0)$ such that $||(\tilde{v}_0, \tilde{w}_0)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} \leq \varepsilon$, then there is a constant C such that

$$||(v - \tilde{v}, w - \tilde{w})||_{\mathcal{X}} \leq C||(v - \tilde{v}_0, w - \tilde{w}_0)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}.$$

Now for the integral system (1.11)-(1.12) we first consider the term $S_\alpha(t)v_0 = e^{-t\Lambda^\alpha}v_0$.

Lemma 3.1 *Let $v_0(x) \in \dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)$ and (3.1) hold true. Then $S_\alpha(t)v_0 \in \mathcal{X}$ and*

$$||S_\alpha(t)v_0||_{\mathcal{X}} \leq C(N, \alpha)||v_0||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}. \tag{3.2}$$

Proof. According to the definition of the norm $||\cdot||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}$ and $L^p - L^q$ estimates for the semigroup operator $S_\alpha(t) = e^{-t\Lambda^\alpha}$, we have

$$\begin{aligned} ||S_\alpha(t)v_0||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} &= \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)S_\alpha(t)v_0||_{L^p} = \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(t)S_\alpha(s)v_0||_{L^p} \\ &\leq C(N, \alpha)\sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)v_0||_{L^p} = C(N, \alpha)||v_0||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}, \end{aligned}$$

and

$$\sup_{t>0} t^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(t)v_0||_{L^p} = ||v_0||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}.$$

Therefore, we have

$$S_\alpha(t)v_0 \in L^\infty((0, \infty), \dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)), \ t^{\frac{1}{m}-\frac{N}{\alpha p}}S_\alpha(t)v_0 \in L^\infty((0, \infty), L^p(\mathbb{R}^N)).$$

Moreover, following the method of [18, Proposition 4.4, P33] we obtain that

$$S_\alpha(t)v_0 \in C_*([0, \infty), \dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)).$$

On the other hand, from $v_0(x) \in \dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)$ and Definition 2.1, we have

$$S_\alpha(t)v_0 \in C((0, \infty), L^p(\mathbb{R}^N)), \quad t^{\frac{1}{m}-\frac{N}{\alpha p}}S_\alpha(t)v_0 \in C((0, \infty), L^p(\mathbb{R}^N)).$$

Hence, we have $S_\alpha(t)v_0 \in \mathcal{X}$ and (3.2) holds true. □

Lemma 3.2 *Let $(v, w) \in \mathcal{X}$ and (3.1) hold true. Then $B(v, \cdots, v, w) \in \mathcal{X}$ and*

$$||B(v, \cdots, v, w)||_{\mathcal{X}} \leq C(N, \alpha, p)||v||_{\mathcal{X}}^m||v - w||_{\mathcal{X}}. \tag{3.3}$$

Proof. According to the definition of the norm $||\cdot||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}$, we have

$$||B(v, \cdots, v, w)(t)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} = \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)B(v, \cdots, v, w)(t)||_{L^p},$$

by the expression (1.12) of $B(v, \cdots, v, w)(t)$, that is,

$$B(\underbrace{v, \cdots, v}_{m}, w) = -\int_0^t S_\alpha(t-\tau)\nabla\cdot(v^m\nabla\phi)(\tau)d\tau, \ \phi = (-\Delta)^{-1}(w - v), \tag{3.4}$$

hence by the Minkowski inequality, we get

$$\begin{aligned} ||B(v,\cdots,v,w)(t)||_{\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)} &= \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)\int_0^t S_\alpha(t-\tau)\nabla\cdot(v^m\nabla\phi)(\tau)d\tau||_{L^p} \\ &\leq \int_0^t \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)S_\alpha(t-\tau)\nabla\cdot(v^m\nabla\phi)(\tau)||_{L^p}d\tau. \end{aligned} \tag{3.5}$$

For $0<s\leq t-\tau$, using the L^p-L^q estimates (2.13) and (2.14) for the semigroup operator $S_\alpha(t)=e^{-t\Lambda^\alpha}$, we have

$$\begin{aligned} &\sup_{0<s\leq t-\tau} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)S_\alpha(t-\tau)\nabla\cdot(v^m\nabla\phi)(\tau)||_{L^p} \leq C(N,\alpha)(t-\tau)^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(t-\tau)\nabla\cdot(v^m\nabla\phi)(\tau)||_{L^p} \\ &= C(N,\alpha)(t-\tau)^{\frac{1}{m}-\frac{N}{\alpha p}}||\nabla\cdot S_\alpha(t-\tau)(v^m\nabla\phi)(\tau)||_{L^p} \leq C(N,\alpha,p)(t-\tau)^{\frac{1}{m}-\frac{N}{\alpha p}}(t-\tau)^{-\frac{mN}{\alpha p}}||(v^m\nabla\phi)](\tau)||_{L^{\frac{Np}{(m+1)N-p}}} \\ &\leq C(N,\alpha,p)(t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}||v||^m_{L^p}||\nabla\phi(\tau)||_{L^{\frac{Np}{N-p}}}, \end{aligned}$$

the last inequality comes from the Hölder inequality for the product $v\cdot v\cdots v\cdot(v-w)$ and $\frac{m}{p}+\frac{N-p}{Np}=\frac{(m+1)N-p}{Np}$. Using the classical Hardy-Littlewood-Sobolev inequality (2.8) and (2.9), we have

$$\begin{aligned} &\sup_{0<s\leq t-\tau} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)S_\alpha(t-\tau)\nabla\cdot[v^m\nabla(-\Delta)^{-1}(v-w)](\tau)||_{L^p} \\ &\leq C(N,\alpha,p)(t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}||v(\tau)||^m_{L^p}||(v-w)(\tau)||_{L^p}. \end{aligned} \tag{3.6}$$

For $s>t-\tau$, using the L^p-L^q estimates (2.13) and (2.14) for the semigroup operator $S_\alpha(t)=e^{-t\Lambda^\alpha}$, we have

$$\begin{aligned} &\sup_{s>t-\tau} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)S_\alpha(t-\tau)\nabla\cdot(v^m\nabla\phi)(\tau)||_{L^p} = \sup_{s>t-\tau} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(t+s-\tau)\nabla\cdot(v^m\nabla\phi)(\tau)||_{L^p} \\ &\leq C(N,\alpha)\sup_{s>t-\tau} s^{\frac{1}{m}-\frac{N}{\alpha p}}(t+s-\tau)^{-\frac{mN}{\alpha p}}||v^m\nabla\phi(\tau)||_{L^{\frac{Np}{(m+1)N-p}}} \leq C(N,\alpha)\sup_{s>t-\tau} s^{\frac{1}{m}-\frac{N}{\alpha p}}(t+s-\tau)^{-\frac{mN}{\alpha p}}||v||^m_{L^p}||\nabla\phi(\tau)||_{L^{\frac{Np}{N-p}}}. \end{aligned}$$

From the condition (3.1): $\max\{1,\frac{mN}{\alpha}\}<p<\min\{N,\frac{m(m+1)N}{\alpha}\}$ and $s>t-\tau$, the function $f(s)=s^{\frac{1}{m}-\frac{N}{\alpha p}}(t+s-\tau)^{-\frac{mN}{\alpha p}}$ has the maximum

$$\max_{s>t-\tau} f(s) = f\left(\frac{\frac{1}{m}-\frac{N}{\alpha p}}{\frac{(m+1)N}{\alpha p}-\frac{1}{m}}(t-\tau)\right) = C(t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}},$$

where C is a constant, by (2.9) we have

$$\begin{aligned} &\sup_{s>t-\tau} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)S_\alpha(t-\tau)\nabla\cdot[v^m\nabla(-\Delta)^{-1}(v-w)](\tau)||_{L^p} \\ &\leq C(N,\alpha,p)(t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}||v(\tau)||^m_{L^p}||(v-w)(\tau)||_{L^p}. \end{aligned} \tag{3.7}$$

Together with (3.6) and (3.7) we have

$$\begin{aligned} &\sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)S_\alpha(t-\tau)\nabla\cdot[v^m\nabla(-\Delta)^{-1}(v-w)](\tau)||_{L^p} \\ &\leq C(N,\alpha,p)(t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}||v(\tau)||^m_{L^p}||(v-w)(\tau)||_{L^p}. \end{aligned} \tag{3.8}$$

Putting (3.8) into (3.5), we have

$$\begin{aligned} ||B(v,\cdots,v,w)(t)||_{\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)} &\leq C(N,\alpha,p)\int_0^t (t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}||v(\tau)||^m_{L^p}||(v-w)(\tau)||_{L^p}d\tau \\ &\leq C(N,\alpha,p)\sup_{\tau>0}(\tau^{\frac{1}{m}-\frac{N}{\alpha p}}||v(\tau)||_{L^p})^m \sup_{\tau>0}(\tau^{\frac{1}{m}-\frac{N}{\alpha p}}||(v-w)(\tau)||_{L^p}) \\ &\qquad \times\int_0^t (t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}\tau^{\frac{(m+1)N}{\alpha p}-\frac{1}{m}-1}d\tau \\ &\leq C(N,\alpha,p)||v||^m_{\mathcal{X}}||v-w||_{\mathcal{X}}\int_0^t (t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}\tau^{\frac{(m+1)N}{\alpha p}-\frac{1}{m}-1}d\tau \\ &\leq C(N,\alpha,p)||v||^m_{\mathcal{X}}||v-w||_{\mathcal{X}}, \end{aligned}$$

in the last inequality we use the fact that the Beta function

$$\int_0^t (t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}} \tau^{\frac{(m+1)N}{\alpha p}-\frac{1}{m}-1} d\tau = \mathcal{B}(\frac{m+1}{m}-\frac{(m+1)N}{\alpha p}, \frac{(m+1)N}{\alpha p}-\frac{1}{m})$$

converges to a constant, since the condition (3.1) implies that $\frac{m+1}{m}-\frac{(m+1)N}{\alpha p}=\frac{m+1}{mp}(p-\frac{mN}{\alpha})>0$ and $\frac{(m+1)N}{\alpha p}-\frac{1}{m}=\frac{1}{mp}(\frac{m(m+1)N}{\alpha}-p)>0$.

Therefore, we have

$$||B(v,\cdots,v,w)(t)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} \leq C(N,\alpha,p)||v||_{\mathcal{X}}^m ||v-w||_{\mathcal{X}}. \tag{3.9}$$

Next, we consider the estimate of $||B(v,\cdots,v,w)(t)||_{L^p}$. From (1.12) we have

$$\begin{aligned}
&||B(v,\cdots,v,w)(t)||_{L^p} = ||\int_0^t S_\alpha(t-\tau)\nabla\cdot(v^m\nabla\phi)(\tau)d\tau||_{L^p}\\
\leq\ & C(N,\alpha)\int_0^t (t-\tau)^{-\frac{mN}{\alpha p}} ||v^m\nabla(-\Delta)^{-1}(v-w)](\tau)||_{L^{\frac{Np}{(m+1)N-p}}} d\tau\\
\leq\ & C(N,\alpha)\int_0^t (t-\tau)^{-\frac{mN}{\alpha p}} ||v||_{L^p}^m ||\nabla(-\Delta)^{-1}(v-w)](\tau)||_{L^{\frac{Np}{N-p}}} d\tau\\
\leq\ & C(N,\alpha,p)||v||_{\mathcal{X}}^m ||v-w||_{\mathcal{X}} \int_0^t (t-\tau)^{-\frac{mN}{\alpha p}} \tau^{-\frac{1}{m}-1+\frac{(m+1)N}{\alpha p}} d\tau\\
\leq\ & C(N,\alpha,p)||v||_{\mathcal{X}}^m ||v-w||_{\mathcal{X}} t^{-\frac{1}{m}+\frac{N}{\alpha p}},
\end{aligned}$$

thus,

$$\sup_{t>0} t^{\frac{1}{m}-\frac{N}{\alpha p}} ||B(v,\cdots,v,w)(t)||_{L^p} \leq C(N,\alpha,p)||v||_{\mathcal{X}}^m ||v-w||_{\mathcal{X}}. \tag{3.10}$$

In order to prove that $B(v,\cdots,v,w)\in\mathcal{X}$, it suffices to prove that $B(v,\cdots,v,w)$ is continuous for $t>0$ and weakly continuous for $t=0$ in $\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)$, and it is continuous for $t\geq 0$ in $L^p(\mathbb{R}^N)$.

For any $0<t_1<t_2$, due to (3.4) we have

$$\begin{aligned}
B(v,\cdots,v,w)(t_2)-B(v,\cdots,v,w)(t_1) &= \int_0^{t_1}[S_\alpha(t_2-\tau)-S_\alpha(t_1-\tau)]\nabla\cdot[v^m\nabla(-\Delta)^{-1}(v-w)](\tau)d\tau\\
&+ \int_{t_1}^{t_2} S_\alpha(t_2-\tau)\nabla\cdot[v^m\nabla(-\Delta)^{-1}(v-w)](\tau)d\tau\\
&:= I(t_1,t_2)+II(t_1,t_2).
\end{aligned} \tag{3.11}$$

Similar to the estimate of $||B(v,\cdots,v,w)(t)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}$, we have

$$\begin{aligned}
&||II(t_1,t_2)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} = \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}} ||S_\alpha(s)II(t_1,t_2)||_{L^p}\\
&\leq \int_{t_1}^{t_2} \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}} ||S_\alpha(s)S_\alpha(t_2-\tau)\nabla\cdot[v^m\nabla(-\Delta)^{-1}(v-w)](\tau)||_{L^p}d\tau\\
&\leq C(N,\alpha,p)||v||_{\mathcal{X}}^m||v-w||_{\mathcal{X}} \int_{t_1}^{t_2}(t_2-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}} \tau^{\frac{(m+1)N}{\alpha p}-\frac{1}{m}-1}d\tau\\
&\leq C(N,\alpha,p)||v||_{\mathcal{X}}^m||v-w||_{\mathcal{X}} t_1^{-1-\frac{1}{m}+\frac{(m+1)N}{\alpha p}} \int_{t_1}^{t_2}(t_2-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}d\tau\\
&\leq C(N,\alpha,p)||v||_{\mathcal{X}}^m||v-w||_{\mathcal{X}} t_1^{-1-\frac{1}{m}+\frac{(m+1)N}{\alpha p}} (t_2-t_1)^{1+\frac{1}{m}-\frac{(m+1)N}{\alpha p}},
\end{aligned}$$

the condition (3.1) implies that $1+\frac{1}{m}-\frac{(m+1)N}{\alpha p}>0$, hence

$$||II(t_1,t_2)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} = \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}} ||S_\alpha(s)II(t_1,t_2)||_{L^p} \to 0 \text{ as } t_2\to t_1. \tag{3.12}$$

According to the property of semigroup,

$$S_\alpha(t_2-\tau)-S_\alpha(t_1-\tau)=[S_\alpha(t_2-t_1)-I]S_\alpha(t_1-\tau), \tag{3.13}$$

for $\phi = (-\Delta)^{-1}(w-v)$ we get

$$
\begin{aligned}
&||I(t_1,t_2)||_{\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)} = \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)I(t_1,t_2)||_{L^p}\\
&\le \int_0^{t_1} \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)[S_\alpha(t_2-t_1)-I]S_\alpha(t_1-\tau)\nabla\cdot(v^m\nabla\phi)(\tau)||_{L^p}d\tau\\
&= \int_0^{t_1} \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||\int_0^{t_2-t_1}\Lambda^\alpha S_\alpha(\mu)S_\alpha(s)S_\alpha(t_1-\tau)\nabla\cdot(v^m\nabla\phi)(\tau)d\mu||_{L^p}d\tau\\
&= \int_0^{t_1} \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||\int_0^{t_2-t_1}\nabla S_\alpha(\mu)\Lambda^\alpha S_\alpha(s)S_\alpha(t_1-\tau)(v^m\nabla\phi)(\tau)d\mu||_{L^p}d\tau\\
&\le \int_0^{t_1} \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}\int_0^{t_2-t_1}||\nabla S_\alpha(\mu)\Lambda^\alpha S_\alpha(s)S_\alpha(t_1-\tau)(v^m\nabla\phi)(\tau)||_{L^p}d\mu d\tau, \qquad (3.14)
\end{aligned}
$$

by the $L^p - L^q$ estimates (2.13) and (2.14) for the semigroup operator $S_\alpha(t) = e^{-t\Lambda^\alpha}$, we have

$$
\begin{aligned}
&\int_0^{t_2-t_1}||\nabla S_\alpha(\mu)\Lambda^\alpha S_\alpha(s)S_\alpha(t_1-\tau)(v^m\nabla\phi)(\tau)||_{L^p}d\mu\\
&\le C(N,\alpha)\int_0^{t_2-t_1}\mu^{-\frac{mN}{\alpha p}}d\mu||\Lambda^\alpha S_\alpha(s)S_\alpha(t_1-\tau)(v^m\nabla\phi)(\tau)||_{L^{\frac{Np}{(m+1)N-p}}}\\
&= C(N,\alpha)(t_2-t_1)^{1-\frac{mN}{\alpha p}}||\Lambda^\alpha S_\alpha(s)S_\alpha(t_1-\tau)(v^m\nabla\phi)(\tau)||_{L^{\frac{Np}{(m+1)N-p}}}. \qquad (3.15)
\end{aligned}
$$

For $0<s\le t_1-\tau$, we have

$$
\begin{aligned}
&\sup_{0<s\le t_1-\tau} s^{\frac{1}{m}-\frac{N}{\alpha p}}||\Lambda^\alpha S_\alpha(s)S_\alpha(t_1-\tau)(v^m\nabla\phi)(\tau)||_{L^{\frac{Np}{(m+1)N-p}}}\\
&= \sup_{0<s\le t_1-\tau} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)\Lambda^\alpha S_\alpha(t_1-\tau)(v^m\nabla\phi)(\tau)||_{L^{\frac{Np}{(m+1)N-p}}}\\
&\le C(N,\alpha)\sup_{0<s\le t_1-\tau} s^{\frac{1}{m}-\frac{N}{\alpha p}}(t_1-\tau)^{-1}||(v^m\nabla\phi)(\tau)||_{L^{\frac{Np}{(m+1)N-p}}}\\
&\le C(N,\alpha)(t_1-\tau)^{\frac{1}{m}-\frac{N}{\alpha p}-1}||v||^m_{L^p}||\nabla\phi||_{L^{\frac{Np}{N-p}}}\\
&\le C(N,\alpha,p)(t_1-\tau)^{\frac{1}{m}-\frac{N}{\alpha p}-1}||v||^m_{L^p}||v-w||_{L^p}. \qquad (3.16)
\end{aligned}
$$

For $s>t_1-\tau$, we have

$$
\begin{aligned}
&\sup_{s>t_1-\tau} s^{\frac{1}{m}-\frac{N}{\alpha p}}||\Lambda^\alpha S_\alpha(s)S_\alpha(t_1-\tau)(v^m\nabla\phi)(\tau)||_{L^{\frac{Np}{(m+1)N-p}}}\\
&= \sup_{s>t_1-\tau} s^{\frac{1}{m}-\frac{N}{\alpha p}}||\Lambda^\alpha S_\alpha(t_1-\tau+s)(v^m\nabla\phi)(\tau)||_{L^{\frac{Np}{(m+1)N-p}}}\\
&\le C(N,\alpha)\sup_{s>t_1-\tau} s^{\frac{1}{m}-\frac{N}{\alpha p}}(t_1-\tau+s)^{-1}||v^m\nabla\phi||_{L^{\frac{Np}{(m+1)N-p}}}\\
&\le C(N,\alpha,p)(t_1-\tau)^{\frac{1}{m}-\frac{N}{\alpha p}-1}||v||^m_{L^p}||v-w||_{L^p}. \qquad (3.17)
\end{aligned}
$$

Putting (3.15)-(3.17) into (3.14), we have

$$
\begin{aligned}
||I(t_1,t_2)||_{\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)} &\le C(N,\alpha,p)(t_2-t_1)^{1-\frac{mN}{\alpha p}}\int_0^{t_1}(t_1-\tau)^{\frac{1}{m}-\frac{N}{\alpha p}-1}||v(\tau)||^m_{L^p}||(v-w)(\tau)||_{L^p}d\tau\\
&\le C(N,\alpha,p)(t_2-t_1)^{1-\frac{mN}{\alpha p}}\sup_{\tau>0}(\tau^{\frac{1}{m}-\frac{N}{\alpha p}}||v(\tau)||_{L^p})^m\sup_{\tau>0}(\tau^{\frac{1}{m}-\frac{N}{\alpha p}}||(v-w)(\tau)||_{L^p})\\
&\quad\times\int_0^{t_1}(t_1-\tau)^{\frac{1}{m}-\frac{N}{\alpha p}-1}\tau^{\frac{(m+1)N}{\alpha p}-\frac{1}{m}-1}d\tau\\
&\le C(N,\alpha,p)(t_2-t_1)^{1-\frac{mN}{\alpha p}}||v||^m_{\mathcal{X}}||v-w||_{\mathcal{X}}Bt_1^{\frac{mN}{\alpha p}-1}, \qquad (3.18)
\end{aligned}
$$

where the Beta function $B = \mathcal{B}(\frac{1}{m}-\frac{N}{\alpha p}, \frac{(m+1)N}{\alpha p}-\frac{1}{m})$ converges due to the condition (3.1), thus we have

$$
||I(t_1,t_2)||_{\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)} \le C||v||^m_{\mathcal{X}}||v-w||_{\mathcal{X}}(t_2-t_1)^{1-\frac{mN}{\alpha p}}t_1^{\frac{mN}{\alpha p}-1}, \qquad (3.19)
$$

that is,

$$||I(t_1,t_2)||_{\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)} = \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)I(t_1,t_2)||_{L^p} \to 0 \text{ as } t_2 \to t_1. \tag{3.20}$$

Putting (3.12) and (3.20) into (3.11) we have

$$||B(v,\cdots,v,w)(t_1) - B(v,\cdots,v,w)(t_2)||_{\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)} \to 0 \text{ as } t_2 \to t_1. \tag{3.21}$$

This means that $B(v,\cdots,v,w)$ is continuous for $t>0$ in $\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)$.

Similarly, we can prove that $B(v,\cdots,v,w)$ is weakly continuous for $t=0$ in $\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)$ and it is continuous for $t\geq 0$ in $L^p(\mathbb{R}^N)$. Therefore, we have

$$B(v,\cdots,v,w) \in C_*([0,\infty), \dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)) \cap C_{\frac{m\alpha p}{\alpha p - mN}}([0,\infty), L^p(\mathbb{R}^N)), \tag{3.22}$$

that is, $B(v,\cdots,v,w)\in\mathcal{X}$ and (3.3) holds true, i.e.,

$$||B(v,\cdots,v,w)||_{\mathcal{X}} \leq C(N,\alpha,p)||v||^m_{\mathcal{X}}||v-w||_{\mathcal{X}}. \tag{3.23}$$

This ends the proof of Lemma 3.2. □

The proof of Theorem 3.1. Now for the integral system (1.11)-(1.12) from the Cauchy problem (1.1), we have

$$(v(t),w(t)) = S_\alpha(t)(v_0,w_0) + (B(v,\cdots,v,w), B(w,\cdots,w,v)), \tag{3.24}$$

in Lemma 3.1 and Lemma 3.2 we deal with the terms $S_\alpha(t)(v_0,w_0)$ and

$$B(v,\cdots,v,w) = \int_0^t S_\alpha(t-\tau)\nabla\cdot[v^m\nabla(-\Delta)^{-1}(v-w)](\tau)d\tau,$$
$$B(w,\cdots,w,v) = \int_0^t S_\alpha(t-\tau)\nabla\cdot[w^m\nabla(-\Delta)^{-1}(w-v)](\tau)d\tau,$$

respectively. For the Banach space $\mathcal{X}$ and multi-linear operator $B(v,\cdots,v,w)$ which satisfies the estimate (3.23), following the Lemma 2.3, for every $(v_0,w_0)\in\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)$, there exists $\varepsilon>0$ such that $(m+1)(2\varepsilon)^mC(N,\alpha,p)<1$, then(3.24) has a unique solution $(v,w)\in\mathcal{X}$ such that $||(v,w)||_X\leq 2\varepsilon$. Therefore, the Cauchy problem (1.1) has a unique global-in-time mild solution in the mixed time-space Besov space. This completes the proof of Theorem 3.1. □

4. Asymptotic stability analysis

Theorem 4.1 *Let N be a positive integer, $1<\alpha\leq 2N$ and (3.1) hold true and (v,w) and $(\tilde{v},\tilde{w})$ be two mild solutions of the Cauchy problem (1.1) described in Theorem 3.1 corresponding to initial conditions (v_0,w_0) and $(\tilde{v}_0,\tilde{w}_0)$, respectively. If $(v_0,w_0),(\tilde{v}_0,\tilde{w}_0)\in\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)$ such that*

$$\lim_{t\to\infty}||S_\alpha(t)(v_0-\tilde{v}_0, w_0-\tilde{w}_0)||_{\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)} = 0, \tag{4.1}$$

then we have the following asymptotic stability

$$\lim_{t\to\infty}\left(||(v-\tilde{v},w-\tilde{w})||_{\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)} + t^{\frac{\alpha}{m}-\frac{N}{p}}||(v-\tilde{v},w-\tilde{w})||_{L^p(\mathbb{R}^N)}\right) = 0. \tag{4.2}$$

Proof. Since $(v_0,w_0),(\tilde{v}_0,\tilde{w}_0)\in\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)$, by Theorem 3.1, there exists a constant $\varepsilon>0$ such that if $||(v_0,w_0),(\tilde{v}_0,\tilde{w}_0)||_{\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}}\leq\varepsilon$, then the mild solutions (v,w) and $(\tilde{v},\tilde{w})$ satisfy that $||(v,w),(\tilde{v},\tilde{w})||_{\mathcal{X}}\leq 2\varepsilon$. From (1.11) and (1.12) we have

$$\begin{cases} v-\tilde{v} = S_\alpha(t)(v_0-\tilde{v}_0) + \sum\limits_{k=0}^{m-1} B_k(v-\tilde{v},v,\tilde{v},v-w) + B_m(\tilde{v},(v-\tilde{v})-(w-\tilde{w})), \\ w-\tilde{w} = S_\alpha(t)(w_0-\tilde{w}_0) + \sum\limits_{k=0}^{m-1} B_k(w-\tilde{w},w,\tilde{w},w-v) + B_m(\tilde{w},(w-\tilde{w})-(v-\tilde{v})), \end{cases}$$

where

$$B_k(v-\tilde v, v, \tilde v, v-w) = B(v-\tilde v, \underbrace{v,\cdots,v}_{k}, \underbrace{\tilde v,\cdots,\tilde v}_{m-1-k}, v-w)$$

$$= \int_0^t S_\alpha(t-\tau)\nabla\cdot[(v-\tilde v)v^k\tilde v^{m-1-k}\nabla(-\Delta)^{-1}(v-w)](\tau)d\tau, \tag{4.3}$$

$$B_m(\tilde v, (v-\tilde v)-(w-\tilde w)) = B(\underbrace{\tilde v,\cdots,\tilde v}_{m}, (v-\tilde v)-(w-\tilde w))$$

$$= \int_0^t S_\alpha(t-\tau)\nabla\cdot[\tilde v^m\nabla(-\Delta)^{-1}((v-\tilde v)-(w-\tilde w))](\tau)d\tau. \tag{4.4}$$

By the definition of $\dot B_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)$−norm, we have

$$||v-\tilde v||_{\dot B_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} \le ||S_\alpha(t)(v_0-\tilde v_0)||_{\dot B_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} + \sum_{k=0}^{m-1} I_k + I_m, \tag{4.5}$$

where

$$(I_k, I_m) = ||(B_k(v-\tilde v, v, \tilde v, v-w), B(\tilde v, (v-\tilde v)-(w-\tilde w)))||_{\dot B_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}.$$

For a constant $\theta\in(0,1)$ determined in later we have

$$\begin{aligned}
I_k &= \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)\int_0^t S_\alpha(t-\tau)\nabla\cdot[(v-\tilde v)v^k\tilde v^{m-1-k}\nabla(-\Delta)^{-1}(v-w)](\tau)d\tau||_{L^p}\\
&\le \int_0^t \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)S_\alpha(t-\tau)\nabla\cdot[(v-\tilde v)v^k\tilde v^{m-1-k}\nabla(-\Delta)^{-1}(v-w)](\tau)||_{L^p}d\tau\\
&\le \Big(\int_0^{\theta t}+\int_{\theta t}^t\Big)\sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(t+s-\tau)\nabla\cdot[(v-\tilde v)v^k\tilde v^{m-1-k}\nabla(-\Delta)^{-1}(v-w)]||_{L^p}d\tau\\
&:= I_{k1}+I_{k2}.
\end{aligned} \tag{4.6}$$

In the procedure of estimate of (3.5), instead of the product $v\cdot v\cdots v\cdot(v-w)$ with $m+1$ exponents such that $\frac{m}{p}+\frac{N-p}{Np}=\frac{(m+1)N-p}{Np}$, use the Hölder inequality for the product $(v-\tilde v)v^k\tilde v^{m-1-k}(v-w)$ with $m+1$ exponents such that $\frac{1}{p}+\frac{k}{p}+\frac{m-1-k}{p}+\frac{N-p}{Np}=\frac{(m+1)N-p}{Np}$, we can prove that

$$\begin{aligned}
I_{k1} &\le C\int_0^{\theta t}(t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}||v-\tilde v||_{L^p}||v||_{L^p}^k||\tilde v||_{L^p}^{m-1-k}||v-w||_{L^p}d\tau\\
&\le C\varepsilon^m\int_0^{\theta}(1-\eta)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}\eta^{-1-\frac{1}{m}+\frac{(m+1)N}{\alpha p}}((t\eta)^{\frac{1}{m}-\frac{N}{\alpha p}}||v(t\eta)-\tilde v(t\eta)||_{L^p})d\eta,
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
I_{k2} &\le C\int_{\theta t}^{t}(t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}||v-\tilde v||_{L^p}||v||_{L^p}^k||\tilde v||_{L^p}^{m-1-k}||v-w||_{L^p}d\tau\\
&\le C\varepsilon^m\int_{\theta t}^{t}(t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}\tau^{-1-\frac{1}{m}+\frac{(m+1)N}{\alpha p}}(\tau^{\frac{1}{m}-\frac{N}{\alpha p}}||v-\tilde v||_{L^p})d\tau\\
&\le C\varepsilon^m[\sup_{\theta t\le\tau\le t}\tau^{\frac{1}{m}-\frac{N}{\alpha p}}||v(\tau)-\tilde v(\tau)||_{L^p}].
\end{aligned} \tag{4.8}$$

Together (4.7) with (4.8) we have

$$\begin{aligned}
I_k &\le C\varepsilon^m\int_0^{\theta}(1-\eta)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}\eta^{-1-\frac{1}{m}+\frac{(m+1)N}{\alpha p}}((t\eta)^{\frac{1}{m}-\frac{N}{\alpha p}}||v(t\eta)-\tilde v(t\eta)||_{L^p})d\eta\\
&+ C\varepsilon^m[\sup_{\theta t\le\tau\le t}\tau^{\frac{1}{m}-\frac{N}{\alpha p}}||v(\tau)-\tilde v(\tau)||_{L^p}],\ k=1,2,\cdots,m-1.
\end{aligned} \tag{4.9}$$

Similarly we have

$$\begin{aligned}
I_m &\le C\varepsilon^m\int_0^{\theta}\frac{(1-\eta)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}}{\eta^{1+\frac{1}{m}-\frac{(m+1)N}{\alpha p}}}((t\eta)^{\frac{1}{m}-\frac{N}{\alpha p}}||((v-\tilde v)(t\eta),(w-\tilde w)(t\eta))||_{L^p})d\eta\\
&+ C\varepsilon^m[\sup_{\theta t\le\tau\le t}\tau^{\frac{1}{m}-\frac{N}{\alpha p}}||((v-\tilde v)(\tau),(w-\tilde w)(\tau))||_{L^p}].
\end{aligned} \tag{4.10}$$

We next consider the term $||v-\tilde{v}||_{L^p(\mathbb{R}^N)}$:

$$||v-\tilde{v}||_{L^p(\mathbb{R}^N)} \le ||S_\alpha(t)(v_0-\tilde{v}_0)||_{L^p(\mathbb{R}^N)} + \sum_{k=0}^{m-1} J_k + J_m, \tag{4.11}$$

where

$$(J_k, J_m) = ||(B_k(v-\tilde{v}, v, \tilde{v}, v-w), B(\tilde{v}, (v-\tilde{v})-(w-\tilde{w})))||_{L^p(\mathbb{R}^N)}.$$

For the first term we have

$$\begin{aligned} t^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(t)(v_0-\tilde{v}_0)||_{L^p(\mathbb{R}^N)} &\le 2^{\frac{1}{m}-\frac{N}{\alpha p}} \sup_{t>0}(\frac{t}{2})^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(\frac{t}{2})(v_0-\tilde{v}_0)||_{L^p(\mathbb{R}^N)} \\ &\le 2^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(t)(v_0-\tilde{v}_0)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}. \end{aligned} \tag{4.12}$$

For the term J_k and $\phi=(-\Delta)^{-1}(w-v)$, we have

$$\begin{aligned} J_k &= ||\int_0^t S_\alpha(t-\tau)\nabla\cdot[(v-\tilde{v})v^k\tilde{v}^{m-1-k}\nabla\phi](\tau)d\tau||_{L^p} \\ &\le C\Big(\int_0^{\theta t}+\int_{\theta t}^t\Big)(t-\tau)^{-\frac{mN}{\alpha p}}||v-\tilde{v}||_{L^p}||v||_{L^p}^k||\tilde{v}||_{L^p}^{m-1-k}||\nabla\phi||_{L^{\frac{Np}{N-p}}}d\tau \\ &\le C\Big(\int_0^{\theta t}+\int_{\theta t}^t\Big)(t-\tau)^{-\frac{mN}{\alpha p}}||v-\tilde{v}||_{L^p}||v||_{L^p}^k||\tilde{v}||_{L^p}^{m-1-k}||v-w||_{L^p}d\tau \\ &\le C\varepsilon^m\Big(\int_0^{\theta t}+\int_{\theta t}^t\Big)(t-\tau)^{-\frac{mN}{\alpha p}}\tau^{-1-\frac{1}{m}+\frac{(m+1)N}{\alpha p}}(\tau^{\frac{1}{m}-\frac{N}{\alpha p}}||v-\tilde{v}||_{L^p})d\tau \\ &\le C\varepsilon^m t^{-\frac{1}{m}+\frac{N}{\alpha p}}\int_0^\theta(1-\eta)^{-\frac{mN}{\alpha p}}\eta^{-1-\frac{1}{m}+\frac{(m+1)N}{\alpha p}}((t\eta)^{\frac{1}{m}-\frac{N}{\alpha p}}||v(t\eta)-\tilde{v}(t\eta)||_{L^p})d\eta \\ &+C\varepsilon^m t^{-\frac{1}{m}+\frac{N}{\alpha p}}[\sup_{\theta t\le\tau\le t}\tau^{\frac{1}{m}-\frac{N}{\alpha p}}||v(\tau)-\tilde{v}(\tau)||_{L^p}],\ k=1,2,\cdots,m-1. \end{aligned} \tag{4.13}$$

Similarly, for the term J_m we have

$$\begin{aligned} J_m &\le C\varepsilon^m t^{-\frac{1}{m}+\frac{N}{\alpha p}}\int_0^\theta\frac{(1-\eta)^{-\frac{mN}{\alpha p}}}{\eta^{1+\frac{1}{m}-\frac{(m+1)N}{\alpha p}}}((t\eta)^{\frac{1}{m}-\frac{N}{\alpha p}}||((v-\tilde{v})(t\eta),(w-\tilde{w})(t\eta))||_{L^p})d\eta \\ &+ C\varepsilon^m t^{-\frac{1}{m}+\frac{N}{\alpha p}}[\sup_{\theta t\le\tau\le t}\tau^{\frac{1}{m}-\frac{N}{\alpha p}}||((v-\tilde{v})(\tau),(w-\tilde{w})(\tau))||_{L^p}]. \end{aligned} \tag{4.14}$$

Together (4.5) with (4.11) we have

$$\begin{aligned} &||v-\tilde{v}||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} + t^{\frac{1}{m}-\frac{N}{\alpha p}}||v-\tilde{v}||_{L^p(\mathbb{R}^N)} \le C||S_\alpha(t)(v_0-\tilde{v}_0)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} \\ &+C\varepsilon^m\int_0^\theta\frac{(1-\eta)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}}{\eta^{1+\frac{1}{m}-\frac{(m+1)N}{\alpha p}}}((t\eta)^{\frac{1}{m}-\frac{N}{\alpha p}}||((v-\tilde{v})(t\eta),(w-\tilde{w})(t\eta))||_{L^p})d\eta \\ &+C\varepsilon^m\int_0^\theta\frac{(1-\eta)^{-\frac{mN}{\alpha p}}}{\eta^{1+\frac{1}{m}-\frac{(m+1)N}{\alpha p}}}((t\eta)^{\frac{1}{m}-\frac{N}{\alpha p}}||((v-\tilde{v})(t\eta),(w-\tilde{w})(t\eta))||_{L^p})d\eta \\ &+C\varepsilon^m[\sup_{\theta t\le\tau\le t}\tau^{\frac{1}{m}-\frac{N}{\alpha p}}||((v-\tilde{v})(\tau),(w-\tilde{w})(\tau))||_{L^p}]. \end{aligned} \tag{4.15}$$

For $w-\tilde{w}$ we can get the same estimate similar to (4.15).

For the convenience we denote

$$\begin{aligned} Q(\theta) &= \int_0^\theta(1-\eta)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}\eta^{-1-\frac{1}{m}+\frac{(m+1)N}{\alpha p}}d\eta + \int_0^\theta(1-\eta)^{-\frac{mN}{\alpha p}}\eta^{-1-\frac{1}{m}+\frac{(m+1)N}{\alpha p}}d\eta, \\ F(t) &= ||S_\alpha(t)(v_0-\tilde{v}_0, w_0-\tilde{w}_0)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}, \\ G(t) &= ||v-\tilde{v}||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} + t^{\frac{1}{m}-\frac{N}{\alpha p}}||v-\tilde{v}||_{L^p(\mathbb{R}^N)}. \end{aligned}$$

Due to the condition (3.1), $\max\{1, \frac{mN}{\alpha}\} < p < \min\{N, \frac{m(m+1)N}{\alpha}\}$, we have

$$1+\frac{1}{m}-\frac{(m+1)N}{\alpha p}=\frac{m+1}{mp}(p-\frac{mN}{\alpha})>0, \quad -\frac{1}{m}+\frac{(m+1)N}{\alpha p}=\frac{1}{mp}(\frac{m(m+1)N}{\alpha}-p)>0,$$
$$1-\frac{mN}{\alpha p}=\frac{1}{p}(p-\frac{mN}{\alpha})>0,$$

then we obtain that $Q(\theta)$ converges and $\lim_{\theta\to 0} Q(\theta)=0$.

Due to the condition (4.1) we have $\lim_{t\to+\infty} F(t)=0$ and $F(t)\in L^\infty[0,+\infty)$. Passing the limit in (4.15) we get

$$M=\limsup_{t\to+\infty} G(t)\le C(N,\alpha,p)\varepsilon^m(Q(\theta)+1)M, \tag{4.16}$$

Choosing θ and ε small enough such that $Q(\theta)<1$ and $2C(N,\alpha,p)\varepsilon^m<1$ respectively, then (4.16) implies that $M=0$. That is, (4.2) holds true. The proof is complete. □

5. Regularizing-decay rate estimates

In this section we consider the regularizing-decay rate estimates of the mild solutions to the system (1.1). Compared to the case $m=1$, the main difficulty is caused by the power-law nonlinearity term v^m as $m>1$ in the first two equations of (1.1). To overcome this difficulty, we will apply multiple Leibniz's rule. For the regularizing-decay rate estimates of mild solutions to the Navier-Stokes equations, we refer the reader to [6, 28, 29, 30].

In what follows, for $x=(x_1,\cdots,x_N)\in\mathbb{R}^N$ and $\beta=(\beta_1,\cdots,\beta_N)\in\mathbb{N}_0^N$, where $\mathbb{N}_0=\mathbb{N}\bigcup\{0\}$ and $\mathbb{N}=\{1,2,\cdots\}$, we denote $\partial_x^\beta=\partial_{x_1}^{\beta_1}\cdots\partial_{x_N}^{\beta_N}$ and $|\beta|=\beta_1+\cdots+\beta_N$.

We first describe the main result on regularizing-decay rate estimates of the mild solutions to the system (1.1).

Theorem 5.1 *Let $N\ge 2$ be a positive integer, $1<\alpha\le 2N$. Assume that p satisfies (3.1) and $(v_0,w_0)\in \dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)$, and (v,w) is the mild solution to the system (1.1) with initial data (v_0,w_0). Furthermore, assume that there exist two positive constants M_1 and M_2 such that*

$$\sup_{0\le t<T}\|(v(t),w(t))\|_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}\le M_1, \tag{5.1}$$
$$\sup_{0<t<T} t^{\frac{1}{m}-\frac{N}{\alpha p}}\|(v(t),w(t))\|_{L^p(\mathbb{R}^N)}\le M_2. \tag{5.2}$$

Then, there exist two positive constants K_1 and K_2 depending only on M_1, M_2, N, α, m and p, such that

$$\|(\partial_x^\beta v(t),\partial_x^\beta w(t))\|_{L^q(\mathbb{R}^N)}\le K_1(K_2|\beta|)^{2|\beta|}t^{-\frac{|\beta|}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}} \tag{5.3}$$

for all $p\le q\le\infty$, $t\in(0,T)$ and $\beta\in\mathbb{N}_0^N$.

Remark 5.1 *In fact, (5.3) is equivalent to the claim*

$$\|(\partial_x^\beta v(t),\partial_x^\beta w(t))\|_{L^q}\le K_1(K_2|\beta|)^{2|\beta|-\delta}t^{-\frac{|\beta|}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}} \tag{5.4}$$

for some $\delta\in(1,2]$ and sufficiently large constants K_1 and K_2.

Let us first prepare the refined L^p-L^q estimate for semigroup operator $S_\alpha(t)$.

Lemma 5.1 *Let $1\le p\le q\le\infty$. Then for any $f\in\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)$, we have*

$$\|\partial_x^\beta S_\alpha(t)f\|_{L^q(\mathbb{R}^N)}\le C_0^{|\beta|}|\beta|^{\frac{|\beta|}{\alpha}}t^{-\frac{|\beta|}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}}\|f\|_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} \tag{5.5}$$

for all $t>0,\beta\in\mathbb{N}_0^N$, and C_0 is a constant depending only on N and α.

Proof. As $S_\alpha(t)$ is the convolution operator with fractional heat kernel $K_t(x)=\mathcal{F}^{-1}(e^{-t|\xi|^\alpha})$, by scaling we see that

$$K_t(x)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^N}e^{ix\cdot\xi}e^{-t|\xi|^\alpha}d\xi=t^{-\frac{N}{\alpha}}K(xt^{-\frac{1}{\alpha}}),$$

where $K(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^N} e^{ix\cdot\xi} e^{-|\xi|^\alpha} d\xi$. It is clear that [15, Lemma 2.2]

$$\nabla K(x) \in L^p(\mathbb{R}^N),\ \nabla K_t(x) \in L^p(\mathbb{R}^N),\ \forall\ t \in (0,\infty),\ \forall\ p \in [1,\infty],$$

thus the Young inequality implies that

$$\|\partial_x S_\alpha(t) f\|_{L^q} \le \|\partial_x K_t(x)\|_{L^1} \|f\|_{L^q} \le C_0(N,\alpha) t^{-\frac{1}{\alpha}} \|f\|_{L^q}. \tag{5.6}$$

By the semigroup property of $S_\alpha(t)$ and the commutativity between semigroup and differential operators, we get

$$\partial_x^\beta S_\alpha(t) f = \prod_{i=1}^N \left(\partial_{x_i} S_\alpha\left(\frac{t}{2|\beta|}\right)\right)^{\beta_i} S_\alpha\left(\frac{t}{2}\right) f. \tag{5.7}$$

Combining (5.6) and (5.7), and using Definition 2.1, we obtain

$$\begin{aligned}
\|\partial_x^\beta S_\alpha(t) f\|_{L^q(\mathbb{R}^N)} &\le \prod_{i=1}^N \left\|\partial_{x_i} S_\alpha\left(\frac{t}{2|\beta|}\right)\right\|^{\beta_i}_{\mathcal{L}(L^q,L^q)} \left\|S_\alpha\left(\frac{t}{2}\right) f\right\|_{L^q} \\
&\le \left(C_0(N,\alpha)\left(\frac{t}{2|\beta|}\right)^{-\frac{1}{\alpha}}\right)^{|\beta|} \left(\frac{t}{4}\right)^{-\frac{N}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)} \left\|S_\alpha\left(\frac{t}{4}\right) f\right\|_{L^p} \\
&\le C_0(N,\alpha)^{|\beta|} |\beta|^{\frac{|\beta|}{\alpha}} t^{-\frac{|\beta|}{\alpha} - \frac{1}{m} + \frac{N}{\alpha q}} \sup_{t>0} \left(\frac{t}{4}\right)^{\frac{1}{m} - \frac{N}{\alpha p}} \left\|S_\alpha\left(\frac{t}{4}\right) f\right\|_{L^p} \\
&\le C_0(N,\alpha)^{|\beta|} |\beta|^{\frac{|\beta|}{\alpha}} t^{-\frac{|\beta|}{\alpha} - \frac{1}{m} + \frac{N}{\alpha q}} \|f\|_{\dot{B}^{-\frac{\alpha}{m}+\frac{N}{p}}_{p,\infty}(\mathbb{R}^N)},
\end{aligned}$$

where $\|\mathbf{T}\|_{\mathcal{L}(L^p,L^q)}$ denotes the norm of linear operator $\mathbf{T}$ from L^p to L^q. This proves the Lemma 5.1. □

Next we recall some useful results.

Lemma 5.2 *[31, Lemma 2.1] Let $\delta > \frac{1}{2}$. Then there exists a positive constant C depending only on δ, such that*

$$\sum_{\alpha<\beta} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} |\alpha|^{|\alpha|-\delta} |\beta-\alpha|^{|\beta-\alpha|-\delta} \le C(\delta) |\beta|^{|\beta|-\delta},\ \forall\, \beta \in \mathbb{N}_0^N. \tag{5.8}$$

Here the notation $\alpha < \beta$ means that $\alpha_i < \beta_i, \forall i \in \mathbb{N}$, $\begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \prod_{i=1}^N \frac{\beta_i!}{\alpha_i!(\beta_i-\alpha_i)!}$, and the dependence of $C(\delta)$ on δ is merely of the form $\sum_{j=1}^\infty j^{-\delta-\frac{1}{2}}$.

Lemma 5.3 *[28] Let ψ_0 be a measurable and locally bounded function in $(0,\infty)$ and $\{\psi_j\}_{j=1}^\infty$ be a sequence of measurable functions in $(0,\infty)$. Assume that $\alpha \in \mathbb{R}$ and $\mu, \nu > 0$ satisfying $\mu + \nu = 1$. Let $B_\eta > 0$ be a number depending on $\eta \in (0,1)$ and B_η be non-increasing with respect to η. Assume that there is a positive constant σ such that*

$$0 \le \psi_0(t) \le B_\eta t^{-\alpha} + \sigma \int_{(1-\eta)t}^t (t-\tau)^{-\mu} \tau^{-\nu} \psi_0(\tau) d\tau, \tag{5.9}$$

$$0 \le \psi_{j+1}(t) \le B_\eta t^{-\alpha} + \sigma \int_{(1-\eta)t}^t (t-\tau)^{-\mu} \tau^{-\nu} \psi_j(\tau) d\tau \tag{5.10}$$

for all $j \ge 0$, $t > 0$ and $\eta \in (0,1)$. Let η_0 be a unique positive number such that

$$I(\eta_0) = \min\{\frac{1}{2\sigma}, I(1)\} \text{ with } I(\eta) = \int_{1-\eta}^1 (1-\tau)^{-\mu} \tau^{-\alpha-\nu} d\tau.$$

Then for any $0 < \eta \le \eta_0$, we have

$$\psi_j(t) \le 2B_\eta t^{-\alpha},\ \forall\, j \ge 0, t > 0.$$

We now prove the Theorem 5.1. Following the idea in Giga-Sawada [28], we first prove the Remark 5.1, a variant of Theorem 5.1 under extra regularity assumption.

Proposition 5.1 *Under the same assumptions in Theorem 5.1. Assume further that*

$$\left(\partial_x^\beta v(t), \partial_x^\beta w(t)\right) \in C\left((0,T), L^q(\mathbb{R}^N)\right) \tag{5.11}$$

for all $p \le q \le \infty$ and $\beta \in \mathbb{N}_0^N$. Then for any $\delta \in (1,2]$, there exist two positive constants K_1 and K_2 depending only on M_1, M_2, N, α, m and p, such that

$$\|(\partial_x^\beta v(t), \partial_x^\beta w(t))\|_{L^q} \le K_1(K_2|\beta|)^{2|\beta|-\delta} t^{-\frac{|\beta|}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}} \tag{5.12}$$

for all $p \le q \le \infty$, $t \in (0,T)$ and $\beta \in \mathbb{N}_0^N$.

Proof. We split the proof into the following two steps by an induction $|\beta| = m$.
Step 1. We will prove (5.12) for $m = 0$. (5.2) implies that (5.12) is trivial if $q = p$, thus it suffices to consider $q \in (p, \infty]$. Let $\eta \in (0,1)$ be a constant to be determined later, we take L^q-norm of the first equation in (1.11) and split the time integral into two parts as follows

$$\begin{aligned} \|v(t)\|_{L^q} &\le \|S_\alpha(t)v_0\|_{L^q} + \Big(\int_0^{t(1-\eta)} + \int_{t(1-\eta)}^t\Big)\|S_\alpha(t-\tau)\nabla\cdot[v^m\nabla(-\triangle)^{-1}(w-v)(\tau)]\|_{L^q}d\tau \\ &:= E_1 + E_2 + E_3. \end{aligned} \tag{5.13}$$

We will estimate term by term.
For E_1, by Lemma 5.1 and (5.1), one can easily see that

$$E_1 \le C_1(N,\alpha)t^{-\frac{\alpha}{m}+\frac{N}{\alpha q}}\|v_0\|_{B_{p,\infty}^{-\frac{1}{m}+\frac{N}{p}}} \le C_1(N,\alpha,M_1)t^{-\frac{1}{m}+\frac{N}{\alpha q}}. \tag{5.14}$$

For E_2 and E_3, by Lemma 2.1, Lemma 2.2 and (5.2), we have

$$\begin{aligned} E_2 &= \int_0^{t(1-\eta)} \|S_\alpha(t-\tau)\nabla\cdot[v^m\nabla(-\triangle)^{-1}(w-v)(\tau)]\|_{L^q}d\tau \\ &\le C_2(N,\alpha,p)\int_0^{t(1-\eta)}(t-\tau)^{-\frac{(m+1)N}{\alpha p}+\frac{N}{\alpha q}}\|v(\tau)\|_{L^p}^m\|(v(\tau),w(\tau))\|_{L^p}d\tau \\ &\le C_2(N,\alpha,p)M_2^{m+1}\int_0^{t(1-\eta)}(t-\tau)^{-\frac{(m+1)N}{\alpha p}+\frac{N}{\alpha q}}\tau^{-1-\frac{1}{m}+\frac{(m+1)N}{\alpha p}}d\tau \\ &\le C_2(N,\alpha,p,M_2)\eta^{-1-\frac{1}{m}}t^{-\frac{1}{m}+\frac{N}{\alpha q}}, \end{aligned} \tag{5.15}$$

$$\begin{aligned} E_3 &= \int_{t(1-\eta)}^t \|S_\alpha(t-\tau)\nabla\cdot[v^m\nabla(-\triangle)^{-1}(w-v)(\tau)]\|_{L^q}d\tau \\ &\le C_3(N,\alpha,p)\int_{t(1-\eta)}^t(t-\tau)^{-\frac{mN}{\alpha p}}\|v(\tau)\|_{L^q}\|v(\tau)\|_{L^p}^{m-1}\|(v(\tau),w(\tau))\|_{L^p}d\tau \\ &\le C_3(N,\alpha,p,M_2)\int_{t(1-\eta)}^t(t-\tau)^{-\frac{mN}{\alpha p}}\tau^{-1+\frac{mN}{\alpha p}}\|v(\tau)\|_{L^q}d\tau. \end{aligned} \tag{5.16}$$

Combining (5.14)-(5.16), and setting $\bar{B}_\eta = C_1(N,\alpha,M_1) + C_2(N,\alpha,p,M_2)\eta^{-1-\frac{1}{m}}$, the inequality (5.13) yields that

$$\|v(t)\|_{L^q} \le \bar{B}_\eta t^{-\frac{1}{m}+\frac{N}{\alpha q}} + C_3\int_{t(1-\eta)}^t(t-\tau)^{-\frac{mN}{\alpha p}}\tau^{-1+\frac{mN}{\alpha p}}\|v(\tau)\|_{L^q}d\tau. \tag{5.17}$$

The estimate for $w(t)$ can be done analogously as (5.17). Hence, we have

$$\|(v(t),w(t))\|_{L^q} \le B_\eta t^{-\frac{1}{m}+\frac{N}{\alpha q}} + C_4\int_{t(1-\eta)}^t(t-\tau)^{-\frac{mN}{\alpha p}}\tau^{-1+\frac{mN}{\alpha p}}\|(v(\tau),w(\tau))\|_{L^q}d\tau, \tag{5.18}$$

where $B_\eta = 2\bar{B}_\eta$ and $C_4 = 2C_3(N,\alpha,p,M_2)$. By applying Lemma 5.3, we get the desired estimate (5.12) for $|\beta| = k = 0$ with $K_1 = 2B_{\eta_0}$ for some $\eta_0 = \eta_0(N,\alpha,p,m,M_1,M_2) \in (0,1)$.
Step 2. Next we prove (5.12) for $|\beta| = k \ge 1$. Due to the appearance of nonlocal function ϕ, we use a different argument to prove (5.12) for $p \le q < N$ and $N \le q \le \infty$, thus we split the proof into the following two cases.
Case 1: $p \le q < N$. In this case, we first differentiate the first equation of (1.11) to obtain the identity

$$\partial_x^\beta v(t) = \partial_x^\beta S_\alpha(t)v_0 - \int_0^t \partial_x^\beta S_\alpha(t-\tau)\nabla\cdot[v^m\nabla(-\Delta)^{-1}(w-v)(\tau)]d\tau. \tag{5.19}$$

We take the L^q-norm of $\partial_x^\beta v$, for some $\eta \in (0,1)$ to be chosen later, we split the time integral into the following two parts:

$$\begin{aligned} \|\partial_x^\beta v(t)\|_{L^q} &\le \|\partial_x^\beta S_\alpha(t)v_0\|_{L^q} + \Big(\int_0^{t(1-\eta)} + \int_{t(1-\eta)}^t\Big)\|\partial_x^\beta S_\alpha(t-\tau)\nabla\cdot[v^m\nabla(-\Delta)^{-1}(w-v)(\tau)]\|_{L^q}d\tau \\ &:= F_1 + F_2 + F_3. \end{aligned} \tag{5.20}$$

We next estimate $F_i(i=1,2,3)$ term by term.

For F_1, Lemma 5.1 implies that

$$F_1 \le C_0^k k^{\frac{k}{\alpha}} t^{-\frac{k}{\alpha}-\frac{1}{m}+\frac{n}{\alpha q}} \|v_0\|_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{n}{p}}} \le M_1 C_0^k k^{\frac{k}{\alpha}} t^{-\frac{k}{\alpha}-\frac{1}{m}+\frac{n}{\alpha q}}. \tag{5.21}$$

For F_2, using Lemma 5.1, Lemma 2.2 and (5.2), we have

$$\begin{aligned}
F_2 &= \int_0^{t(1-\eta)} \|\partial_x^\beta S_\alpha(t-\tau)\nabla\cdot[v^m\nabla(-\Delta)^{-1}(w-v)(\tau)]\|_{L^q} d\tau \\
&\le C_5(N,\alpha)\int_0^{t(1-\eta)} \Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}} \|\partial_x^\beta S_\alpha\Big(\frac{t-\tau}{2}\Big)[v^m\nabla(-\Delta)^{-1}(w-v)(\tau)]\|_{L^q} d\tau \\
&\le C_5(N,\alpha)\int_0^{t(1-\eta)} \Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}} \prod_{i=1}^N \|\partial_{x_i} S_\alpha\Big(\frac{t-\tau}{4k}\Big)\|^{k_i}_{\mathcal{L}(L^q,L^q)} \\
&\quad \times \|S_\alpha\Big(\frac{t-\tau}{4}\Big)[v^m\nabla(-\Delta)^{-1}(w-v)(\tau)]\|_{L^q} d\tau \\
&\le C_5(N,\alpha)\int_0^{t(1-\eta)} \Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}} \Big[C_0(\frac{t-\tau}{4k})^{-\frac{1}{\alpha}}\Big]^k \Big(\frac{t-\tau}{4}\Big)^{-\frac{(m+1)N-p}{\alpha p}+\frac{N}{\alpha q}} \\
&\quad \times \|v^m\nabla(-\Delta)^{-1}(w-v)(\tau)\|_{L^{\frac{Np}{(m+1)N-p}}} d\tau \\
&\le C_5(N,\alpha,p)C_0^k k^{\frac{k}{\alpha}} \int_0^{t(1-\eta)} \Big(\frac{t-\tau}{4}\Big)^{-\frac{k}{\alpha}-\frac{N}{\alpha}(\frac{m+1}{p}-\frac{1}{q})} \|v(\tau)\|^m_{L^p}\|(v(\tau),w(\tau))\|_{L^p} d\tau \\
&\le C_5(N,\alpha,p)M_2^{m+1}C_0^k k^{\frac{k}{\alpha}} \int_0^{t(1-\eta)} \Big(\frac{t-\tau}{4}\Big)^{-\frac{k}{\alpha}-\frac{N}{\alpha}(\frac{m+1}{p}-\frac{1}{q})} \tau^{-1-\frac{1}{m}+\frac{(m+1)N}{\alpha p}} d\tau \\
&\le C_5(N,\alpha,p,M_2)C_0^k k^{\frac{k}{\alpha}} \eta^{-\frac{k}{\alpha}-1-\frac{1}{m}} t^{-\frac{k}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}},
\end{aligned} \tag{5.22}$$

where $k=k_1+k_2+\cdots+k_N$ and $k_i=|\beta_i|(i=1,2,\ldots,N)$.

Using Leibniz's rule, we split F_3 into the following three parts:

$$\begin{aligned}
F_3 &= \int_{t(1-\eta)}^t \|\partial_x^\beta S_\alpha(t-\tau)\nabla\cdot[v^m\nabla(-\Delta)^{-1}(w-v)(\tau)]\|_{L^q} d\tau \\
&\le C_6(N,\alpha)\int_{t(1-\eta)}^t \Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}} \|S_\alpha\Big(\frac{t-\tau}{2}\Big)\partial_x^\beta[v^m\nabla(-\Delta)^{-1}(w-v)(\tau)]\|_{L^q} d\tau \\
&\le C_6(N,\alpha)\int_{t(1-\eta)}^t \Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}} \|S_\alpha\Big(\frac{t-\tau}{2}\Big)[(\partial_x^\beta v^m)\nabla(-\Delta)^{-1}(w-v)(\tau)]\|_{L^q} d\tau \\
&+ C_6(N,\alpha)\int_{t(1-\eta)}^t \Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}} \|S_\alpha\Big(\frac{t-\tau}{2}\Big)\sum_{0<\gamma<\beta}\begin{pmatrix}\beta\\ \gamma\end{pmatrix}(\partial_x^\gamma v^m)(\partial_x^{\beta-\gamma}\nabla(-\Delta)^{-1}(w-v)(\tau))\|_{L^q} d\tau \\
&+ C_6(N,\alpha)\int_{t(1-\eta)}^t \Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}} \|S_\alpha\Big(\frac{t-\tau}{2}\Big)[v^m\partial_x^\beta\nabla(-\Delta)^{-1}(w-v)(\tau)]\|_{L^q} d\tau \\
&:= F_{31}+F_{32}+F_{33}.
\end{aligned} \tag{5.23}$$

Here, the notation $\gamma<\beta$ means that $\gamma\le\beta$ and $|\gamma|<|\beta|$.

Now we establish the estimates for $F_{3j}(j=1,2,3)$. For F_{31}, using Leibniz's rule again, we can split F_{31} into two parts as follows:

$$\begin{aligned}
F_{31} &= C_7(N,\alpha)\int_{t(1-\eta)}^t \Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}} \|S_\alpha\Big(\frac{t-\tau}{2}\Big)[(\partial_x^\beta v^m)\nabla(-\Delta)^{-1}(w-v)]\|_{L^q} d\tau \\
&= C_7(N,\alpha)\int_{t(1-\eta)}^t \Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}} \|S_\alpha\Big(\frac{t-\tau}{2}\Big)\Big[\sum_\beta \begin{pmatrix}\beta_m\\ \beta_{m-1}\end{pmatrix}\begin{pmatrix}\beta_{m-1}\\ \beta_{m-2}\end{pmatrix}\cdots\begin{pmatrix}\beta_2\\ \beta_1\end{pmatrix} \\
&\quad \times(\partial_x^{\beta_1}v)(\partial_x^{\beta_2-\beta_1}v)\ldots(\partial_x^{\beta_m-\beta_{m-1}}v)+mv^{m-1}(\partial_x^\beta v)\nabla(-\Delta)^{-1}(w-v)\Big]\|_{L^q} d\tau \\
&= C_7(N,\alpha)\sum_\beta\prod_{i=1}^m \begin{pmatrix}\beta_i\\ \beta_{i-1}\end{pmatrix}\int_{t(1-\eta)}^t \Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}} \|S_\alpha\Big(\frac{t-\tau}{2}\Big)\prod_{i=1}^m(\partial_x^{\beta_i-\beta_{i-1}}v)\nabla(-\Delta)^{-1}(w-v)(\tau)\|_{L^q} d\tau \\
&\quad +C_7(N,\alpha,m)\int_{t(1-\eta)}^t \Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}} \|S_\alpha\Big(\frac{t-\tau}{2}\Big)v^{m-1}(\partial_x^\beta v)\nabla(-\Delta)^{-1}(w-v)\|_{L^q} d\tau \\
&:= G_1+G_2,
\end{aligned} \tag{5.24}$$

where we denote $\sum_\beta = \sum_{0=\beta_0\leq\beta_1\leq\cdots\leq\beta_{m-1}<\beta_m=\beta}$.

For G_2, using Lemma 2.1, Lemma 2.2 and (5.2), we have

$$
\begin{aligned}
G_2 &\leq C_8(N,\alpha,m,p)\int_{t(1-\eta)}^{t}\Big(\frac{t-\tau}{2}\Big)^{-\frac{mN}{\alpha p}}\|v\|_{L^p}^{m-1}\|\partial_x^\beta v\|_{L^q}\|(v(\tau),w(\tau))\|_{L^p}d\tau\\
&\leq C_8(N,\alpha,m,p)M_2^m\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{mN}{\alpha p}}\tau^{-1+\frac{mN}{\alpha p}}\|\partial_x^\beta v\|_{L^q}d\tau.
\end{aligned}\tag{5.25}
$$

For G_1, using Lemma 2.1, Lemma 2.2, Lemma 5.2, (5.2) and (5.12), we have

$$
\begin{aligned}
G_1 &\leq C_9(N,\alpha,p)\sum_\beta\prod_{i=1}^{m}\binom{\beta_i}{\beta_{i-1}}\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{(m-1)N}{\alpha q}-\frac{N}{\alpha p}}\\
&\quad\times\prod_{i=1}^{m}\|\partial_x^{\beta_i-\beta_{i-1}}v\|_{L^q}\|(v(\tau),w(\tau))\|_{L^p}d\tau\\
&\leq C_9(N,\alpha,p)M_2\sum_\beta\prod_{i=1}^{m}\binom{\beta_i}{\beta_{i-1}}\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{(m-1)N}{\alpha q}-\frac{N}{\alpha p}}\\
&\quad\times\prod_{i=1}^{m}\Big[K_1(K_2|\beta_i-\beta_{i-1}|^{2|\beta_i-\beta_{i-1}|-\delta})\tau^{-\frac{|\beta_i-\beta_{i-1}|}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}}\Big]\tau^{-\frac{1}{m}+\frac{N}{\alpha p}}d\tau\\
&\leq C_9(N,\alpha,p,M_2)\sum_\beta\prod_{i=1}^{m}\binom{\beta_i}{\beta_{i-1}}\prod_{i=1}^{m}\Big[K_1(K_2|\beta_i-\beta_{i-1}|^{2|\beta_i-\beta_{i-1}|-\delta})\Big]\\
&\quad\times\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{(m-1)N}{\alpha q}-\frac{N}{\alpha p}}\tau^{-\frac{k}{\alpha}-1+\frac{mN}{\alpha q}-\frac{1}{m}+\frac{N}{\alpha p}}d\tau\\
&\leq C_9(N,\alpha,p,M_2)(C(\delta))^{2(m-1)}k^{2k-\delta}K_1^mK_2^{2k-m\delta}I(\eta)t^{-\frac{k}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}},
\end{aligned}\tag{5.26}
$$

where

$$
I(\eta)=\int_{1-\eta}^{1}(1-\tau)^{-\frac{(m-1)N}{\alpha q}-\frac{N}{\alpha p}}\tau^{-\frac{k}{\alpha}-1+\frac{mN}{\alpha q}-\frac{1}{m}+\frac{N}{\alpha p}}d\tau.\tag{5.27}
$$

For F_{32}, using the same arguments as G_1, we have

$$
\begin{aligned}
F_{32} &\leq C_{10}(N,\alpha)\int_{t(1-\eta)}^{t}\Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}}\|S_\alpha\Big(\frac{t-\tau}{2}\Big)\Big[\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}(\partial_x^\gamma v^m)\\
&\quad\times(\partial_x^{\beta-\gamma}\nabla(-\Delta)^{-1}(w-v)(\tau))\Big]\|_{L^q}d\tau\\
&\leq C_{10}(N,\alpha)\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}\int_{t(1-\eta)}^{t}\Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}}\|S_\alpha\Big(\frac{t-\tau}{2}\Big)(\partial_x^\gamma v^m)\\
&\quad\times(\partial_x^{\beta-\gamma}\nabla(-\Delta)^{-1}(w-v)(\tau))\|_{L^q}d\tau\\
&= C_{10}(N,\alpha)\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}\int_{t(1-\eta)}^{t}\Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}}\|S_\alpha\Big(\frac{t-\tau}{2}\Big)\Big[\sum_\gamma\prod_{i=1}^{m}\binom{\gamma_i}{\gamma_{i-1}}\\
&\quad\times\prod_{j=1}^{m}(\partial_x^{\gamma_j-\gamma_{j-1}}v^m)\Big](\partial_x^{\beta-\gamma}\nabla(-\Delta)^{-1}(w-v)(\tau))\|_{L^q}d\tau\\
&\leq C_{10}(N,\alpha)\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}\sum_\gamma\prod_{i=1}^{m}\binom{\gamma_i}{\gamma_{i-1}}\int_{t(1-\eta)}^{t}\Big(\frac{t-\tau}{2}\Big)^{-\frac{1}{\alpha}}\\
&\quad\times\|S_\alpha\Big(\frac{t-\tau}{2}\Big)\prod_{j=1}^{m}(\partial_x^{\gamma_j-\gamma_{j-1}}v^m)(\partial_x^{\beta-\gamma}\nabla(-\Delta)^{-1}(w-v)(\tau))\|_{L^q}d\tau,
\end{aligned}
$$

according to the property of semigroup we get

$$
\begin{aligned}
F_{32} \leq\ & C_{10}(N,\alpha,p)\sum_{0<\gamma<\beta}\begin{pmatrix}\beta\\ \gamma\end{pmatrix}\sum_{\gamma}\prod_{i=1}^{m}\begin{pmatrix}\gamma_i\\ \gamma_{i-1}\end{pmatrix}\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{N(m-1)}{\alpha q}-\frac{N}{\alpha p}}\\
&\times\prod_{j=1}^{m}\|\partial_x^{\gamma_j-\gamma_{j-1}}v\|_{L^q}\|\partial_x^{\beta-\gamma}(v(\tau),w(\tau))\|_{L^p}d\tau\\
\leq\ & C_{10}(N,\alpha,p)\sum_{0<\gamma<\beta}\begin{pmatrix}\beta\\ \gamma\end{pmatrix}\sum_{\gamma}\prod_{i=1}^{m}\begin{pmatrix}\gamma_i\\ \gamma_{i-1}\end{pmatrix}\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{N(m-1)}{\alpha q}-\frac{N}{\alpha p}}\\
&\times\prod_{j=1}^{m}\Big[K_1(K_2|\gamma_j-\gamma_{j-1}|)^{2|\gamma_j-\gamma_{j-1}|-\delta}\tau^{-\frac{|\gamma_j-\gamma_{j-1}|}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}}\Big]\\
&\times\Big[K_1(K_2|\beta-\gamma|)^{2|\beta-\gamma|-\delta}\tau^{-\frac{|\beta-\gamma|}{\alpha}-\frac{1}{m}+\frac{N}{\alpha p}}\Big]d\tau\\
\leq\ & C_{10}(N,\alpha,p)(C(\delta))^mK_1^{m+1}K_2^{2k-(m+1)\delta}k^{2k-\delta}I(\eta)t^{-\frac{k}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}},
\end{aligned}
\tag{5.28}
$$

where $\sum_\gamma$ is defined the same as that in estimating F_{31} and $I(\eta)$ is defined in (5.27).

For F_{33}, analogously we have

$$
\begin{aligned}
F_{33} \leq\ & C_{11}(N,\alpha)\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{N(m-1)}{\alpha q}-\frac{N}{\alpha p}}\|v\|_q^m\|\partial_x^{\beta}\nabla(-\triangle)^{-1}(w-v)(\tau)\|_{L^{\frac{Np}{N-p}}}d\tau\\
\leq\ & C_{11}(N,\alpha)\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{N(m-1)}{\alpha q}-\frac{N}{\alpha p}}\|v\|_q^m\|\partial_x^{\beta-1}(v(\tau),w(\tau))\|_{L^{\frac{Np}{N-p}}}d\tau\\
\leq\ & C_{11}(N,\alpha)\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{N(m-1)}{\alpha q}-\frac{N}{\alpha p}}[K_1\tau^{-\frac{1}{m}+\frac{N}{\alpha q}}]^m\\
&\times\Big[K_1(K_2(k-1))^{2(k-1)-\delta}\tau^{\frac{k-1}{\alpha}-\frac{1}{m}+\frac{N(N-p)}{\alpha Np}}\Big]d\tau\\
\leq\ & C_{11}(N,\alpha)K_1^{m+1}K_2^{2(k-1)-\delta}k^{2k-\delta}I(\eta)t^{-\frac{k}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}},
\end{aligned}
\tag{5.29}
$$

where $I(\eta)$ is defined in (5.27).

Combining the above estimates (5.20)–(5.29) and setting $\bar{B}_\eta$ by

$$
\bar{B}_\eta=M_1C_0^kk^{\frac{k}{\alpha}}+C_5C_0^kk^{\frac{k}{\alpha}}\eta^{-\frac{k}{\alpha}-1-\frac{1}{m}}+C_{12}k^{2k-\delta}I(\eta),
$$

and

$$
C_{12}=C_9K_1^mK_2^{2k-m\delta}+C_{10}K_1^{m+1}K_2^{2k-(m+1)\delta}+C_{11}K_1^{m+1}K_2^{2(k-1)-\delta},
\tag{5.30}
$$

we obtain

$$
\|\partial_x^\beta v(t)\|_{L^q}\leq\bar{B}_\eta t^{-\frac{k}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}}+C_8\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{mN}{\alpha p}}\tau^{-1+\frac{mN}{\alpha p}}\|\partial_x^\beta v(\tau)\|_{L^q}d\tau.
\tag{5.31}
$$

Similarly, we can deal with $\partial_x^\beta w(t)$. Hence, we conclude that

$$
\|(\partial_x^\beta v(t),\partial_x^\beta w(t))\|_{L^q}\leq B_\eta t^{-\frac{k}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}}+C_{13}\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{mN}{\alpha p}}\tau^{-1+\frac{mN}{\alpha p}}\|(\partial_x^\beta v(\tau),\partial_x^\beta w(\tau))\|_{L^q}d\tau,
\tag{5.32}
$$

where $B_\eta=2\bar{B}_\eta$ and $C_{13}=2C_8(N,\alpha,m,p)$.

Let $\eta_k=\frac{1}{2k}$. It is clear that $I(\eta_k)$ is strictly monotone decreasing in k and $I(\eta_k)\to 0$ as $k\to\infty$. Choosing k_0 sufficiently large, such that $I(\frac{1}{2k})\leq\frac{1}{2C_{13}}$ for all $k\geq k_0$, applying Lemma 5.3, we get

$$
\|(\partial_x^\beta v(t),\partial_x^\beta w(t))\|_{L^q} \quad\leq\quad 2B_{\frac{1}{2k}}t^{-\frac{k}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}}
\tag{5.33}
$$

for all $t>0$ and $|\beta|=k$. Note that from (5.33), we can choose K_1 and K_2 sufficiently large such that (5.12) holds for all β satisfying $|\beta|\leq k_0$. Hence, it suffices to prove that it is possible to choose K_1 and K_2 such that $2B_{\frac{1}{2k}}\leq K_1(K_2k)^{2k-\delta}$ for all $k>k_0$. Since

$$
I(\frac{1}{2k})=\int_{1-\frac{1}{2k}}^{1}(1-\tau)^{-\frac{(m-1)N}{\alpha q}-\frac{N}{\alpha p}}\tau^{-\frac{k}{\alpha}-1+\frac{mN}{\alpha q}-\frac{1}{m}+\frac{N}{\alpha p}}d\tau\leq(1-\frac{1}{2k})^{-\frac{k}{\alpha}-1-\frac{1}{m}}\leq e^{\frac{1}{2\alpha}}(1-\frac{1}{2k})^{-1-\frac{1}{m}}\leq 16,
$$

we can calculate $2B_{\frac{1}{2k}}$ as follows:

$$\begin{aligned} 2B_{\frac{1}{2k}} = 4\bar{B}_{\frac{1}{2k}} &\leq 4[M_1 C_0^k k^{\frac{k}{\alpha}} + C_5 C_0^k k^{\frac{k}{\alpha}} (2k)^{\frac{k}{\alpha}+1+\frac{1}{m}} + 16C_{12} k^{2k-\delta}] \\ &\leq 4[M_1 C_0^k + 2^{\frac{k}{\alpha}+1+\frac{1}{m}} C_5 C_0^k k^{1+\frac{1}{m}+\delta} + 16C_{12}] k^{2k-\delta}. \end{aligned}$$

Obviously, there exists a constant $C_{14} > C_0$ such that $C_0^k + 2^{\frac{k}{\alpha}+1+\frac{1}{m}} C_0^k k^{1+\frac{1}{m}+\delta} \leq C_{14}^{2k-\delta}$. Hence,

$$2B_{\frac{1}{2k}} \leq 4[(M_1 + C_5) C_{14}^{2k-\delta} + 16C_{12}] k^{2k-\delta}, \tag{5.34}$$

where C_{12} is defined in (5.30).

Choosing $K_1 := 8(M_1 + C_5)$ and $K_2 := \max\{C_{14}, 32(C_9 + C_{10})K_1, 32C_{11}K_1^{\frac{m}{2}}\}$, we obtain (5.12). This completes the proof of Proposition 5.1 for $p \leq q < N$.

Case 2: $N \leq q \leq \infty$. Now we are in a position to establish the estimate of $\|\partial_x^\beta v(t)\|_{L^q}$ for $N \leq q \leq \infty$. For p satisfying (3.1), using the Gagliardo-Nirenberg inequality [32], we have

$$\|\partial_x^\beta v(t)\|_{L^q} \leq C(N,p) \|\partial_x^\beta v(t)\|_{L^p}^{\theta} \|\partial_x^2 \partial_x^\beta v(t)\|_{L^p}^{1-\theta}, \ \theta = 1 - \frac{N}{2p} + \frac{N}{2q}. \tag{5.35}$$

Now from (5.35) and the result of Case 1 we see that

$$\begin{aligned} \|\partial_x^\beta v(t)\|_{L^q} &\leq C(N,p)[K_1 (K_2 k)^{2k-\delta} t^{-\frac{k}{\alpha}-\frac{1}{m}+\frac{N}{\alpha p}}]^{\theta} [K_1 (K_2(k+2))^{2(k+2)-\delta} t^{-\frac{k+2}{\alpha}-\frac{1}{m}+\frac{N}{\alpha p}}]^{1-\theta} \\ &\leq C(N,p) K_1 (K_2(k+2))^{2k+4-\delta} t^{-\frac{k}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}}. \end{aligned} \tag{5.36}$$

It is clear that there exists a constant $C_{15} \geq 2$ such that $k^4 \leq C_{15}^{2k-\delta}$, thus we have

$$(K_2(k+2))^{2k+4-\delta} = K_2^4 k^4 (1 + \frac{2}{k})^{2k+4-\delta} (K_2 k)^{2k-\delta} \leq 81 e^4 K_2^4 (C_{15} K_2 k)^{2k-\delta}.$$

Hence, we can choose K_1 and K_2 sufficiently large such that (5.12) holds for all $p \leq q \leq \infty$. This completes the proof of Proposition 5.1. □

Finally, let us show that under the assumptions of Theorem 5.1, the mild solution $(v(t), w(t))$ of (1.1) always satisfies the regularity condition (5.12).

Proposition 5.2 *Under the assumptions of Theorem 5.1, the mild solution $(v(t), w(t))$ satisfies that*

$$t^{\frac{|\beta|}{\alpha}+\frac{1}{m}-\frac{N}{\alpha q}} \|(\partial_x^\beta v(t), \partial_x^\beta w(t))\|_{L^q} \leq \tilde{K}_1 (\tilde{K}_2 |\beta|)^{2|\beta|-\delta} \tag{5.37}$$

for all $p \leq q \leq \infty$, $t \in (0,T)$ and $\beta \in \mathbb{N}_0^N$, where $\tilde{K}_1$ and $\tilde{K}_2$ are constants depending only on $M_1, M_2, m, N, \alpha, p$ and δ.

Proof. Since the mild solution $(v(t), w(t))$ is the limit function of the sequence $(v_j(t), w_j(t))$ of appropriate Picard iterations as follows:

$$\begin{aligned} &(v_1(t), w_1(t)) = (S_\alpha(t) v_0, S_\alpha(t) w_0), \\ &v_j(t) = S_\alpha(t) v_0 + \int_0^t S_\alpha(t-\tau) \nabla \cdot [v_{j-1}^m \nabla (-\Delta)^{-1} (v_{j-1} - w_{j-1})](\tau) d\tau, \ j \geq 2, \\ &w_j(t) = S_\alpha(t) w_0 + \int_0^t S_\alpha(t-\tau) \nabla \cdot [w_{j-1}^m \nabla (-\Delta)^{-1} (w_{j-1} - v_{j-1})](\tau) d\tau, \ j \geq 2. \end{aligned}$$

Step 1. We first show that

$$\sup_{j \geq 1} \sup_{0<t<T} t^{\frac{1}{m}-\frac{N}{\alpha p}} \|(v_j(t), w_j(t))\|_{L^p} \leq M_2. \tag{5.38}$$

When $j = 1$, following from (5.1) we have

$$\begin{aligned} \|(v_1, w_1)\|_{L^p} &= \|(S_\alpha(t) v_0, S_\alpha(t) w_0)\|_{L^p} \leq t^{-\frac{1}{m}+\frac{N}{\alpha p}} \sup_{0<t<T} t^{\frac{1}{m}-\frac{N}{\alpha p}} \|(S_\alpha(t) v_0, S_\alpha(t) w_0)\|_{L^p} \\ &\leq t^{-\frac{1}{m}+\frac{N}{\alpha p}} \|(v_0, w_0)\|_{B_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}} \leq M_1 t^{-\frac{1}{m}+\frac{N}{\alpha p}}. \end{aligned} \tag{5.39}$$

Hence (5.38) holds for $j = 1$.

When $j \geq 2$, using Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
\|v_j(t)\|_{L^p} &\leq \|S_\alpha(t)v_0\|_{L^p} + \int_0^t \|S_\alpha(t-\tau)\nabla\cdot[v_{j-1}^m\nabla(-\Delta)^{-1}(v_{j-1}-w_{j-1})]\|_{L^p}(\tau)d\tau \\
&\leq M_1 t^{-\frac{1}{m}+\frac{N}{\alpha p}} + C(N,\alpha,p)\int_0^t (t-\tau)^{-\frac{mN}{\alpha p}}\|v_{j-1}(\tau)\|_{L^p}^m \|(v_{j-1}(\tau), w_{j-1}(\tau))\|_{L^p} d\tau \\
&\leq M_1 t^{-\frac{1}{m}+\frac{N}{\alpha p}} + C(N,\alpha,p)\Big[\sup_{0<s<T} s^{\frac{1}{m}-\frac{N}{\alpha p}}\|(v_{j-1}(s), w_{j-1}(s))\|_{L^p}\Big]^{m+1} t^{-\frac{1}{m}+\frac{N}{\alpha p}} B,
\end{aligned}$$

where $B = \int_0^1 (1-\tau)^{-\frac{mN}{\alpha p}}\tau^{-1-\frac{1}{m}+\frac{(m+1)N}{\alpha p}}d\tau = \mathcal{B}(1-\frac{mN}{\alpha p}, -\frac{1}{m}+\frac{(m+1)N}{\alpha p})$ is the standard Beta function which is obviously finite.

For $w_j(t)$ we have the analogous estimate. Then, for $j = 2, 3, \cdots$, we get

$$\|(v_j(t), w_j(t))\|_{L^p} \leq C(N,\alpha,p,m,M_1,B)t^{-\frac{1}{m}+\frac{N}{\alpha p}} := M_2 t^{-\frac{1}{m}+\frac{N}{\alpha p}}, \tag{5.40}$$

where the constant $C(N,\alpha,p,m,M_1,B)$ is always finite. Therefore (5.38) holds true.

Step 2. To apply the Lemma 5.3, we need to show that $\|(\partial_x^\beta v_1(t), \partial_x^\beta w_1(t))\|_{L^q}$ is locally bounded in $(0,T)$. Using Lemma 2.2 and (5.1), we have

$$\begin{aligned}
\|\partial_x^\beta v_1(t)\|_{L^q} &= \|\partial_x^\beta S_\alpha\Big(\frac{t}{2}\Big)S_\alpha\Big(\frac{t}{2}\Big)v_0\|_{L^q} \leq C(N,\alpha)\Big(\frac{t}{2}\Big)^{-\frac{|\beta|}{\alpha}-\frac{N}{\alpha}(\frac{1}{p}-\frac{1}{q})}\|S_\alpha\Big(\frac{t}{2}\Big)v_0\|_{L^p} \\
&\leq C(N,\alpha)\Big(\frac{t}{2}\Big)^{-\frac{|\beta|}{\alpha}-\frac{N}{\alpha}(\frac{1}{p}-\frac{1}{q})}\Big(\frac{t}{2}\Big)^{-\frac{1}{m}+\frac{N}{\alpha p}}\sup_{t>0}\Big(\frac{t}{2}\Big)^{\frac{1}{m}-\frac{N}{\alpha p}}\|S_\alpha\Big(\frac{t}{2}\Big)v_0\|_{L^p} \\
&\leq C(N,\alpha)M_1\Big(\frac{t}{2}\Big)^{-\frac{|\beta|}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}}.
\end{aligned}$$

Similarly we have a similar estimate on $w_j(t)$. Then $\|(\partial_x^\beta v_1(t), \partial_x^\beta w_1(t))\|_{L^q}$ is locally bounded in $(0,T)$.

Step 3. Similarly to the proof of Proposition 5.1, let $\psi_j(t) = \|\partial_x^\beta v_j(t)\|_{L^q}$, for all $j \geq 1$ and $t \in (0,T)$, we have

$$\psi_{j+1}(t) \leq \bar{B}_\eta t^{-\frac{|\beta|}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}} + C_8\int_{t(1-\eta)}^t (t-\tau)^{-\frac{mN}{\alpha p}}\tau^{-1+\frac{mN}{\alpha p}}\psi_j(\tau)d\tau. \tag{5.41}$$

Using Lemma 5.3 (the version of sequences), we can choose appropriate constants $\tilde{K}_1$ and $\tilde{K}_2$ such that

$$\psi_j(t) \leq \tilde{K}_1(\tilde{K}_2|\beta|)^{2|\beta|-\delta}t^{-\frac{|\beta|}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}}. \tag{5.42}$$

For $w_j(t)$ we have the similar estimate. Hence we complete the proof of Proposition 5.2. □

The proof of Theorem 5.1. Now Theorem 5.1 follows immediately from Proposition 5.1 and Proposition 5.2. We complete the proof of Theorem 5.1. □

6. A generalized fractional drift diffusion system

In this section we consider a fractional drift diffusion system with generalized electric potential equation

$$\begin{cases}
\partial_t v + \Lambda^\alpha v = \nabla\cdot(v^m\nabla\phi), & t>0, x\in\mathbb{R}^N, \\
\partial_t w + \Lambda^\alpha w = \nabla\cdot(w^m\nabla\phi), & t>0, x\in\mathbb{R}^N, \\
\phi = \mathcal{K}(v-w)(x) = c\int_{\mathbb{R}^N} b(x,y)(v-w)(y)dy, & t>0, x\in\mathbb{R}^N, \\
v(x,0) = v_0(x),\ w(x,0) = w_0(x), & x\in\mathbb{R}^N,
\end{cases} \tag{6.1}$$

where c is a constant and $b(x,y)$ is the kernel function of nonlocal linear integral operator $\mathcal{K}$.

For $\mathcal{K} = (-\Delta)^{-1}$ which comes from the Poisson equation $\Delta\phi = v - w$, (6.1) becomes the fractional drift diffusion system (1.1). For instance,

$$\mathcal{K}(u)(x) = c\int_{\mathbb{R}^N}(x-y)u(y)|x-y|^{-N}dy, \tag{6.2}$$

where c is a constant. If $c<0$, the equation $u_t = \Delta u + \nabla\cdot(u\nabla\mathcal{K}(u))$ models the Brownian diffusion of charge carriers interacting via Coulomb forces. If $c>0$, the operator $\mathcal{K}$ reflects the mutual gravitational attraction of particles. Furthermore, Biler-Woyczynski [27] considered the equation $u_t = \Lambda^\alpha u + \nabla\cdot(u\nabla\mathcal{K}(u))$.

We also give the global existence and asymptotic stability of the mild solution to the Cauchy problem (6.1).

Theorem 6.1 *Let N be a positive integer, $1<\alpha\le 2N$ and (3.1) hold true. Assume that $(v_0,w_0)\in \dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)$. If the derivative of kernel function $b(x,y)$ satisfies*

$$|Db(x,y)|\le C|x-y|^{-N+1}, \tag{6.3}$$

then there exists $\varepsilon>0$ such that if $||(v_0,w_0)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}}\le\varepsilon$, the Cauchy problem (6.1) has a unique global mild solution $(v,w)\in\mathcal{X}$ such that $||(v,w)||_{\mathcal{X}}\le 2\varepsilon$. Moreover, the solution depends continuously on initial data in the following sense: let $(\tilde{v},\tilde{w})\in\mathcal{X}$ be the solution of (6.1) with initial data $(\tilde{v}_0,\tilde{w}_0)$ such that $||(\tilde{v}_0,\tilde{w}_0)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}\le\varepsilon$, then there is a constant C such that

$$||(v-\tilde{v},w-\tilde{w})||_{\mathcal{X}}\le C||(v-\tilde{v}_0,w-\tilde{w}_0)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}. \tag{6.4}$$

Proof. After a few modifications of the proof to Theorem 3.1, we can prove this theorem. Here we just give the main difference in the proof.

By the fractional heat semigroup $S_\alpha(t)=e^{-t\Lambda^\alpha}$, we rewrite the system (6.1) as a system of integral equations

$$\begin{cases} v(t)=S_\alpha(t)v_0+B(v,\cdots,v,w),\\ w(t)=S_\alpha(t)w_0+B(w,\cdots,w,v),\end{cases} \tag{6.5}$$

where

$$B(\underbrace{v,\cdots,v}_{m},w)=\int_0^t S_\alpha(t-\tau)\nabla\cdot[v^m\nabla\mathcal{K}(v-w)](\tau)d\tau. \tag{6.6}$$

Similar to (3.4)-(3.8), we have

$$\begin{aligned} ||B(v,\cdots,v,w)(t)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} &= \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)\int_0^t S_\alpha(t-\tau)\nabla\cdot[v^m\nabla\mathcal{K}(v-w)](\tau)d\tau||_{L^p}\\ &\le \int_0^t \sup_{s>0} s^{\frac{1}{m}-\frac{N}{\alpha p}}||S_\alpha(s)S_\alpha(t-\tau)\nabla\cdot[v^m\nabla\mathcal{K}(v-w)](\tau)||_{L^p}d\tau\\ &\le C(N,\alpha)\int_0^t (t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}||v^m\nabla\mathcal{K}(v-w)(\tau)||_{L^{\frac{Np}{(m+1)N-p}}}d\tau\\ &\le C(N,\alpha)\int_0^t (t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}||v(\tau)||_{L^p}^m||\nabla\mathcal{K}(v-w)(\tau)||_{L^{\frac{Np}{N-p}}}d\tau, \end{aligned} \tag{6.7}$$

due to the condition (6.3): $|Db(x,y)|\le C|x-y|^{-N+1}$, use Hardy-Littlewood-Sobolev inequality for the integral $\int_{\mathbb{R}^N}|x-y|^{-N+1}|v-w|dy$, we have

$$||\nabla\mathcal{K}(v-w)||_{L^{\frac{Np}{N-p}}}\le C(N,p)||v-w||_{L^p}. \tag{6.8}$$

then

$$\begin{aligned} &||B(v,\cdots,v,w)(t)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}\\ \le\ & C(N,\alpha,p)\sup_{\tau>0}(\tau^{\frac{1}{m}-\frac{N}{\alpha p}}||v(\tau)||_{L^p})^m\sup_{\tau>0}(\tau^{\frac{1}{m}-\frac{N}{\alpha p}}||(v-w)(\tau)||_{L^p})\int_0^t(t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}\tau^{\frac{(m+1)N}{\alpha p}-\frac{1}{m}-1}d\tau\\ \le\ & C(N,\alpha,p)||v||_{\mathcal{X}}^m||v-w||_{\mathcal{X}}\int_0^t(t-\tau)^{\frac{1}{m}-\frac{(m+1)N}{\alpha p}}\tau^{\frac{(m+1)N}{\alpha p}-\frac{1}{m}-1}d\tau\le C(N,\alpha,p)||v||_{\mathcal{X}}^m||v-w||_{\mathcal{X}}, \end{aligned} \tag{6.9}$$

therefore we have

$$||B(v,\cdots,v,w)(t)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)}\le C(N,\alpha,p)||v||_{\mathcal{X}}^m||v-w||_{\mathcal{X}}. \tag{6.10}$$

Similarly we have

$$\sup_{t>0} t^{\frac{1}{m}-\frac{N}{\alpha p}}||B(v,\cdots,v,w)(t)||_{L^p}\le C(N,\alpha,p)||v||_{\mathcal{X}}^m||v-w||_{\mathcal{X}}. \tag{6.11}$$

Following the main estimates (6.10) and (6.11) and the proof of Theorem 3.1, the Cauchy problem (6.1) has a unique global-in-time mild solution in the mixed time-space Besov space. This completes the proof of Theorem 6.1. □

Using the same method we can prove that the mild solution of the Cauchy problem (6.1) has the following asymptotic stability.

Theorem 6.2 *Let $N \geq 2$ be a positive integer, $1 < \alpha \leq 2N$, (3.1) and (6.3) hold true. Assume that (v, w) and $(\tilde{v}, \tilde{w})$ are two mild solutions of the Cauchy problem (6.1) described in Theorem 6.1 corresponding to initial conditions (v_0, w_0) and $(\tilde{v}_0, \tilde{w}_0)$, respectively. If $(v_0, w_0), (\tilde{v}_0, \tilde{w}_0) \in \dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)$ such that*

$$\lim_{t\to\infty} ||S_\alpha(t)(v_0 - \tilde{v}_0, w_0 - \tilde{w}_0)||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} = 0, \tag{6.12}$$

then we have the following asymptotic stability

$$\lim_{t\to\infty} \Big(||(v - \tilde{v}, w - \tilde{w})||_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} + t^{\frac{\alpha}{m}-\frac{N}{p}} ||(v - \tilde{v}, w - \tilde{w})||_{L^p(\mathbb{R}^N)} \Big) = 0. \tag{6.13}$$

Theorem 6.3 *Let $N \geq 2$ be a positive integer, $1 < \alpha \leq 2N$, (3.1) and (6.3) hold true. Assume that $(v_0, w_0) \in \dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)$, and (v, w) is the mild solution to the system (6.1) with initial data (v_0, w_0). Furthermore, assume that there exist two positive constants M_1 and M_2 such that*

$$\sup_{0\leq t<T} \|(v(t), w(t))\|_{\dot{B}_{p,\infty}^{-\frac{\alpha}{m}+\frac{N}{p}}(\mathbb{R}^N)} \leq M_1, \tag{6.14}$$

$$\sup_{0<t<T} t^{\frac{1}{m}-\frac{N}{\alpha p}} \|(v(t), w(t))\|_{L^p(\mathbb{R}^N)} \leq M_2. \tag{6.15}$$

Then, there exist two positive constants K_1 and K_2 depending only on M_1, M_2, N, α, m and p, such that

$$\|(\partial_x^\beta v(t), \partial_x^\beta w(t))\|_{L^q(\mathbb{R}^N)} \leq K_1 (K_2 |\beta|)^{2|\beta|} t^{-\frac{|\beta|}{\alpha}-\frac{1}{m}+\frac{N}{\alpha q}} \tag{6.16}$$

for all $p \leq q \leq \infty$, $t \in (0, T)$ and $\beta \in \mathbb{N}_0^N$.

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