Global existence and asymptotic behaviour of solutions for a hyperbolic-parabolic model of chemotaxis on network

Yafeng Li[a,*], Chunlai Mu[a,†], Qiao Xin[b‡]

[a] College of Mathematics and Statistics, Chongqing University, Chongqing 401331, PR China

[b] College of Mathematics and Statistics, Yili normal University, Yining 835000, PR China

Abstract

In this paper, we discuss a hyperbolic-parabolic system modeling biological phenomena evolving on a network. The global existence of the is obtained by using energy estimates with suitable the transmission conditions at interior. Moreover, for the case of acyclic network, the existence and uniqueness of stationary solution to the system is proposed and it is proved that these ones are asymptotic profiles for a class of global solutions
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1 Introduction

Chemotaxis is a phenomenon of collective movement of microorganisms in the direction of increasing chemical concentration. The classical PDE model of chemotaxis was introduced in the 1970 and it was named the Keller-Segel model [22, 23]. A lot of adapted and expanded versions of the Keller-Segel models [1, 7, 19, 18] in the last decades.

By contrast, different from the classical Keller-Segel model, the evolution of the density of cells is described by a hyperbolic system coupled with a parabolic equation (or elliptic) equation for the chemoattractant in [28, 29], which in one dimension reads

$$\begin{cases} u_t^+ + \lambda u_x^+ = \mu^+(\phi, \phi_x, u^+, u^-), \\ u_t^- - \lambda u_x^- = \mu^-(\phi, \phi_x, u^+, u^-), & \text{in } I \times (0,\infty), \\ \tau\phi_t = \phi_{xx} + \beta(u^+ + u^-) - h(\phi),\ \tau \geq 0,\ D > 0, \lambda > 0, \end{cases} \tag{1.1}$$

where the cells $u = u^+ + u^-$ has been split into densities for right and left moving cells, λ represents the turning the cell moving speed, and $\mu^\pm$ are turning rates (rates of change of

*E-mail address:lyf911105@163.com
†E-mail address: clmu2005@163.com
‡E-mail address: xinqiaoylsy@163.com

direction from + to $-$), the external chemical signal ϕ is produced or consumed by the cell species itself. This is modeled by the precise form of the reaction term $\beta(u^+ + u^-), h(\phi)$, respectively. In fact, hyperbolic models have been recently introduced [9, 10, 21, 20, 12] since they yield a more realistic finite speed of propagation, in contrast to the parabolic ones, and allow better observation of the phenomena during the initial phase.

Models like (1.1) on network were proposed in [16], these models originated from tissue engineering, when human's tissue is injured, fibroblasts play a very important role in the process of repair which create a new extracellular matrix, essentially made by collagen, and move along it to fill the wound driven by chemotaxis. To accelerate this process, tissue engineers use artificial scaffolds, constituted by a network of crossed polymeric threads, which are inserted within the wound [8, 24].

In this paper, we mainly consider (1.1) which cast on a network formed by n nodes and m oriented arcs connecting the nodes. Each arc is characterized by a typical velocity λ_i, and then, we consider the triple of unknowns (u_i^+, u_i^-, ϕ_i) on each arc I_i. Our mathematical model on networks reproduce this configuration: the arcs mimic the fibers of the scaffold, the nodes the contact points between the fibers, and functions $u_i^{\pm}, \phi_i$ are the densities of fibroblasts and chemoattractant on each of them. We should note that the literature [31] has been experimentally observed that material composition, fiber structure and fiber diameter of scaffolds and so on affect proliferation, cellular organization, and subsequent tissue morphogenesis. The aim of this paper is to propose the global existence and the asymptotic behaviour of solutions to the system (1.1) on the network complemented with initial, boundary, and transmission conditions at nodes. The direct relation between our model and existing experiments will be further considered since the experimental setting was not thought to measure the relevant data needed in our framework.

Before this, let us mention some previous contribution in this direction. The hyperbolic models on networks which were mentioned above, have been previously researched in [26, 15, 30, 3, 2], with different kinds of transmission conditions. In [4], the authors treated the following hyperbolic chemotaxis model on network from a numerical point of view with suitable transmission conditions,

$$\begin{cases} u_{it} + \lambda_i v_{ix} = 0, \\ v_{it} + \lambda_i u_{ix} = u_i \phi_{ix} - \beta_i v_i, \;\; x \in I_i, \;\; t \geq 0, \;\; i = 1, ..., m, \;\; \lambda_i > 0, \\ \phi_{it} = D_i \phi_{ixx} + a_i u_i - b_i \phi_i, \end{cases} \tag{1.2}$$

where u_i stands for the concentration in each arc I_i, v_i is average flux and ϕ_i is the chematractant concentration in each arc I_i. Inspiration from numerical results, Guarguaglini et al proved the global existence and the asymptotic behaviour of solutions to the system (1.2) both for the homogeneous boundary conditions and for nonhomogeneous boundary conditions in [13, 17, 11]. Moreover, in [14], Guarguaglini et al considered the chemotaxis model (1.1) with $\tau = 0$ (hyperbolic-elliptic) on network,

$$\begin{cases} u_{it}^+ + \lambda_i u_{ix}^+ = \mu_i^+(\phi_i, \phi_{ix}, u_i^+, u_i^-) & \text{in } I_i \times (0, \infty), \\ u_{it}^- - \lambda_i u_{ix}^- = \mu_i^-(\phi_i, \phi_{ix}, u_i^+, u_i^-), & \text{in } I_i \times (0, \infty), \\ -\phi_{ixx} + h(\phi_i) = \beta(u_i^+, u_i^-), & \text{in } I_i \times (0, \infty), \end{cases} \tag{1.3}$$

complemented with the suitable transmission conditions, and in case of the function $\mu^{\pm}$ satisfies some assumed conditions, they proved local and global existence of nonnegative solutions with small (in the L^1-norm)initial values. Finally, in [5], the authors treated the same as our model from a numerical point of view with homogeneous Neumann boundary conditions and non-homogeneous Neumann boundary conditions. But subject to the homogeneous Neumann boundary conditions, the chemotaxis model (1.1) on network with $\tau > 0$(hyperbolic-parabolic system) without corresponding theoretical research, hence motivated by [5, 13, 17, 14], we are concerned with the global existence and the asymptotic behaviour of solutions to the following hyperbolic-parabolic chemotaxis system on the oriented graph(network),

$$\begin{cases} u_{it}^{+} + \lambda_i u_{ix}^{+} = \mu_i^{+}(\phi_i, \phi_{ix}, u_i^{+}, u_i^{-}) & \text{in } I_i \times (0, \infty), \\ u_{it}^{-} - \lambda_i u_{ix}^{-} = \mu_i^{-}(\phi_i, \phi_{ix}, u_i^{+}, u_i^{-}), & \text{in } I_i \times (0, \infty), \\ \phi_{it} = \phi_{ixx} - h(\phi_i) + \beta(u_i^{+}, u_i^{-}), & \text{in } I_i \times (0, \infty), \end{cases} \tag{1.4}$$

complemented with the transmission conditions introduced in [14] at each inner node of the oriented graph $G(\mathcal{N}, \mathcal{A})$, and homogeneous Neumann conditions at the external node. To go a step further, we introduce symbols and related definitions about our model, where $G(\mathcal{N}, \mathcal{A})$ is a finite connected graph with vertex set $\mathcal{N}$ of n nodes (vertices) and a set $\mathcal{A}$ of m oriented arcs, $\mathcal{A} = \{I_i : i \in \mathcal{M} = \{1, 2, ..., m\}\}$. More precisely, a set $\mathcal{A}$ is divided into $\mathcal{A}_{in} = \{\tilde{I}_i, i = 1, \ldots, \tilde{m}\}$ with all $\tilde{m}$ oriented arcs connecting two internal nodes and a set $\mathcal{A}_{ex} = \{\bar{I}_i, i = 1, \ldots, \bar{m}\}$ with all $\bar{m}$ oriented arcs incoming or outgoing from the external points, naturally, $\mathcal{A} = \mathcal{A}_{in} \cup \mathcal{A}_{ex}, m = \tilde{m} + \bar{m}$. In addition, a vertex set $\mathcal{N} = \mathcal{N}_{in} \cup \mathcal{N}_{ex}$ is divided into the external point set $\mathcal{N}_{ex} = \{\bar{N}_j, j = 1, \ldots, \bar{n}\}$ and the internal point set $\mathcal{N}_{in} = \{N_p, p = 1, \ldots, \tilde{n}\}$. Moreover, for each (internal or external) node N we shall denote by I_N and O_N the sets of the arcs incoming or outing from the node N, respectively. For every (internal or external) node N we shall also $\mathcal{M}_N = I_N \cup O_N$.

Every arc I_i is considered as a one dimensional interval $[0, L_i]$, L_i is the abscissa of the node N if $I_i \in I_N$ $(i \in I_N)$ whereas, it is 0 if $I_i \in O_N (i \in O_N)$. On the other hand, in this paper of section 4 and 5, we mainly consider the acyclic graph $G(\mathcal{N}, \mathcal{A})$ which does not contain cycles. More specifically, for each couple of vertices, there exists a unique path with no repeated arcs connecting them. On this basis, we define the Banach space and the Hilbert space on the orient graph,

$$L^p(\mathcal{A}) = \{f : f_i \in L^p(I_i)\}, \text{ for } 1 \leq p < \infty; \ H^s(\mathcal{A}) = \{f : f_i \in H^s(I_i)\}, \text{ for } s = 1, 2$$

with norm

$$||f||_{L^p(\mathcal{A})} = \sum_{i \in \mathcal{M}} ||f_i||_{L^p(I_i)}, \ ||f||_\infty = \sum_{i \in \mathcal{M}} ||f_i||_\infty; \ \ ||f||_{H^s(\mathcal{A})} = \sum_{i \in \mathcal{M}} ||f_i||_{H^s(I_i)},$$

and let $X = (L^2(\mathcal{A}))^2, \ \ Y = (H^1(\mathcal{A}))^2$ endowed with the inner products

$$\Big\langle U, V \Big\rangle_X = \sum_{i \in \mathcal{M}} \int_{I_i} (u_i^{+} v_i^{+} + u_i^{-} v_i^{-}), \ \ (U = (u_i^{+}, u_i^{-}), \ V = (v_i^{+}, v_i^{-}))$$

and

$$\Big\langle U, V \Big\rangle_Y = \sum_{i \in \mathcal{M}} \int_{I_i} \Big((u_i^{+} v_i^{+} + u_i^{-} v_i^{-}) + (u_{ix}^{+} v_{ix}^{+} + u_{ix}^{-} v_{ix}^{-}) \Big), \ \ (U, V \in Y).$$

For the sake of brevity, we shall also use the following notations

$$\mu^{\pm}(\phi_i, \phi_{ix}, u_i^{+}, u_i^{-}) = \mu_i^{\pm}(\phi_i, \phi_{ix}, u_i^{+}, u_i^{-}),$$

$$\mu^{\pm}(\phi, \phi_x, u^+, u^-) = \left\{\mu_i^{\pm}(\phi, \phi_x, u^+, u^-)\right\}_{i\in\mathcal{M}}$$

and

$$h(\phi_i) = h_i(\phi_i),\ h(\phi) = \{h(\phi_i)\}_{i\in\mathcal{M}}\,,$$
$$\beta_i(u_i^+, u_i^-) = \beta(u_i^+, u_i^-),\quad \beta(u^+, u^-) = \left\{\beta(u_i^+, u_i^-)\right\}_{i\in\mathcal{M}}\,,$$

for all $\phi, \phi_x, u^{\pm} : \mathcal{A} \to R$ and for every $i \in \mathcal{M}$. For more detail information on notations, notions, and conventions, we refer the reader to [14, 13].

Based on the above basic theory, we introduce the following the boundary conditions and the transmission condition of system (1.4)

(I) for every external node $N \in \mathcal{N}_{ex}$

$$u_i^-(L_i, t) = u_i^+(L_i, t),\ i \in I_N,\quad u_i^+(0, t) = u_i^-(0, t),\ i \in O_N, \tag{1.5}$$

and

$$\phi_{ix}(L_i, t) = 0,\ i \in I_N,\quad \phi_{ix}(0, t) = 0,\ i \in O_N; \tag{1.6}$$

(II) for every internal node $N \in \mathcal{N}_{in}$

$$\begin{cases} u_i^-(L_i, t) = \sum_{j\in I_N} \xi_{ij} u_j^+(L_j, t) + \sum_{j\in O_N} \xi_{ij} u_j^-(0, t), & \text{if } \ i \in I_N, \\ u_i^+(0, t) = \sum_{j\in I_N} \xi_{ij} u_j^+(L_j, t) + \sum_{j\in O_N} \xi_{ij} u_j^-(0, t), & \text{if } \ i \in O_N, \end{cases} \tag{1.7}$$

where the constant $\xi_{ij} \in [0, 1]$ are the transmission coefficients, they represent the probability that a cell at a node decide to move from the ith to the jth arc of the network, also including the turnabout on same arc. Using transmission condition (1.7) and Proposition 2.1 in [4], then ξ_{ij} satisfies

$$\sum_{i\in\mathcal{M}_N} \lambda_i \xi_{ij} = \lambda_j;\ \sum_{i\in\mathcal{M}_N} \xi_{ij} = 1,\quad \text{for every } \ j \in \mathcal{M}_N. \tag{1.8}$$

Specially, the transmission condition implies the continuity of the fluxes at node, meaning that we cannot loose nor gain any cells, which yields

$$\sum_{i\in I_N} \lambda_i(u_i^+(L_i, t) - u_i^-(L_i, t)) - \sum_{i\in O_N} \lambda_i(u_i^+(0, t) - u_i^-(0, t)) = 0,$$

with correspond to the conservation of the total mass

$$\sum_{i\in\mathcal{M}} \int_{I_i} \Big(u_i^+(x, t) + u_i^+(x, t)\Big) = \sum_{i\in\mathcal{M}} \int_{I_i} \Big(u_{i0}^+(x) + u_{i0}^-(x)\Big). \tag{1.9}$$

Now let us consider the transmission condition ϕ. Also in this case, we only consider the continuity of the flux at node N. Using the Kedem-Katchalsky permability conditions [25], which has been first propose in case of fluxes at membranes.

(III) for every internal node $N \in \mathcal{N}_{in}$

$$\begin{aligned} \phi_{ix}(L_i, t) = &\sum_{j\in I_N} \alpha_{ij}(\phi_j(L_j, t) - \phi_i(L_i, t)) \\ &+ \sum_{j\in O_N} \alpha_{ij}(\phi_j(0, t) - \phi_i(L_i, t)),\quad \text{if } \ i \in I_N, \end{aligned} \tag{1.10}$$

$$
\begin{aligned}
-\phi_{ix}(0,t) = &\sum_{j\in I_N} \alpha_{ij}(\phi_j(L_j,t) - \phi_i(0,t)) \\
&+ \sum_{j\in O_N} \alpha_{ij}(\phi_j(0,t) - \phi_i(0,t)), \quad \text{if} \quad i \in O_N,
\end{aligned}
\tag{1.11}
$$

where $\alpha_{ij} \geq 0$ and the condition $\alpha_{ij} = \alpha_{ji}$ for every $i, j \in \mathcal{M}$ yields the conservation of the fluxes at node N, that is to say

$$
\sum_{i\in\mathcal{I}_N} \phi_{ix}(L_i,t) - \sum_{i\in\mathcal{O}_N} \phi_{ix}(0,t) = 0.
$$

Besides, the system (1.4) satisfies the initial condition

$$
(u_0^+, u_0^-) \in (H^1(\mathcal{A}))^2, \ \phi_0 \in H^2(\mathcal{A}) \text{ satisfy } (1.5) - (1.11). \tag{1.12}
$$

Finally, in order to obtain the global existence and the asymptotic behaviour of solutions to $(1.4) - (1.12)$, we introduce some reasonable basic assumptions, $\mu_i^\pm : \mathbb{R}^4 \to \mathbb{R}$, $\beta_i : \mathbb{R}^2 \to \mathbb{R}$ and $h_i : \mathbb{R}^2 \to \mathbb{R}$, $i \in \mathcal{M}$ are summarized below:

(H_1) $\mu_i^\pm \in C^2(\mathbb{R}^4)$ and $\mu_i^\pm(\phi_i, \phi_{ix}, 0, 0) = 0$ for all $(\phi, \phi_{ix}) \in \mathbb{R}^2$ and $i \in \mathcal{M}$;

(H_2) $\beta_i \in C^2(\mathbb{R}^2) \cap \mathrm{Lip}(\mathbb{R}^2)$ and $\beta_i(0,0) = 0$ for all $i \in \mathcal{M}$;

(H_3) for every $i \in \mathcal{M}$, $h_i \in C^2(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$, $h_i(0) = 0$ and there $\eta_{i,1}, \eta_{i,2} \in \mathbb{R}$, $0 < \eta_{i,1} \leq \eta_{i,2}$ such that

$$
\eta_{i,1} \leq h_i'(s) \leq \eta_{i,2} \quad \text{for every} \quad s \in \mathbb{R}.
$$

Let us observe that by the substitution $u_i = u_i^+ + u_i^-$ and $v_i = u_i^+ - u_i^-$, then the model (1.2) can be turn into a special case of the our chemotaxis model (1.4)(the model (1.2) is obviously satisfied assumptions $(H_1) - (H_3)$ of the function $\mu^\pm, h(\phi), \beta(u^+, u^-)$). However, the methods in the literature [13, 17, 14] are not enough to solve this problems. In order to deal with our problem, we need some innovation in methods and handling skills in this paper. To go a step further, we emphasize that the methods of proof in this paper, and consequently the results obtained, differ from those in [13, 17, 14], particularly due to the following facts:

(1)The structure of system (1.4) is different from [14], (hyperbolic-parabolic instead of hyperbolic-elliptic).

(2) The transmission conditions for the hyperbolic part, which are different from those in [13, 17], and strongly influence the problem.

(3) In particular, the approach in the proof of Proposition 4.2 pointed out in section 4 is deeply different from the one followed in [17]. In fact, in the case of acyclic network, Guarguaglini et al proved the question of existence and uniqueness of stationary solutions of problem (1.2) with fixed mass $\Theta = \sum_{i\in\mathcal{M}} \int_{I_i} u_i(x)dx$ using the Banach Fixed point Theorem in [17]. Obviously, the existence and uniqueness of stationary solutions of problem (1.2) imply that v is constant on each arcs and has to be null on the external edges. Moreover, for the case of acyclic graph, the flux $v_i = 0$ for all $x \in I_i$, $i \in \mathcal{M}$. Hence, the first equation of stationary problem to the problem (1.2) is a variables separated ordinary differential equation, it's easy to get $u_i(x) = C_i \exp(\frac{\phi_i}{\lambda_i})$. In this case, it is easy to verify that the function u_i, ϕ_i satisfies the transmission condition and the fix mass $\Theta = \sum_{i\in\mathcal{M}} \int_{I_i} C_i \exp(\frac{\phi_i}{\lambda_i})$. Moreover, using the Banach fixed point theorem, it show that the problem (1.2) has a unique stationary solution $(C_i \exp(\frac{\phi_i}{\lambda_i}), 0, \phi_i(x))$, and if $\frac{a_i}{b_i} = Q$, where Q is a positive constant, then the unique stationary solution to (1.2) is the constant solution without any restrictions on the structure of the network, and also

gets the same theoretical result on general network (in details, please refer to Theorem 3.1 and Proposition 3.1 of [17]). In addition, Theorem 4.2 of [17] is devote to study the unique constant solution to problem (1.2) which are asymptotic profiles for a class of global solutions. The theoretical results are very beautiful.

Because, for our model has strong nonlinearity and the transmission conditions for the hyperbolic part, which are different from those in [13], it is very difficult to achieve such a satisfactory result. In order to prove the existence and unique solution to the stationary problem of problem (1.4) on the acyclic network by using the contraction mapping principle and the Banach fixed point theorem (see Theorem 1.3). The key to transform the first two equation of stationary problem into an integral operator. Moreover, we show that the operator is contraction mapping, and also verify that the operator satisfies the transmission condition in each interior vertices which is equivalent to solving a system of equations consisting of the number of m equations. This is the difficulty in dealing with this problem (see Proposition 4.2) comparing with the literature [17]. Besides, the stationary solution of our mold is a constant solution, which requires us to have strict requirements on the function $\mu^{\pm}, \beta(u^+, u^-), h(\phi)$. Hence, for general network, we only give the decision condition that the unique stationary solution of problem (1.4) is a constant solution(see Remark 1.1). Our results stated as follows:

Theorem 1.1. ***(Local existence)*** *Let assumptions* $(H_1)-(H_3)$ *hold. Then, there exists* $T_0 > 0$ *such that problem* $(1.4) - (1.12)$ *has unique local solution* (u^+, u^-, ϕ),

$$(u^+, u^-) \in C^1([0, T_0]; X) \cap C([0, T_0]; Y),$$

$$\phi \in C([0, T_0]; H^2(\mathcal{A})) \cap C^1([0, T_0]; L^2(\mathcal{A})).$$

Moreover, $\phi \in H^1([0, T_0]; H^1(\mathcal{A}))$.

Theorem 1.2. ***(Global existence)*** *Let assumptions* $(H_1) - (H_3)$ *hold. There exists a positive constant* ϵ_0 *such that, if*

$$||u_0^+||_{H^1(\mathcal{A})},\ ||u_0^-||_{H^1(\mathcal{A})},\ ||\phi_0||_{H^2(\mathcal{A})} \le \epsilon_0, \tag{1.13}$$

then there exists a unique global solution (u^+, u^-, ϕ) *to problem* $(1.4) - (1.12)$,

$$(u^+, u^-) \in C^1([0, \infty); X) \cap C([0, \infty); Y), \tag{1.14}$$

$$\phi \in C([0, \infty); H^2(\mathcal{A})) \cap C^1([0, \infty); L^2(\mathcal{A})) \cap H^1([0, \infty); H^1(\mathcal{A})). \tag{1.15}$$

Next, we restrict our attention to acyclic graphs, and approach the results of existence of stationary solution of $(1.4) - (1.12)$ with fixed mass

$$\Theta = ||u^+(x)||_{L^1(\mathcal{A})} + ||u^-(x)||_{L^1(\mathcal{A})}, \tag{1.16}$$

in the content of the following theorem.

Theorem 1.3. *Let* $G(\mathcal{N}, \mathcal{A})$ *be an acyclic graph and assumptions* $(H_1) - (H_3)$ *hold, there exists* $\varepsilon > 0$ *such that, if* $0 \le \Theta \le \varepsilon$ *where* Θ *is fixed mass which defines in* (1.16)*, then problem* (1.4)$-$(1.12) *has a unique stationary solution* $(U^+(x), U^-(x), \Psi(x)) \in (H^1(\mathcal{A}))^2 \times H^2(\mathcal{A})$ *satisfying* (1.16).

Remark 1.1: Theorem 1.3 ensures the existence and uniqueness stationary solution to the problem (1.4). At this abasia, we discuss the unique stationary solution of the problem (1.4) which is the constant solutions $(\mathrm{U}_c^+, U_c^-, \Psi_c)$. To do this, we need to add conditions for the function $\mu^{\pm}$,

(H_1') $\mu_i^{\pm} \in C^2(\mathbb{R}^4)$ and $\mu_i^{\pm}(\phi_i, 0, u_i^+, u_i^-) = 0$ for all $(\phi, u_i^+, u_i^-) \in \mathbb{R}^3$ and $i \in \mathcal{M}$.
Notice that, if (Ψ_c, U_c^+, U_c^-) satisfies

$$\mu^{\pm}(\Psi_c, 0, U_c^+, U_c^-) = 0 \ , h(\Psi_c) = \beta(U_c^+, U_c^+) \text{ and } (1.16), \tag{1.17}$$

then the unique stationary solution to the problem (1.4) is constant solutions (U_c^+, U_c^-, Ψ_c) without any restrictions on the structure of the network. Hence, on general network, we also can obtain the same theoretical results.

Theorem 1.4. ***(Asymptotic behavior)*** *Let $G(\mathcal{N}, \mathcal{A})$ be an acyclic graph and assumptions $(H_1) - (H_3)$ hold, let $(U^+(x), U^-(x), \Psi(x))$ be a stationary solution to problem $(1.4) - (1.12)$. There exists $\epsilon_0, \epsilon_1 > 0$ such that, if*

$$||U^+||_{H^1(\mathcal{A})} + ||U^-||_{H^1(\mathcal{A})}, \ ||\Psi||_{H^2(\mathcal{A})} \leq \epsilon_0 \tag{1.18}$$

and

$$||u_0^+ - U^+||_{H^1(\mathcal{A})}, \ ||u_0^- - U^-||_{H^1(\mathcal{A})}, \ ||\phi_0 - \Psi||_{H^2(\mathcal{A})} \leq \epsilon_1, \tag{1.19}$$

then the problem $(1.4) - (1.12)$ has a unique global solution (u^+, u^-, ϕ),

$$(u^+, u^-) \in C^1([0, \infty); X) \cap C([0, \infty); Y),$$

$$\phi \in C([0, \infty); H^2(\mathcal{A})) \cap C^1([0, \infty); L^2(\mathcal{A})) \cap H^1([0, \infty); H^1(\mathcal{A})).$$

Moreover

$$\lim_{t\to+\infty} ||u_i^+(,t) - U_i^+||_{C(I_i)}, \ \lim_{t\to+\infty} ||u_i^-(,t) - U_i^-||_{C(I_i)}, \ \lim_{t\to+\infty} ||\phi_i^-(,t) - \Psi_i||_{C(I_i)} = 0. \tag{1.20}$$

The paper is organized as follows: In section 2, we establish the existence and uniqueness of local solution result by the fixed point technique. The section 3 is devoted to the consequent the proof of global existence of solutions of the problem $(1.4) - (1.12)$ under small (in a suitable norm) initial data. In section 4 and 5, for the case of acyclic graphs, we prove that there exist a unique stationary solution of the problem $(1.4) - (1.12)$ under suitable condition and show that these ones are asymptotic profiles for a class of global solutions.

2 Local existence

In this section, we prove the existence and uniqueness of the local solution of the system $(1.4) - (1.12)$. First, we consider the linear operator $A_1 : D(A_1) \to L^2(\mathcal{A})$,

$$D(A_1) = \{\phi \in H^2(\mathcal{A}) : (1.6), (1.10) - (1.11) \text{ hold}\},$$

$$A_1(\phi) = \{\phi_{ixx} - h(\phi_i)\}_{i\in\mathcal{M}},$$

and the linear operator $A_2 : D(A_2) \to L^2(\mathcal{A})$,

$$D(A_2) = \{U = (u^+, u^-) \in (H^1(\mathcal{A}))^2 : (1.5), (1.7) - (1.8) \text{ hold}\},$$

$$A_2U = \{-\lambda_i u^+_{ix}, \lambda_i u^-_{ix}\}_{i\in\mathcal{M}}.$$

Next, we are going to prove the existence of unique local solution of $(1.4)-(1.12)$ by the fixed point technique, combining the local solutions of the two disjointed problems

$$\begin{cases} \phi \in C([0,T];H^2(\mathcal{A})) \cap C^1([0,T];L^2(\mathcal{A})) \cap H^1([0,T];H^1(\mathcal{A})), \\ \phi'(t) = A_1\phi + g(t), \quad \mathcal{A}\times[0,T], \\ (1.6) \ \text{ and } \ (1.10)-(1.11) \ \text{ hold for every } \ t\in[0,T], \end{cases} \tag{2.1}$$

where $g \in C^1([0,T];L^2(\mathcal{A})) \cap C([0,T],H^1(\mathcal{A}))$ and

$$\begin{cases} U = (u^+,u^-) \in C^1([0,T];X) \cap C([0,T];D(A_2)), \\ U'(t) = A_2U(t) + F(t,U(t)), \quad t\in[0,T], \\ U(0) = (u_0^+,u_0^-) \in D(A_2), \end{cases} \tag{2.2}$$

where $F(t,U(t)) \in C^1([0,T],X)$.

Before proving the existence of local solution to the problem $(1.4)-(1.12)$, we shall show the following lemmas.

Lemma 2.1 ([14])**.** *Let $\phi \in H^2(\mathcal{A})$ satisfy* (1.6) *and* $(1.10)-(1.11)$*. Then, for every nondecreasing Lipschitz continuous function $F:\mathbb{R}\to\mathbb{R}$ there holds*

$$-\sum_{i\in\mathcal{M}}\int_{I_i}\phi_{ixx}F(\phi_i)dx \geq \sum_{i\in\mathcal{M}}\int_{I_i}(\phi_{ix})^2F'(\phi_i)dx.$$

Lemma 2.2 ([14])**.** *Let $U=(u^+,u^-)\in Y$ satisfy* (1.5) *and* $(1.7)-(1.8)$*. Then, for every convex function $G\in C^1(\mathbb{R})$ there holds*

$$\sum_{i\in\mathcal{M}}\lambda_i\int_{I_i}\left\{[G(u_i^+)]_x - [G(u_i^-)]_x\right\} \geq 0.$$

Proposition 2.1. *Let assumption (H_3) holds and $g\in C([0,T];H^1(\mathcal{A}))\cap C^1([0,T],L^2(\mathcal{A}))$, there exist $M,K\geq 0$ such that , if $M\geq \sup_{[0,T]}||g(t)||_{H^1(\mathcal{A})}$ and $K\geq ||\phi_0||_{H^2(\mathcal{A})}+4M$, then there exists a unique solution to problem* (2.1) *with*

$$\sup_{[0,T]}||\phi(t)||_{H^2(\mathcal{A})} \leq K. \tag{2.3}$$

Moreover, $\phi\in H^1([0,T],H^1(\mathcal{A}))$ and

$$\sum_{i\in\mathcal{M}}||\phi_{it}(t)||_2^2 + \sum_{i\in\mathcal{M}}\int_0^T||\phi_x'(t)||_2^2dt \leq K. \tag{2.4}$$

Proof. Due to the fact that A_1 generates a contraction semigroup in X in [14], we obtain

$$\phi(t) = \mathcal{T}_1(t)\phi_0 + \int_0^t\mathcal{T}_1(t-s)g(s)ds. \tag{2.5}$$

Moreover, we set

$$\mathcal{F}_1(t) = \int_0^t\mathcal{T}_1(s)g(s)ds, \tag{2.6}$$

since $\mathcal{F}_1(t) \in C^1([0,T]; L^2(\mathcal{A}))$, then we have

$$\mathcal{F}_1'(t) = \int_0^t \mathcal{T}_1(s) g'(t-s) ds + \mathcal{T}_1 g(0). \tag{2.7}$$

Moreover, $\mathcal{F}_1 \in C([0,T]; D(A_1))$ and $A_1\mathcal{F}_1(t) = \mathcal{F}_1'(t) - g(t)$, hence, we obtain

$$\begin{aligned} ||\phi(t)||_{D(A_1)} &\leq ||\phi_0||_{D(A_1)} + ||\mathcal{F}_1(t)||_X + ||A_1\mathcal{F}_1(t)||_X \\ &\leq ||\phi_0||_{D(A_1)} + \int_0^t ||g(s)||_X ds + ||\mathcal{F}_1'(t)||_X + ||g(t)||_X. \end{aligned} \tag{2.8}$$

Now, using (2.5) we obtain

$$\begin{aligned} &||\phi(t)||_{D(A_1)} \leq ||\phi_0||_{D(A_1)} + ||g(0)||_X + ||g(t)||_X \\ &+ t\bigg(\sup_{t\in[0,T]} ||g'(t)||_X + \sup_{t\in[0,T]} ||g(t)||_X \bigg), \end{aligned} \tag{2.9}$$

ie $\sup_{t\in[0,T]} ||\phi||_{H^2} \leq K$.

To prove the last claim, letting $\Delta^k f = f(x,t+k) - f(x,t)$, employing the equation (2.1), for all $k \in \mathbb{R}$ with $|k| \leq \min\{\delta, T-\tau\}$, we have

$$\int_\delta^\tau \int_{I_i} \bigg(\Delta^k\phi_{it}\Delta^k\phi_i - \Delta^k\phi_{ixx}\Delta^k\phi_i + \Delta^k h(\phi_i)\Delta^h\phi_i + \Delta^k\phi_i\Delta^k g_i \bigg) dxdt = 0. \tag{2.10}$$

Moreover, using the Lemma 2.1 we conclude $J_1 \leq 0$, then there holds

$$\begin{aligned} &\sum_{i\in\mathcal{M}} \bigg\{ \int_{I_i} (\Delta^k\phi_i)^2 dx + \eta_{i,1} \int_\delta^\tau \int_{I_i} (\Delta^k\phi_{ix})^2 dxdt \bigg\} \\ &\leq \underbrace{\int_\delta^t \int_{I_i} \Delta^k\phi_{ixx}\Delta^k\phi_i dxdt}_{J_1} + C \sum_{i\in\mathcal{M}} \bigg(\int_{I_i} (\Delta^k\phi_i(\delta))^2 dx + \int_\delta^t \int_{I_i} (\Delta^k g_i)^2 dxdt \bigg) \\ &\leq C \sum_{i\in\mathcal{M}} \bigg(\int_{I_i} (\Delta^k\phi_i(\delta))^2 dx + \int_\delta^t \int_{I_i} (\Delta^k g_i)^2 dxdt \bigg). \end{aligned} \tag{2.11}$$

where $C > 0$ is constant, we divide the equalities (2.11) by k^2, and letting $h, \delta \to 0$, since $\phi, g \in C^1([0,T]; L^2(\mathcal{A}))$, then the inequality (2.11) implies that $\phi \in H^1([0,T]; H^1(\mathcal{A}))$ and (2.4).

Next, we prove the well-posedness results of problem (2.2), the operator A_2 generates a contraction semigroup in Y in [14, 6], then $F(t, U(t)) \in C^1([0,T]; X)$, the problem (2.2) has a unique solution U

$$U(t) = \mathcal{T}_2(t)U_0 + \int_0^t \mathcal{T}_2(t-s) F(s, U(s)) ds, \tag{2.12}$$

where $\mathcal{T}_2(t)$ is the contraction semigroup generated by A_2.

Let X and Y be the Hilbert spaces defined in the first part of this paper. For $T > 0$, we set

$$\mathbb{H}_{\phi,T} = C([0,T]; H^2(\mathcal{A})) \cap C^1([0,T]; L^2(\mathcal{A})) \cap H^1([0,T]; H^1(\mathcal{A}))$$

and

$$\mathbb{X} = C^1([0,T], X), \quad \mathbb{Y} = C([0,T], Y), \quad \mathbb{H}_{U,T} = \mathbb{X} \cap \mathbb{Y},$$

with the norm

$$||U||_{\mathbb{Y}} = \sup_{t\in[0,T]} ||U(t)||_Y, \ ||U||_{\mathbb{X}} = \sup_{t\in[0,T]} (||U(t)||_X + ||U'(t)||_X),$$

$$||U||_{\mathbb{H}_{U,T}} = \sup_{t\in[0,T]} (||U(t)||_Y + ||U'(t)||_X).$$

We introduce the function $F : [0,T] \to X$,

$$F(t, U(t)) = \Big(\mu^+(f(t), f_x(t), u^+, u^-), \mu^-(f(t), f_x(t), u^+, u^-)\Big),$$

Moreover, given $K, Q > 0$, for all $f \in \mathbb{H}_{\phi,T}$ with $\sup_{[0,T]} ||f(t)||_{H^2(\mathcal{A})} \le K$, and $U_1 = (u_1^+, u_1^-), U_2 = (u_2^+, u_2^-) \in \mathbb{H}_{U,T}$ with $||U_1||_{\mathbb{H}_{U,T}}, \ ||U_2||_{\mathbb{H}_{U,T}} \le Q$, then there exist positive constants $R(K), R(Q)$, such that

$$\sup_{t\in[0,T]} ||f(t)||_{L^\infty(\mathcal{A})}, \ \sup_{t\in[0,T]} ||f_x(t)||_{L^\infty(\mathcal{A})} \le R(K), \tag{2.13}$$

$$\sup_{t\in[0,T]} ||u_1^\pm(t)||_{L^\infty(\mathcal{A})} \le R(Q), \ \sup_{t\in[0,T]} ||u_2^\pm(t)||_{L^\infty(\mathcal{A})} \le R(Q), \tag{2.14}$$

using the assumption (H_1), then the following inequality holds

$$\sup_{t\in[0,T]} ||F(t, U_1(t)) - F(t, U_2(t))||_X \le C_1(K, Q) \sup_{t\in[0,T]} ||U_1 - U_2||_X, \tag{2.15}$$

where $C_1(K, Q)$ is positive constant depending on D_μ, and

$$D_\mu = \max_{i\in\mathcal{M}} \left\{ \sup_{[-R(K),R(K)]\times[-R(Q),R(Q)]} \Big(|\nabla\mu_i^+| + |\nabla\mu_i^-|\Big) \right\}. \tag{2.16}$$

Let $\Delta^h f = f(t+h) - f(t)$, $|h| \le \min\{\delta, T-\tau\}$, using (2.15) we obtain

$$\begin{aligned} &\int_\delta^{T-\tau} \left\| \frac{1}{h}\Big(\Delta^h F(t, U_1(t)) - \Delta^h F(t, U_2(t))\Big) \right\|_X dt \\ &\le \int_\delta^{T-\tau} \left\| \frac{1}{h}\Big(\Delta^h U_1(t) - \Delta^h U_2(t)\Big) \right\|_X dt. \end{aligned} \tag{2.17}$$

Since $F(t) \in C^1([0,T]; X)$ and $U_1, U_2 \in \mathbb{H}_{U,T}$, letting $h, \delta, \tau \to 0$ yields

$$\sup_{t\in[0,T]} ||F'(t, U_1(t)) - F'(t, U_2(t))||_X \le C_2(K, Q) \sup_{t\in[0,T]} ||U_1(t) - U_2(t)||_{\mathbb{H}_{U,T}}, \tag{2.18}$$

where $C_2(K, Q)$ is positive constant depending on D_μ, K, Q.

Proposition 2.2. *If assumption (H_1) holds. Then for every $U_0 = (u_0^+, u_0^-) \in D(A_2)$ and for every $T > 0$, $f \in \mathbb{H}_{\phi,T}$, $v \in \mathbb{H}_{U,T}$, there exists a unique solution U to problem (2.2), where $F(t)$ is given in (2.12). Moreover, $Q_1 > 2||U_0||_{D(A_2)} + ||F(0)||_X$ and $Q, K > 0$ there exists $T > 0$ such that*

$$||U||_{\mathbb{X}} \le Q_1 \textit{ and } ||U||_{\mathbb{Y}} \le (1 + \frac{1}{\underline{\lambda}})Q_1 \textit{ for all } T_0 \in [0,T), \tag{2.19}$$

wherever $\sup_{t\in[0,T]} ||f(t)||_{H^2(\mathcal{A})} \le K$, $||v||_{\mathbb{H}_{U,T}} \le Q$ and $\underline{\lambda} = \min_{i\in\mathcal{M}} \lambda_i$.

Proof. Let $Q, K > 0$ be fixed arbitrarily and $\sup_{t\in[0,T]} ||f(t)||_{H^2(\mathcal{A})} \le K,\ ||v||_{\mathbb{H}_{U,T}} \le Q$, using (2.15) and (2.18) with $U_1(t) = v(t), F(U_2(t)) = 0$, we derive

$$\sup_{t\in[0,T_0]} ||F(t)||_X \le C_1(K,Q) \sup_{t\in[0,T_0]} ||v(t)||_X, \quad \sup_{t\in[0,T_0]} ||F'(t)||_X \le C_2(K,Q)||v||_{\mathbb{H}_{U,T_0}} \quad (2.20)$$

with $C_1(K,Q), C_2(K,Q)$ are positive constants. Thanks to (2.12), we see that

$$||U(t)||_X \le ||\mathcal{T}_2(t)U_0|| + \int_0^t ||\mathcal{T}_2(t-s)F(s)||_X ds. \quad (2.21)$$

Since $\mathcal{T}_2(t)$ is contraction semigroup, then $||\mathcal{T}_2(t)|| \le 1$, thus for all $t \in [0, T_0]$, we obtain

$$\begin{aligned} ||U(t)||_X &\le ||U_0||_X + \int_0^t ||(F(s)||_X ds \\ &\le ||U_0||_X + tC_1(K,Q) \sup_{t\in[0,T_0]} ||v||_X. \end{aligned} \quad (2.22)$$

Moreover, we have

$$\mathcal{F}_2(t) = \int_0^t \mathcal{T}_2(t-s)F(s)ds. \quad (2.23)$$

The regularity of F implies $\mathcal{F}_2 \in C^1([0,T_0];X) \cap C([0,T_0];D(A_2))$,

$$A_2\mathcal{F}_2 = \mathcal{F}_2'(t) - F(t) \quad (2.24)$$

and

$$\mathcal{F}_2'(t) = \int_0^t \mathcal{T}(t-s)F'(s)ds + \mathcal{T}_2(t)F(0). \quad (2.25)$$

Then, using Lemma 2.1 and (2.24), for all $t \in [0, T_0]$ we have

$$\begin{aligned} ||U'(t)||_X &\le ||A_2U_0||_X + \int_0^t ||(F'(s)||_X ds + ||F(0)||_X \\ &\le ||A_2U_0||_X + tC_2(K,Q)||v||_{\mathbb{H}_{U,T_0}} + ||F(0)||_X. \end{aligned} \quad (2.26)$$

Combining (2.22) and (2.26), by choosing T_0 is small enough, we get

$$\sup_{t\in[0,T]} (||U(t)||_X + ||U'(t)||_X) \le Q_1. \quad (2.27)$$

By direct calculations, for every $t \in [0, T_0]$, there holds

$$\begin{aligned} ||A_2U(t)||_X &\le ||A_2\mathcal{T}_2(t)U_0||_X + ||A_2\mathcal{F}_2(t)||_X \\ &\le ||A_2U_0||_X + \int_0^t ||F'(s)||_X ds + ||F(t) - \mathcal{T}_2(t)(F(0)||_X \\ &\le ||A_2U_0||_X + \int_0^t ||F'(s)||ds + ||F(0) - \mathcal{T}_2(t)F(0)||_X \\ &\quad + ||F(t) - F(0)||_X \\ &\le ||A_2U_0||_X + 2tC_2(K,Q)||v||_{\mathbb{H}_{U,T_0}} + ||F(0)||_X. \end{aligned} \quad (2.28)$$

Taking T_0 sufficiently small and setting $\underline{\lambda} = \min_{i\in\mathcal{M}} \lambda_i$. Combining (2.21) and (2.28) give (2.19).

Proof of Theorem 1.1.(i)(**Uniqueness**) Suppose that (U,ϕ) and $(\hat{U},\hat{\phi})$ are two solutions to problem $(1.4)-(1.12)$ in $[0,T]$ with the same initial date $U_0=(u_0^+,u_0^-)\in D(A_2)$, such that

$$\sup_{t\in[0,T]}||u^{\pm}(t)||_{L^\infty(\mathcal{A})},\quad \sup_{t\in[0,T]}||\hat{u}^{\pm}(t)||_{L^\infty(\mathcal{A})}\le R,$$

$$\sup_{t\in[0,T]}||\phi(t)||_{L^\infty(\mathcal{A})},\quad \sup_{t\in[0,T]}||\hat{\phi}(t)||_{L^\infty(\mathcal{A})};\quad \sup_{t\in[0,T]}||\phi_x(t)||_{L^\infty(\mathcal{A})},\quad \sup_{t\in[0,T]}||\hat{\phi}_x(t)||_{L^\infty(\mathcal{A})}\le R,$$

then, arguing as in Lemma 2.2, we obtain

$$\langle A_2(U(s)-\hat{U}(s)),U(s)-\hat{U}(s)\rangle_X\le 0,\text{for all } s\in[0,T], \tag{2.29}$$

where $U=(u^+,u^-),\hat{U}=(\hat{u}^+,\hat{u}^-)$.

In view of the above inequality, multiplying the first two equations in (1.4) by $U-\hat{U}$ and then integrating in $[0,T]$, we show

$$\begin{aligned}&||u^+(t)-\hat{u}^+(t)||^2_{L^2(\mathcal{A})}+||u^-(t)-\hat{u}^-(t)||^2_{L^2(\mathcal{A})}\\ \le& C_1(R)\int_0^t\Big(||\phi(s)-\hat{\phi}(s)||^2_{L_2(\mathcal{A})}+||u^+(s)-\hat{u}^+(s)||^2_{L^2(\mathcal{A})}\\ &+||u^-(s)-\hat{u}^-(s)||^2_{L^2(\mathcal{A})}\Big)ds\end{aligned} \tag{2.30}$$

for some $C_1(R)>0$ only depending on $\sum_{i\in\mathcal{M}}\Big(\max_{[-R,R]^4}|\nabla\mu_i^+|+\max_{[-R,R]^4}|\nabla\mu_i^-|\Big)$. Moreover, from (2.1) it follows that for every $t\in[0,T]$ and $i\in\mathcal{M}$ there holds

$$\begin{aligned}(\phi_i(t)-\hat{\phi}_i(t))_t=&(\phi_i(t)-\hat{\phi}_i(t))_{xx}-(h(\phi_i)-h(\hat{\phi}_i))\\ &+\Big(\beta(u_i^+(t),u_i^-(t))-\beta(\hat{u}_i^+(t),\hat{u}_i^-(t))\Big).\end{aligned} \tag{2.31}$$

Multiplying the above equation by $\phi_i-\hat{\phi}_i$ and then integrating in $[0,T]$, we obtain

$$\begin{aligned}&||\phi(t)-\hat{\phi}(t)||^2_2+\int_0^t\Big(||\phi(t)-\hat{\phi}(t)||^2_2+||\phi_x(t)-\hat{\phi}_x(t)||^2_2\Big)\\ &\le C_2(R)(||u^+(t)-\hat{u}^+(t)||^2_{L^2(\mathcal{A})}+||u^-(t)-\hat{u}^-(t)||^2_{L^2(\mathcal{A})}),\end{aligned} \tag{2.32}$$

for some $C_2(R)>0$ independent on t. Then the uniqueness results follow from (2.30) and (2.32) by the Gronwall Lemma.

(ii)(**Existence**) Now we are going to prove existence results. Let $U_0=(u_0^+,u_0^-)\in D(A_2)$ and ϕ_0 be the solution to problem (2.1) with $g(t)=\beta(u_{i0}^+,u_{i0}^-)$ and let Q_1 be a quantity such that

$$Q_1>||U_0||_{D(A_2)}+||\mu(\phi_0,\phi_{0x},u_0^+,u_0^-)||_X. \tag{2.33}$$

Moreover, let $T>0$ to be chosen below and let $R(Q_1)$ be a quantity such that

$$\sup_{[0,T]}||U||_{(L^\infty(\mathcal{A}))^2}\le R(Q_1),\quad\text{for all }\ U\in\mathbb{H}_{U,T}\ \text{ such that } ||U||_{\mathbb{Y}}\le(1+\frac{1}{\lambda})Q_1. \tag{2.34}$$

By arguing as in Proposition 2.1, we know that there exists a constant $K>0$ such that

$$\sup_{t\in[0,T]}||\phi(t)||_{H^2(\mathcal{A})}\le K \tag{2.35}$$

and

$$\sum_{i\in\mathcal{M}} ||\phi_{it}(t)||_2^2 + \sum_{i\in\mathcal{M}} \int_0^T ||\phi_x'(t)||_2^2 dt \le K, \tag{2.36}$$

whenever $\phi \in \mathbb{H}_{\phi,T}$ is the solution of problem (2.1) with $g_i = \beta_i(u^+(t), u^-(t))$ and $U = (u^+, u^-) \in \mathbb{X}$ satisfying $||U||_{\mathbb{X}} \le Q_1$. Moreover, let $R(K)$ be a quantity such that

$$\sup_{[0,T]} ||\phi_x(t)||_{L^\infty(\mathcal{A})} \le R(K), \;\; \sup_{[0,T]} ||\phi(t)||_{L^\infty(\mathcal{A})} \le R(K)$$

for all $\phi \in \mathbb{H}_{\phi,T}$ such that $||\phi||_{\mathbb{H}_{\phi,T}} \le K$. Now, we consider

$$B_{Q_1K} = \begin{cases} (U,\phi) \in \mathbb{H}_{U,T} \times \mathbb{H}_{\phi,T}, \quad (U(0),\phi(0)) = (U_0,\phi_0), \\ ||U||_{\mathbb{X}} \le Q_1, \quad ||U||_{\mathbb{Y}} \le (1+\frac{1}{\underline{\lambda}})Q_1, \\ \sup\limits_{t\in[0,T]} ||\phi||_{H^2(\mathcal{A})}, \sum\limits_{i\in\mathcal{M}} \left(||\phi_{it}(t)||_2^2 + \int_0^T ||\phi_{ixt}(t)||_2^2 dt \right) \le K. \end{cases}$$

We equip $B_{Q_1,K}$ with the distance generated by the norms of $\mathbb{H}_{U,T}$ and $\mathbb{H}_{\phi,T}$ and define a map G on $B_{Q_1,K}$ in the following way: $(V,\psi) = (v^+, v^-, \psi) \in B_{Q_1,K}$, then $(U,\phi) = G(V,\psi)$, such that ϕ is the solution to problem (2.1) where $g_i = \beta_i(u_i^+, u_i^-)$ and U is the solution to problem (2.2) where $F(t,U(t)) = \left\{\mu_i^+(\phi,\phi_x,u^+,u^-), \mu_i^-(\phi,\phi_x+,u^+,u^-)\right\}_{i\in\mathcal{M}}$.

Proposition 2.1 and Proposition 2.2 ensure that the map is well defined on $B_{Q_1,K}$. Moreover, we are going to prove that the operator G is contraction mapping. In view of $(2.12), 2.15)$ and (2.18), for $t \in [0,T_0]$, there holds

$$\begin{aligned} ||U(t) - \hat{U}(t)||_X &\le \int_0^t ||\mathcal{T}_2(t-s)|| ||F(s) - \hat{F}(s)|| \\ &\le T_0 \bar{C}_1(K,Q_1) \sup_{[0,T]} ||V - \hat{V}||_X, \end{aligned} \tag{2.37}$$

$$\begin{aligned} ||U'(t) - \hat{U}'(t)||_X &\le \int_0^t |\mathcal{T}_2(t-s)|| ||F'(s) - \hat{F}'(s)|| ds \\ &\le T_0 \bar{C}_2(K,Q_1) \sup_{[0,T_0]} ||V - \hat{V}||_{\mathbb{H}_{U,T_0}} \end{aligned} \tag{2.38}$$

and

$$\begin{aligned} ||A_2U(t) - A_2\hat{U}(t)||_X &\le ||\tilde{U}(t) - \hat{U}(t)||_X + ||F'(t) - \hat{F}'(t)||_X \\ &\le T_0 \bar{C}_3(K,Q_1) \sup_{[0,T_0]} ||V - \hat{V}||_{\mathbb{H}_{U,T_0}}, \end{aligned} \tag{2.39}$$

where $\bar{C}_1(K,Q_1), \bar{C}_2(K,Q_2)$ are suitable positive constants depending on $K,\ Q_1$.

Next, by discussing about the previous estimates for ϕ, we have

$$\begin{aligned} ||g'(t) - \hat{g}'(t)||_{L^2(\mathcal{A})} &= ||\beta'(u_i^+, u_i^-(t)) - \beta'(\hat{u}_i^+(t), \hat{u}_i^-(t))||_{L^2(\mathcal{A})} \\ &\le \tilde{D}_1^{Q_1} ||U(t) - \hat{U}(t)||_Y^2 + \tilde{D}_2^{Q_1} ||U'(t) - \hat{U}'(t)||_X^2 \end{aligned} \tag{2.40}$$

and

$$\sup_{[0,T_0]} ||\phi(t) - \hat{\phi}(t)||_{H^2(\mathcal{A})} \le D_1 T_0 (\sup_{[0,T_0]} ||U(t) - \hat{U}(t)||_Y + \sup_{[0,T_0]} ||U'(t) - \hat{U}'(t)||_X), \tag{2.41}$$

where $\tilde{D}_1^{Q_1}$ and $\tilde{D}_2^{Q_1}$ are suitable positive constant depending on Q_1, D_1 only depends $D_\beta = \sum_{i\in\mathcal{M}} ||\nabla\beta||_{L^\infty(R^2)}$.

From (2.41) and the above discussion, we obtain

$$\begin{aligned}
&\frac{1}{2}\sup_{[0,T_0]} ||\phi'(t) - \tilde{\phi}'(t)||_2^2 + \int_0^{T_0} ||\phi_{xt}(t) - \tilde{\phi}_{xt}(t)||_2^2 dt \\
&\le D_2 T_0 \Big(\sup_{[0,T_0]} ||U(t) - \hat{U}(t)||_Y^2 + \sup_{[0,T_0]} ||U'(t) - \hat{U}'(t)||_X^2 \Big),
\end{aligned} \tag{2.42}$$

$$||\tilde{U} - \hat{U}||_{\mathbb{H}_{U,T_0}} \le T_0 \bar{C}_4(K, Q_1) ||V - \hat{V}||_{\mathbb{H}_{U,T_0}},$$

for a suitable constant $\bar{C}_4(K, Q_1)$ independent of T_0. If T_0 is suitable small, then the operator G is a contraction mapping on $B_{Q_1,K}$, then the unique fixed point $(U,\phi) \in B_{Q_1,K}$ is the solution to the problem $(1.4) - (1.12)$.

3 Global existence

In this section, we mainly prove the existence of the global solution to the problem $(1.4) - (1.12)$. The local solution to problem $(1.4) - (1.12)$ is given by Theorem 1.1,

$$\begin{aligned}
&(u^+, u^-) \in C^1([0,T]; X) \cap C([0,T]; Y), \\
&\phi \in C([0,T]; H^2(\mathcal{A})) \cap C^1([0,T]; L^2(\mathcal{A})) \cap H^1([0,T]; H^1(\mathcal{A})),
\end{aligned} \tag{3.1}$$

which can be extended to time interval $[0,+\infty)$ by the Continuation Principle [27]. First we introduce the functional

$$\begin{aligned}
&N_T^2(u^+, u^-, \phi) \\
=&\sum_{i\in\mathcal{M}} \Big(\sup_{[0,T]} ||u_i^+(t)||_{H^1}^2 + \sup_{[0,T]} ||u_i^-(t)||_{H^1}^2 + \sup_{[0,T]} ||\phi_i^-(t)||_{H^2}^2 \Big) \\
&+ \sum_{i\in\mathcal{M}} \int_0^T \Big(||u_i^-(t)||_{H^1}^2 + ||u_i^+(t)||_{H^1}^2 + ||u_{it}^-(t)||_2^2 + ||u_{it}^+(t)||_2^2 \Big) \\
&+ \sum_{i\in\mathcal{M}} \int_0^T \Big(||\phi_{ix}(t)||_{H^1}^2 + ||\phi_{ixt}(t)||_2^2 \Big).
\end{aligned} \tag{3.2}$$

If we prove that the functional satisfies the following inequality

$$N_T^2(u^+, u^+, \phi) \le \hat{C}_1 N_0^2(u^+, u^+, \phi) + \hat{C}_2 N_T^3(u^+, u^+, \phi), \tag{3.3}$$

then according to Nishida's Lemma in [27], this fact proves Theorem 1.2. Hence, in order to get (3.3), we need some prior estimates.

Proposition 3.1. *Let assumption* (H_1) *holds and* (u^+, u^+, ϕ) *be a local solution* (3.1) *to problem* $(1.4) - (1.12)$. *Then*

$$\begin{aligned}
&\sum_{i\in\mathcal{M}} \Big(\sup_{[0,T]} ||u_i^+(t)||_2^2 + \sup_{[0,T]} ||u_i^+(t)||_2^2 \Big) \\
\le& C_1 \sum_{i\in\mathcal{M}} \Big\{ ||u_{0i}^+||_2^2 + ||u_{0i}^-||_2^2 + \int_0^T (||u_i^+||_2^2 + ||u_i^-||_2^2) dt \\
&+ \sup_{[0,T]} \Big(||u_i^+(t)||_{H^1} + ||u_i^-(t)||_{H^1} \Big) \int_0^T \Big(||\phi_i(t)||_2^2 + ||\phi_{ix}(t)||_2^2 \Big) dt \Big\},
\end{aligned} \tag{3.4}$$

for a suitable positive constant C_1.

Proof. Multiplying the first equation and the second equation in (1.4) by u_i^+, u_i^-, respectively, and summing up I_i, $i \in \mathcal{M}$; after by (H_1) we have estimated

$$\begin{aligned}
&\frac{1}{2}\sum_{i\in\mathcal{M}}\int_0^T\int_{I_i}\frac{d}{dt}\left\{(u_i^+)^2+(u_i^-)^2\right\}dxdt+\underbrace{\sum_{i\in\mathcal{M}}\int_0^T\int_{I_i}\left\{\lambda_i u_{ix}^+u_i^+-\lambda_i u_{ix}^-u_i^-\right\}dxdt}_{J_1}\\
&=\sum_{i\in\mathcal{M}}\int_0^T\int_{I_i}\left\{\mu^+(\phi_i,\phi_{ix},u_i^+,u_i^-)u_i^++\mu^-(\phi_i,\phi_{ix},u_i^+,u_i^-)u_i^-\right\}dxdt\\
&\leq\sum_{i\in\mathcal{M}}\int_0^T\int_{I_i}D_\mu\left(\left|(\phi_i+\phi_{ix}+u_i^++u_i^-)u_i^+\right|+\left|(\phi_i+\phi_{ix}+u_i^++u_i^-)u_i^-\right|\right)dxdt\\
&\leq\sum_{i\in\mathcal{M}}\int_0^T\int_{I_i}D_\mu\left|\left\{(\phi_i+\phi_{ix})(u_i^++u_i^-)+2(u_i^+)^2+2(u_i^-)^2\right\}\right|dxdt,\\
&\leq\sum_{i\in\mathcal{M}}\left\{\sup_{[0,T]}\left(||u_i^+(t)||_{H^1}+||u_i^-(t)||_{H^1}\right)\int_0^T\left(||\phi_i(t)||_2^2+||\phi_{ix}(t)||_2^2\right)dt\right.\\
&\left.+\int_0^T(||u_i^+||_2^2+||u_i^-||_2^2)dt\right\},
\end{aligned}\tag{3.5}$$

where D_μ is the coefficient in (2.16). Due to Lemma 2.2, it implies that $J_1 \geq 0$. By above discussion, we obtain (3.4). This completes the proof.

Proposition 3.2. *Let assumption* (H_1) *holds and* (u^+, u^+, ϕ) *be a local solution* (3.1) *to problem* $(1.4)-(1.12)$. *Then*

$$\begin{aligned}
&\sum_{i\in\mathcal{M}}\sup_{[0,T]}\left(||u_{it}^+(t)||_2^2+||u_{it}^-(t)||_2^2\right)\\
\leq&C_2\sum_{i\in\mathcal{M}}\left\{||u_{i0}^+||_{H^1}^2+||u_{i0}^-||_{H^1}^2+||\phi_{i0}||_{H^2}^2\right.\\
&\left.+\int_0^T\left[(||u_{it}^+(t)||_2^2+||u_{it}^-(t)||_2^2)+(||\phi_{it}(t)||_2^2+||\phi_{ixt}(t)||_2^2)\right]dt\right\},
\end{aligned}\tag{3.6}$$

for a suitable positive constant C_2.

Proof. Let $\triangle^h u_i^\pm = u_i^\pm(x,t+h) - u_i^\pm(x,t)$, $i \in \mathcal{M}$, we have

$$\begin{cases}\left(\Delta^h u_{it}^++\lambda_i\Delta^h u_{ix}^+\right)\Delta^h u_i^+=\Delta^h\mu_i^+(\phi_i,\phi_{ix},u_i^+,u_i^-)\Delta^h u_i^+,\\ \left(\Delta^h u_{it}^--\lambda_i\Delta^h u_{ix}^-\right)\Delta^h u_i^-=\Delta^h\mu_i^-(\phi_i,\phi_{ix},u_i^+,u_i^-)\Delta^h u_i^-.\end{cases}\tag{3.7}$$

Summing the above two equations and integrating over $I_i \times (\delta, \tau)$, for $0 < \tau < T$, $|h| \leq$

$\min\{\delta, T-\tau\}$; after using Lemma 2.2 yields

$$\sum_{i\in\mathcal{M}}\int_\delta^\tau\int_{I_i}\left(\frac{(\Delta^h u_i^+)^2+(\Delta^h u_i^-)^2}{2}\right)_t dxdt+\underbrace{\sum_{i\in\mathcal{M}}\int_\delta^\tau\int_{I_i}\lambda_i\left(\frac{(\Delta^h u_i^+)^2-(\Delta^h u_i^-)^2}{2}\right)_x dxdt}_{J_2}$$
$$=\sum_{i\in\mathcal{M}}\int_\delta^\tau\int_{I_i}\left\{\Delta^h\mu^+(\phi_i,\phi_{ix},u_i^+,u_i^-)\Delta^h u_i^++\Delta^h\mu^-(\phi_i,\phi_{ix},u_i^+,u_i^-)\Delta^h u_i^-\right\}dxdt$$
$$\leq\sum_{i\in\mathcal{M}}\int_\delta^\tau\int_{I_i}D_\mu\left\{\left|(\Delta^h\phi+\Delta^h\phi_{ix})(\Delta^h u_i^++\Delta^h u_i^-)\right|+2(\Delta^h u_i^+)^2+2(\Delta^h u_i^-)^2\right\}dxdt. \tag{3.8}$$

Now we divide the inequality (3.8) by h^2, and letting h and δ go to zero, we obtain the claim.

Proposition 3.3. *Let assumption* (H_1) *holds and* (u^+,u^+,ϕ) *be a local solution* (3.1) *to problem* $(1.4)-(1.12)$*. Then*

$$\sum_{i\in\mathcal{M}}\int_0^T\left(||u_{ix}^+(t)||_2^2+||u_{ix}^-(t)||_2^2\right)dt$$
$$\leq C_3\sum_{i\in\mathcal{M}}\left\{\int_0^T\left(||u_{it}^+(t)||_2^2+||u_{it}^-(t)||_2^2+||u_i^+(t)||_{H^1}^2+||u_i^-(t)||_{H^1}^2\right)dt\right. \tag{3.9}$$
$$\left.+\int_0^T\left(||\phi_{ix}(t)||_2^2+||\phi_i(t)||_2^2\right)dt\right\},$$

for a suitable positive constant C_3.

Proof. Multiplying the first and the second equation in (1.4) by u_{ix}^+, u_{ix}^-, respectively, and summing up $i\in\mathcal{M}$; after using assumption (H_1) we obtain

$$\sum_{i\in\mathcal{M}}\int_{I_i}\int_0^T\left(u_{it}^+u_{ix}^++u_{it}^-u_{ix}^-\right)dxdt$$
$$=\sum_{i\in\mathcal{M}}\int_{I_i}\int_0^T\left(\mu^+(\phi,\phi_{ix},u_i^+,u_i^-)u_{ix}^++\mu^-(\phi,\phi_{ix},u_i^+,u_i^-)u_{ix}^-\right)dxdt$$
$$\leq\sum_{i\in\mathcal{M}}\int_{I_i}\int_0^T D_\mu\left(\left|(\phi+\phi_{ix})(u_{ix}^-+u_{ix}^+)\right|+|u_i^+u_{ix}^-|+|u_i^-u_{ix}^-|+|u_i^+u_{ix}^+|+|u_i^-u_{ix}^-|\right)dxdt$$
$$\leq C_3\sum_{i\in\mathcal{M}}\left\{\int_0^T\left(||u_{it}^+(t)||_2^2+||u_{it}^-(t)||_2^2+||u_i^+(t)||_{H^1}^2+||u_i^-(t)||_{H^1}^2\right)dt\right.$$
$$\left.+\int_0^T\left(||\phi_{ix}(t)||_2^2+||\phi_i(t)||_2^2\right)dt\right\}, \tag{3.10}$$

where D_μ is the coefficients in (2.16), we obtain the claim.

Proposition 3.4. *Let assumption* (H_1) *holds and* (u^+,u^+,ϕ) *be a local solution* (3.1) *to* *problem* $(1.4)-(1.12)$*. Then*

$$
\begin{aligned}
&\sum_{i\in\mathcal{M}}\sup_{[0,T]}\left(||u_{ix}^{+}(t)||_2^2+||u_{ix}^{-}(t)||_2^2\right)\\
&\leq C_4\sum_{i\in\mathcal{M}}\sup_{[0,T]}\left(||u_{it}^{+}(t)||_2^2+||u_{it}^{-}(t)||_2^2+||u_{i}^{+}(t)||_{H^1}^2+||u_{i}^{-}(t)||_{H^1}^2\right)\\
&+C_4\sum_{i\in\mathcal{M}}\sup_{[0,T]}\left(||\phi_{ix}(t)||_2^2+||\phi_i(t)||_2^2\right),
\end{aligned}
\tag{3.11}
$$

for a suitable positive constant C_4*.*

Proof. Multiplying the first and the second equation in (1.4) by u_{ix}^{+}, u_{ix}^{-}, respectively, and summing up $i\in\mathcal{M}$. Moreover, using the Cauchy-Schwarz inequality, we obtain the claim.

Proposition 3.5. *Let assumptions* $(H_2),(H_3)$ *hold and* (u^+,u^+,ϕ) *be a local solution* (3.1) *to problem* $(1.4)-(1.12)$*. Then*

$$
\begin{aligned}
&\sum_{i\in\mathcal{M}}\sup_{[0,T]}||\phi_{it}(t)||_2^2+\sum_{i\in\mathcal{M}}\int_0^T(||\phi_{it}(t)||_2^2+||\phi_{itx}(t)||_2^2)dt\\
&\leq C_5\bigg(\sum_{i\in\mathcal{M}}||\phi_{i0}||_{H^2}^2+||u_{i0}^{+}||_{H^1}^2+||u_{i0}^{-}||_{H^1}^2+\int_0^T(||u_{it}^{+}(t)||_2^2+||u_{it}^{-}(t)||_2^2)dt\bigg),
\end{aligned}
\tag{3.12}
$$

for a suitable positive constant C_5*.*

Proof. Using the same notation as in the proof of Proposition 3.2, by the third equation in (1.4) and assumptions $(H_2),(H_3)$, we get

$$
\begin{aligned}
&\sum_{i\in\mathcal{M}}\int_\delta^\tau\int_{I_i}\frac{(\Delta^h\phi)_{it}^2}{2}dxdt\\
&\leq\underbrace{\sum_{i\in\mathcal{M}}\int_\delta^\tau\int_{I_i}(\Delta^h\phi_{ixx})\Delta^h\phi_i dxdt}_{J_3}\\
&-\sum_{i\in\mathcal{M}}\int_\delta^\tau\int_{I_i}\eta_{i,1}(\Delta^h\phi_i)^2dxdt+\sum_{i\in\mathcal{M}}\int_\delta^\tau\int_{I_i}D_\beta|(\Delta^h u_i^{+}+\Delta^h u_i^{-})||\Delta^h\phi_i|dxdt,
\end{aligned}
\tag{3.13}
$$

where $D_\beta=\max_{i\in\mathcal{M}}(||\nabla\beta_i||_{L^\infty(R^2)})$.

Using Lemma 2.1, we see that

$$
\begin{aligned}
&\sum_{i\in\mathcal{M}}\int_\delta^\tau\int_{I_i}\frac{(\Delta^h\phi)_t^2}{2}dxdt+\sum_{i\in\mathcal{M}}\int_\delta^\tau\int_{I_i}(\Delta^h\phi_{ix})^2dxdt\\
&\leq\sum_{i\in\mathcal{M}}\int_\delta^\tau\int_{I_i}\left\{D_\beta|(\Delta^h u_i^{+}+\Delta^h u_i^{-})\Delta^h\phi_i|-\eta_{i,1}(\Delta^h\phi_i)^2\right\}dxdt.
\end{aligned}
\tag{3.14}
$$

Moreover, we divide the inequality (3.14) by h^2, by the Cauchy-Schwarz inequality and letting $h,\delta\to 0$, there holds

$$
\begin{aligned}
&\sum_{i\in\mathcal{M}}||\phi_{it}(t)||_2^2+\sum_{i\in\mathcal{M}}\int_0^T\left(||\phi_{it}(t)||_2^2+||\phi_{itx}(t)||_2^2\right)dt\\
&\leq C_5\sum_{i\in\mathcal{M}}\left\{||\phi_{it}(0)||_2^2+\int_0^t\left(||u_{it}^{+}(t)||_2^2+||u_{it}^{-}(t)||_2^2\right)\right\},
\end{aligned}
\tag{3.15}
$$

which implies (3.12). This completes the proof of Proposition 3.5.

Proposition 3.6. *Let assumptions* $(H_2),(H_3)$ *hold and* (u^+,u^+,ϕ) *be a local solution* (3.1) *to problem* (1.4) – (1.12). *Then*

$$\sum_{i\in\mathcal{M}}\sup_{[0,T]}\Big(||\phi_{ixx}(t)||_2^2+||\phi_{ix}(t)||_2^2\Big)\leq C_6\sum_{i\in\mathcal{M}}\sup_{[0,T]}\Big(||\phi_{it}(t)||_2^2+||u_i^+(t)||_2^2+||u_i^-(t)||_2^2\Big),\quad (3.16)$$

for a suitable positive constant C_6.

Proof. Multiplying the third equation in (1.4) by ϕ_{ixx}, and summing $i\in\mathcal{M}$; after, by Lemma 2.2 and assumptions (H_3), (H_2) we show that

$$\begin{aligned}&\sum_{i\in\mathcal{M}}\int_{I_i}\phi_{it}\phi_{ixx}dx\\&=\sum_{i\in\mathcal{M}}\int_{I_i}\beta(u_i^+,u_i^-)\phi_{ixx}dx+\sum_{i\in\mathcal{M}}\int_{I_i}(\phi_{ixx})^2dx-\sum_{i\in\mathcal{M}}\int_{I_i}h(\phi_i)\phi_{ixx}dx\\&\leq\sum_{i\in\mathcal{M}}\int_{I_i}D_\beta(|(u_i^+|+|u_i^-|)|\phi_{ixx}|dx+\sum_{i\in\mathcal{M}}\int_{I_i}\eta_{i,1}(\phi_{ix})^2dx+\sum_{i\in\mathcal{M}}\int_{I_i}(\phi_{ixx})^2dx,\end{aligned}\quad (3.17)$$

where $D_\beta=\max_{i\in\mathcal{M}}(||\nabla\beta_i||_{L^\infty(R^2)})$.

From (3.17) it follows that

$$\begin{aligned}&\sum_{i\in\mathcal{M}}\sup_{[0,T]}\Big(||\phi_{ixx}(t)||_2^2+||\phi_{ix}(t)||_2^2\Big)\\&\leq C_6\sum_{i\in\mathcal{M}}\sup_{[0,T]}\Big(||\phi_{it}(t)||_2^2+||u_i^+(t)||_2^2+||u_i^-(t)||_2^2\Big),\end{aligned}\quad (3.18)$$

we obtain the estimate (3.16).

Proposition 3.7. *Let assumptions* $(H_2),(H_3)$ *hold and* (u^+,u^+,ϕ) *be a local solution* (3.1) *to problem* (1.4) – (1.12). *Then*

$$\begin{aligned}&\sum_{i\in\mathcal{M}}\int_0^T\Big(||\phi_{ix}(t)||_2^2+||\phi_{ixx}(t)||_2^2\Big)dt\\&\leq C_7\sum_{i\in\mathcal{M}}\int_0^T\Big(||u_i^+(t)||_{H^1}^2+||u_i^-(t)||_{H^1}^2+||\phi_{it}(t)||_2^2\Big),\end{aligned}\quad (3.19)$$

for a suitable positive constant C_7.

Proof. Multiplying the third equation in (1.4) by ϕ_{ixx}, and summing $i\in\mathcal{M}$, we obtain

$$\begin{aligned}&\sum_{i\in\mathcal{M}}\int_0^T\int_{I_i}\phi_{it}\phi_{ixx}dxdt\\&=\sum_{i\in\mathcal{M}}\int_{I_i}\int_0^T\left\{\beta(u_i^+,u_i^-)\phi_{ixx}-h(\phi_i)\phi_{ixx}+(\phi_{ixx})^2\right\}dxdt\\&=\underbrace{\sum_{i\in\mathcal{M}}\int_0^T\int_{I_i}\Big(\beta(u_i^+,u_i^-)\phi_{ix}\Big)_x dxdt}_{J_4}-\sum_{i\in\mathcal{M}}\int_0^T\int_{I_i}h(\phi_i)\phi_{ixx}dxdt\\&-\sum_{i\in\mathcal{M}}\int_0^T\int_{I_i}(\beta(u_i^+,u_i^-))_x\phi_{ix}dxdt+\sum_{i\in\mathcal{M}}\int_0^T\int_{I_i}(\phi_{ixx})^2dxdt.\end{aligned}\quad (3.20)$$

Dealing with J_4 by (1.7) and (1.6), we find

$$
\begin{aligned}
J_4 &= \sum_{i\in\mathcal{M}} \int_0^T \int_{I_i} \left\{\beta(u_i^+, u_i^-)\phi_{ix}\right\}_x dxdt \\
&= \sum_{i\in\mathcal{M}} \int_0^T \left\{\beta(u_i^+(L_i,t), u_i^-(L_i,t))\phi_{ix}(L_i,t) - \beta(u_i^+(0,t), u_i^-(0,t))\phi_{ix}(0,t)\right\} dt \\
&= \sum_{i\in\mathcal{M}} \int_0^T D_\beta \left\{\left(|u_i^+(L_i,t)| + |u_i^-(L_i,t)|\right)|\phi_{ix}(L_i,t)|\right\} \\
&+ \sum_{i\in\mathcal{M}} \int_0^T D_\beta \left\{\left(|(u_i^+(0,t)| + |u_i^-(0,t))|\right)|\phi_{ix}(0,t)|\right\} dt, \\
&\leq \sum_{i\in\mathcal{M}} \int_0^T \left(||u_i^+(t)||_{L^\infty(I_i)} + ||u_i^-(t)||_{L^\infty(I_i)}\right)||\phi_{ix}||_{L^\infty(I_i)} dt \\
&\leq \sum_{i\in\mathcal{M}} \int_0^T C_7\left(||u_i^+||_{H^1}^2 + ||u_i^-||_{H^1}^2 + ||\phi_{ix}||_{H^1}^2\right) dt,
\end{aligned}
\tag{3.21}
$$

where $D_\beta = \max_{i\in\mathcal{M}}(||\nabla\beta_i||_{L^\infty(R^2)})$.

Hence, combining (3.20), (3.21) and using Lemma 2.1 entails that

$$
\begin{aligned}
&\sum_{i\in\mathcal{M}} \int_0^T \left(||\phi_{ix}(t)||_2^2 + ||\phi_{ixx}(t)||_2^2\right) dt \\
&\leq\leq C_7 \sum_{i\in\mathcal{M}} \int_0^T \left(||u_i^+(t)||_{H^1}^2 + ||u_i^-(t)||_{H^1}^2 + ||\phi_{it}(t)||_2^2\right),
\end{aligned}
\tag{3.22}
$$

we obtain the claim.

Proof of Theorem 1.2. Collecting all the energy estimates of Proposition 3.1-Proposition 3.7, we obtain the inequality (3.3) holding for the functional N_T introduced at the beginning of the section. Applying Lemma 4.4.1 in [16] and [27], then we can conclude that for suitably small N_0, N_T remains bounded for all $T > 0$; this fact proves Theorem 1.2.

4 Stationary solution on acyclic network

In this section, we restrict our attention to acyclic graphs and research the question of the existence and uniqueness of stationary solutions to problem (1.4) − (1.12), with fixed mass (1.16). Let $v_i = \lambda_i(u_i^+ - u_i^-)$, then the flux conservation at node N,

$$
\sum_{i\in I_N} v(L_i,t) - \sum_{i\in O_N} v(0,t) = 0, \tag{4.1}
$$

and the existence and uniqueness of stationary solutions of problem (1.4) − (1.12) imply that v is constant on each arcs and has to be null on the external edges. Moreover, for the case of the acyclic graph, the above equality reduces to

$$
v_i(x) = 0,\ x \in I_i,\ i \in \mathcal{M}. \tag{4.2}
$$

Next, we consider the stationary problem to problem (1.4) – (1.12)

$$\begin{cases} \lambda_i u_{ix}^+ = \mu_i^+(\phi_i, \phi_{ix}, u_i^+, u_i^-), & x \in I_i, i \in \mathcal{M}, \\ -\lambda_i u_{ix}^- = \mu_i^-(\phi_i, \phi_{ix}, u_i^+, u_i^-), & x \in I_i, i \in \mathcal{M}, \\ -\phi_{ixx} + h(\phi_i) = \beta(u_i^+, u_i^-), & x \in I_i, i \in \mathcal{M}, \end{cases} \tag{4.3}$$

with the boundary conditions and the transmission conditions:

(I) for every external node $N \in \mathcal{N}_{ex}$

$$\phi_{ix}(L_i) = 0, \text{ if } i \in I_N; \quad \phi_{ix}(0) = 0, \text{ if } i \in O_N \tag{4.4}$$

and

$$u_i^-(L_i) = u_i^+(L_i), \text{ if } i \in I_N, \quad u_i^+(0) = u_i^-(0), \text{ if } i \in O_N; \tag{4.5}$$

(II) for every internal node $N \in \mathcal{N}_{in}$

$$\begin{cases} u_i^-(L_i) = \sum_{j\in I_N} \xi_{ij} u_j^+(L_j) + \sum_{j\in O_N} \xi_{ij} u_j^-(0), & \text{if } i \in I_N, \\ u_i^+(0) = \sum_{j\in I_N} \xi_{ij} u_j^+(L_j) + \sum_{j\in O_N} \xi_{ij} u_j^-(0), & \text{if } i \in O_N, \end{cases} \tag{4.6}$$

where ξ_{ij} are the coefficients in (1.8);

(III) for every internal node $N \in \mathcal{N}_{in}$

$$\begin{aligned} \phi_{ix}(L_i) = &\sum_{j\in I_N} \alpha_{ij}(\phi_j(L_j) - \phi_i(L_i)) \\ &+ \sum_{j\in O_N} \alpha_{ij}(\phi_j(0) - \phi_i(L_i)), \quad \text{if } i \in I_N, \end{aligned} \tag{4.7}$$

$$\begin{aligned} -\phi_{ix}(0) = &\sum_{j\in I_N} \alpha_{ij}(\phi_j(L_j) - \phi_i(0)) \\ &+ \sum_{j\in O_N} \alpha_{ij}(\phi_j(0) - \phi_i(0)), \quad \text{if } i \in O_N, \end{aligned} \tag{4.8}$$

where $\alpha_{ij} \geq 0$ and $\alpha_{ij} = \alpha_{ji}$ for every $i, j \in \mathcal{M}$. Then again, by the definition of v_i and (4.2), we obtain

$$\mu^+(\phi_i(x), \phi_{ix}(x), u_i^+(x), u_i^-(x)) = -\mu^-(\phi_i(x), \phi_{ix}(x), u_i^+(x), u_i^-(x)) \tag{4.9}$$

and

$$\begin{cases} \sum_{j\in I_N} \xi_{ij} u_j^+(L_j) + \sum_{j\in O_N} \xi_{ij} u_j^-(0) - u_i^+(L_i) = 0, & \text{if } i \in I_N, \\ \sum_{j\in I_N} \xi_{ij} u_j^+(L_j) + \sum_{j\in O_N} \xi_{ij} u_j^-(0) - u_i^-(0) = 0, & \text{if } i \in O_N. \end{cases} \tag{4.10}$$

Moreover, by (4.6), (4.10) and (1.8), which imply that

$$u_i^+(L_i) + u_i^-(L_i) = u_j^-(0) + u_j^+(0), \; i \in I_N, \; j \in O_N. \tag{4.11}$$

Now, we consider the linear operator $\mathcal{L} : D(\mathcal{L}) \to L^2(\mathcal{A})$,

$$D(\mathcal{L}) = \{U = (u_i^+(x), u_i^-(x)) \in (H^1(\mathcal{A}))^2 : (4.5), (4.6) \text{ hold}\},$$

$$\mathcal{L}U = \{\lambda_i u_{ix}^+, -\lambda_i u_{ix}^-\}_{i\in\mathcal{M}}.$$

In order to obtain the existence of solutions to problem (4.3) by the fixed point technique, combining the solutions of two disjoint problems

$$\begin{cases} \mathcal{L}U(x) = F_1(x, U(x)), \quad x \in I_i, i \in \mathcal{M}, \\ (4.5) \ \text{ and } \ (4.6) \ \text{ hold for every } \ x \in I_i, \end{cases} \tag{4.12}$$

where $F(x) \in (H^1(\mathcal{A}))^2$, and

$$\begin{cases} -\phi_{ixx} + h(\phi_i) = g_i(x), \quad x \in I_i, i \in \mathcal{M}, \\ (4.4) \ \text{ and } \ (4.7) - (4.8) \ \text{ hold.} \end{cases} \tag{4.13}$$

where $g(x) \in L^2(\mathcal{A})$.

Proposition 4.1. *Let G be an acyclic graph and assumption (H_3) holds, let $g(x) \in L^2(\mathcal{A})$ there exists a unique solution $\phi \in H^2(\mathcal{A})$ which solves problem (4.13). Moreover, there exists a constant $D_1 > 0$ such that*

$$||\phi||_{H^2} \le D_1 ||g(x)||_{L^2(\mathcal{A})}. \tag{4.14}$$

Proof. Notice that, the existence and uniqueness of the solution $\phi \in H^2(\mathcal{A})$ to the problem (4.13) (for a general $g(x) \in L^2(\mathcal{A})$ and a general network) is showed by the Lax-Milgram theorem in [14].

Finally, we only need to prove last claim. Multiplying the equation in (4.13) by ϕ_i and applying Lemma 2.1, we obtain

$$\sum_{i\in\mathcal{M}} \int_{I_i} \{\phi_{ix}^2 + \eta_{i,1}\phi_i^2\}\, dx \le \sum_{i\in\mathcal{M}} ||g_i||_{L^2(I_i)} ||\phi||_{L^2(I_i)}, \tag{4.15}$$

whence

$$\sum_{i\in\mathcal{M}} \int_{I_i} \phi_i^2 + \phi_{ix}^2 \le C_1 ||g(x)||_{L^2(\mathcal{A})} ||\phi||_{H^1(\mathcal{A})}, \tag{4.16}$$

ie

$$||\phi||_{H^1(\mathcal{A})} \le C_2 ||g(x)||_{L^2(\mathcal{A})}, \tag{4.17}$$

where C_1, C_2 are positive constants.

Using (4.13), for every $i \in \mathcal{M}$, there also holds

$$[\phi_{ixx}]^2 \le 2[h(\phi)]^2 + 2[g_i]^2 \le C_3 ||g(x)||_{L^2(\mathcal{A})}. \tag{4.18}$$

Hence there exists $C_4 > 0$ such

$$||\phi_{xx}||_{L^2(\mathcal{A})} \le C_4 ||g(x)||_{L^2(\mathcal{A})}. \tag{4.19}$$

This completes the proof of Proposition 4.1.

Next, we prove the well-posedness results of problem (4.12). To do this, we define the following the Banach space and the operator T

$$\mathbb{M}_0 = \left\{ U = (u^+, u^-) \in (H^1(\mathcal{A}))^2 \text{ with } ||U||_{(H^1(\mathcal{A}))^2} \le M \right\}$$

and

$$(TU)(x) = U_0 + \int_{\mathcal{A}} F(x, U(x))dx, \tag{4.20}$$

where $F(x, U(x)) = \Big(\mu^+(f, f_x, u^+, u^-), \mu^-(f, f_x, u^+, u^-)\Big)$.

Moreover, given $K, M > 0$, for $f \in H^2(\mathcal{A})$ with $||f||_{H^2(\mathcal{A})} \leq K$ and $V_1 = (v_1^+, v_1^-), V_2 = (v_2^+, v_2^-) \in (H^1(\mathcal{A}))^2$ with

$$||V_1||_{(H^1(\mathcal{A}))^2},\ ||V_2||_{(H^1(\mathcal{A}),)^2} \leq M.$$

using assumption (H_1), we know that the Lipschitz continuity of $F \in (H^1(\mathcal{A}))^2$, then there exists a positive constant $C(K, M)$ depending also on K, M, and $C(K, M)$ non-decreasing with K, M such that

$$||F_1(x, V_1(x) - F_1(x, V_2(x))||_{(H^1(\mathcal{A}))^2} \leq C(K, M)||V_1(x) - V_2(x)||_{(H^1(\mathcal{A}))^2}. \tag{4.21}$$

Proposition 4.2. *Let G be a acyclic graph and assumption (H_1), for $f \in H^2(\mathcal{A})$, there exists $K > 0$ such that, if $||f||_{H^2(\mathcal{A})} \leq K$, then there exists a unique solution U to problem* (4.12) *and*

$$||U||_{(H^1(\mathcal{A}))^2} \leq M_\Theta, \tag{4.22}$$

where M_Θ is constant depending on Θ and Θ gives in (1.16).

Proof. Firstly, we prove the existence a unique solution to problem (4.12). This is divided into two steps.

Step 1. We show that operator T maps $\mathbb{M}_0$ into $\mathbb{M}_0$. By (4.20) and (4.21), there holds

$$\begin{aligned} &||(TU)(x) - U_0||_{(H^1(\mathcal{A})^2)} \\ &\leq C_1(K, M)||U(x)||_{(H^1(\mathcal{A}))^2}. \end{aligned} \tag{4.23}$$

Thus, the operator T is continuous for any $x \in I_i,\ i \in \mathcal{M}$.

Moreover, for $x \in I_i$, $U(x), V(x) \in \mathbb{M}_0$, we have

$$\begin{aligned} &||TU - TV||_{H^1(\mathcal{A})} \\ &\leq ||F_1(x, U(x) - F_1(x, V(x))||_{(H^1(\mathcal{A}))^2} \\ &\leq C_2(K, M)||U(x) - V(x)||_{(H^1(\mathcal{A}))^2}. \end{aligned} \tag{4.24}$$

Letting $C_1(K, M), C_2(K, M) \leq \frac{1}{2}$, then the nonlinear operator T is a strict contraction in $\mathbb{M}_0$. We conclude that there exists a unique $U(x) \in \mathbb{M}_0$ such that $TU(x) = U(x)$.

Step 2. We need to verify the operator T satisfying the transmission condition (4.6) and the boundary condition (4.5), for every $I_i = [0, L_i] \in \mathcal{A},\ i \in \mathcal{M}$,

$$\begin{cases} u_{ix}^+ = \frac{1}{\lambda_i}\mu_i^+(f_i(x), f_{ix}(x), u_i^+(x), u_i^-(x)), \\ u_i^+(0) = u_{i0}^+, \end{cases} \tag{4.25}$$

and

$$\begin{cases} u_{ix}^- = -\frac{1}{\lambda_i}\mu_i^-(f_i(x), f_{ix}(x), u_i^+(x), u_i^-(x)), \\ u_i^-(0) = u_{i0}^-. \end{cases} \tag{4.26}$$

Moreover, we obtain

$$u_i^+(L_i) = u_{0i}^+ + \frac{1}{\lambda_i}\int_{I_i} \mu^+(f_i, f_{ix}, u_i^+, u_i^-)dx \tag{4.27}$$

and

$$u_i^-(L_i) = u_{0i}^- - \frac{1}{\lambda_i}\int_{I_i} \mu^-(f_i, f_{ix}, u_i^+, u_i^-)dx. \tag{4.28}$$

To this purpose, fix any edge $I_i = [0, L_i]$ connecting two (internal or external)nodes N_1 and N_2, we obtain

$$I_i \in O_{N_1}, \quad \text{and} \quad I_i \in I_{N_2}. \tag{4.29}$$

We distinguish three cases:
case(i) $N_1, N_2 \in \mathcal{N}_{in}$ are internal vertices of the graph. Since $i \in O_{N_1}$, we impose that

$$u_i^+(0) = \sum_{j\in O_{N_1}} \xi_{ij} u_j^-(0) + \sum_{j\in I_{N_1}} \xi_{ij} \underbrace{\left(u_j^+(0) + \frac{1}{\lambda_j}\int_{I_j} \mu^+(f_j, f_{jx}, u_j^+, u_j^-)dx\right)}_{u_j^+(L_j)}. \tag{4.30}$$

Rewriting the equation (4.30), we get

$$u_i^+(0) - \sum_{j\in O_{N_1}} \xi_{ij} u_j^-(0) - \sum_{j\in I_{N_1}} \xi_{ij} u_j^+(0) = B_{i,in}^+, \quad (i \in O_{N_1}, N_1 \in \mathcal{N}_{in}), \tag{4.31}$$

where $B_{i,in}^+ = \sum_{j\in I_{N_1}} \frac{1}{\lambda_j}\xi_{ij} \int_{I_j} \mu^+(f_j, f_{jx}, u_j^+, u_j^-)dx$.

Similarity, we obtain

$$\begin{aligned} &\underbrace{\left(u_i^-(0) - \frac{1}{\lambda_i}\int_{I_i} \mu^-(f_i, f_{ix}, u_i^+, u_i^-)dx\right)}_{u_i^-(L_i)} \\ &= \sum_{j\in O_{N_2}} \xi_{ij} u_j^-(0) + \sum_{j\in I_{N_2}} \xi_{ij} \underbrace{\left(u_j^+(0) + \frac{1}{\lambda_j}\int_{I_j} \mu^+(f_j, f_{jx}, u_j^+, u_j^-)dx\right)}_{u_j^+(L_j)}, \end{aligned} \tag{4.32}$$

consequently

$$u_i^-(0) - \sum_{j\in O_{N_2}} \xi_{ij} u_j^-(0) - \sum_{j\in I_{N_2}} \xi_{ij} u_j^+(0) = B_{i,in}^-, \quad (i \in I_{N_2}, N_2 \in \mathcal{N}_{in}), \tag{4.33}$$

where

$$B_{i,in}^- = \sum_{j\in I_{N_2}} \frac{1}{\lambda_j}\xi_{ij} \int_{I_j} \mu^+(f_j, f_{ix}, u_j^+, u_j^-)dx + \frac{1}{\lambda_i}\int_{I_i} \mu^-(f_i, f_{ix}, u_i^+, u_i^-)dx. \tag{4.34}$$

Case(ii) Let $N_2 \in \mathcal{N}_{ex}$, $I_i \in I_{N_2}$, by the boundary condition (4.5), we have

$$\begin{aligned} &\underbrace{\left(u_i^+(0) + \frac{1}{\lambda_i}\int_{I_i} \mu^+(f_i, f_{ix}, u_i^+, u_i^-)dx\right)}_{u_i^+(L_i)} \\ &= \underbrace{\left(u_i^-(0) - \frac{1}{\lambda_i}\int_{I_i} \mu^-(f_i, f_{ix}, u_i^+, u_i^-)dx\right)}_{u_i^-(L_i)}. \end{aligned} \tag{4.35}$$

Moreover, we get

$$u_i^-(0) - u_i^+(0) = B_{i,ex}^-, \tag{4.36}$$

using (4.9) yields $B_{i,ex}^- = 0$.
Case(iii) Let $N_1 \in \mathcal{N}_{ex}$, $I_i \in O_{N_1}$, by the boundary condition, we get

$$u_i^+(0) = u_i^-(0), \quad I_{N_1} \in O_{N_1}. \tag{4.37}$$

Moreover, by $u_i^+(L_i) - u_i^-(L_i) = u_j^+(0) - u_j^-(0), i \in I_N,\ j \in O_N$ and (4.11), we obtain

$$\begin{cases} u_i^+(L_i) + u_i^-(L_i) = u_j^-(0) + u_j^+(0),\ i \in I_N,\ j \in O_N, \\ u_i^+(L_i) - u_i^-(L_i) = u_j^+(0) - u_j^-(0),\ i \in I_N,\ j \in O_N. \end{cases} \tag{4.38}$$

Consequently,

$$\frac{1}{\lambda_i}\int_{I_i} \mu^+(f_i, f_{ix}, u_i^+, u_i^-)dx = u_j^+(0) - u_i^+(0),\ i \in I_N,\ j \in O_N \tag{4.39}$$

and

$$\frac{1}{\lambda_i}\int_{I_i} \mu^-(f_i, f_{ix}, u_i^+, u_i^-)dx = u_i^-(0) - u_j^-(0), \quad i \in I_N,\ j \in O_N. \tag{4.40}$$

Using (4.39) and (4.40), one has

$$B_{i,in}^+ = \sum_{j\in I_{N_1}} \xi_{ij}\Big(u_i^+(0) - u_j^+(0)\Big),\ i \in O_{N_1}. \tag{4.41}$$

Hence, we see that

$$(1 - (\sum_{j\in I_{N_1}} \xi_{ij}))u_i^+(0) - \sum_{j\in O_{N_1}} u_j(0) = 0, \quad (i \in O_{N_1}, N_1 \in \mathcal{N}_{in}). \tag{4.42}$$

Similarity, we have

$$\begin{aligned} B_{i,in}^- &= \sum_{j\in I_{N_2}} \frac{1}{\lambda_j}\xi_{ij}\int_{I_j} \mu^+(f_i, f_{ix}, u_j^+, u_j^-)dx \\ &+ \frac{1}{\lambda_i}\int_{I_i} \mu^-(f_i, f_{ix}, u_i^+, u_i^-)dx \\ &= \sum_{j\in I_{N_2}} \xi_{ij}(u_{j*}^+(0) - u_j^+(0)) + u_i^-(0) - u_{j*}^-(0),\ j^* \in O_{N_2}. \end{aligned} \tag{4.43}$$

By (4.43), we write

$$(1 - \xi_{ij_*})u_{j_*}^-(0) - \sum_{j\neq j_*\in O_{N_2}} \xi_{ij}u_j^-(0) - (\sum_{j\in I_{N_2}} \xi_{ij})u_{j_*}^+(0) = 0, \tag{4.44}$$

where $i \in I_{N_2}, j_* \in O_{N_2},\ N_2 \in \mathcal{N}_{in}$.

Therefore, we derive

$$\begin{cases} (1 - (\sum_{j\in I_{N_1}} \xi_{ij}))u_i^+(0) - \sum_{j\in O_{N_1}} \xi_{ij}u_j^-(0) = 0,\ i \in O_{N_1}, N_1 \in \mathcal{N}_{in}, \\ (1 - \xi_{ij_*})u_{j_*}^-(0) - \sum_{j\neq j_*\in O_{N_2}} \xi_{ij}u_j^-(0) - (\sum_{j\in I_{N_2}} \xi_{ij})u_{j_*}^+(0) = 0, \\ i \in I_{N_2}\ , j_* \in O_{N_2}, N_2 \in \mathcal{N}_{in}, \\ u_i^-(0) - u_i^+(0) = 0, \quad \text{if } i \in I_{N_2}, N_2 \in N_{ex}, \\ u_i^-(0) - u_i^+(0) = 0, \quad \text{if } i \in O_{N_1}, N_1 \in N_{ex}. \end{cases} \tag{4.45}$$

The rest of the proof is devote to show that (4.45) is uniquely solvable. In the literature [14] of Lemma 5.3, authors proved that the operator A_2 is m-dissipative operator, it has involved the proof that the similarity equation system (4.45) has a unique solution. Consequently, we omit this proof here.

Now, we only need to prove last claim. Multiplying the first and the second equation in (4.12) by u^+_{ix}, u^-_{ix}, respectively, and summing up I_i, $i \in \mathcal{M}$, we have

$$\begin{aligned}
&\sum_{i\in\mathcal{M}} \int_{I_i} \Big([u^+_{ix}]^2 + [u^-_{ix}]^2 \Big) dx \\
&= \sum_{i\in\mathcal{M}} \frac{1}{\lambda_i} \int_{I_i} \Big(\mu^+(f_i, f_{ix}, u^+_i, u^-_i) u^+_{ix} - \mu^-(f_i, f_{ix}, u^+_i, u^-_i) u^-_{ix} \Big) dx \\
&\le \frac{D_\mu}{\min_{i\in\mathcal{M}}(\lambda_i)} \sum_{i\in\mathcal{M}} \int_{I_i} \Big(|u^+_i| + |u^-_i| + |f_{ix}| + |f_i| \Big) |u^+_{ix}| dx \\
&+ \frac{D_\mu}{\min_{i\in\mathcal{M}}(\lambda_i)} \sum_{i\in\mathcal{M}} \int_{I_i} \Big(|u^+_i| + |u^-_i| + |f_{ix}| + |f_i| \Big) |u^-_{ix}| dx \\
&\le \frac{2D_\mu}{\min_{i\in\mathcal{M}}(\lambda_i)} \sum_{i\in\mathcal{M}} \Big\{ \Big(||f_{ix}||_2^2 + ||f_i||_2^2 \Big) + \Big(||u^+_i||_2^2 + ||u^-_i||_2^2 \Big) \Big\},
\end{aligned} \tag{4.46}$$

where D_μ gives in (2.16).

In addition, by the first two equation (4.3) and assumption (H_1), there holds

$$\begin{aligned}
&\sum_{i\in\mathcal{M}} \int_{I_i} \Big(u^+_{ix} + u^-_{ix} \Big) dx \\
&= \sum_{i\in\mathcal{M}} \int_{I_i} \frac{1}{\lambda_i} \Big(\mu^+_i(f_i, f_{ix}, u^+_i, u^-_i) - \mu^+_i(f_i, f_{ix}, u^+_i, u^-_i)) dx \\
&\le \sum_{i\in\mathcal{M}} \int_{I_i} \frac{1}{\min_{i\in\mathcal{M}}(\lambda_i)} 2D_\mu \Big(|u^+_i| + |u^-_i| + |f_i| + |f_{ix}| \Big) dx \\
&\le \frac{1}{\min_{i\in\mathcal{M}}(\lambda_i)} \Big(2D_\mu\Theta + 2|\mathcal{A}| D_\mu (||f||_{L^\infty(\mathcal{A})} + ||f_x||_{L^\infty(\mathcal{A})}) \Big) \\
&\le \frac{2D_\mu}{\min_{i\in\mathcal{M}}(\lambda_i)}\Theta + \frac{2|\mathcal{A}|D_\mu}{\min_{i\in\mathcal{M}}(\lambda_i)} \Big(\max_{i\in\mathcal{M}}(S_{i,1}) ||f||_{H^1(\mathcal{A})} + ||f_x||_{L^\infty(\mathcal{A})} \Big) \\
&\le \frac{2D_\mu}{\min_{i\in\mathcal{M}}(\lambda_i)}\Theta + \frac{4|\mathcal{A}|D_\mu \max_{i\in\mathcal{M}}(S_{i,1})K}{\min_{i\in\mathcal{M}}(\lambda_i)}
\end{aligned} \tag{4.47}$$

and

$$\begin{aligned}
&||u^+||_{W^{1,1}(\mathcal{A})} + ||u^-||_{W^{1,1}(\mathcal{A})} \\
&\le \Big(\frac{2D_\mu}{\min_{i\in\mathcal{M}}(\lambda_i)} + 1\Big)\Theta + \frac{4|\mathcal{A}|D_\mu \max_{i\in\mathcal{M}}(S_{i,1})K}{\min_{i\in\mathcal{M}}(\lambda_i)}, \\
&\le C_5\Theta + C_6,
\end{aligned} \tag{4.48}$$

where $C_5 = \frac{2D_\mu}{\min_{i\in\mathcal{M}}(\lambda_i)} + 1$ and $C_6 = \frac{4|\mathcal{A}|D_\mu \max_{i\in\mathcal{M}}(S_{i,1})K}{\min_{i\in\mathcal{M}}(\lambda_i)}$.

Using (4.48), we obtain

$$\sum_{i\in\mathcal{M}} \int_{I_i} \left\{ (u^+_i)^2 + (u^-_i)^2 \right\} dx \le 2|\mathcal{A}| [\max_{i\in\mathcal{M}}(S_{i,2})]^2 (C_5\Theta + C_6)^2. \tag{4.49}$$

Moreover,

$$\begin{aligned}
&\sum_{i\in\mathcal{M}}\int_{I_i}\left([u_{ix}^+]^2+[u_{ix}^-]^2\right)dx \\
&\le \frac{2D_\mu}{\min_{i\in\mathcal{M}}(\lambda_i)}K^2+\frac{4|\mathcal{A}|D_\mu[\max_{i\in\mathcal{M}}(S_{i,1})]^2}{\min_{i\in\mathcal{M}}(\lambda_i)}(C_5\Theta+C_6)^2,
\end{aligned}\tag{4.50}$$

where $C_7=\frac{2D_\mu}{\min_{i\in\mathcal{M}}(\lambda_i)}K^2$ and $C_8=\frac{4|\mathcal{A}|D_\mu[\max_{i\in\mathcal{M}}(S_{i,2})]^2}{\min_{i\in\mathcal{M}}(\lambda_i)}$
and

$$\begin{aligned}
&||u_i^+||_{H^1(\mathcal{A})}+||u_i^+||_{H^1(\mathcal{A})} \\
&\le 2\sqrt{C_7+C_8(C_5\Theta+C_6)^2}=M_\Theta.
\end{aligned}\tag{4.51}$$

Proof of Theorem 1.3. (i) (**Uniqueness**) It's easy to prove a uniqueness results for solution to the problem (4.3) $-$ (4.8) by similar method of the proof of Theorem 1.1. Hence, we omit the proof here.
(ii)(**Existence**) By arguing as in Proposition 4.1 and 4.2, we known that exists $K_0>0$, such that

$$\begin{aligned}
||\phi||_{H^2(\mathcal{A})} &\le D_1||\beta(u^+,u^-)||_{L^2(\mathcal{A})} \\
&\le D_1D_\beta\left(||u^+||_{L^2(\mathcal{A})}+||u^-||_{L^2(\mathcal{A})}\right) \\
&\le K_0M_\Theta=K_\Theta,
\end{aligned}\tag{4.52}$$

whenever $\phi\in H^2(\mathcal{A})$ is the solution of problem (4.13) with $g_i(u_i^+,u_i^-)$ and $U=(u^+,u^-)\in(H^1(\mathcal{A}))^2$ satisfying $||U||_{H^1(\mathcal{A})}\le M_\Theta$ and where $D_\beta=\sum_{i\in\mathcal{M}}||\nabla\beta_i||_{L^\infty}(R^2)$. Based on the above discussion, we consider following the set

$$B_{M_\Theta,K_\Theta}=\begin{cases} U(x)=(u^+(x),u^-(x))\in(H^1(\mathcal{A}))^2, & \phi(x)\in H^2(\mathcal{A}),\\ ||U||_{(H^1(\mathcal{A}))^2}\le M_\Theta, ||\phi||_{H^2(\mathcal{A})}\le K_\Theta, \end{cases}$$

equipped with the distance generated by the norms of $(H^1(\mathcal{A}))^2$ and $H^2(\mathcal{A})$. We define a map G on B_{M_Θ,K_Θ} as follow: $(V(x),\varphi(x))=(v^+(x),v^-(x),\varphi(x))\in B_{M_\Theta,K_\Theta}$, then $(U,\phi)=G(V,\psi)$ is such that $U(x)$ is the solution to problem (4.13) where $F=(\mu^+(\psi,\psi_x,v^+,v^-),\mu^-(\psi,\psi_x,v^+,v^-))$ and $\phi(x)$ is the solution to problem (4.12) where $g_i=\beta(u_i^+,u_i^-)$.

Proposition 4.1 and Proposition 4.2 ensure that the map G is well defined on B_{M_Θ,K_Θ}. Next, we are going to prove that G is contraction B_{M_Θ,K_Θ}, then

$$\begin{aligned}
&(V(x),\psi(x)),(\tilde{V}(x),\tilde{\psi}(x))\in B_{M_\Theta,K_\Theta}, \\
&(U(x),\phi(x))=G(V(x),\psi(x)),\ (\tilde{U}(x),\tilde{\phi}(x))=G(\tilde{V}(x),\tilde{\psi}(x)).
\end{aligned}\tag{4.53}$$

By (4.21) and (4.14), we have

$$\begin{aligned}
||U(x)-\tilde{U}(x)||_{(H^1(\mathcal{A}))^2} &\le ||F-\tilde{F}||_{(H^1(\mathcal{A}))^2} \\
&\le C(M_\Theta,K_\Theta)||V-\tilde{V}||_{(H^1(\mathcal{A}))^2}
\end{aligned}\tag{4.54}$$

and

$$\begin{aligned}
||\phi(x)-\tilde{\phi}(x)||_{H^2(\mathcal{A})} &\le ||\beta(u_i^+(x),u_i^+(x))-\beta(\tilde{u}_i^+(x),\tilde{u}_i^+(x)))||_X \\
&\le D_\beta||U(x)-\tilde{U}(x)||_X \\
&\le C(M_\Theta,K_\Theta)||V-\tilde{V}||_{(H^1(\mathcal{A}))^2},
\end{aligned}\tag{4.55}$$

where $C(M_\Theta, K_\Theta)$ is positive constants depending on Θ, and $C(M_\Theta, K_\Theta)$ increases with Θ. If $C(M_\Theta, K_\Theta)$ is suitable small, then the map G is a contraction mapping on B_{M_Θ, K_Θ}, then there exists unique fixed point $(U(x), \phi(x)) \in B_{M_\Theta, K_\Theta}$ which is the solution to the problem $(4.3) - (4.8)$.

5 Asymptotic behaviour

In this section, in case of acyclic oriented graphs, we are going to show that the stationary solution previous introduced, provide the asymptotic profiles for a class of solution to problem $(1.4) - (1.12)$.

If (u^+, u^-, ϕ) is the solution to problem $(1.4) - (1.12)$ with initial data $(u^+_{i0}, u^-_{i0}, \phi_{i0})$ and the $(U^+(x), U^-(x), \Psi(x))$ is the stationary solution to the problem $(1.4) - (1.12)$. Then we set following the triple

$$(\tilde{u}^+(x,t), \tilde{u}^-(x,t), \tilde{\phi}(x,t)) = (u^+(x,t) - U^+(x), u^-(x,t) - U^-(x), \phi(x,t) - \Psi(x)), \quad (5.1)$$

which is the local solution of the following problem

$$\begin{cases} \tilde{u}^+_{it} + \lambda_i \tilde{u}^+_{ix} = \mu^+_i(\tilde{\phi}_i + \Psi_i, (\tilde{\phi} + \Psi)_{ix}, \tilde{u}^+_i + U^+_i, \tilde{u}^-_i + U^-_i) - \lambda_i U^+_{ix}, \\ \tilde{u}^-_{it} - \lambda_i \tilde{u}^-_{ix} = \mu^+_i(\tilde{\phi}_i + \Psi_i, (\tilde{\phi} + \Psi)_{ix}, \tilde{u}^+_i + U^+_i, \tilde{u}^-_i + U^-_i) + \lambda_i U^-_{ix}, \\ \tilde{\phi}_{it} = \tilde{\phi}_{ixx} - h(\tilde{\phi} + \Psi_i) + \beta_i(\tilde{u}^+_i + U^+_i, \tilde{u}^-_i + U^-_i) + \Psi_{ixx}, \\ \tilde{u}^+_i(x,0) = \tilde{u}^+_{i0} = u^+_i(x,0) - U^+_i(x), \tilde{u}^-_i(x,0) = \tilde{u}^-_{i0} = u^+_i(x,0) - U^-_i(x), \\ \tilde{\phi}_i(x,0) = \tilde{\phi}_{i0} = \phi_i(x,0) - \Psi_i(x), \\ x \in I_i, t \geq 0, i \in \mathcal{M}, \end{cases} \quad (5.2)$$

complemented with the condition $(1.5) - (1.11)$ and $(\tilde{u}^+_{i0}, \tilde{u}^-_{i0}, \tilde{\phi}_{i0})$ defined above.

The existence and uniqueness of local solution to problem (5.2) can be achieved by means of semigroup theory, following the similar method used in the section 2, with a little modifications.

Proposition 5.1. *Let $G(\mathcal{N}, \mathcal{A})$ be an acyclic graph and assumptions $(H_1) - (H_3)$ hold, let $(U^+(x), U^-(x), \Psi(x))$ be a stationary solution to problem $(1.4) - (1.12)$ and $(\tilde{u}^+, \tilde{u}^-, \tilde{\phi})$ be the local solution to $(5.2), (1.5) - (1.11)$,*

$$(\tilde{u}^+, \tilde{u}^-) \in C([0,T]; H^1(\mathcal{A})) \cap C^1([0,T]; L^2(\mathcal{A})), \quad (5.3)$$

$$\tilde{\phi} \in C([0,T]; H^2(\mathcal{A})) \cap C^1([0,T]; L^2(\mathcal{A})) \cap H^1([0,T]; H^1(\mathcal{A})). \quad (5.4)$$

There exists $\varepsilon > 0$ such that, if $||U^+||_{H^1(\mathcal{A})} + ||U^-||_{H^1(\mathcal{A})}, ||\Psi||_{H^2(\mathcal{A})} \leq \varepsilon$, then

(a)

$$\begin{aligned} &\sum_{i \in \mathcal{M}} \left(\sup_{[0,T]} ||\tilde{u}^+_i(t)||_2^2 + \sup_{[0,T]} ||\tilde{u}^-_i(t)||_2^2 \right) \\ \leq &\tilde{C}_1 \sum_{i \in \mathcal{M}} \Bigg\{ ||\tilde{u}^+_{0i}||_2^2 + ||\tilde{u}^-_{0i}||_2^2 + \int_0^T (||\tilde{u}^+_i(t)||_2^2 + ||\tilde{u}^-_i(t)||_2^2) dt \\ &+ \sup_{[0,T]} \left(||\tilde{u}^+_i(t)||_{H^1} + ||\tilde{u}^-_i(t)||_{H^1} \right) \int_0^T \left(||\tilde{\phi}_i(t)||_2^2 + ||\tilde{\phi}_{ix}(t)||_2^2 \right) dt \Bigg\}, \end{aligned} \quad (5.5)$$

(b)

$$\begin{aligned}
&\sum_{i\in\mathcal{M}}\sup_{[0,T]}\Big(||\tilde{u}_{it}^{+}(t)||_2^2+||\tilde{u}_{it}^{-}(t)||_2^2\Big)\\
\leq&\tilde{C}_2\sum_{i\in\mathcal{M}}\Big\{||\tilde{\phi}_{i0}||_{H^2}^2+||\tilde{u}_{i0}^{+}||_{H^1}^2+||\tilde{u}_{i0}^{-}||_{H^1}^2\\
&+\int_0^T\Big[(||\tilde{u}_{it}^{+}(t)||_2^2+||\tilde{u}_{it}^{-}(t)||_2^2)+(||\tilde{\phi}_{it}(t)||_2^2+||\tilde{\phi}_{ixt}(t)||_2^2)\Big]dt\Big\},
\end{aligned}\tag{5.6}$$

(c)

$$\begin{aligned}
&\sum_{i\in\mathcal{M}}\Big\{\int_0^T\Big(||\tilde{u}_{ix}^{+}(t)||_2^2+||\tilde{u}_{ix}^{-}(t)||_2^2\Big)dt\Big\}\\
&\leq\tilde{C}_3\sum_{i\in\mathcal{M}}\Big\{\int_0^T\Big(||\tilde{u}_{it}^{+}(t)||_2^2+||\tilde{u}_{it}^{-}(t)||_2^2+||\tilde{u}_{i}^{+}(t)||_{H^1}^2+||\tilde{u}_{i}^{-}(t)||_{H^1}^2\Big)dt\\
&+\int_0^T\Big[\Big(||\tilde{\phi}_{ix}(t)||_2^2+||\tilde{\phi}_{i}(t)||_2^2\Big)\Big]dt\Big\},
\end{aligned}\tag{5.7}$$

(d)

$$\begin{aligned}
&\sum_{i\in\mathcal{M}}\sup_{[0,T]}\Big(||\tilde{u}_{ix}^{+}(t)||_2^2+||\tilde{u}_{ix}^{-}(t)||_2^2\Big)\\
&\leq\tilde{C}_4\sum_{i\in\mathcal{M}}\Big\{\sup_{[0,T]}\Big(||\tilde{u}_{it}^{+}(t)||_2^2+||\tilde{u}_{it}^{-}(t)||_2^2+||\tilde{u}_{i}^{+}(t)||_{H^1}^2+||\tilde{u}_{i}^{-}(t)||_{H^1}^2\Big)\\
&+\sup_{[0,T]}\Big(||\tilde{\phi}_{ix}(t)||_2^2+||\tilde{\phi}_{i}(t)||_2^2\Big)\Big\},
\end{aligned}\tag{5.8}$$

(e)

$$\begin{aligned}
&\sum_{i\in\mathcal{M}}\sup_{[0,T]}||\tilde{\phi}_{it}(t)||_2^2+\sum_{i\in\mathcal{M}}\int_0^T\Big(||\tilde{\phi}_{it}(t)||_2^2+||\tilde{\phi}_{itx}(t)||_2^2\Big)dt\\
&\leq\tilde{C}_5\sum_{i\in\mathcal{M}}\Big(||\tilde{\phi}_{i0}||_{H^2}^2+||u_{i0}^{+}||_{H^1}^2+||u_{i0}^{-}||_{H^1}^2+\int_0^T(||\tilde{u}_{it}^{+}(t)||_2^2+||\tilde{u}_{it}^{-}(t)||_2^2)dt\Big),
\end{aligned}\tag{5.9}$$

(f)

$$\begin{aligned}
&\sum_{i\in\mathcal{M}}\sup_{[0,T]}\Big(||\tilde{\phi}_{ixx}(t)||_2^2+||\tilde{\phi}_{ix}(t)||_2^2\Big)\\
&\leq\tilde{C}_6\sum_{i\in\mathcal{M}}\sup_{[0,T]}\Big(||\tilde{\phi}_{it}(t)||_2^2+||\tilde{u}_{i}^{+}(t)||_2^2+||\tilde{u}_{i}^{-}(t)||_2^2\Big),
\end{aligned}\tag{5.10}$$

(g)

$$\begin{aligned}
&\sum_{i\in\mathcal{M}}\int_0^T\Big(||\tilde{\phi}_{ix}(t)||_2^2+||\tilde{\phi}_{ixx}(t)||_2^2\Big)dt\\
&\leq\tilde{C}_7\sum_{i\in\mathcal{M}}\int_0^T\Big(||\tilde{u}_{i}^{+}(t)||_{H^1}^2+||\tilde{u}_{i}^{-}(t)||_{H^1}^2+||\tilde{\phi}_{it}(t)||_2^2\Big)dt,
\end{aligned}\tag{5.11}$$

where $\tilde{C}_1,\tilde{C}_2,\tilde{C}_3,\tilde{C}_4,\tilde{C}_5,\tilde{C}_6,\ \tilde{C}_7$ *are positive constants.*

Proposition 5.2. *Let $G(\mathcal{N}, \mathcal{A})$ be an acyclic graph and assumptions $(H_1) - (H_3)$ hold, let $(U^+(x), U^-(x), \Psi(x))$ be a stationary solution to problem $(1.4) - (1.12)$. There exist positive constants ϵ_0, ϵ_1 such that, if*

$$||U^+||_{H^1(\mathcal{A})} + ||U^-||_{H^1(\mathcal{A})}, ||\Psi||_{H^2(\mathcal{A})} \le \epsilon_0, \ ||\tilde{u}_0^+||_{H^1(\mathcal{A})}, \ ||\tilde{u}_0^-||_{H^1(\mathcal{A})}, \ ||\tilde{\phi}_0||_{H^2(\mathcal{A})} \le \epsilon_1, \tag{5.12}$$

then there exists a unique global solution $(\tilde{u}^+, \tilde{u}^-, \tilde{\phi})$ to problem $(5.2), (1.5) - (1.11)$,

$$(\tilde{u}^+, \tilde{u}^-) \in C([0,\infty); H^1(\mathcal{A})) \cap C^1([0,\infty); L^2(\mathcal{A})), \tag{5.13}$$

$$\tilde{\phi} \in C([0,\infty); H^2(\mathcal{A})) \cap C^1([0,\infty); L^2(\mathcal{A})) \cap H^1([0,\infty); H^1(\mathcal{A})). \tag{5.14}$$

Moreover, $N_T(\tilde{u}^+, \tilde{u}^-, \tilde{\phi})$ is bounded, uniformly in T.

Proof. *This proof is similar with Theorem 1.2. Hence, we omit the proof here.*

Proof of Theorem 1.4. Combining Proposition 5.1 and Proposition 5.2, we easily prove Theorem 1.4. This proof is similar with [17] of Theorem 4.2, we omit the proof here.

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