BLOW-UP VERSUS GLOBAL WELL-POSEDNESS FOR THE FOCUSING INLS WITH INVERSE-SQUARE POTENTIAL

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ABSTRACT. We study the focusing inhomogeneous nonlinear Schrödinger equation with inverse-square potential

$$i\partial_t u + \Delta u - \frac{a}{|x|^2}u + |x|^{-b}|u|^2 u = 0,$$

where $a > -\frac{1}{4}$ and $0 < b < 1$ in dimension three. We extend the results of Campos-Cardoso [3] to inhomogeneous nonlinear Schrödinger equation with inverse-square potential, and the proof is based on the method from Duyckaerts-Roudenko [8]. Furthermore, our result compensates for the one of Campos-Guzmán [4], obtaining blow-up versus global existence dichotomy for solutions beyond the threshold.

1. INTRODUCTION

We consider the inhomogeneous nonlinear Schrödinger equation with inverse-square potential

$$\begin{cases} i\partial_t u - \mathcal{L}_a u = \mu|x|^{-b}|u|^\alpha u, \ (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}, 0 < b < 2, \mu \neq 0, \mathcal{L}_a = -\Delta + \frac{a}{|x|^2}$ with $a > -\frac{(d-2)^2}{4}$. The restriction on a can ensure the self-adjoint operator $\mathcal{L}_a$ to be positive by the sharp Hardy inequality. The operator $\mathcal{L}_a$ arises as models in many problems of physics and geometry. For example, in the theory of combustion (see [23]), and in quantum mechanics (see [20]).

The solution to (1.1) has the conservation of mass and energy

$$M(u) = \int_{\mathbb{R}^d} |u(t,x)|^2 \mathrm{d}x \equiv M(u_0), \tag{1.2}$$

$$E_a(u) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{2}\frac{a}{|x|^2}|u(t,x)|^2 + \frac{\mu}{\alpha+2}\frac{|u(t,x)|^{\alpha+2}}{|x|^b} \right) \mathrm{d}x \equiv E_a(u_0). \tag{1.3}$$

(1.1) is focusing for $\mu < 0$ and defocusing for $\mu > 0$. Moreover, (1.1) is invariant under the scaling

$$u(t,x) \longmapsto u_\lambda(t,x) := \lambda^{\frac{2-b}{\alpha}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

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One can check out $\|u_\lambda\|_{\dot{H}^{s_c}_x} = \|u\|_{\dot{H}^{s_c}_x}$ only when $s_c := \frac{d}{2} - \frac{2-b}{\alpha}$. Therefore, (1.1) is called mass-critical when $s_c = 0$ ($\alpha = \frac{4-2b}{d}$), it corresponds to inter-critical when $0 < s_c < 1$ ($\frac{4-2b}{d} < \alpha < 2^*$), and (1.1) is known as energy-critical problem when $s_c = 1$ ($\alpha = 2^*$), where

$$2^* = \begin{cases} \dfrac{4-2b}{d-2}, & d \geqslant 3, \\ \infty, & d = 1, 2. \end{cases}$$

Before talking about (1.1), we firstly recall the general inhomogeneous nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \mu |x|^{-b}|u|^\alpha u, \ (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^d. \end{cases} \tag{1.4}$$

Similar to (1.1), (1.4) also has the following conserved quantities

$$M(u) = \int_{\mathbb{R}^d} |u(t,x)|^2 \mathrm{d}x \equiv M(u_0), \tag{1.5}$$

$$E_0(u) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u(t,x)|^2 + \frac{\mu}{\alpha+2} \frac{|u(t,x)|^{\alpha+2}}{|x|^b} \right) \mathrm{d}x \equiv E_0(u_0). \tag{1.6}$$

It is worth mentioning that a huge literature has devoted to studying the general inhomogeneous nonlinear Schrödinger equation. The local well-posedness theory was obtained in [6, 11, 12, 13] with $0 < \alpha < 2^*$. In particular, for the focusing case, Genoud [11] proved that (1.4) is global well-posed in $H^1(\mathbb{R}^d)$ if $u_0 \in H^1(\mathbb{R}^d)$ and $\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q_1\|_{L^2(\mathbb{R}^d)}$ when $\alpha = \frac{4-2b}{d}$, where Q_1 is the ground state solution to the elliptic equation

$$\Delta Q_1 - Q_1 + |x|^{-b}|Q_1|^\alpha Q_1 = 0. \tag{1.7}$$

Combet and Genoud [2] gave the classification of minimal mass blow-up solutions for (1.4) with $\alpha = \frac{4-2b}{d}$ as $\|u_0\|_{L^2} = \|Q_1\|_{L^2}$. As for $\frac{4-2b}{d} < \alpha < 2^*$, Farah and Guzmán [9, 10] and Dinh [5] established the global behavior of solutions to (1.4), and they proved a criteria between blow-up and scattering for $M(u)^{1-s_c}E_0(u)^{s_c} < M(Q_1)^{1-s_c}E_0(Q_1)^{s_c}$. Miao-Murphy-Zheng in [21] extended the radial result on scattering for the 3D cubic inhomogeneous NLS to the non-radial setting. Later, Campos and Cardoso [3] established the scattering and blowing-up dichotomy for $M(u)^{1-s_c}E_0(u)^{s_c} \geqslant M(Q_1)^{1-s_c}E_0(Q_1)^{s_c}$.

There are also some work to study the inhomogeneous nonlinear Schrödinger equation with a potential V

$$\begin{cases} i\partial_t u + \Delta u - Vu = \mu |x|^{-b}|u|^\alpha u, \ (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^3, \end{cases} \tag{1.8}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is assumed to satisfy

$$V \in K_0 \cap L^{\frac{3}{2}}, \quad \|V_-\|_K < 4\pi, \tag{1.9}$$

where the potential class K_0 is the closure of bounded compactly supported function with respect to the global Kato norm

$$\|V\|_K = \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|} \mathrm{d}x, \quad V_-(x) := \min\{V(x), 0\}. \tag{1.10}$$

And $V \geqslant 0, x \cdot \nabla V \leqslant 0$ and $x \cdot \nabla V \in L^{\frac{3}{2}}$. Hong [15] proved that global solution scatters by the use of concentration-compactness method for $M(u)^{1-s_c}E_V(u)^{s_c} <$

$M(Q_1)^{1-s_c}E_0(Q_1)^{s_c}$ with $\alpha = 2, \mu = -1$ and $b = 0$. Hamano-Ikeda [16] extended Hong's results to $\frac{4}{3} < \alpha < 4$ with radially symmetric initial data. Combining the results from [10] and [15], Guo-Wang-Yao [14] established the local well-posedness. They also showed scattering theory in the energy space for (1.8) with $\alpha = 2$ and $0 < b < 1$. Besides, Dinh [6] proved the global existence for $\frac{4-2b}{3} < \alpha < 4 - 2b$ and $0 < b < 1$, and established scattering theory only for $\frac{4-2b}{3} < \alpha < 3 - 2b$.

Moreover, there are many approaches to research a critical case of the Kato class potential, i.e. the equation (1.1). For $b = 0$, Burq-Planchon-Stalker-Tahvildar-Zadeh in [1] studied the Strichartz estimates for the equation (1.1). Zhang-Zheng [24] proved scattering in H^1 in the regime

$$\begin{cases} a \geqslant 0 & d = 3, \\ a > -\frac{(d-2)^2}{4} + \frac{4}{(\alpha+2)^2} & d \geqslant 4 \end{cases}$$

for the defocusing inter-critical case to (1.1). In the defocusing energy-critical case, Killip-Miao-Visan-Zhang-Zheng [18] showed global well-posedness and scattering for $a > -\frac{1}{4} + \frac{1}{25}$ in $\mathbb{R}^3$, and carried out the variational analysis needed to treat the focusing case. Later, for the focusing inter-critical case, Killip-Murphy-Visan-Zheng in [17] constructed a solution to the elliptic equation

$$-\mathcal{L}_a Q_2 - Q_2 + |Q_2|^\alpha Q_2 = 0,$$

then they obtained scattering and blow-up when $M(u)E_a(u) < M(Q_2)E_a(Q_2)$, $-\frac{1}{4} < a < 0$ and $\alpha = 2$ in $\mathbb{R}^3$. Moreover, using the method from Dodson-Murphy [7], Zheng in [25] gave a new proof of scattering below the ground state that avoids the use of concentration compactness for the focusing radial inter-critical case in $d \geqslant 3$. For $b \neq 0$, recently Miao-Murphy-Zheng in [22] showed a scattering result at the sharp threshold $M(u)E_a(u) = M(Q_1)E_0(Q_1)$ for the equation (1.1) with $a > 0$, and they also obtained the same result for equation (1.8). Besides, if $u_0 \in H^1_a(\mathbb{R}^d)$, Campos and Guzman [4] established the global existence and blow-up in $H^1_a(\mathbb{R}^d)$ for the equation (1.1) when $\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}$, $\alpha = \frac{4-2b}{d}$ and $\frac{4-2b}{d} < \alpha < \frac{4-2b}{d-2}$, $M(u)^{1-s_c}E_a(u)^{s_c} < M(Q)^{1-s_c}E_a(Q)^{s_c}$, where Q is the ground state solution to the elliptic equation

$$-\mathcal{L}_a Q - Q + |x|^{-b}|Q|^\alpha Q = 0. \tag{1.11}$$

Then we define the mass-energy $\mathcal{ME}$ for (1.1) as

$$\mathcal{ME} = \frac{M(u)^{1-s_c}E_a(u)^{s_c}}{M(Q)^{1-s_c}E_a(Q)^{s_c}}. \tag{1.12}$$

Inspired by the above works, our aim is to describe the influence of inverse-square potential in (1.1) when $\mathcal{ME} \geqslant 1$. Based on the method from Duyckaerts-Roudenko [8], our result compensates for the one of Campos-Guzmán [4] in 3D, obtaining blow-up vs global existence criteria for the solution to (1.1) under the assumption of $\mathcal{ME} \geqslant 1$. We define a continuous function

$$I(t) = \int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 \mathrm{d}x \tag{1.13}$$

for $xu \in L^2(\mathbb{R}^d)$.

Our main results in this paper are as follows:

Theorem 1.1. *Let $(d,\alpha)=(3,2)$, $a>-\frac{1}{4}$ and $0<b<1$. Assuming $I(0)<\infty, u_0\in H_a^1(\mathbb{R}^3)$, $\mathcal{ME}\geqslant 1$, and*

$$\mathcal{ME}\left(1-\frac{(I'(0))^2}{32E_a(u)I(0)}\right)^{s_c}\leqslant 1 \tag{1.14}$$

for the solution u to (1.1), we have

(1) (Blow-up) If $I'(0)\leqslant 0$ and

$$M(u_0)^{1-s_c}\left(\int_{\mathbb{R}^3}|x|^{-b}|u_0|^4\mathrm{d}x\right)^{s_c}>M(Q)^{1-s_c}\left(\int_{\mathbb{R}^3}|x|^{-b}Q^4\mathrm{d}x\right)^{s_c}, \tag{1.15}$$

then $u(t,x)$ blows-up in finite time.

(2) (Global well-posedness) If $I'(0)\geqslant 0$ and

$$M(u_0)^{1-s_c}\left(\int_{\mathbb{R}^3}|x|^{-b}|u_0|^4\mathrm{d}x\right)^{s_c}<M(Q)^{1-s_c}\left(\int_{\mathbb{R}^3}|x|^{-b}Q^4\mathrm{d}x\right)^{s_c}, \tag{1.16}$$

then $u(t,x)$ exists globally. Moreover

$$\limsup_{t\to\infty}M(u)^{1-s_c}\left(\int_{\mathbb{R}^3}|x|^{-b}|u|^4\mathrm{d}x\right)^{s_c}<M(Q)^{1-s_c}\left(\int_{\mathbb{R}^3}|x|^{-b}Q^4\mathrm{d}x\right)^{s_c}. \tag{1.17}$$

Remark 1.2.

(1) The reason of $0<b<1$ instead of $0<b<2$ is that the existence of ground state requires $\alpha<\frac{4-2b}{d-2}$, while we focus our attention on the case of $(d,\alpha)=(3,2)$ in this paper.

(2) We believe that the scattering results can be achieved for the Global well-posedness case in Theorem 1.1 by using the concentrate-compactness argument, which will be considered later.

This paper is organized as follows. In Section 2, we give some preliminaries including the Gagliardo-Nirenberg inequality, Virial identities and some related estimates. Finally, we prove the main results Theorem 1.1 in Section 3.

2. PRELIMINARIES

First, we give some notations which will be used throughout this paper. If X,Y are nonnegative quantities, we use $X\lesssim Y$ to denote the estimate $X\leqslant CY$ for some C. We also use $X\sim Y$ if $X\lesssim Y\lesssim X$. We use $L^q(\mathbb{R}^3)$ to denote the Banach space of the measurable functions $f:\mathbb{R}^3\to\mathbb{C}$ whose norm

$$\|f\|_{L^q(\mathbb{R}^3)}:=\left(\int_{\mathbb{R}^3}|f(x)|^q\mathrm{d}x\right)^{\frac{1}{q}}$$

is finite, with a usual modification when $q=\infty$. Similar to [19], we define Sobolev spaces $\dot{H}_a^{s,r}(\mathbb{R}^3)$ and $H_a^{s,r}(\mathbb{R}^3)$ associated to $\mathcal{L}_a$ by the closure of $C_0^\infty(\mathbb{R}^3\backslash\{0\})$ under the norms

$$\|u\|_{\dot{H}_a^{s,r}(\mathbb{R}^3)}:=\|(\mathcal{L}_a)^{\frac{a}{2}}u\|_{L^r(\mathbb{R}^3)}\quad\text{and}\quad\|f\|_{H_a^{s,r}(\mathbb{R}^3)}:=\|(1+\mathcal{L}_a)^{\frac{a}{2}}u\|_{L^r(\mathbb{R}^3)}.$$

We abbreviate $\dot{H}_a^s(\mathbb{R}^3):=\dot{H}_a^{s,2}(\mathbb{R}^3)$ and $H_a^s(\mathbb{R}^3):=H_a^{s,2}(\mathbb{R}^3)$.

Next, we recall a Gagliardo-Nirenberg inequality which is established in [4],

Lemma 2.1. *Let $0<b<1$. Then the Gagliardo-Nirenberg inequality*

$$\big\||x|^{-b}|u|^4\big\|_{L^1} \leqslant C_{GN}\|u\|_{L^2}^{1-b}\|u\|_{\dot H_a^1}^{3+b}, \qquad u\in H_a^1 \tag{2.1}$$

holds, and the sharp constant C_{GN} is attained by a function $Q\in H_a^1$, i.e. $C_{GN}=\frac{\||x|^{-b}Q^4\|_{L^1}}{\|Q\|_{L^2}^{1-b}\|Q\|_{\dot H_a^1}^{3+b}}$, where Q is the positive real solution, ground state, to (1.11).

Rewriting (2.1) as

$$\left(\int_{\mathbb{R}^3}|x|^{-b}|u|^4\mathrm{d}x\right)^{\frac{2}{3+b}} \leqslant C_Q\left(M(u)\right)^{\frac{1-b}{3+b}}\|u\|_{\dot H_a^1}^2, \tag{2.2}$$

and it follows from (2.1) that

$$C_Q := \frac{\left(\int_{\mathbb{R}^3}|x|^{-b}Q^4\mathrm{d}x\right)^{\frac{2}{3+b}}}{(M(Q))^{\frac{1-b}{3+b}}\|Q\|_{\dot H_a^1}^2}.$$

Furthermore, we can integrate the products of (1.11) with Q and $x\cdot\nabla Q$ respectively, which results in

$$\|Q\|_{L^2}^2=\frac{1-b}{3+b}\|Q\|_{\dot H_a^1}^2=\frac{1-b}{4}\int_{\mathbb{R}^3}|x|^{-b}Q^4\mathrm{d}x, \tag{2.3}$$

then we obtain

$$E_a(Q)=\frac{1+b}{8}\int_{\mathbb{R}^3}|x|^{-b}Q^4\mathrm{d}x. \tag{2.4}$$

Besides, we give the Virial identities and some related estimates.

Lemma 2.2. *(Virial identities, see* [4]*). Let u be the solution to* (1.1) *with $(d,\alpha)=(3,2)$, $0<b<1$ and $a>-\frac14$. Assuming $I(0)<\infty$ (refereed to as finite variance), then the following Virial identities hold:*

$$I'(t)=4\,\mathrm{Im}\int_{\mathbb{R}^3}x\bar u\cdot\nabla u\mathrm{d}x, \tag{2.5}$$

$$I''(t)=8\|u\|_{\dot H_a^1}^2-(6+2b)\int_{\mathbb{R}^3}|x|^{-b}|u|^4\mathrm{d}x. \tag{2.6}$$

Lemma 2.3. *Let $u\in H_a^1(\mathbb{R}^3)$ and $xu\in L^2(\mathbb{R}^3)$, then we have*

$$(I'(t))^2\leqslant 16I(t)\left[\|u\|_{\dot H_a^1}^2-\frac{1}{C_QM(u)^{\frac{1-b}{3+b}}}\left(\int_{\mathbb{R}^3}|x|^{-b}|u|^4\mathrm{d}x\right)^{\frac{2}{3+b}}\right], \tag{2.7}$$

$$\|u\|_{\dot H_a^1}^2=\frac{8(b+3)E_a(u)-I''(t)}{4(b+1)}, \tag{2.8}$$

$$\int_{\mathbb{R}^3}|x|^{-b}|u|^4\mathrm{d}x=\frac{16E_a(u)-I''(t)}{2(b+1)}. \tag{2.9}$$

Proof. The proof is similar to [8]. First, for $\alpha\in\mathbb{R}$, we can obtain

$$\int_{\mathbb{R}^3}\left|\sqrt{\mathcal{L}_a}\left(e^{i\alpha|x|^2}u\right)\right|^2\mathrm{d}x=4\alpha^2\int_{\mathbb{R}^3}|x|^2|u|^2\mathrm{d}x+4\alpha\,\mathrm{Im}\int_{\mathbb{R}^3}x\cdot\nabla u\bar u\mathrm{d}x + \int_{\mathbb{R}^3}|\sqrt{\mathcal{L}_a}u|^2\mathrm{d}x. \tag{2.10}$$

Applying Lemma 2.1 and the rewriting Gagliardo-Nirenberg inequality (2.2) to $e^{i\alpha|x|^2}u$, then we have

$$C_Q M(u)^{\frac{1-b}{3+b}}\left(4\alpha^2\int_{\mathbb{R}^3}|x|^2|u|^2\mathrm{d}x+4\alpha\,\mathrm{Im}\int_{\mathbb{R}^3}x\cdot\nabla u\bar{u}\mathrm{d}x+\int_{\mathbb{R}^3}|\sqrt{\mathcal{L}_a}u|^2\mathrm{d}x\right)$$
$$-\left(\int_{\mathbb{R}^3}|x|^{-b}|u|^4\mathrm{d}x\right)^{\frac{2}{3+b}}\geqslant 0 \tag{2.11}$$

for all $\alpha\in\mathbb{R}$. We can see that the left-hand side of (2.11) is a quadratic in α. The discriminant of this quadratic in α must be negative, which yields the inequality (2.7). After using the equality (2.6), we can easily obtain (2.8) and (2.9). ☐

3. Proof the main theorem

Now let us concentrate on proving the main result, Theorem 1.1.

Let $z(t)=\sqrt{I(t)}\in C(\mathbb{R})$. If we substitute (2.8) and (2.9) into (2.7), then

$$(z'(t))^2=\frac{(I'(t))^2}{4I(t)}\leqslant 4f(I''(t)), \tag{3.1}$$

where

$$f(y)=-\frac{y}{4(b+1)}+\frac{2(b+3)}{b+1}E_a(u)-\frac{1}{C_Q M(u)^{\frac{1-b}{3+b}}}\left[\frac{1}{2(b+1)}\left(16E_a(u)-y\right)\right]^{\frac{2}{3+b}}$$

for any $y\in(-\infty,16E_a(u))$ by (2.9).

So we have

$$f'(y)=-\frac{1}{4(b+1)}+\frac{1}{C_Q M(u)^{\frac{1-b}{3+b}}}\frac{2}{3+b}\left(\frac{1}{2(b+1)}\right)^{\frac{2}{3+b}}\left(16E_a(u)-y\right)^{\frac{2}{3+b}-1}.$$

Since $\frac{2}{3+b}-1<0(s_c>0)$, $f(y)$ is decreasing on $(-\infty,y_0)$ and increasing on $(y_0,16E_a(u))$, where y_0 satisfies

$$\frac{1}{4(b+1)}=\frac{1}{C_Q M(u)^{\frac{1-b}{3+b}}}\frac{2}{3+b}\left(\frac{1}{2(b+1)}\right)^{\frac{2}{3+b}}\left(16E_a(u)-y_0\right)^{\frac{2}{3+b}-1}. \tag{3.2}$$

By (3.2) and a simple calculation, we get $f(y_0)=\frac{y_0}{8}$.

Moreover, using (2.4), (2.3) and the expression of C_Q , we rewrite (3.2) as

$$\left(\frac{M(u)}{M(Q)}\right)^{1-s_c}\left(\frac{E_a(u)-\frac{y_0}{16}}{E_a(Q)}\right)^{s_c}=1. \tag{3.3}$$

Next, by using (3.3) and $y_0\in(-\infty,16E_a(u))$, we can rewrite $\mathcal{ME}\geqslant 1$ as $y_0\geqslant 0$, and rewrite (1.14) as

$$(z'(0))^2\geqslant\frac{y_0}{2}=4f(y_0). \tag{3.4}$$

Blow − up. We know the assumption $I'(0)\leqslant 0$ means

$$z'(0)\leqslant 0. \tag{3.5}$$

By using (3.3), the assumption (1.15) is equivalent to

$$\left(\frac{M(u_0)}{M(Q)}\right)^{1-s_c}\left(\frac{\int_{\mathbb{R}^3}|x|^{-b}|u_0|^4\mathrm{d}x}{\frac{8}{b+1}E_a(Q)}\right)^{s_c}>\left(\frac{M(u)}{M(Q)}\right)^{1-s_c}\left(\frac{E_a(u)-\frac{y_0}{16}}{E_a(Q)}\right)^{s_c}.$$

Then by (2.9), we get

$$I''(0) < y_0. \tag{3.6}$$

Now we **claim** : $z''(t) < 0, \forall\, t \in [0, T_+(u))$.

If the claim holds, we assume $T_+(u) = \infty$. By using (3.5) and Taylor expression of $z(t)$ around $t = 0$, we obtain

$$\begin{aligned} z(t) =& z(0) + z'(0)t + z''(\theta)t^2 \\ <& z(0) + z'(0)t \end{aligned}$$

for $\theta \in (0, t)$, which implies that $z(t)$ will arrive 0 in a finite time. Then we deduce a contradiction since $z(t) > 0$.

Indeed, by (3.4) and (3.6), we obtain that

$$z''(0) = \frac{1}{z(0)} \left(\frac{I''(0)}{2} - (z'(0))^2 \right) < 0. \tag{3.7}$$

We assume that Claim does not hold, then there exists $t_0 \in (0, T_+(u))$ such that $t_0 = \sup\{t \in (0, T_+(u)), z''(0) \geqslant 0\}$, by the continuity of $z''(t)$, we have

$$z''(t_0) = 0$$

and

$$z''(t) < 0, \quad \forall\, t \in [0, t_0).$$

Using (3.4) and (3.5), we deduce

$$z'(t) < z'(0) \leqslant -\sqrt{\frac{y_0}{2}} = -2\sqrt{f(y_0)}, \qquad t \in (0, t_0]. \tag{3.8}$$

Therefore, $(z'(t))^2 > \frac{y_0}{2} = 4f(y_0), \forall t \in (0, t_0]$. Using (3.1), we get $f(y_0) < f(I''(t)), \forall\, t \in (0, t_0]$. So,

$$I''(t) < y_0, \qquad \forall t \in [0, t_0]. \tag{3.9}$$

Combining this with (3.7), (3.8), we have $z''(t_0) = \frac{1}{z(t_0)} \left(\frac{I''(t_0)}{2} - (z'(t_0))^2 \right) < 0$, which is a contradiction with $z''(t_0) = 0$. Thus the claim holds.

Global well − posedness. The assumptions $I'(0) \geqslant 0$ and (1.16) are equivalent to following inequalities $z'(0) \geqslant 0$, and $I''(0) > y_0$.

Using (3.4) and $z'(0) \geqslant 0$, we have

$$z'(0) \leqslant \sqrt{\frac{y_0}{2}}. \tag{3.10}$$

Then there will exists $t_1 \geqslant 0$ such that

$$z'(t_1) > \sqrt{\frac{y_0}{2}} = 2\sqrt{f(y_0)}. \tag{3.11}$$

Indeed, if (3.10) is strict, we choose $t_1 = 0$. If $z'(0) = \sqrt{\frac{y_0}{2}}$ and using (3.7), we have $z''(0) > 0$, so (3.11) follows for small $t_1 > 0$. So we can choose a small parameter $\varepsilon_1 > 0$ such that

$$z'(t_1) \geqslant 2\sqrt{f(y_0)} + 2\varepsilon_1. \tag{3.12}$$

Now we **claim** : $z'(t) > 2\sqrt{f(y_0)} + \varepsilon_1, \forall\, t \geqslant t_1$.

If the claim does not hold, there exists t_2 such that $t_2 = \inf\{t > t_1 : z'(t) \leqslant 2\sqrt{f(y_0)} + \varepsilon_1\}$. By the continuity of $z'(t)$, we get

$$z'(t_2) = 2\sqrt{f(y_0)} + \varepsilon_1 \tag{3.13}$$

and

$$z'(t) \geqslant 2\sqrt{f(y_0)} + \varepsilon_1, \qquad \forall\, t \in [t_1, t_2]. \tag{3.14}$$

Using (3.1)

$$\left(2\sqrt{f(y_0)} + \varepsilon_1\right)^2 \leqslant (z'(t))^2 \leqslant 4f(I''(t)), \qquad \forall\, t \in [t_1, t_2], \tag{3.15}$$

we get that $f(I''(t)) > f(y_0)$ for all $t \in [t_1, t_2]$ if ε_1 small enough, which implies $I''(t) \neq y_0$. Similar the method to prove (3.9) and $I''(0) > y_0$, we get $I''(t) > y_0$, for $t \in [t_1, t_2]$.

We will prove that there exists a constant C such that

$$I''(t) \geqslant y_0 + \frac{\sqrt{\varepsilon_1}}{C}, \qquad \forall\, t \in [t_1, t_2]. \tag{3.16}$$

In fact, if $I''(t) \geqslant y_0+1$, then (3.16) holds (for C large enough). If $y_0 < I''(t) \leqslant y+1$, by the Taylor expansion of f around $y = y_0$, there exists $a > 0$ such that

$$f(y) \leqslant f(y_0) + a(y - y_0)^2 \quad \text{when} \quad |y - y_0| \leqslant 1.$$

Next by (3.15), we get

$$\left(2\sqrt{f(y_0)} + \varepsilon_1\right)^2 \leqslant (z'(t))^2 \leqslant 4f(I''(t)) \leqslant 4f(y_0) + 4a(I''(t) - y_0)^2.$$

Therefore, we have $C = 2\sqrt{a}\left(4(f(y_0))^{\frac{1}{2}} + \varepsilon_1\right)^{-\frac{1}{2}}$ in (3.16).

However, by (3.13) and (3.16) we have

$$\begin{aligned} z''(t_2) =& \frac{1}{z(t_2)}\left(\frac{I''(t_2)}{2} - (z'(t_2))^2\right) \\ \geqslant& \frac{1}{z(t_2)}\left(\frac{\sqrt{\varepsilon_1}}{2C} - 4\varepsilon_1\sqrt{f(y_0)} - \varepsilon_1^2\right) \\ >& \frac{1}{z(t_2)}\frac{\sqrt{\varepsilon}}{4C}, \end{aligned}$$

where $\varepsilon < \varepsilon_1$ is small enough. Then we get $z''(t_2) > 0$, which contradicts with (3.13) and (3.14). So we obtain the claim.

We note that (3.16) holds for all $t \in [t_1, T_+(u))$. Hence, we obtain

$$\begin{aligned} M(u)^{1-s_c}\left(\int_{\mathbb{R}^3}\frac{|u|^4}{|x|^b}\mathrm{d}x\right)^{s_c} &= M(u)^{1-s_c}\left(\frac{1}{2(b+1)}\left(16E_a(u) - I''(t)\right)\right)^{s_c} \\ &\leqslant M(u)^{1-s_c}\left(\frac{1}{2(b+1)}\left(16E_a(u) - y_0 - \frac{\sqrt{\varepsilon_0}}{C}\right)\right)^{s_c} \\ &\leqslant \left(\frac{8}{(b+1)}\right)^{s_c} M(Q)^{1-s_c}E_a(Q)^{s_c} \\ &= M(Q)^{1-s_c}\left(\int_{\mathbb{R}^3}|x|^{-b}Q^4\mathrm{d}x\right)^{s_c}. \end{aligned}$$

Then by mass and energy conservation, we have

$$\|u\|_{\dot{H}_a^1}^2 = E_a(u) + \int_{\mathbb{R}^3} \frac{|u|^4}{|x|^b} \mathrm{d}x < C$$

for all $t \in [t_1, T_+(u))$, where C depending on $M(u_0), E_a(u_0), M(Q)$, and $E_a(Q)$. So $u(t,x)$ exists globally.

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