NUMERICAL SOLUTION FOR TIME-FRACTIONAL MURRAY REACTION-DIFFUSION EQUATIONS VIA REDUCED DIFFERENTIAL TRANSFORM METHOD

Serkan Okur[1], Muhammed Yiğider [2]

Abstract

In this study, solutions of time-fractional differential equations that emerge from science and engineering have been investigated by employing reduced differential transform method. Initially, the definition of the derivatives with fractional order and their important features are given. Afterwards, by employing the Caputo derivative, reduced differential transform method has been introduced. Finally, the numerical solutions of the fractional order Murray equation have been obtained by utilizing reduced differential transform method and results have been compared through graphs and tables.

Keywords: Time-fractional differential equations, Reduced differential transform methods, Murray equations, Caputo fractional derivative.

1. Introduction

The mathematical modelling is highly important for defining and creating solutions for problems, which are encountered in science and engineering. As we know that there is no exact solution for most partial differential equations. The fact that these differential equations have fractional derivatives and non-linearity makes it difficult to produce analytical solutions. At this juncture, numerical methods come to the aid. Firstly, Zhou used the differential transformation method for solving linear and nonlinear initial value problems in electrical circuit analysis [1]. In 1980, Adomian introduced some definitions and theorems about the Adomian decomposition method which he named after himself, also he demonstrated how to apply this method to some differential equations [2]. Chen and Ho used the transform method to obtain eigen-values and eigen-functions [3]. He, introduced a new analytical solution called the method of variational iterations for the solution of nonlinear problems. Initial value problems are solved with the help of Lagrange multiplier in variational theory via this method. This method, converges faster to the analytical solutions compared to the Adomian method [4]. Chen and Ho introduced the novel two-dimensional

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differential transform method to solve Partial Differential Equations (PDE) and for the first-time differential transform method was applied to partial differential equations [5]. Two-dimensional differential transform method for solving the initial value problems for partial differential equations have been studied by Ayaz. Novel theorems have been introduced and some linear and non-linear PDEs solved by employing this method [6]. The RDTM, which was first proposed by the Keskin [7], has received much attention due to its applications to solve a wide variety of problems. In this study, RDTM and HPM are successfully applied to the fractional Benney-Lin equation and solutions are obtained [8]. In the Srivastava's study, the solution of two and three dimensional time-fractional telegraph equation with RDTM is presented. As a result, it has been observed that RDTM technique is efficient and, by this method, the numerical solution quickly convergences to the analytical solution [9].

In this study, the solution of time-fractional differential Murray equation is tackled. The Burgers' equation, which is a member of the reaction-diffusion equation is defined as:

$$u_t = u_{xx} + \lambda_1 u u_x$$

The Burgers' equation has many applications in applied mathematics, modeling fluid dynamics, modeling of gas dynamics, boundary layer behavior, turbulence and shock wave formation [10]. Fisher's equation, another member of the reaction diffusion equation, is defined as:

$$u_t = u_{xx} + \lambda_2 u - \lambda_3 u^2.$$

It was introduced by Fisher to describe the dynamics of the spread of a mutant gene. The Fisher equation has wide applications in many fields. They explain the spread of biological populations, branching Brownian motion processes, logistic population growth, neurophysiology, flame propagation, neutron population in a nuclear reaction, chemical kinetics, autocatalytic chemical reactions and nuclear reactor theory. Murray equation is the generalized form of the Fisher and Burgers equations. Nonlinear Reaction-Diffusion Equations are very important due to their ease of use in various fields in Engineering and Science [10]. The Murray equation, which is a member of the reaction-diffusion equation family, is as follows;

$$u_t = u_{xx} + \lambda_1 u u_x + \lambda_2 u - \lambda_3 u^2, 0 \le x < 1, 0 \le t < 1 \quad [11,12].$$

Reaction-Diffusion equations have a wide range of applications in science and engineering and have gained attention in recent years due to their interesting properties and rich variety of solutions. The fractional version of Murray equation define as

$$D_t^{\alpha} u = u_{xx} + \lambda_1 u u_x + \lambda_2 u - \lambda_3 u^2, 0 \le x < 1, 0 \le t < 1$$

In this study, we apply reduced differential transform method to obtain the numerical solution of fractional Murray differential equation.

The paper is organized as follows; in section 2, we present some essential definitions of the fractional calculus theory. In section 3, we demonstrate the definition and some features of fractional reduced differential transform method. Also we define the time-fractional Murray equation with initial condition. In section 4, we illustrate a numerical example and we apply the numerical method to solve the test problem. The tables and graphs are created for this example. Finally, in section 5 the data obtained in the previous sections are compared and comments are made on the method.

2. Basic definitions

In this section, some notations and definitions, which are necessary for the solution of the problem, are given. Fractional analysis theory is as old as classical analysis theory and interest in fractional analysis has increased in the last two decades. In addition, more than one definition has emerged. In this study, Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative and Caputo derivative are given.

Definition 2.1. [13] A real valued function $f(x)$, $x>0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p>\mu$, such that $f(x)=x^p f_1(x)$ where $f_1(x) \in C[0,\infty)$ and a function $f(x)$, $x>0$ is said to be in the space C_μ^m, $m \in N \cup \{0\}$, if $f^{(m)} \in C_\mu$.

$f_1(x) \in C$¿ olmak üzere, $f(x)=x^p f_1(x)$ olacak şekilde $p>\mu$, $\mu \in R$ reel sayısı varsa $f(x)$, $x>0$ reel fonksiyonu C_α uzayındadır denir ve $f^{(m)} \in C_\mu$ ise, bu takdirde $f(x)$, $x>0$ fonksiyonu C_μ^muzayındadır denir.

Definition 2.2 The Riemann-Liouville fractional integral operator [14] of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$ is defined as

$$\left(J_a^\alpha f\right)(x)=\frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau \,.\, x>a, x>0, \tag{2.1}$$

The properties of the operator J^α can be found in [15], and here we only mention the following (in case, $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\nu>-1$):

$$\left(J_a^0 f\right)(x)=f(x) \tag{2.2}$$

$$\left(J_a^{\alpha} J_a^{\beta} f\right)(x)=\left(J_a^{\alpha+\beta} f\right)(x)$$
(2.3)

$$J_a^{\alpha} x^{v}=\frac{\Gamma(v+1)}{\Gamma(\alpha+v+1)} x^{\alpha+v}$$
(2.4)

where $\alpha, \beta \geq 0, x>0$ and $v>-1$.

Definition 2.3 Let the function f be continuous and integrable in every finite (a, x) range. Let $m \in N, m-1<\alpha \leq m$ and $x>a, a \in R$. Therefore, the Riemann-Liouville fractional derivative [15] of the function f is defined as

$$D^{\alpha} f(x)=D^{m} J^{m-\alpha} f(x)$$
(2.5)

$$\left(D_a^{\alpha} f\right)(x)=\frac{d^m}{d x^m}\left[\frac{1}{\Gamma(m-\alpha)} \int_a^x (x-\tau)^{m-\alpha-1} f(\tau)\, d\tau\right],$$
(2.6)

Definition 2.4 The fractional derivative of $f(x)$ in the Caputo sense [16,17] is defined as

$$D^{\alpha} f(x)=J^{m-\alpha} D^{m} f(x)$$
(2.7)

$$\left(D_a^{\alpha} f\right)(x)=\frac{1}{\Gamma(m-\alpha)} \int_a^x (x-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau,$$
(2.8)

for $m-1<\alpha \leq m$, $m \in N$, $x>0$, $f \in C_{-1}^{m}$.

The following two properties of this operator will be used in what follows.

Lemma If $m-1<\alpha\leq m, m\in N$ and $f\in C_{\mu}^{m}$, $\mu\geq-1$, then

a) $$\left(D_a^{\alpha} J_a^{\alpha} f\right)(x)=f(x)$$

(2.9)

b) $$\left(J_a^{\alpha} D_a^{\alpha} f\right)(x)=f(x)-\sum_{k=0}^{m-1} f^{(k)}(a)\frac{(x-a)^k}{k!}, a\geq 0 \tag{2.10}$$

The Caputo fractional derivative is considered here, because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

3. 2-dimensional fractional reduced differential transform method

In this section, reduced differential transform method will be given for the solution of fractional partial differential equations. By using the reduced differential transform method, we can decrease the processes density of the generalized differential transform method. Therefore, it will be possible to obtain solutions of fractional differential equations quickly.

Let us consider a function of two individual variables $u(x,t)$, and suppose that it can be represented as a product of two single-variable functions, i.e., $u(x,t)=\phi(x)\psi(t)$. On the basis of the properties of the one-dimensional differential transformation , the function $u(x,t)$ can be represented as [18]

$$u(x,t)=\sum_{k=0}^{\infty}\Phi(k)x^k\sum_{h=0}^{\infty}\Psi_{\alpha}(h)t^{\alpha h}=\sum_{h=0}^{\infty}U_{\alpha h}(x)t^{\alpha h}$$

Where $0<\alpha\leq 1$, $U_{\alpha h}(x)=\Psi_{\alpha}(h)\Phi(k)$ is called spectrum function of $u(x,t)$. The basic definitions and operations of the reduced differential transform are introduced as follows

Definition 3.1 [19,20] Let $u(x,t)$ be an analytic function that continuously differentiable with respect to time t and space x in domain of interest . Define

$$U_h(x)=\frac{1}{\Gamma(\alpha h+1)}\left[D_t^{\alpha h}u(x,t)\right]_{t=t_0}$$

(3.1)

where α is the parameter that describes the order of time fractional derivative in the Caputo sense and t-dimensional spectrum function $U_h(x)$ is defined as

$$u(x,t)=\sum_{h=0}^{\infty} U_h(x)(t-t_0)^{\alpha h} \tag{3.2}$$

Combining equation (3.1) and (3.2), we have

$$u(x,t)=\sum_{h=0}^{\infty} \frac{1}{\Gamma(\alpha h+1)}\left[D_t^{\alpha h} u(x,t)\right]_{t=t_0}(t-t_0)^{\alpha h}.$$

(3.3)

When $t_0=0$, Eqs. (3.3) reduces to

$$u(x,t)=\sum_{h=0}^{\infty} \frac{1}{\Gamma(\alpha h+1)}\left[D_t^{\alpha h} u(x,t)\right]_{t=t_0} t^{\alpha h}$$

From the above definition, it can be found that the concept of the reduced differential transform is derived from the power series expansion of a function.

Table 1: Reduced fractional differential Transformations [7,21]

Original function	**Transformed function**
$u(x,t)$	$U_h(x)=\frac{1}{\Gamma(1+h\alpha)}\left[\frac{\partial^{ha}}{\partial t^{ha}} u(x,t)\right]_{t=0}$
$u(x,t)=l_1 w(x,t) \pm l_2 v(x,t)$	$U_k(x)=l_1 W_h(x) \pm l_2 V_h(x)\, l_1, l_2 \in R$
$u(x,t)=cw(x,t)(c\in R)$	$U_h(x)=c\,W_h(x)(c\in R)$

$u(x,t)=\frac{\partial^r}{\partial x^r}w(x,t)$	$U_h(x)=\frac{\partial^r}{\partial x^r}W_h(x)$
$u(x,t)=\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}w(x,t)$	$U_h(x)=\frac{\Gamma(h\alpha+N\alpha+1)}{\Gamma(h\alpha+1)}W_{h+N}(x)$
$u(x,t)=w(x,t)v(x,t)$	$U_h(x)=\sum_{s=0}^{h}V_s(x)W_{h-s}(x)$
$\psi(x,t)=u(x,t)w(x,t)v(x,t)$	$\Psi_k=\sum_{r=0}^{k}\sum_{i=0}^{r}U_i(x)V_{r-i}(x)W_{k-r}(x)$

We illustrate the reduced differential transform method by employing the fractional Murray equation in standard form,

$$L(u(x,t))+R(u(x,t))+N(u(x,t))+F(u(x,t))=0 \tag{3.4}$$

with initial condition

$$u(x,0)=f(x)$$
(3.5)

where $L=\frac{\partial^\alpha}{\partial t^\alpha}, R=\frac{\partial^2}{\partial x^2}, F=u$ and $N=u\frac{\partial u}{\partial x}$ are the linear operators which has partial derivatives.

According to RDTM formulas in Table 1, we can derive the following iteration formulas;

$$\frac{\Gamma(\alpha(h+N)+1)}{\Gamma(\alpha h+1)}U_{h+1}(x)=-N(U_h(x))-R(U_h(x))-U_h(x)$$
(3.6)

where $N(U_h(x))$, $R(U_h(x))$ and $U_h(x)$ are the transformations of $N(u(x,t)),R(u(x,t))$ and $F(u(x,t))$, respectively. From the initial condition, we write

$$U_0(x)=f(x)$$
(3.7)

Substituting eqs. (3.7) into (3.6) and by a straight forward iterative formula, we get the following $U_h(x)$ values. Then, we apply the inverse transformation to all the values $\{U_h(x)\}_{h=0}^{n}$ to obtain the approximation solution as

$$\tilde{u}_n(x,t)=\sum_{h=0}^{n} U_h(x)t^{\alpha h} \tag{3.8}$$

where n is order of approximation solution. Thus, the exact solution of the problem is obtained by

$$u(x,t)=\lim_{n\to\infty}\tilde{u}_n(x,t).$$

4. Numerical example

Example The Murray equation with initial condition and analytical solution is given by

$$D_t^{\alpha}u=u_{xx}+\lambda_1 u u_x+\lambda_2 u-\lambda_3 u^2, 0\le x<1, 0\le t<1 \tag{4.1}$$

$$u(x,0)=\frac{1}{2}\left(1+\tanh\left(\frac{x}{4}\right)\right)$$

(4.2)

$$u(x,t)=\frac{\lambda_2}{2\lambda_3}\left(1+\tanh\left[\frac{\lambda_2}{8\lambda_3^2}\left(2\lambda_1\lambda_3 x+\left(\lambda_1^2+4\lambda_3^2\right)t\right)\right]\right).[10]$$

(4.3)

If $\lambda_1=1, \lambda_2=1$ ve $\lambda_3=1$ ($\lambda_1, \lambda_2, \lambda_3 \in R$) in equation (4.1), the time-fractional partial differential equation turns into

$$D_t^{\alpha}u=u_{xx}+u u_x+u-u^2, 0\le x<1, 0\le t<1 \tag{4.4}$$

Using the initial condition at (4.2), we apply the reduced differential transform method to (4.4) Murray equation and obtained:

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}U_{h+1}(x)=\frac{\partial^2}{\partial x^2}U_h(x)+\sum_{s=0}^{h}U_s(x)\frac{\partial}{\partial x}U_{h-s}(x)+U_h(x)-\sum_{s=0}^{h}U_s(x)U_{h-s}(x)$$
(4.5)

If we iterate for $h=0,1,2,3\ldots$;

$$U_0(x)=\frac{1}{2}\left(1+\tanh\left(\frac{x}{4}\right)\right)$$

$$U_1=\frac{4+sech^2\left(\frac{x}{4}\right)-4\tanh^2\left(\frac{x}{4}\right)}{16\Gamma(\alpha+1)}$$

$$U_2=\frac{1}{64\Gamma(2\alpha+1)}¿$$

$$-2\,sech^2\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)+16\tanh^3\left(\frac{x}{4}\right)¿$$

$$U_3=\frac{-1}{512\Gamma(3\alpha+1)\Gamma^2(\alpha+1)}¿$$

$$+16\Gamma(2\alpha+1)sech^2\left(\frac{x}{4}\right)+9\Gamma^2(\alpha+1)sech^4\left(\frac{x}{4}\right)+2\Gamma(2\alpha+1)sech^4\left(\frac{x}{4}\right)$$

$$+36\Gamma^2(\alpha+1)sech^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)+20\Gamma(2\alpha+1)sech^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$-86\Gamma^2(\alpha+1)sech^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)+5\Gamma(2\alpha+1)sech^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$-128\Gamma^2(\alpha+1)\tanh^2\left(\frac{x}{4}\right)-64\Gamma(2\alpha+1)\tanh^2\left(\frac{x}{4}\right)$$

$$-138\Gamma^2(\alpha+1)sech^2\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)-16\Gamma(2\alpha+1)sech^2\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)$$

$$-36\,\Gamma^2(\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^3\left(\frac{x}{4}\right)-20\,\Gamma(2\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^3\left(\frac{x}{4}\right)$$

$$+128\,\Gamma^2(\alpha+1)\tanh^4\left(\frac{x}{4}\right)+32\,\Gamma(2\alpha+1)\tanh^4\left(\frac{x}{4}\right)¿$$

$$U_4=\frac{-1}{4096\,\Gamma(4\alpha+1)\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)}¿$$

$$+128\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^2\left(\frac{x}{4}\right)-92\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^4\left(\frac{x}{4}\right)$$

$$-44\,\Gamma^2(2\alpha+1)\,sech^4\left(\frac{x}{4}\right)+4\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^4\left(\frac{x}{4}\right)$$

$$-233\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^6\left(\frac{x}{4}\right)-13\,\Gamma^2(\alpha+1)\,sech^6\left(\frac{x}{4}\right)$$

$$-7\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^6\left(\frac{x}{4}\right)-256\,\Gamma^2(2\alpha+1)\tanh\left(\frac{x}{4}\right)$$

$$-512\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\tanh\left(\frac{x}{4}\right)-416\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$-288\,\Gamma^2(2\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)-400\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$-456\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)-96\,\Gamma^2(2\alpha+1)\,sech^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$-64\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$+322\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^6\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)-85\,\Gamma^2(2\alpha+1)\,sech^6\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$-24\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^6\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$-488\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)$$

$$-264\,\Gamma^{2}(2\alpha+1)\,sech^{2}\left(\frac{x}{4}\right)\tanh^{2}\left(\frac{x}{4}\right)-616\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^{2}\left(\frac{x}{4}\right)\tanh^{2}\left(\frac{x}{4}\right)$$

$$+2280\,\Gamma(2\alpha+1)\,\Gamma^{2}(\alpha+1)\,sech^{4}\left(\frac{x}{4}\right)\tanh^{2}\left(\frac{x}{4}\right)$$

$$+2280\,\Gamma(2\alpha+1)\,\Gamma^{2}(\alpha+1)\,sech^{4}\left(\frac{x}{4}\right)\tanh^{2}\left(\frac{x}{4}\right)$$

$$+144\,\Gamma^{2}(2\alpha+1)\,sech^{4}\left(\frac{x}{4}\right)\tanh^{2}\left(\frac{x}{4}\right)-144\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^{4}\left(\frac{x}{4}\right)\tanh^{2}\left(\frac{x}{4}\right)$$

$$+1024\,\Gamma(2\alpha+1)\,\Gamma^{2}(\alpha+1)\tanh^{3}\left(\frac{x}{4}\right)+512\,\Gamma^{2}(2\alpha+1)\tanh^{3}\left(\frac{x}{4}\right)$$

$$+1024\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\tanh^{3}\left(\frac{x}{4}\right)$$

$$+1892\,\Gamma(2\alpha+1)\,\Gamma^{2}(\alpha+1)\,sech^{2}\left(\frac{x}{4}\right)\tanh^{3}\left(\frac{x}{4}\right)$$

$$+288\,\Gamma^{2}(2\alpha+1)\,sech^{2}\left(\frac{x}{4}\right)\tanh^{3}\left(\frac{x}{4}\right)+384\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^{2}\left(\frac{x}{4}\right)\tanh^{3}\left(\frac{x}{4}\right)$$

$$-128\,\Gamma(2\alpha+1)\,\Gamma^{2}(\alpha+1)\,sech^{4}\left(\frac{x}{4}\right)\tanh^{3}\left(\frac{x}{4}\right)$$

$$+140\,\Gamma^{2}(2\alpha+1)\,sech^{4}\left(\frac{x}{4}\right)\tanh^{3}\left(\frac{x}{4}\right)-8\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^{4}\left(\frac{x}{4}\right)\tanh^{3}\left(\frac{x}{4}\right)$$

$$+488\,\Gamma(2\alpha+1)\,\Gamma^{2}(\alpha+1)\,sech^{2}\left(\frac{x}{4}\right)\tanh^{4}\left(\frac{x}{4}\right)$$

$$+232\,\Gamma^{2}(2\alpha+1)\,sech^{2}\left(\frac{x}{4}\right)\tanh^{4}\left(\frac{x}{4}\right)+488\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^{2}\left(\frac{x}{4}\right)\tanh^{4}\left(\frac{x}{4}\right)$$

$$-1024\,\Gamma(2\alpha+1)\,\Gamma^{2}(\alpha+1)\tanh^{5}\left(\frac{x}{4}\right)-256\,\Gamma^{2}(2\alpha+1)\tanh^{5}\left(\frac{x}{4}\right)$$

$$-512\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\tanh^{5}\left(\frac{x}{4}\right)+16\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^{2}\left(\frac{x}{4}\right)\tanh^{5}\left(\frac{x}{4}\right)\text{¿}$$

From here, the approximate solution is found from the inverse transformation of the values of the set $\{U_k(x)\}_{k=0}^{4}$.

In order to obtain the approximate solution of this equation, if the below terms are written on the total series,

$$\tilde{u}_n(x,t)=\sum_{h=0}^{n} U_h(x)t^{\alpha h}$$

and we then arrive at the following solution:

$$\tilde{u}_4(x,t)=\sum_{h=0}^{4} U_h(x)t^{\alpha h}=\frac{1}{2}\left(1+\tanh\left(\frac{x}{4}\right)\right)+\frac{4+sech^2\left(\frac{x}{4}\right)-4\tanh^2\left(\frac{x}{4}\right)}{16\Gamma(\alpha+1)}t^{\alpha}$$

$$\frac{+1}{64\Gamma(2\alpha+1)}¿$$

$$-2sech^2\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)+16\tanh^3\left(\frac{x}{4}\right)¿t^{2\alpha}$$

$$\frac{+-1}{512\Gamma(3\alpha+1)\Gamma^2(\alpha+1)}¿$$

$$+16\Gamma(2\alpha+1)sech^2\left(\frac{x}{4}\right)+9\Gamma^2(\alpha+1)sech^4\left(\frac{x}{4}\right)+2\Gamma(2\alpha+1)sech^4\left(\frac{x}{4}\right)$$

$$+36\Gamma^2(\alpha+1)sech^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)+20\Gamma(2\alpha+1)sech^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$-86\Gamma^2(\alpha+1)sech^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)+5\Gamma(2\alpha+1)sech^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$-128\Gamma^2(\alpha+1)\tanh^2\left(\frac{x}{4}\right)-64\Gamma(2\alpha+1)\tanh^2\left(\frac{x}{4}\right)$$

$$-138\Gamma^2(\alpha+1)sech^2\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)-16\Gamma(2\alpha+1)sech^2\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)$$

$$-36\,\Gamma^2(\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^3\left(\frac{x}{4}\right)-20\,\Gamma(2\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^3\left(\frac{x}{4}\right)$$

$$+128\,\Gamma^2(\alpha+1)\tanh^4\left(\frac{x}{4}\right)+32\,\Gamma(2\alpha+1)\tanh^4\left(\frac{x}{4}\right)¿\,t^{3\alpha}$$

$$\frac{+-1}{4096\,\Gamma(4\alpha+1)\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)}¿$$

$$+128\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^2\left(\frac{x}{4}\right)-92\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^4\left(\frac{x}{4}\right)$$

$$-44\,\Gamma^2(2\alpha+1)\,sech^4\left(\frac{x}{4}\right)+4\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^4\left(\frac{x}{4}\right)$$

$$-233\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^6\left(\frac{x}{4}\right)-13\,\Gamma^2(\alpha+1)\,sech^6\left(\frac{x}{4}\right)$$

$$-7\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^6\left(\frac{x}{4}\right)-256\,\Gamma^2(2\alpha+1)\tanh\left(\frac{x}{4}\right)$$

$$-512\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\tanh\left(\frac{x}{4}\right)-416\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$-288\,\Gamma^2(2\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)-400\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$-456\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)-96\,\Gamma^2(2\alpha+1)\,sech^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$-64\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$+322\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^6\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)-85\,\Gamma^2(2\alpha+1)\,sech^6\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$-24\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^6\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$-488\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)$$

$$-264\,\Gamma^2(2\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)-616\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)$$

$$+2280\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^4\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)$$

$$+2280\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^4\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)$$

$$+144\,\Gamma^2(2\alpha+1)\,sech^4\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)-144\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^4\left(\frac{x}{4}\right)\tanh^2\left(\frac{x}{4}\right)$$

$$+1024\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\tanh^3\left(\frac{x}{4}\right)+512\,\Gamma^2(2\alpha+1)\tanh^3\left(\frac{x}{4}\right)$$

$$+1024\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\tanh^3\left(\frac{x}{4}\right)$$

$$+1892\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^3\left(\frac{x}{4}\right)$$

$$+288\,\Gamma^2(2\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^3\left(\frac{x}{4}\right)+384\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^3\left(\frac{x}{4}\right)$$

$$-128\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^4\left(\frac{x}{4}\right)\tanh^3\left(\frac{x}{4}\right)$$

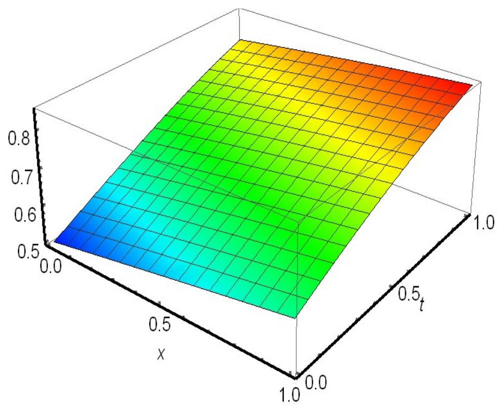
$$+140\,\Gamma^2(2\alpha+1)\,sech^4\left(\frac{x}{4}\right)\tanh^3\left(\frac{x}{4}\right)-8\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^4\left(\frac{x}{4}\right)\tanh^3\left(\frac{x}{4}\right)$$

$$+488\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^4\left(\frac{x}{4}\right)$$

$$+232\,\Gamma^2(2\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^4\left(\frac{x}{4}\right)+488\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^4\left(\frac{x}{4}\right)$$

$$-1024\,\Gamma(2\alpha+1)\,\Gamma^2(\alpha+1)\tanh^5\left(\frac{x}{4}\right)-256\,\Gamma^2(2\alpha+1)\tanh^5\left(\frac{x}{4}\right)$$

$$-512\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\tanh^5\left(\frac{x}{4}\right)+16\,\Gamma(3\alpha+1)\,\Gamma(\alpha+1)\,sech^2\left(\frac{x}{4}\right)\tanh^5\left(\frac{x}{4}\right)¿\,t^{4\alpha}$$



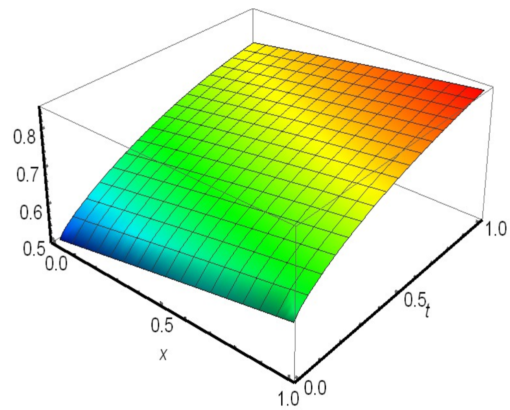
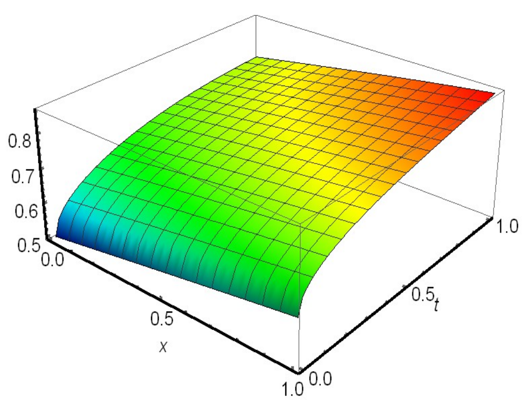


Fig. 4.a Murray equation graph for $\alpha = 1$

Fig. 4.b Murray equation graph for $\alpha = 0{,}75$



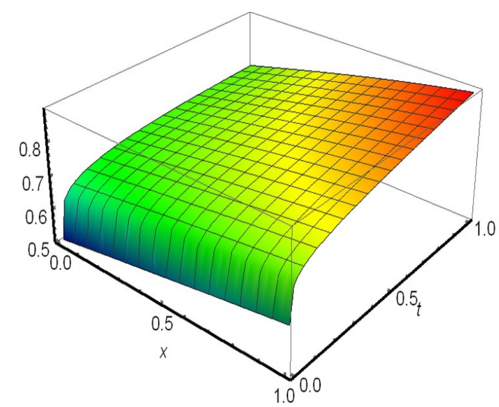


Fig. 4.c Murray equation graph for $\alpha = 0{,}50$

Fig. 4.d Murray equation graph for $\alpha = 0{,}25$

Table 2 When $\alpha = 1$ and $t = 0{,}125$, the $u(x,t)$ numerical solution of time-fractional differential equation (4.4)

x value	*t value*	*Numerical solution*	*Analytical solution*	*Absolute error*
0,125	0,125	0,5544702743	0,5544704649	$1{,}9 \times 10^{-7}$

0,225	0,125	0,5667858200	0,5667860060	$1{,}8\times10^{-7}$
0,325	0,125	0,5790194078	0,5790195874	$1{,}7\times10^{-7}$
0,425	0,125	0,5911567286	0,5911568998	$1{,}7\times10^{-7}$
0,525	0,125	0,6031839359	0,6031840971	$1{,}6\times10^{-7}$
0,625	0,125	0,6150877057	0,6150878555	$1{,}4\times10^{-7}$
0,725	0,125	0,6268552909	0,6268554280	$1{,}3\times10^{-7}$
0,825	0,125	0,6384745708	0,6384746941	$1{,}2\times10^{-7}$
0,925	0,125	0,6499340935	0,6499342022	$1{,}0\times10^{-7}$

Table 3 When $t=0{,}125$ and $\alpha=1$, $\alpha=0{,}75$, $\alpha=0{,}50$, $\alpha=0{,}25$, the $u(x,t)$ numerical solution of time-fractional differential equation (4.4)

x	t	$\alpha=1$	$\alpha=0{,}75$	$\alpha=0{,}50$	$\alpha=0{,}25$
0,125	0,125	0,5544702743	0,5859103286	0,6334478288	0,6894731167
0,225	0,125	0,5667858200	0,5979239246	0,6446632874	0,7000300566
0,325	0,125	0,5790194078	0,6098216121	0,6557299695	0,7104476486
0,425	0,125	0,5911567286	0,6215906289	0,6666378629	0,7207052979
0,525	0,125	0,6031839359	0,6332187959	0,6773775486	0,7307833264
0,625	0,125	0,6150877057	0,6446945618	0,6879402342	0,7406632610
0,725	0,125	0,6268552909	0,6560070412	0,6983177834	0,7503280899
0,825	0,125	0,6384745708	0,6671460477	0,7085022739	0,7597624789
0,925	0,125	0,6499340935	0,6781021211	0,7184883430	0,7689529449

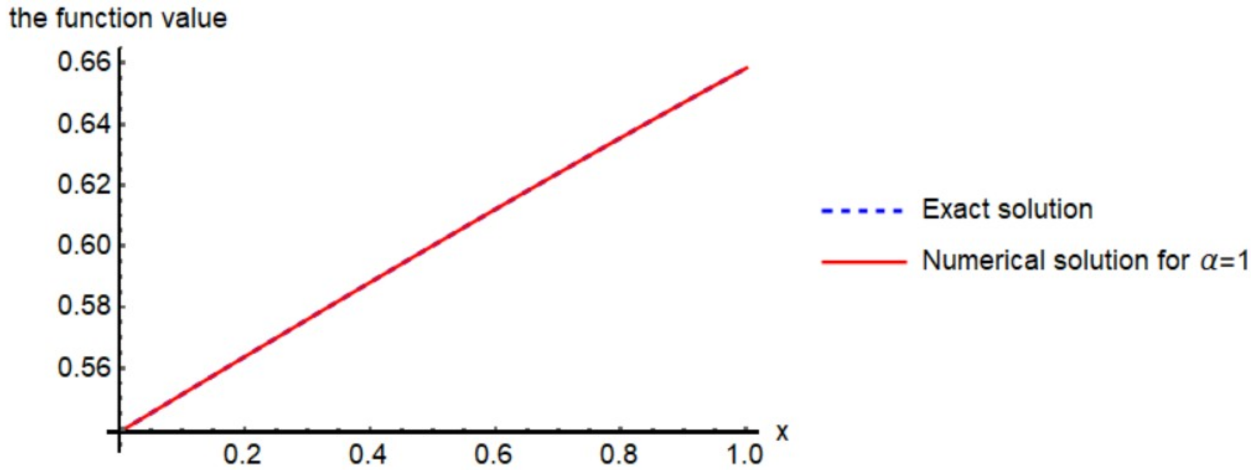


Fig. 5.a 2D graphic of the exact and numerical solution of $u(x,0.125)$ of the (4.4) time-fractional differential equation for $\alpha=1$.

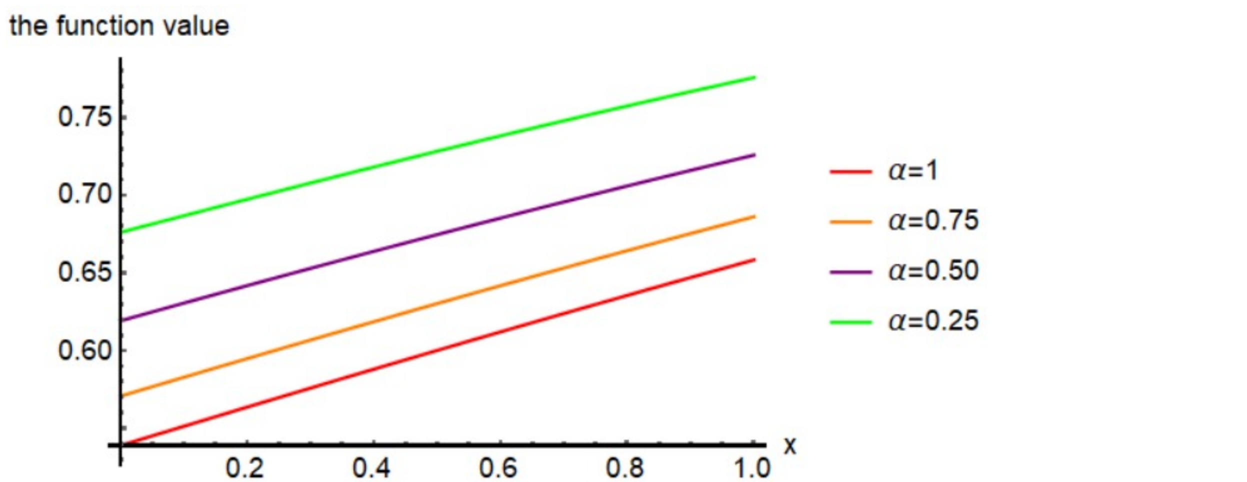


Fig. 5.b 2D graphic of the numerical solution of $u(x,0.125)$ of the (4.4) time-fractional differential equation for $\alpha=1$, $\alpha=0{,}75$, $\alpha=0{,}50$, $\alpha=0{,}25$ of the (4.4) time fractional differential equation.

Table 4 When $\alpha=1$ and $t=0{,}325$, the $u(x,t)$ numerical solution of time-fractional differential equation (4.4)

x value	*t value*	*Numerical solution*	*Analytical solution*	*Absolute error*
0,125	0,325	0,6150657708	0,6150878555	0,0000220847
0,225	0,325	0,6268340714	0,6268554281	0,0000213566
0,325	0,325	0,6384542799	0,6384746942	0,0000204143

0,425	0,325	0,6499149279	0,6499342022	0,0000192743
0,525	0,325	0,6612052504	0,6612232068	0,0000179564
0,625	0,325	0,6723152161	0,6723316992	0,0000164831
0,725	0,325	0,6832355529	0,6832504316	0,0000148788
0,825	0,325	0,6939577659	0,6939709354	0,0000131695
0,925	0,325	0,7044741505	0,7044855324	0,0000113819

Table 5 When $t=0{,}325$ and$\alpha=1$, $\alpha=0{,}75$, $\alpha=0{,}50$, $\alpha=0{,}25$, the $u(x,t)$ numerical solution of time-fractional differential equation (4.4)

x	t	$\alpha=1$	$\alpha=0{,}75$	$\alpha=0{,}50$	$\alpha=0{,}25$
0,125	0,325	0,6150657708	0,6541561282	0,6919781039	0,7192267946
0,225	0,325	0,6268340714	0,6651658729	0,7021446159	0,7307671602
0,325	0,325	0,6384542799	0,6760042568	0,7121570774	0,7421752989
0,425	0,325	0,6499149279	0,6866625171	0,7220040752	0,7534040021
0,525	0,325	0,6612052504	0,6971325	0,7316746799	0,7644082702
0,625	0,325	0,6723152161	0,7074069155	0,7411585762	0,7751460837
0,725	0,325	0,6832355529	0,7174788827	0,7504461818	0,7855790777
0,825	0,325	0,6939577659	0,7273424109	0,7595287530	0,7956731036
0,925	0,325	0,7044741505	0,7369921707	0,7683984747	0,8053986682

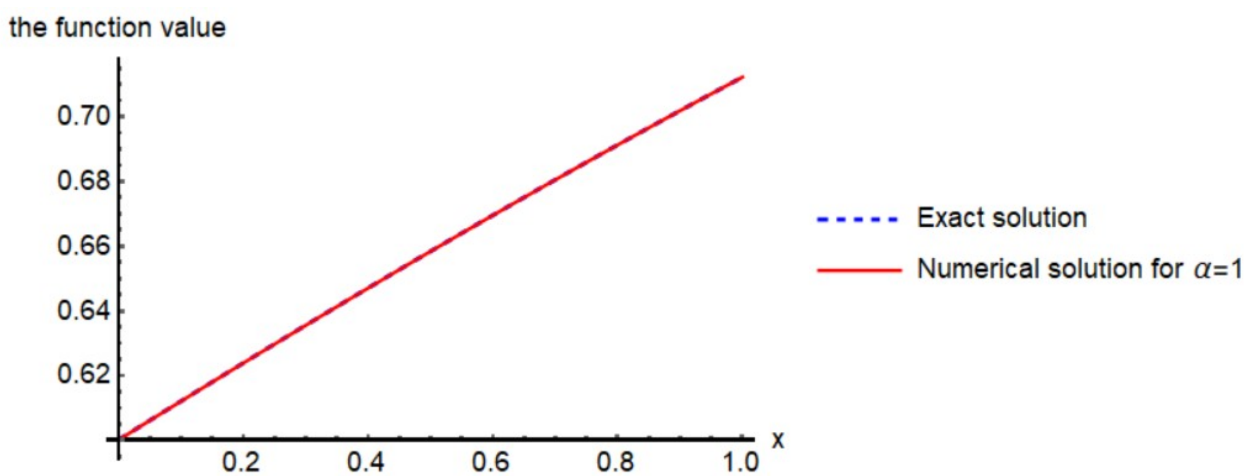


Fig. 6.a 2D graphic of the exact and numerical solution of $u(x, 0.325)$ of the (4.4) time-fractional differential equation for $\alpha = 1$.

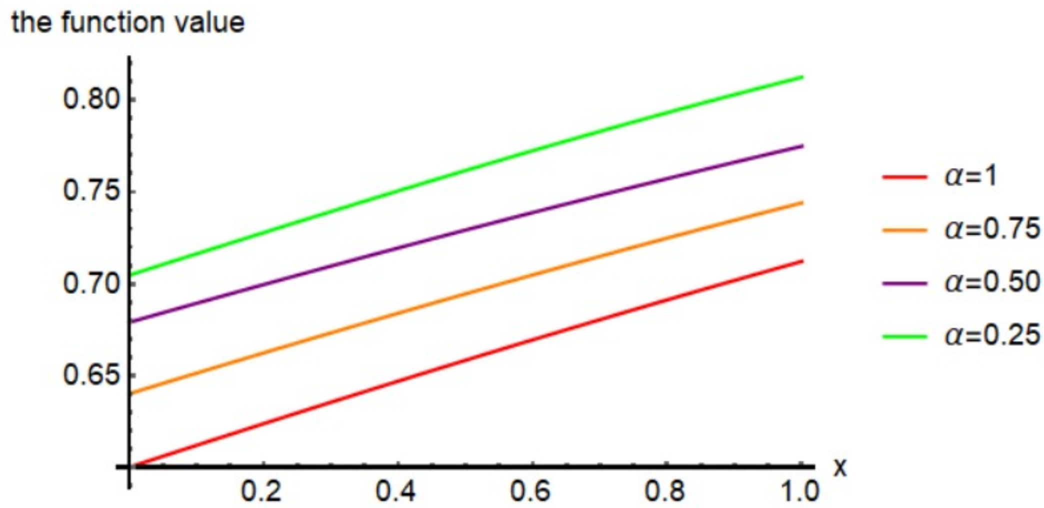


Fig. 6.b 2D graphic of the numerical solution of $u(x, 0.325)$ of the (4.4) time-fractional differential equation for $\alpha = 1$, $\alpha = 0{,}75$, $\alpha = 0{,}50$, $\alpha = 0{,}25$ of the (4.4) time fractional differential equation.

It was observed that the acquired for t=0,125 taken in tables 2 and 3 are consistent with the 2D plots in figure 5a and 5b. At the same time, for the t=0,325 value taken in tables 4 and 5, it was acquired that it is consistent with the 2D plots in figures 6a and 6b. In our studies, it has been observed that there is a convergence for the other values of t in the table and the variable values of x, as in the above graphs.

5 Conclusions

In this study, the reduced differential transform method was applied to the time-fractional Murray equation, which is a member of Reaction-Diffusion equation family. A series solution for the problem was obtained by using the initial condition. These solutions were compared with tables and graphs. For $\alpha = 1$ these solutions were compared with the analytical solutions. As a result, this method was found to be very useful for solving nonlinear equations. Reduced differential transform method converges to the solution faster than classical or generalized differential transform methods. Furthermore, this method eliminates the intensive processing load.

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