Decay rate of generalized solutions for equations of a viscous heat-conducting gas

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Abstract

The large time behavior is considered for the solutions of the Navier-Stokes equations for one-dimensional viscous polytropic ideal gas in unbounded domains. Using the *local* anti-derivatives functions technique, we obtain the power type decay estimates for the generalized solutions as time goes to infinity.

Keywords: Compressible Navier-Stokes; Decay rates; Unbounded domains; Generalized solutions; Large initial data.

1 Introduction

The compressible Navier-Stokes system describing the one-dimensional motion of a viscous heat-conducting perfect polytropic gas is governed by the equations in the Lagrange variables(cf.[2])

$$v_t = u_x, \tag{1.1}$$

$$u_t + \left(\frac{R\theta}{v}\right)_x = \left(\mu\frac{u_x}{v}\right)_x, \tag{1.2}$$

$$\left(c_v\theta + \frac{1}{2}u^2\right)_t + \left(\frac{R\theta}{v}u\right)_x = \left(\kappa\frac{\theta_x}{v} + \mu\frac{uu_x}{v}\right)_x. \tag{1.3}$$

Here, $t > 0$ is time, $x \in \mathbb{R} = (-\infty, +\infty)$ denotes the Lagrange mass coordinate; the unknown functions $v > 0, u, \theta > 0$ are, respectively, the specific volume of the gas, fluid velocity, absolute temperature; the positive constants μ and κ are the viscosity and heat conductivity coefficients; the heat capacity $c_v = \frac{R}{\gamma-1}$ is constant with $\gamma > 1$ being the adiabatic exponent and $R > 0$ being the given constant.

Since the first work of Kazhikhov-Shelukhin [14] on the global existence of large solutions for the equations (1.1)-(1.3) in bounded intervals, significant progress has been achieved on the mathematical aspect. See, for example, [1, 6, 19, 22, 20]. For the Cauchy problem of equations (1.1)-(1.3), the global existence of large solutions

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is originally due to Kazhikhov-Shelukhin [13]. Jiang [10, 11] proved that the specific volume is point-wise bounded for all x,t and studied some partial results on the large-time behavior of solutions. Later on, Li-Liang[12] obtained the uniform in x,t estimate on the temperature, and prove that the global solution is asymptotically stable as time tends to infinity. We also refer readers to the papers [1, 9, 8, 18] and the references cited therein.

We study the equations (1.1)-(1.3) with the initial conditions

$$(v(x,0),u(x,0),\theta(x,0)) = (v_0(x),u_0(x),\theta_0(x)), \quad x\in\mathbb{R} \tag{1.4}$$

and the far-field behaviors

$$\lim_{|x|\to\infty}(v(x,t),u(x,t),\theta(x,t)) = (1,0,1), \quad t>0. \tag{1.5}$$

For the sake of convenience, we first collect some known existence results.

Proposition 1.1 ([1, 13, 10, 11, 12]) *Assume that the initial functions* (v_0,u_0,θ_0) *satisfy*

$$v_0-1,u_0,\theta_0-1\in H^1(\mathbb{R}), \quad \inf_{x\in\mathbb{R}} v_0(x)>0,\ \inf_{x\in\mathbb{R}}\theta_0(x)>0. \tag{1.6}$$

Then, there exists a unique global (large) generalized solution (v,u,θ) *to the problem* (1.1)-(1.5)*, satisfying the following properties:*

$$C_0^{-1}\le v,\,\theta\le C_0 \quad \forall\ (x,t)\in\mathbb{R}\times[0,+\infty), \tag{1.7}$$

$$v-1,u,\theta-1\in L^\infty(0,\infty;H^1(\mathbb{R}))\cap C\left(\overline{\mathbb{R}\times[0,+\infty)}\right), \tag{1.8}$$

$$v_t,v_x,v_{xt},u_t,\theta_t,u_{xx},\theta_{xx}\in L^2(0,\infty;L^2(\mathbb{R})), \tag{1.9}$$

$$\lim_{t\to+\infty}\left(\|(v-1,u,\theta-1)(\cdot,t)\|_{L^p(\mathbb{R})}+\|(v_x,u_x,\theta_x)\|_{L^2(\mathbb{R})}\right)=0, \quad p\in(2,\infty], \tag{1.10}$$

where the constant C_0 *is positive and depends only on* μ,κ,R,c_v, $\|(v_0-1,u_0,\theta_0-1)\|_{H^1(\mathbb{R})}$, $\inf_{x\in\mathbb{R}}v_0(x)$ *and* $\inf_{x\in\mathbb{R}}\theta_0(x)$.

1.1 Main results

In this paper, we establish the decay rates for the generalized solution (v,u,θ) of the problem (1.1)-(1.5) as time goes to infinity.

Theorem 1.1 *Let the initial functions* (v_0,u_0,θ_0) *satisfy* (1.6)*, and* (v,u,θ) *be the solution to the problem* (1.1)-(1.5) *stated in Proposition 1.1. Then, for all* $t\ge 0$,

$$\begin{aligned}
&\|(v-1,u,\theta-1)(\cdot,t)\|_{L^2(\mathbb{R})}\le C(1+t)^{-\frac{1}{2}},\\
&\|(v-1,u,\theta-1)(\cdot,t)\|_{L^\infty(\mathbb{R})}\le C(1+t)^{-\frac{1}{2}},\\
&\|(v_x,u_x,\theta_x)(\cdot,t)\|_{L^2(\mathbb{R})}\le C(1+t)^{-\frac{1}{2}}.
\end{aligned} \tag{1.11}$$

The decay rates in (1.11) require no restriction other than Proposition 1.1. In the second Theorem 1.2, we shall improve the decay rate of the $\|(v_x,u_x,\theta_x)(\cdot,t)\|_{L^2(\mathbb{R})}$.

Theorem 1.2 *Under the same assumptions listed in Theorem 1.1, the decay rates in* (1.11) *can be improved as*

$$\begin{aligned}
&\|(v-1,u,\theta-1)(\cdot,t)\|_{L^2(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}},\\
&\|(v-1,u,\theta-1)(\cdot,t)\|_{L^\infty(\mathbb{R})} \le C(1+t)^{-\frac{3}{4}},\\
&\|(v_x,u_x,\theta_x)(\cdot,t)\|_{L^2(\mathbb{R})} \le C(1+t)^{-1},
\end{aligned} \tag{1.12}$$

provided that the functions

$$(\Phi_0,\Psi_0,W_0)(x) = \int_{-\infty}^{x} \left(v_0-1,\, u_0,\, \theta_0-1+\frac{1}{2}u_0^2\right)(y)dy \tag{1.13}$$

belong to $L^2(\mathbb{R})$.

Some remarks are in order:

Remark 1.1 *Our results apply to the equations* (1.1)-(1.3) *in half-line* $\mathbb{R}_+ = (0,+\infty)$, *equipped with one type of the following boundary conditions*

$$u(0,t)=0,\ \theta_x(0,t)=0 \quad \text{and} \quad \lim_{x\to+\infty}(v,u,\theta)=(1,0,1),$$

or

$$u(0,t)=0,\ \theta(0,t)=1 \quad \text{and} \quad \lim_{x\to+\infty}(v,u,\theta)=(1,0,1).$$

Remark 1.2 *In analogy to the linearized system of equations, we obtained in Theorem 1.2 the optimal decay estimates for the solutions to nonlinear problem* (1.1)-(1.5).

Remark 1.3 *For bounded intervals in dimension one, the solutions of equations* (1.1)-(1.3) *are shown to be nonlinearly exponentially stable as time tends to infinity, see, e.g., [20].*

Remark 1.4 *We refer to the papers, e.g., [3, 4, 21, 16, 17], for related results in high dimensions, in the case when the initial functions are smooth enough and have small oscillation around an equilibrium state.*

1.2 Methodology.

We comment on the analysis of the proof of Theorem 1.1 and Theorem 1.2.

First, the standard energy estimates on equations (1.1)-(1.3) provide that

$$t\|(v-1,u,\theta-1)\|^2_{H^1(\mathbb{R})} \le C + C\int_0^t \|(v-1,u,\theta-1)\|^2_{L^2(\mathbb{R})}. \tag{1.14}$$

To proceed, a crucial step is to get the uniform boundedness of $\int_0^t \|(v-1,u,\theta-1)\|^2_{L^2(\mathbb{R})}$. In the light of the papers [7, 15], we come up with the *local* anti-derivatives functions:

$$(\Phi,\Psi,W)(x,t) = \int_{-\infty}^{x} \mathbf{1}(y)\left(v-1,u,c_v(\theta-1)+\frac{1}{2}u^2\right)(y,t)dy,$$

where $\mathbf{1}(x) = \mathbf{1}_{[kd,(k+1)d]}(x)$ is the characteristic in interval $[kd,(k+1)d]$ with integer k and the given small constant $d > 0$. We remark that the main function of $\mathbf{1}(x)$ is to localize the whole space $\mathbb{R}$, such that the Poincaré inequality could be applied, that is,

$$
\begin{aligned}
&\|(\Phi,\Psi,W)(\cdot,t)\|_{L^2([kd,(k+1)d])} + \|(\Phi,\Psi,W)(\cdot,t)\|_{L^\infty([kd,(k+1)d])} \\
&\le C\sqrt{d}\,\|(\Phi_x,\Psi_x,W_x)(\cdot,t)\|_{L^2([kd,(k+1)d])}, \quad \forall\, t \ge 0.
\end{aligned}
$$

Thereafter, we formulate (by approximation) the system of equations in (Φ,Ψ,W), and control the local (in space) integral quantity

$$
\int_0^t \|(v-1,u,c_v(\theta-1)+\frac{1}{2}u^2)\|^2_{L^2([kd,(k+1)d])}
$$

by good terms; see inequality (2.44). In this regards, we are able to derive the uniform upper bound of $\int_0^t \|(v-1,u,\theta-1)\|^2_{L^2(\mathbb{R})}$ by repeating the argument for every integer $k = 1, \pm 1, \pm 2, ...$

In Theorem 1.2, we use the Fourier transform and give the L^p-L^q estimates on the linearized equations in terms of anti-derivatives functions. This allows us to derive the optimal decay rates for the solutions, as long as the additional restriction (1.13) is made.

1.3 Notation:

Let $p \in [1,+\infty]$ and integer $m \ge 0$. We denote by $W^{m,p}(\Omega)$ the usual Sobolev space defined in $\Omega \subseteq \mathbb{R}$ with norm $\|\cdot\|_{W^{m,p}(\Omega)}$. For convenience, we use the abbreviated conventions $W^{m,2}(\Omega) = H^m(\Omega)$, $W^{0,p}(\Omega) = L^p(\Omega)$. During this paper, the capital letter $C > 0$ symbols a generic constant which depends only on μ,κ,R,c_v, $\|(v_0-1,u_0,\theta_0-1)\|_{H^1(\mathbb{R})}$, $\inf_{x\in\mathbb{R}} v_0(x)$ and $\inf_{x\in\mathbb{R}} \theta_0(x)$, additionally, C_α is used to emphasize the dependence of C on α.

The rest Section 2 and Section 3 are devoted to proving Theorem 1.1 and Theorem 1.2 respectively.

2 Proof of Theorem 1.1

2.1 Standard energy estimates

Lemma 2.1 *There exists a constant C such that for all $t \ge 0$*

$$
\begin{aligned}
&t\|(v-1,u,\theta-1)\|^2_{H^1(\mathbb{R})} + \int_0^t s\left(\|(v_x,u_x,\theta_x)\|^2_{L^2(\mathbb{R})} + \|(u_{xx},\theta_{xx})\|^2_{L^2(\mathbb{R})}\right) \\
&\le C + C\int_0^t \|(v-1,u,\theta-1)\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{2.1}
$$

Proof. By (1.1), integrating (1.2) after multiplied by tv_x/v gives

$$
\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}(t\int_{\mathbb{R}}\frac{v_x^2}{v^2})\\
&=\frac{\mu}{2}\int_{\mathbb{R}}\frac{v_x^2}{v^2}+t\int_{\mathbb{R}}(\frac{R\theta}{v})_x\frac{v_x}{v}+t\int_{\mathbb{R}}u_t\frac{v_x}{v}\\
&=\frac{\mu}{2}\int_{\mathbb{R}}\frac{v_x^2}{v^2}+t\int_{\mathbb{R}}\frac{R\theta_x}{v}\frac{v_x}{v}-t\int_{\mathbb{R}}\frac{R\theta v_x^2}{v^3}+\frac{d}{dt}(t\int_{\mathbb{R}}u\frac{v_x}{v})-\int_{\mathbb{R}}u\frac{v_x}{v}+t\int_{\mathbb{R}}\frac{u_x^2}{v},
\end{aligned}
\tag{2.2}
$$

which, together with (1.7), (1.9), and the Cauchy inequality, leads to

$$
\begin{aligned}
&t\|v_x\|^2_{L^2(\mathbb{R})}+\int_0^t s\|v_x\|^2_{L^2(\mathbb{R})}\\
&\le Ct\|u\|^2_{L^2(\mathbb{R})}+C\int_0^t\left(\|u\|^2_{L^2(\mathbb{R})}+\|v_x\|^2_{L^2(\mathbb{R})}\right)+C\int_0^t s\|(u_x,\theta_x)\|^2_{L^2(\mathbb{R})}\\
&\le C+Ct\|u\|^2_{L^2(\mathbb{R})}+C\int_0^t\|u\|^2_{L^2(\mathbb{R})}+C\int_0^t s\|(u_x,\theta_x)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{2.3}
$$

Integrating (1.2) after multiplied by tu_{xx} yields

$$
\frac{1}{2}\frac{d}{dt}(t\int_{\mathbb{R}}u_x^2)+t\int_{\mathbb{R}}\frac{\mu u_{xx}^2}{v}=\frac{1}{2}\int_{\mathbb{R}}u_x^2+t\int_{\mathbb{R}}(\frac{R\theta}{v})_x u_{xx}+t\int_{\mathbb{R}}\frac{\mu u_x v_x}{v^2}u_{xx}.
\tag{2.4}
$$

Making use of (1.7) and (1.9), we integrate (2.4) to deduce

$$
\begin{aligned}
&t\|u_x\|^2_{L^2(\mathbb{R})}+\int_0^t s\|u_{xx}\|^2_{L^2(\mathbb{R})}\\
&\le C\int_0^t\|u_x\|^2_{L^2(\mathbb{R})}+C\int_0^t s\left(\|\theta_x\|^2_{L^2(\mathbb{R})}+\|v_x\|^2_{L^2(\mathbb{R})}+\|u_xv_x\|^2_{L^2(\mathbb{R})}\right)\\
&\le C+C\int_0^t s\left(\|v_x\|^2_{L^2(\mathbb{R})}+\|(u_x,\theta_x)\|^2_{L^2(\mathbb{R})}\right)+\frac{1}{2}\int_0^t s\|u_{xx}\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

This combining with (2.3) yields

$$
\begin{aligned}
&t\|(v_x,u_x)\|^2_{L^2(\mathbb{R})}+\int_0^t s\left(\|v_x\|^2_{L^2(\mathbb{R})}+\|u_{xx}\|^2_{L^2(\mathbb{R})}\right)\\
&\le C+Ct\|u\|^2_{L^2(\mathbb{R})}+C\int_0^t\|u\|^2_{L^2(\mathbb{R})}+C\int_0^t s\|(u_x,\theta_x)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{2.5}
$$

It follows from (1.2) and (1.3) that

$$
c_v\theta_t+\frac{R\theta}{v}u_x=\left(\frac{\kappa\theta_x}{v}\right)_x+\frac{\mu u_x^2}{v}.
\tag{2.6}
$$

Integrating (2.6) after multiplied by $t\theta_{xx}$ yields

$$
\begin{aligned}
&\frac{c_v}{2}\frac{d}{dt}(t\int_{\mathbb{R}}\theta_x^2)+t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v}\\
&=\frac{c_v}{2}\int_{\mathbb{R}}\theta_x^2+t\int_{\mathbb{R}}\frac{R\theta}{v}u_x\theta_{xx}+t\int_{\mathbb{R}}\frac{\kappa\theta_xv_x}{v^2}\theta_{xx}-t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}\theta_{xx}.
\end{aligned}
\tag{2.7}
$$

Utilizing (1.7) and (1.9) once again, we integrate (2.7) to deduce

$$
\begin{aligned}
& t\|\theta_x\|_{L^2(\mathbb{R})}^2 + \int_0^t s\|\theta_{xx}\|_{L^2(\mathbb{R})}^2 \\
& \leq C + C\int_0^t s\left(\|u_x\|_{L^2(\mathbb{R})}^2 + \|\theta_x v_x\|_{L^2(\mathbb{R})}^2 + \|u_x\|_{L^4(\mathbb{R})}^4\right) \\
& \leq C + C\int_0^t s\left(\|u_x\|_{L^2(\mathbb{R})}^2 + \|\theta_x\|_{L^2(\mathbb{R})}^2\right) + \frac{1}{2}\int_0^t s\left(\|u_{xx}\|_{L^2(\mathbb{R})}^2 + \|\theta_{xx}\|_{L^2(\mathbb{R})}^2\right).
\end{aligned}
\tag{2.8}
$$

As a result of (2.5) and (2.8), we get

$$
\begin{aligned}
& t\|(v_x, u_x, \theta_x)\|_{L^2(\mathbb{R})}^2 + \int_0^t s\left(\|v_x\|_{L^2(\mathbb{R})}^2 + \|(u_{xx}, \theta_{xx})\|_{L^2(\mathbb{R})}^2\right) \\
& \leq C + Ct\|u\|_{L^2(\mathbb{R})}^2 + C\int_0^t \|u\|_{L^2(\mathbb{R})}^2 + C\int_0^t s\|(u_x, \theta_x)\|_{L^2(\mathbb{R})}^2.
\end{aligned}
\tag{2.9}
$$

Next, multiplying (1.1) by $R(1-\frac{1}{v})$, (1.2) by u, (2.6) by $(1-\frac{1}{\theta})$, adding the resulting expression together, we obtain

$$
\begin{aligned}
& \left(R(v - \ln v - 1) + c_v(\theta - \ln\theta - 1) + \frac{1}{2}u^2\right)_t + \frac{\mu u_x^2}{v\theta} + \frac{\kappa\theta_x^2}{v\theta^2} \\
& = \left(\frac{\mu u u_x - Ru\theta}{v} + Ru + \frac{\kappa(\theta-1)\theta_x}{v\theta}\right)_x.
\end{aligned}
\tag{2.10}
$$

By (1.7), integrating (2.10) after multiplied by t yields

$$
t\|(v-1, u, \theta-1)\|_{L^2(\mathbb{R})}^2 + \int_0^t s\|(u_x, \theta_x)\|_{L^2(\mathbb{R})}^2 \leq C\int_0^t \|(v-1, u, \theta-1)\|_{L^2(\mathbb{R})}^2.
\tag{2.11}
$$

Therefore, the (2.1) follows from (2.9) and (2.11). The proof of Lemma 2.1 is thus finished.

2.2 The local anti-derivatives technique

In view of Lemma 2.1, it remains need to control the last term $\int_0^t \|(v-1, u, \theta-1)\|_{L^2(\mathbb{R})}^2$ in (2.11).

We have the following

Claim 2.1 *Under the assumptions made in Theorem 1.1, there is a constant C such that*

$$
\int_0^t \|(v-1, u, \theta-1)\|_{L^2(\mathbb{R})}^2 \leq C.
\tag{2.12}
$$

Substituting (2.12) into (2.1) yields directly the desired (1.11). So, to complete the proof of Theorem 1.1, the only left task is to verify Claim 2.1.

Proof of Claim 2.1. We divide the process into several steps.

Step 1. In the light of [7, 15], we introduce the *local* anti-derivative functions:

$$(\Phi, \Psi, W)(x,t) = \int_{-\infty}^{x} \mathbf{1}(y)\left(v-1,\, u,\, c_v(\theta-1)+\frac{1}{2}u^2\right)(y,t)dy, \tag{2.13}$$

where $\mathbf{1}(x) = \mathbf{1}_{[kd,\,(k+1)d]}(x)$ is the characteristic function in $[kd,\,(k+1)d]$ with integer $k \in \mathbb{Z}$ and $d > 0$ is a given small constant.

By (2.13), it is clear that $\Phi(x,t)$ is continuous, and moreover,

$$\Phi(x,t) \le \int_{kd}^{(k+1)d} |\Phi_x|, \quad \forall\, x \in [kd, (k+1)d].$$

This implies

$$\begin{aligned}
&\|\Phi(\cdot,t)\|_{L^2([kd,\,(k+1)d])} + \|\Phi(\cdot,t)\|_{L^\infty([kd,\,(k+1)d])} \\
&\le C\sqrt{d}\|\Phi_x(\cdot,t)\|_{L^2([kd,\,(k+1)d])} \\
&= C\sqrt{d}\|\mathbf{1}\Phi_x(\cdot,t)\|_{L^2(\mathbb{R})}, \quad t \ge 0.
\end{aligned} \tag{2.14}$$

Similarly,

$$\begin{aligned}
&\|(\Psi,W)(\cdot,t)\|_{L^2([kd,\,(k+1)d])} + \|(\Psi,W)(\cdot,t)\|_{L^\infty([kd,\,(k+1)d])} \\
&\le C\sqrt{d}\|(\Psi_x,W_x)(\cdot,t)\|_{L^2([kd,\,(k+1)d])} \\
&= C\sqrt{d}\|\mathbf{1}(\Psi_x,W_x)(\cdot,t)\|_{L^2(\mathbb{R})}, \quad t \ge 0.
\end{aligned} \tag{2.15}$$

Step 2. For small constant $\varepsilon > 0$, we define the approximating function

$$\mathbf{1}^\varepsilon(x) = \mathbf{1}^\varepsilon_{[kd,\,(k+1)d]}(x) = \begin{cases} \dfrac{x-kd}{\varepsilon}, & x \in [kd, kd+\varepsilon], \\ 1, & x \in [kd+\varepsilon,\,(k+1)d-\varepsilon], \\ \dfrac{(k+1)d-x}{\varepsilon}, & x \in [(k+1)d-\varepsilon,\,(k+1)d], \\ 0, & \textit{others}. \end{cases} \tag{2.16}$$

Multiply (1.1)-(1.3) against $\mathbf{1}^\varepsilon$, to find

$$\partial_t(\mathbf{1}^\varepsilon(v-1)) - (\mathbf{1}^\varepsilon u)_x = -\partial_x \mathbf{1}^\varepsilon u, \tag{2.17}$$

$$\partial_t(\mathbf{1}^\varepsilon u) + \left(\mathbf{1}^\varepsilon(\frac{R\theta}{v}-1)\right)_x = \left(\mathbf{1}^\varepsilon \frac{\mu u_x}{v}\right)_x - \partial_x \mathbf{1}^\varepsilon\left(\frac{\mu u_x}{v} - (\frac{R\theta}{v}-1)\right), \tag{2.18}$$

$$\begin{aligned}
\partial_t\left(\mathbf{1}^\varepsilon(c_v(\theta-1)+\frac{1}{2}u^2)\right) + \left(\mathbf{1}^\varepsilon\frac{R\theta}{v}u\right)_x &= \left(\mathbf{1}^\varepsilon\frac{\kappa\theta_x+\mu u u_x}{v}\right)_x \\
&\quad - \partial_x\mathbf{1}^\varepsilon\left(\frac{\kappa\theta_x+\mu u u_x}{v} - \frac{R\theta}{v}u\right).
\end{aligned} \tag{2.19}$$

Set

$$(\Phi^\varepsilon, \Psi^\varepsilon, W^\varepsilon)(x,t) = \int_{-\infty}^{x} \mathbf{1}^\varepsilon(y)\left(v-1,\, u,\, c_v(\theta-1)+\frac{1}{2}u^2\right)(y,t)dy. \tag{2.20}$$

Integrating (2.17)-(2.19) over $(-\infty, x)$, we compute

$$\partial_t\Phi^\varepsilon - \Psi^\varepsilon_x = -\int_{-\infty}^{x} \partial_x\mathbf{1}^\varepsilon u dy, \tag{2.21}$$

$$\partial_t\Psi^\varepsilon - R\Phi^\varepsilon_x + \frac{R}{c_v}W^\varepsilon_x = \mu\Psi^\varepsilon_{xx} + Q^\varepsilon_1 - \int_{-\infty}^x \partial_x\mathbf{1}^\varepsilon\left(\frac{\mu u_x}{v} - (\frac{R\theta}{v}-1)\right), \tag{2.22}$$

$$\partial_t W^\varepsilon + R\Psi^\varepsilon_x = \frac{\kappa}{c_v}W^\varepsilon_{xx} + Q^\varepsilon_2 - \int_{-\infty}^x \partial_x\mathbf{1}^\varepsilon\left(\frac{\kappa\theta_x+\mu u u_x}{v} - \frac{R\theta}{v}u\right), \tag{2.23}$$

where

$$Q^\varepsilon_1 = \left(-\frac{\mu(v-1)u_x}{v} + \frac{R}{2c_v}u^2 + \frac{R(\theta-1)(v-1)}{v} - \frac{R(v-1)^2}{v}\right)\mathbf{1}^\varepsilon - \mu\partial_x\mathbf{1}^\varepsilon u$$

and

$$\begin{aligned} Q^\varepsilon_2 = &\left(-\frac{\kappa(v-1)\theta_x}{v} - \frac{\kappa}{c_v}uu_x + \frac{\mu u u_x}{v} - \frac{R(\theta-1)u}{v} + \frac{R(v-1)u}{v}\right)\mathbf{1}^\varepsilon \\ &- \frac{\kappa}{c_v}\partial_x\mathbf{1}^\varepsilon(c_v(\theta-1)+\frac{1}{2}u^2). \end{aligned}$$

Step 3. Multiply (2.21) by Φ^ε, (2.22) by $\frac{1}{R}\Psi^\varepsilon$, (2.23) by $\frac{1}{c_vR}W^\varepsilon$, respectively, to discover

$$\begin{aligned} &\partial_t\left(\frac{1}{2}(\Phi^\varepsilon)^2 + \frac{1}{2R}(\Psi^\varepsilon)^2 + \frac{1}{2c_vR}(W^\varepsilon)^2\right) + \frac{\mu}{R}(\Psi^\varepsilon_x)^2 + \frac{\kappa}{c_v^2R}(W^\varepsilon_x)^2 \\ &= \left(\Phi^\varepsilon\Psi^\varepsilon - \frac{1}{c_v}\Psi^\varepsilon W^\varepsilon + \frac{\mu}{R}\Psi^\varepsilon\Psi^\varepsilon_x + \frac{\kappa}{c_v^2R}W^\varepsilon W^\varepsilon_x\right)_x + \frac{1}{R}Q^\varepsilon_1\Psi^\varepsilon + \frac{1}{c_vR}Q^\varepsilon_2W^\varepsilon \\ &\quad - \Phi^\varepsilon\int_{-\infty}^x\partial_y\mathbf{1}^\varepsilon u - \frac{1}{R}\Psi^\varepsilon\int_{-\infty}^x\partial_y\mathbf{1}^\varepsilon\left(\frac{\mu u_x}{v} - (\frac{R\theta}{v}-1)\right) \\ &\quad - \frac{1}{c_vR}W^\varepsilon\int_{-\infty}^x\partial_y\mathbf{1}^\varepsilon\left(\frac{\kappa\theta_x+\mu uu_x}{v} - \frac{R\theta}{v}u\right). \end{aligned} \tag{2.24}$$

After multiplied by $\mathbf{1}^\varepsilon$, it gives from (2.24) that

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}}\mathbf{1}^\varepsilon\left(\frac{1}{2}(\Phi^\varepsilon)^2 + \frac{1}{2R}(\Psi^\varepsilon)^2 + \frac{1}{2c_vR}(W^\varepsilon)^2\right) + \int_{\mathbb{R}}\mathbf{1}^\varepsilon\left(\frac{\mu}{R}(\Psi^\varepsilon_x)^2 + \frac{\kappa}{c_v^2R}(W^\varepsilon_x)^2\right) \\ &= -\int_{\mathbb{R}}\left(\Phi^\varepsilon\Psi^\varepsilon - \frac{1}{c_v}\Psi^\varepsilon W^\varepsilon + \frac{\mu}{R}\Psi^\varepsilon\Psi^\varepsilon_x + \frac{\kappa}{c_v^2R}W^\varepsilon W^\varepsilon_x\right)\partial_x\mathbf{1}^\varepsilon \\ &\quad + \int_{\mathbb{R}}\mathbf{1}^\varepsilon\left(\frac{1}{R}Q^\varepsilon_1\Psi^\varepsilon + \frac{1}{c_vR}Q^\varepsilon_2W^\varepsilon\right) \\ &\quad - \int_{\mathbb{R}}\mathbf{1}^\varepsilon\Phi^\varepsilon\int_{-\infty}^x\partial_y\mathbf{1}^\varepsilon u - \frac{1}{R}\int_{\mathbb{R}}\mathbf{1}^\varepsilon\Psi^\varepsilon\int_{-\infty}^x\partial_y\mathbf{1}^\varepsilon\left(\frac{\mu u_x}{v} - (\frac{R\theta}{v}-1)\right) \\ &\quad - \frac{1}{c_vR}\int_{\mathbb{R}}\mathbf{1}^\varepsilon W^\varepsilon\int_{-\infty}^x\left(\frac{\kappa\theta_x+\mu uu_x}{v} - \frac{R\theta}{v}u\right). \end{aligned} \tag{2.25}$$

We shall estimate the terms in (2.25) as follows:

Making use of (2.14)-(2.16), (2.20), and the Cauchy inequality, we deduce

$$\begin{aligned} &-\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\left(\Phi^\varepsilon\Psi^\varepsilon - \frac{1}{c_v}\Psi^\varepsilon W^\varepsilon + \frac{\mu}{R}\Psi^\varepsilon\Psi^\varepsilon_x + \frac{\kappa}{c_v^2R}W^\varepsilon W^\varepsilon_x\right)\partial_x\mathbf{1}^\varepsilon \\ &\le C\||\Phi\Psi| + |\Psi W| + |\Psi\Psi_x| + |WW_x|\|_{L^\infty([kd,(k+1)d])} \\ &\le C\frac{1}{\sqrt{d}}\|(\Phi,\Psi,W)\|^2_{L^\infty([kd,(k+1)d])} + \sqrt{d}\|(\Psi_x,W_x)\|^2_{L^\infty([kd,(k+1)d])} \\ &\le C\sqrt{d}\|\mathbf{1}(\Phi_x,\Psi_x,W_x)\|^2_{L^2(\mathbb{R})} + \sqrt{d}\|\mathbf{1}(u_x,\theta_x)\|^2_{L^2(\mathbb{R})}, \end{aligned} \tag{2.26}$$

where, for the last two inequalities, we also used the embedding inequality

$$\|(\Psi_x, W_x)\|^2_{L^\infty([kd,(k+1)d])} \le C\|(\Psi_x, W_x)\|^2_{L^2([kd,(k+1)d])} + \|(u_x, \theta_x)\|^2_{L^2([kd,(k+1)d])},$$

which are valid owes to (1.8) and (2.13).

Similar method runs that

$$\begin{aligned}
-\frac{1}{R}\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \mathbf{1}^\varepsilon \Phi^\varepsilon \int_{-\infty}^x \partial_y \mathbf{1}^\varepsilon u &\le C\||\Phi||u|\|_{L^\infty([kd,(k+1)d])} \\
&\le C\left(\|\Phi\|^2_{L^\infty([kd,(k+1)d])} + \|u\|^2_{L^\infty([kd,(k+1)d])}\right) \\
&\le C\sqrt{d}\|\mathbf{1}(\Phi_x, \Psi_x)\|^2_{L^2(\mathbb{R})} + \sqrt{d}\|\mathbf{1}u_x\|^2_{L^2(\mathbb{R})}
\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}
&-\frac{1}{R}\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \mathbf{1}^\varepsilon \Psi^\varepsilon \int_{-\infty}^x \partial_y \mathbf{1}^\varepsilon \left((\frac{R\theta}{v} - 1)\frac{\mu u_x}{v}\right) \\
&\quad + \frac{1}{c_v R}\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \mathbf{1}^\varepsilon W^\varepsilon \int_{-\infty}^x \left(\frac{R\theta}{v}u - \frac{\kappa\theta_x}{v} - \frac{\mu u u_x}{v}\right) \\
&\le C\sqrt{d}\|(\Psi, W)\|^2_{L^\infty([kd,(k+1)d])} + \sqrt{d}\|(u, u_x, \theta_x)\|^2_{L^\infty([kd,(k+1)d])} \\
&\le C\sqrt{d}\|\mathbf{1}(\Psi_x, W_x)\|^2_{L^2(\mathbb{R})} + \sqrt{d}\|\mathbf{1}(u_x, \theta_x)\|^2_{H^1(\mathbb{R})}.
\end{aligned} \tag{2.28}$$

Using (2.14)-(2.16) and (2.20) once again, one deduces

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \mathbf{1}^\varepsilon \left(\frac{1}{R}Q_1^\varepsilon \Psi^\varepsilon + \frac{1}{c_v R}Q_2^\varepsilon W^\varepsilon\right) \\
&= \int_{\mathbb{R}} \mathbf{1}\left(\frac{1}{R}Q_1\Psi + \frac{1}{c_v R}Q_2 W\right) \\
&\quad + \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \mathbf{1}^\varepsilon \left(\frac{\mu}{R}\Psi^\varepsilon \partial_x \mathbf{1}^\varepsilon u + \frac{\kappa}{c_v^2 R}W^\varepsilon \partial_x \mathbf{1}^\varepsilon (c_v(\theta - 1) + \frac{1}{2}u^2)\right) \\
&\le \int_{\mathbb{R}} \mathbf{1}\left(\frac{1}{R}Q_1\Psi + \frac{1}{c_v R}Q_2 W\right) + \frac{C}{\sqrt{d}}\|\mathbf{1}(\Psi, W)\|^2_{L^\infty(\mathbb{R})} + \sqrt{d}\|\mathbf{1}(u, \theta - 1)\|^2_{L^\infty(\mathbb{R})} \\
&\le \int_{\mathbb{R}} \mathbf{1}\left(Q_1^2 + Q_2^2\right) + C\sqrt{d}\|\mathbf{1}(\Psi_x, W_x)\|^2_{L^2(\mathbb{R})} + \sqrt{d}\|\mathbf{1}(u_x, \theta_x)\|^2_{L^2(\mathbb{R})},
\end{aligned} \tag{2.29}$$

where

$$Q_1 = -\frac{\mu(v-1)u_x}{v} + \frac{R}{2c_v}u^2 + \frac{R(\theta - 1)(v-1)}{v} - \frac{R(v-1)^2}{v} \tag{2.30}$$

and

$$Q_2 = -\frac{\kappa(v-1)\theta_x}{v} - \frac{\mu}{c_v}u u_x + \frac{\mu u u_x}{v} - \frac{R(\theta - 1)u}{v} + \frac{R(v-1)u}{v}. \tag{2.31}$$

Therefore, if we substitute (2.26)-(2.29) into (2.25), utilize Proposition 1.1, we obtain by passing $\varepsilon \to 0$

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \left(\frac{1}{2}\Phi^2 + \frac{1}{2R}\Psi^2 + \frac{1}{2c_v R}W^2\right)\mathbf{1} + \int_{\mathbb{R}} \mathbf{1}\left(\frac{\mu}{R}\Psi_x^2 + \frac{\kappa}{c_v R}W_x^2\right) \\
&\le C\sqrt{d}\|\mathbf{1}(\Phi_x, \Psi_x, W_x)\|^2_{L^2(\mathbb{R})} + \sqrt{d}\|\mathbf{1}(u_x, \theta_x)\|^2_{H^1(\mathbb{R})} + \int_{\mathbb{R}} \mathbf{1}\left(Q_1^2 + Q_2^2\right).
\end{aligned} \tag{2.32}$$

Next to control the $\|\mathbf{1}\Phi_x\|^2_{L^2(\mathbb{R})}$ in (2.32). Owing to (2.21), we multiply (2.22) by Φ^ε_x and compute

$$
\begin{aligned}
&\partial_t\left(\frac{\mu}{2}(\Phi^\varepsilon_x)^2-\Psi^\varepsilon\Phi^\varepsilon_x\right)+R(\Phi^\varepsilon_x)^2\\
&=-\left(\Psi^\varepsilon\Psi^\varepsilon_x-\Psi^\varepsilon\int_{-\infty}^x\partial_x\mathbf{1}^\varepsilon u\right)_x+\Psi^\varepsilon_x\left(\Psi^\varepsilon_x-\int_{-\infty}^x\partial_x\mathbf{1}^\varepsilon u\right)\\
&\quad+\Phi^\varepsilon_x\left(\frac{R}{c_v}W^\varepsilon_x-\partial_x\mathbf{1}u-Q^\varepsilon_1-\int_{-\infty}^x\partial_x\mathbf{1}^\varepsilon\left((\frac{R\theta}{v}-1)+\frac{\mu u_x}{v}\right)\right),
\end{aligned}
$$

which implies after multiplied by $\mathbf{1}^\varepsilon$

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\mathbf{1}^\varepsilon\left(\frac{\mu}{2}(\Phi^\varepsilon_x)^2-\Psi^\varepsilon\Phi^\varepsilon_x\right)+R\int_{\mathbb{R}}\mathbf{1}^\varepsilon(\Phi^\varepsilon_x)^2\\
&=\int_{\mathbb{R}}\left(\Psi^\varepsilon\Psi^\varepsilon_x-\Psi^\varepsilon\int_{-\infty}^x\partial_x\mathbf{1}^\varepsilon u\right)\partial_x\mathbf{1}^\varepsilon+\int_{\mathbb{R}}\mathbf{1}^\varepsilon\Psi^\varepsilon_x\left(\Psi^\varepsilon_x-\int_{-\infty}^x\partial_x\mathbf{1}^\varepsilon u\right)\\
&\quad+\int_{\mathbb{R}}\mathbf{1}^\varepsilon\Phi^\varepsilon_x\left(\frac{R}{c_v}W^\varepsilon_x-\partial_x\mathbf{1}u-Q^\varepsilon_1-\int_{-\infty}^x\partial_x\mathbf{1}^\varepsilon\left((\frac{R\theta}{v}-1)+\frac{\mu u_x}{v}\right)\right).
\end{aligned}\tag{2.33}
$$

Thanks to (2.14)-(2.16) and (2.20), similar deduction as (2.26) and (2.29) shows

$$
\begin{aligned}
&\lim_{\varepsilon\to0}\int_{\mathbb{R}}\left(\Psi^\varepsilon\Psi^\varepsilon_x-\Psi^\varepsilon\int_{-\infty}^x\partial_x\mathbf{1}^\varepsilon u\right)\partial_x\mathbf{1}^\varepsilon\\
&\quad+\lim_{\varepsilon\to0}\int_{\mathbb{R}}\mathbf{1}^\varepsilon\Psi^\varepsilon_x\left(\Psi^\varepsilon_x-\int_{-\infty}^x\partial_x\mathbf{1}^\varepsilon u\right)-\lim_{\varepsilon\to0}\int_{\mathbb{R}}\mathbf{1}^\varepsilon\Phi^\varepsilon_x\partial_x\mathbf{1}^\varepsilon u\\
&\le C\left(\|\mathbf{1}\Psi_x\|^2_{L^2(\mathbb{R})}+\|\mathbf{1}u_x\|^2_{L^2(\mathbb{R})}\right),
\end{aligned}\tag{2.34}
$$

$$
\begin{aligned}
&\lim_{\varepsilon\to0}\int_{\mathbb{R}}\mathbf{1}^\varepsilon\Phi^\varepsilon_x\left(\frac{R}{c_v}W^\varepsilon_x-\int_{-\infty}^x\partial_x\mathbf{1}^\varepsilon\left((\frac{R\theta}{v}-1)+\frac{\mu u_x}{v}\right)\right)\\
&\le\frac{R}{4}\|\mathbf{1}\Phi_x\|^2_{L^2(\mathbb{R})}+C\left(\|\mathbf{1}(\Psi_x,W_x)\|^2_{L^2(\mathbb{R})}+\|\mathbf{1}(u_x,\theta_x)\|^2_{H^1(\mathbb{R})}\right),
\end{aligned}\tag{2.35}
$$

and

$$
\begin{aligned}
&-\lim_{\varepsilon\to0}\int_{\mathbb{R}}\mathbf{1}^\varepsilon\Phi^\varepsilon_xQ^\varepsilon_1\\
&=-\int_{\mathbb{R}}\mathbf{1}\Phi_xQ_1+\lim_{\varepsilon\to0}\int_{\mathbb{R}}\mathbf{1}^\varepsilon\Phi^\varepsilon\partial_x\mathbf{1}^\varepsilon u\\
&\le\frac{R}{4}\|\mathbf{1}\Phi_x\|^2_{L^2(\mathbb{R})}+C\|\mathbf{1}\Psi_x\|^2_{L^2(\mathbb{R})}+C\|\mathbf{1}u_x\|^2_{L^2(\mathbb{R})}+C\int_{\mathbb{R}}\mathbf{1}Q_1^2.
\end{aligned}\tag{2.36}
$$

Hence, insert inequalities (2.34)-(2.36) back into (2.33) concludes that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\mathbf{1}\left(\frac{\mu}{2}\Phi_x^2-\Psi\Phi_x\right)+\frac{R}{2}\int_{\mathbb{R}}\mathbf{1}\Phi_x^2\\
&\le C\|\mathbf{1}(\Psi_x,W_x)\|^2_{L^2(\mathbb{R})}+C\|\mathbf{1}(u_x,\theta_x)\|^2_{H^1(\mathbb{R})}+C\int_{\mathbb{R}}\mathbf{1}Q_1^2,
\end{aligned}\tag{2.37}
$$

by sending $\varepsilon\to0$.

In summary, select the constant $\lambda > 0$ large and then d small, (2.32) $\times \lambda +$ (2.37) guarantees that, for some constant C,

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \mathbf{1}\left\{\lambda\left(\frac{1}{2}\Phi^2 + \frac{1}{2R}\Psi^2 + \frac{1}{2c_vR}W^2\right) + \frac{\mu}{2}\Phi_x^2 - \Psi\Phi_x\right\} \\
&\quad + C^{-1}\int_{\mathbb{R}} \mathbf{1}\left(\Phi_x^2 + \Psi_x^2 + W_x^2\right) \\
&\leq C\|\mathbf{1}(u_x, \theta_x)\|_{H^1(\mathbb{R})}^2 + C\int_{\mathbb{R}} \mathbf{1}\left(Q_1^2 + Q_1^2\right).
\end{aligned} \tag{2.38}$$

Observe that, if λ is taken large enough,

$$\int_{\mathbb{R}} \mathbf{1}\left\{\lambda\left(\frac{1}{2}\Phi^2 + \frac{1}{2R}\Psi^2 + \frac{1}{2c_vR}W^2\right) + \frac{\mu}{2}\Phi_x^2 - \Psi\Phi_x\right\} \geq C\|\mathbf{1}(\Phi, \Psi, W, \Phi_x)\|_{L^2(\mathbb{R})}^2.$$

By this, we integrate (2.38) and receive

$$\begin{aligned}
&\|\mathbf{1}(\Phi, \Psi, W, \Phi_x)(\cdot, t)\|_{L^2(\mathbb{R})}^2 + \int_0^t \|\mathbf{1}(\Phi_x, \Psi_x, W_x)\|_{L^2(\mathbb{R})}^2 \\
&\leq C\|\mathbf{1}(\Phi, \Psi, W, \Phi_x)(\cdot, 0)\|_{L^2(\mathbb{R})}^2 + C\int_0^t \|\mathbf{1}(u_x, \theta_x)\|_{H^1(\mathbb{R})}^2 + C\int_0^t\int_{\mathbb{R}} \mathbf{1}\left(Q_1^2 + Q_1^2\right).
\end{aligned} \tag{2.39}$$

Observe from (2.13)-(2.15) that

$$\begin{aligned}
&\|\mathbf{1}(\Phi, \Psi, W, \Phi_x)(\cdot, 0)\|_{L^2(\mathbb{R})}^2 \\
&\leq C\|\mathbf{1}(\Phi, \Psi, W)(\cdot, 0)\|_{L^2(\mathbb{R})}^2 + C\|\mathbf{1}(v_0 - 1)\|_{L^2(\mathbb{R})}^2 \\
&\leq C\|\mathbf{1}(\Phi_x, \Psi_x, W_x)(\cdot, 0)\|_{L^2(\mathbb{R})}^2 + C\|\mathbf{1}(v_0 - 1)\|_{L^2(\mathbb{R})}^2 \\
&\leq C\|\mathbf{1}(v_0 - 1, u_0, \theta_0 - 1)\|_{H^1(\mathbb{R})}^2,
\end{aligned} \tag{2.40}$$

and from (1.8), (2.13), (2.30)-(2.31) that

$$\begin{aligned}
&C\int_0^t\int_{\mathbb{R}} \mathbf{1}\left(Q_1^2 + Q_1^2\right) \\
&\leq C\int_0^t \|(v-1, u, \theta-1)\|_{L^\infty(\mathbb{R})}^2 \int_{\mathbb{R}} \mathbf{1}\left((v-1)^2 + u^2 + (\theta-1)^2 + u_x^2 + \theta_x^2\right) \\
&\leq C\left(\int_0^{t_0} + \int_{t_0}^t\right) \|(v-1, u, \theta-1)\|_{L^\infty(\mathbb{R})}^2 \int_{\mathbb{R}} \mathbf{1}\left((v-1)^2 + u^2 + (\theta-1)^2\right) \\
&\quad + C\int_0^t \|(v-1, u, \theta-1)\|_{L^\infty(\mathbb{R})}^2 \int_{\mathbb{R}} \mathbf{1}\left(u_x^2 + \theta_x^2\right) \\
&\leq C\int_0^{t_0}\int_{\mathbb{R}} \mathbf{1}\left((v-1)^2 + u^2 + (\theta-1)^2\right) \\
&\quad + C\int_{t_0}^t \|(v-1, u, \theta-1)\|_{L^\infty(\mathbb{R})}^2 \int_{\mathbb{R}} \mathbf{1}\left(\Phi_x^2 + \Psi_x^2 + W_x^2\right) \\
&\quad + C\int_0^t\int_{\mathbb{R}} \mathbf{1}\left(u_x^2 + \theta_x^2\right).
\end{aligned} \tag{2.41}$$

Remember that (1.10), there is a large point t_0 such that

$$C\|(v-1,u,\theta-1)(\cdot,s)\|^2_{L^\infty(\mathbb{R})}\le\frac{1}{2},\quad \forall\ s\in[t_0,\infty). \tag{2.42}$$

With the aid of (2.40)-(2.42), we estimate (2.39) as

$$\begin{aligned}&\int_0^t\|\mathbf{1}(\Phi_x,\Psi_x,W_x)\|^2_{L^2(\mathbb{R})}\\&\le C\|\mathbf{1}(v_0-1,u_0,\theta_0-1)\|^2_{H^1(\mathbb{R})}+C\int_0^{t_0}\|\mathbf{1}(v-1,u,\theta-1)\|^2_{L^2(\mathbb{R})}\\&\quad+C\int_0^t\|\mathbf{1}(u_x,\theta_x)\|^2_{H^1(\mathbb{R})}.\end{aligned} \tag{2.43}$$

Step 4. Recalling the definition of $\mathbf{1}(x)=\mathbf{1}_{[kd,(d+1)d]}(x)$, we repeat the deduction of (2.43) for every $k=0,\pm1,\pm2,...$, sum them up, we infer

$$\begin{aligned}&\int_0^t\|(v-1,u,c_v(\theta-1)+\frac{1}{2}u^2)\|^2_{L^2(\mathbb{R})}\\&=\int_0^t\sum_{k=-\infty}^{+\infty}\|\mathbf{1}(v-1,u,c_v(\theta-1)+\frac{1}{2}u^2)\|^2_{L^2(\mathbb{R})}\\&=\int_0^t\sum_{k=-\infty}^{+\infty}\|\mathbf{1}(\Phi_x,\Psi_x,W_x)\|^2_{L^2(\mathbb{R})}\\&\le C\sum_{k=-\infty}^{+\infty}\|\mathbf{1}(v_0-1,u_0,\theta_0-1)\|^2_{H^1(\mathbb{R})}+C\int_0^{t_0}\sum_{k=-\infty}^{+\infty}\|\mathbf{1}(v-1,u,\theta-1)\|^2_{L^2(\mathbb{R})}\\&\quad+C\int_0^t\sum_{k=-\infty}^{+\infty}\|\mathbf{1}(u_x,\theta_x)\|^2_{H^1(\mathbb{R})}\\&=C\|(v_0-1,u_0,\theta_0-1)\|^2_{H^1(\mathbb{R})}+C\max_{0\le s\le t_0}\|(v-1,u,\theta-1)(\cdot,s)\|^2_{L^2(\mathbb{R})}\\&\quad+C\int_0^t\|(u_x,\theta_x)\|^2_{H^1(\mathbb{R})}\\&\le C,\end{aligned} \tag{2.44}$$

where the last inequality is valid owes to Proposition 1.1.

For another hand, inequalities (2.44) and (1.8) implies

$$\int_0^t\|(v-1,u,c_v(\theta-1)+\frac{1}{2}u^2)\|^2_{L^2(\mathbb{R})}\ge\int_0^t\|(v-1,u,\theta-1)\|^2_{L^2(\mathbb{R})}-C. \tag{2.45}$$

The combination of (2.45) with (2.44) yields directly the required (2.12). SThis is the end of the proof of Claim 2.1.

3 Proof of Theorem 1.2

3.1 The linearized equations of anti-derivatives

Without causing confusion with (2.13), we still denote by

$$(\Phi, \Psi, W)(x,t) = \int_{-\infty}^{x} \left(v-1,\, u,\, c_v(\theta-1) + \frac{1}{2}u^2 \right)(y,t)dy. \tag{3.1}$$

Hence, integrating (1.1)-(1.3) over $(-\infty, x)$ yields

$$\partial_t \mathbb{U} + \mathbb{A}\mathbb{U} = \mathbb{F}, \tag{3.2}$$

where

$$\mathbb{U} = \begin{pmatrix} \Phi \\ \Psi \\ W \end{pmatrix}, \quad \mathbb{A} = \begin{pmatrix} 0 & -\partial_x & 0 \\ -R\partial_x & -\mu\partial_{xx} & \frac{R}{c_v}\partial_x \\ 0 & R\partial_x & -\frac{\kappa}{c_v}\partial_{xx} \end{pmatrix}, \quad \mathbb{F} = \begin{pmatrix} 0 \\ Q_1 \\ Q_2 \end{pmatrix}, \tag{3.3}$$

in which, Q_1 and Q_2 are taken from (2.30) and (2.31) respectively.

3.2 Fourier transform for linearized equations

Consider the homogeneous problem of equations (3.2):

$$\partial_t \mathbb{U} + \mathbb{A}\mathbb{U} = 0, \quad \mathbb{U}(x,0) = (\Phi_0, \Psi_0, W_0)^{tr}. \tag{3.4}$$

Denote by $\widehat{f}$ the Fourier transform of f and by $\vee$ the inverse. Taking Fourier transform of (3.4) in x variable to receive the ordinary differential equations in t variable, i.e.,

$$\partial_t \widehat{\mathbb{U}} + \widehat{\mathbb{A}}\widehat{\mathbb{U}} = 0, \tag{3.5}$$

where the coefficient matrix

$$\widehat{\mathbb{A}} = \begin{pmatrix} 0 & -i\xi & 0 \\ -iR\xi & \mu\xi^2 & i(\gamma-1)\xi \\ 0 & iR\xi & \frac{\kappa(\gamma-1)}{R}\xi^2 \end{pmatrix}. \tag{3.6}$$

Direct calculation shows

$$|\lambda\mathbb{I} + \widehat{\mathbb{A}}| = \lambda^3 + \left(\frac{\kappa(\gamma-1)}{R} + \mu \right)\xi^2\lambda^2 + \left(\frac{\mu\kappa(\gamma-1)}{R}\xi^2 + R\gamma \right)\xi^2\lambda + \kappa(\gamma-1)\xi^4. \tag{3.7}$$

We have the following assertions:

- Equation (3.7) has three real roots for large ξ. In particular, if $\mu \neq \frac{\kappa(\gamma-1)}{R}$,

  $$\lambda_1 = -\mu\xi^2 + O(1), \quad \lambda_2 = -\frac{\kappa(\gamma-1)}{R}\xi^2 + O(1), \quad \lambda_3 = -\frac{R}{\mu} + O(\xi^{-2}),$$

  while if $\mu = \frac{\kappa(\gamma-1)}{R}$,

  $$\lambda_1 = \lambda_2 = -\mu\xi^2 + O(\xi), \quad \lambda_3 = -\frac{R}{\mu} + O(\xi^{-2}).$$

- Equation (3.7) has one real root and a pair complex roots for small ξ.

$$\lambda_1=\overline{\lambda_2}=i\sqrt{R\gamma}|\xi|-\frac{R\gamma\mu+\kappa(\gamma-1)^2}{2R\gamma}\xi^2+O(\xi^3),\quad \lambda_3=-\frac{\kappa(\gamma-1)}{R\gamma}\xi^2+O(\xi^4),$$

with overline " $-$ " stands for the complex conjugate.

- Equation (3.7) has a triple root for at most two points of ξ.

Let the matrix $\mathbb{P}(\xi)=(P_1,P_2,P_3)$ satisfy $\mathbb{P}^{-1}(\xi)\widehat{\mathbb{A}}(\xi)\mathbb{P}(\xi)=\mathbb{J}(\xi)=\oplus_i J_i(\lambda_i)$, where $\mathbb{J}(\xi)$ is the Jordan matrix. Then, the solution takes

$$\widehat{\mathbb{U}}(\xi,t)=e^{-t\widehat{\mathbb{A}}}\widehat{\mathbb{U}}(\xi,0),$$

where $e^{-t\widehat{\mathbb{A}}(\xi)}=\mathbb{P}(\xi)e^{-t\mathbb{J}(\xi)}\mathbb{P}^{-1}(\xi)$ is the solution semigroup generated by $\widehat{\mathbb{A}}$ (cf.[16]). More precisely, $\widehat{\mathbb{U}}=(\widehat{\Phi},\widehat{\Psi},\widehat{W})$ has the explicit form:

$$\begin{aligned}
\widehat{\Phi}=&\sum_{k=1}^{3}e^{t\lambda_k}\frac{(\lambda_k+\mu\xi^2)(\lambda_k+\kappa(\gamma-1)R^{-1}\xi^2)+R(\gamma-1)\xi^2}{(\lambda_k-\lambda_l)(\lambda_k-\lambda_j)}\widehat{\Phi_0}\\
&+\sum_{k=1}^{3}e^{t\lambda_k}\frac{i(\lambda_k+\kappa(\gamma-1)R^{-1}\xi^2)\xi}{(\lambda_k-\lambda_l)(\lambda_k-\lambda_j)}\widehat{\Psi_0}+\sum_{k=1}^{3}e^{t\lambda_k}\frac{(\gamma-1)\xi^2}{(\lambda_k-\lambda_l)(\lambda_k-\lambda_j)}\widehat{W_0},\\
\widehat{\Psi}=&\sum_{k=1}^{3}e^{t\lambda_k}\frac{iR(\lambda_k+\kappa(\gamma-1)R^{-1}\xi^2)\xi}{(\lambda_k-\lambda_l)(\lambda_k-\lambda_j)}\widehat{\Phi_0}+\sum_{k=1}^{3}e^{t\lambda_k}\frac{\lambda_k(\lambda_k+\kappa(\gamma-1)R^{-1}\xi^2)}{(\lambda_k-\lambda_l)(\lambda_k-\lambda_j)}\widehat{\Psi_0}\\
&+\sum_{k=1}^{3}e^{t\lambda_k}\frac{-i\lambda_k(\gamma-1)\xi}{(\lambda_k-\lambda_l)(\lambda_k-\lambda_j)}\widehat{W_0},\\
\widehat{W}=&\sum_{k=1}^{3}e^{t\lambda_k}\frac{R^2\xi^2}{(\lambda_k-\lambda_l)(\lambda_k-\lambda_j)}\widehat{\Phi_0}+\sum_{k=1}^{3}e^{t\lambda_k}\frac{-i\lambda_k R\xi}{(\lambda_k-\lambda_l)(\lambda_k-\lambda_j)}\widehat{\Psi_0}\\
&+\sum_{k=1}^{3}e^{t\lambda_k}\frac{\lambda_k(\lambda_k+\mu\xi^2)+R\xi^2}{(\lambda_k-\lambda_l)(\lambda_k-\lambda_j)}\widehat{W_0}.
\end{aligned}$$

Having the above information in hand, we follow the same (but simpler) argument as that in [17, Theorem 3.1] and conclude

Lemma 3.1 *Under the same assumptions in Theorem 1.2, we have*

$$\left\|\partial_x^m\left[e^{-t\widehat{\mathbb{A}}}\widehat{\mathbb{U}}(\xi,0)\right]^{\vee}(\cdot,t)\right\|_{L^p(\mathbb{R})}\le C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{m}{2}}\|\mathbb{U}(\cdot,0)\|_{L^q(\mathbb{R})},\tag{3.8}$$

where $1\le q\le 2\le p\le\infty$ *and integer* $m\ge 0$.

Remark 3.1 *In case of half-line* $\mathbb{R}_+=[0,+\infty)$, *we obtain Lemma 3.1 by using the cut-off technique. See [17, Theorem1.3].*

Utilizing the Duhamel's principle, the solution of (3.2) takes the form

$$\mathbb{U}(x,t) = \left[e^{-t\widehat{\mathbb{A}}}\widehat{\mathbb{U}}(\xi,0)\right]^{\vee} + \int_0^t \left[e^{-(t-s)\widehat{\mathbb{A}}}\widehat{\mathbb{F}}(\xi,s)\right]^{\vee} ds.$$

This, along with Lemma 3.1, implies

$$\|\partial_x^2\mathbb{U}(\cdot,t)\|_{L^2(\mathbb{R})} \le C(1+t)^{-1}\|\mathbb{U}(\cdot,0)\|_{L^2(\mathbb{R})} + C\int_0^t (1+t-s)^{-\frac{5}{4}}\|\mathbb{F}(\cdot,s)\|_{L^1(\mathbb{R})}ds. \tag{3.9}$$

3.3 Proof of inequality (1.12)

By (2.30)-(2.31) and Proposition 1.1, it gives from (3.3) that

$$\|\mathbb{F}\|_{L^1(\mathbb{R})} \le C\left(\|(v-1,u,\theta-1)\|^2_{L^2(\mathbb{R})} + \|(v_x,u_x,\theta_x)\|^2_{L^2(\mathbb{R})}\right). \tag{3.10}$$

Inequality (3.10), along with (1.11), ensures that

$$\begin{aligned}\int_0^t (1+t-s)^{-\frac{5}{4}}\|\mathbb{F}(\cdot,s)\|_{L^1(\mathbb{R})}ds &\le C\int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-1}ds \\ &= C\left(\int_0^{t/2} + \int_{t/2}^t\right)(1+t-s)^{-\frac{5}{4}}(1+s)^{-1}ds \\ &\le C(1+t)^{-1}.\end{aligned} \tag{3.11}$$

Recalling (3.1) and (1.13), we use (3.11) and (3.9) to deduce

$$\begin{aligned}\|(v_x,u_x,\theta_x)(\cdot,t)\|_{L^2(\mathbb{R})} &\le C\|\partial_x^2\mathbb{U}(\cdot,t)\|_{L^2(\mathbb{R})} \\ &\le C(1+t)^{-1}\|\mathbb{U}(\cdot,0)\|_{L^2(\mathbb{R})} + C\int_0^t (1+t-s)^{-\frac{5}{4}}\|\mathbb{F}(\cdot,s)\|_{L^1(\mathbb{R})}ds \\ &\le C(1+t)^{-1}.\end{aligned}$$

This together with the embedding theorem give birth to (1.12), the required.

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