

# THE LIMIT PROBLEM OF THE PATLAK-KELLER-SEGEL-STOKES SYSTEM IN SCALLING CRITICAL SPACE

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**ABSTRACT.** In this paper, we consider a singular limit problem of Cauchy problem for Patlak-Keller-Segel equation coupled with Stokes equation in scalling critical space. Precisely, by taking advantage of a coupling structure of equations and using a scale decomposition technique, it is shown that when the relaxation time parameter  $\epsilon \rightarrow \infty$ , a solution of Patlak-Keller-Segel system coupled with nonstationary Stokes equation

$$\begin{cases} \epsilon^{-1}u_t^\epsilon - \Delta u^\epsilon = -\nabla \mathbf{P}^\epsilon + n^\epsilon \nabla c^\epsilon, & x \in \mathbb{R}^d, t > 0, \\ -\Delta c^\epsilon = n^\epsilon, \quad \nabla \cdot u^\epsilon = 0, & x \in \mathbb{R}^d, t > 0, \\ n_t^\epsilon + u^\epsilon \cdot \nabla n^\epsilon - \Delta n^\epsilon = -\nabla \cdot (n^\epsilon \nabla c^\epsilon), & x \in \mathbb{R}^d, t > 0, \\ (u^\epsilon, n^\epsilon)|_{t=0} = (u_0, n_0), & x \in \mathbb{R}^d \end{cases}$$

converges to that of Patlak-Keller-Segel system coupled with stationary Stokes equation

$$\begin{cases} -\Delta u = -\nabla \mathbf{P} + n \nabla c, & x \in \mathbb{R}^d, t > 0, \\ \nabla \cdot u = 0, \quad -\Delta c = n, & x \in \mathbb{R}^d, t > 0, \\ n_t + u \cdot \nabla n - \Delta n = -\nabla \cdot (n \nabla c), & x \in \mathbb{R}^d, t > 0, \\ n|_{t=0} = n_0, & x \in \mathbb{R}^d \end{cases}$$

in the critical Fourier-Besov space under certain conditions.

## 1. INTRODUCTION

Motived by recent studies by Lorz [8, 10], we consider the Stokes-Patlak-Keller-Segel equations modeling the chemotaxis phenomenon in moving fluid:

$$(1.1) \quad \begin{cases} \epsilon^{-1}u_t^\epsilon - \Delta u^\epsilon = -\nabla \mathbf{P}^\epsilon + n^\epsilon \nabla c^\epsilon, & x \in \mathbb{R}^d, t > 0, \\ -\Delta c^\epsilon = n^\epsilon, \quad \nabla \cdot u^\epsilon = 0, & x \in \mathbb{R}^d, t > 0, \\ n_t^\epsilon + u^\epsilon \cdot \nabla n^\epsilon - \Delta n^\epsilon = -\nabla \cdot (n^\epsilon \nabla c^\epsilon), & x \in \mathbb{R}^d, t > 0, \\ (u^\epsilon, n^\epsilon)|_{t=0} = (u_0, n_0), & x \in \mathbb{R}^d, \end{cases}$$

where  $n^\epsilon, c^\epsilon$  and  $u^\epsilon$  denote the cell density, the chemical density and the ambient fluid velocity, respectively.  $\mathbf{P}^\epsilon = \mathbf{P}^\epsilon(x, t) \in \mathbb{R}$  is the pressure in the fluid,  $u_0, n_0$  are the given initial data with  $\nabla \cdot u_0 = 0$ . The first equation describes the fluid motion subject to forcing induced by the cell migration. The deterministic relation between  $n^\epsilon$  and  $c^\epsilon$  given in the second equation implies that the cells secrete the chemo-attractants. The third equation characterizes the evolution of

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the cell density subject to the chemotaxis-induced aggregation, the diffusion raised by random Brownian motion, and the transportation by ambient fluid flow  $u^\epsilon$ . Moreover, by considering that the chemo-attractant diffuses much faster than the fluid advection and cell aggregation, and reaches equilibrium in a faster time-scale, a relaxation time parameter  $0 < \epsilon \rightarrow \infty$  is introduced to determine the tardy evolution undergone by  $u^\epsilon$ ; see [8]. We refer to [5, 6, 7, 8, 9, 10, 11, 13] for derivation of this model and more detailed discussions on its physical background and biological significance.

One of the most interesting mathematical questions one may ask of (1.1) is whether there exists a limit system as  $\epsilon \rightarrow \infty$ . Formally, it is easy to see that a solution of (1.1) converges to that of the corresponding Patlak-Keller-Segel system coupled with stationary Stokes flows:

$$(1.2) \quad \begin{cases} -\Delta u = -\nabla \mathbf{P} + n \nabla c, & x \in \mathbb{R}^d, t > 0, \\ \nabla \cdot u = 0, \quad -\Delta c = n, & x \in \mathbb{R}^d, t > 0, \\ n_t + u \cdot \nabla n - \Delta n = -\nabla \cdot (n \nabla c), & x \in \mathbb{R}^d, t > 0, \\ n|_{t=0} = n_0, & x \in \mathbb{R}^d, \end{cases}$$

when the relaxation time parameter  $\epsilon \rightarrow \infty$ . The model (1.2) comes from experiments for the case of bacteria consuming the chemical, see [8, 10], and is also interesting since some aspects of its analysis are much better understood than the system (1.1). We refer readers to see [1, 3, 12, 14, 15] and the related references therein for such singular limit problems related to some other systems. Moreover, we would like to point out that if  $(u, n) = (u^\epsilon(x, t), n^\epsilon(x, t))$  is a solution to (1.1), then  $(u_\lambda, n_\lambda) = (\lambda u^\epsilon(\lambda x, \lambda^2 t), \lambda^2 n^\epsilon(\lambda x, \lambda^2 t))$  is also a solution to (1.1) for each  $\lambda > 0$ . This scaling property further motivates the choice of a scaling critical space as the space of initial data.

We believe that the study of this singular limit problem can improve our understanding those of (1.2). We introduce the following Banach space

$$(1.3) \quad \|n\|_\Theta := \sup_{t>0} \|n(t)\|_{\dot{\mathfrak{B}}_{p,\infty}^{-2+d-d/p}(\mathbb{R}^d)} + \sup_{t>0} \left( t^{\frac{\alpha}{2}} \|n(t)\|_{\dot{\mathfrak{B}}_{p,\infty}^{-2+\alpha+d-d/p}(\mathbb{R}^d)} \right) := \|n\|_{\mathbb{Z}_0} + \|n\|_{\mathbb{Z}_\alpha}.$$

Now our main result asserts temporally uniform convergence of these solutions when  $\epsilon \rightarrow +\infty$  as follows:

**Theorem 1.1.** *Assume that  $d \geq 2$ ,  $\max\{4 - d + \frac{d}{p}, 0\} < \alpha < 1$ . Let  $n_0 \in \dot{\mathfrak{B}}_{p,\infty}^{-2+d-d/p}(\mathbb{R}^d)$  and  $u_0 \in \dot{\mathfrak{B}}_{p,\infty}^{-1+d-d/p}(\mathbb{R}^d) \cap \dot{\mathfrak{B}}_{1,1}^{-1}(\mathbb{R}^d)$ . There exist positive constants  $\ell > 0$  and  $\varepsilon > 0$ . If*

$$\|n_0\|_{\dot{\mathfrak{B}}_{p,\infty}^{-2+d-d/p}(\mathbb{R}^d)} < \varepsilon.$$

(1) *Then the system (1.2) admits a unique solution  $n \in \Theta$ . Furthermore, for  $\epsilon > 1$  and*

$$\epsilon^{-\alpha/2} \|u_0\|_{\dot{\mathfrak{B}}_{p,\infty}^{-1+d-d/p}(\mathbb{R}^d) \cap \dot{\mathfrak{B}}_{1,1}^{-1}(\mathbb{R}^d)} < \ell,$$

*then the system (1.1) has a unique solution  $n^\epsilon \in \Theta$ .*

(2) *Let  $n^\epsilon$  and  $n$  be the solutions of (1.1) and (1.2), respectively. Then one can find a function  $C(\epsilon)$  such that*

$$\|n - n^\epsilon\|_\Theta \leq C(\epsilon),$$

*where*

$$C(\epsilon) \rightarrow 0, \text{ when } \epsilon \rightarrow +\infty.$$

The layout of the paper is as follows. In Section 2, we collect and proof some technical Lemmas. Our main results are proved in Section 3. Throughout this paper, we adopt the convention that a nonessential constant  $C$  may change from line to line. Given two quantities  $a$  and  $b$ , we denote  $a \lesssim b$  as  $a \leq Cb$ .

## 2. PRELIMINARIES

Firstly, we recall the Littlewood-Paley decomposition and the homogeneous Fourier-Besov spaces. Let  $\mathcal{S}(\mathbb{R}^d)$  be the Schwartz class of rapidly decreasing functions, and  $\mathcal{S}'(\mathbb{R}^d)$  be the space of tempered distributions. Here  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  denote Fourier and inverse Fourier transforms of  $L^1(\mathbb{R}^d)$ , respectively, defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx$$

and

$$\mathcal{F}^{-1}f(x) = \check{f}(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(\xi) d\xi.$$

We take a couple of smooth functions  $(\chi, \psi)$  supported on  $\{\xi : |\xi| \leq 1\}$  with values in  $[0, 1]$  such that for all  $\xi \in \mathbb{R}^d$ ,

$$\chi(\xi) + \sum_{j \geq 0} \psi(2^{-j}\xi) = 1, \quad \varphi(\xi) = \psi\left(\frac{\xi}{2}\right) - \psi(\xi),$$

and we denote  $\varphi(2^{-j}\xi)$  by  $\varphi_j(\xi)$ . We write  $h = \mathcal{F}^{-1}\varphi$  and  $\tilde{h} = \mathcal{F}^{-1}\chi$ . The homogeneous dyadic blocks  $\Delta_j$  and  $S_j$  are defined by

$$\Delta_j u = \varphi(2^{-j}D)u = 2^{dj} \int_{\mathbb{R}^d} h(2^j y) u(x - y) dy, \quad \forall j \in \mathbb{Z}$$

and

$$S_j u = \sup_{j' \leq j-1} \Delta_{j'} u = \chi(2^{-j}D)u = 2^{dj} \int_{\mathbb{R}^d} \tilde{h}(2^j y) u(x - y) dy, \quad \forall j \in \mathbb{Z}.$$

We denote by  $\mathcal{S}'_h(\mathbb{R}^d)$  a space of tempered distribution  $f$  such that

$$\lim_{j \rightarrow \infty} S_j f = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}^d).$$

Then, we can define homogeneous Littlewood-Paley decomposition by

$$u = \sum_{j \in \mathbb{Z}} \Delta_j u, \quad \text{in } \mathcal{S}'_h(\mathbb{R}^d).$$

The concept of paraproduct enables us to deal with the interaction of two functions in terms of low or high frequency parts [2]. The homogeneous paraproduct of  $v$  and  $u$  is defined by

$$T_u v = \sum_{i \leq j-2} \Delta_i u \Delta_j v = \sum_j S_{j-1} u \Delta_j v.$$

The homogeneous remainder of  $v$  and  $u$  is defined by

$$R(u, v) = \sum_{|j-j'| \leq 1} \Delta_j u \Delta_{j'} v.$$

In Sections two and three, we will frequently use the following decomposition:

$$(2.1) \quad uv = \sum_{j \in \mathbb{Z}} S_j u \Delta_j v + \sum_{j \in \mathbb{Z}} S_j v \Delta_j u.$$

Again, up to finitely many terms [2], we have

$$(2.2) \quad \Delta_j(uv) = \sum_{k \geq j-2} S_k u \Delta_k v + \sum_{k \geq j-2} \Delta_k u S_k v.$$

Let  $s \in \mathbb{R}$ ,  $(p, r) \in [1, \infty]^2$  and  $f \in \mathcal{S}'_h(\mathbb{R}^d)$ . The homogeneous Fourier-Besov space is defined by

$$\dot{\mathfrak{B}}_{p,r}^s(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'_h(\mathbb{R}^d) : \|f\|_{\dot{\mathfrak{B}}_{p,r}^s(\mathbb{R}^d)} < +\infty \right\},$$

where

$$(2.3) \quad \|f\|_{\dot{\mathfrak{B}}_{p,r}^s(\mathbb{R}^d)} = \begin{cases} \left[ \sum_{k \in \mathbb{Z}} \left( 2^{ks} \|\mathcal{F}(\Delta_k f)\|_{L^p(\mathbb{R}^d)} \right)^r \right]^{\frac{1}{r}}, & \text{if } 1 \leq r < \infty, \\ \sup_{k \in \mathbb{Z}} \left( 2^{ks} \|\mathcal{F}(\Delta_k f)\|_{L^p(\mathbb{R}^d)} \right), & \text{if } r = \infty. \end{cases}$$

**2.1. The estimate of system (1.2).** The goal of this subsection is to investigate linear and nonlinear terms of the system (1.2) in the framework of Fourier-Besov space. By the Duhamel formula, a solution of system (1.2) can be written as

$$(2.4) \quad n = e^{t\Delta} n_0 + \mathbb{B}^{11}(n, n) + \mathbb{B}^{12}(n, n, n),$$

with

$$(2.5) \quad \begin{cases} \mathbb{B}^{11}(n, n) := - \int_0^t e^{(t-s)\Delta} \nabla \cdot (n \nabla (-\Delta)^{-1} n) ds, \\ \mathbb{B}^{12}(n, n, n) := - \int_0^t e^{(t-s)\Delta} \left[ \mathbb{P}(-\Delta)^{-1} (n \nabla (-\Delta)^{-1} n) \right] \cdot \nabla n ds, \end{cases}$$

where  $\mathbb{P} = (\mathbb{P}_{jk})_{1 \leq j, k \leq d}$  is defined as a matrix-valued operator, and  $\mathbb{R}_j = \frac{\partial}{\partial x_j} (-\Delta)^{-\frac{1}{2}} (j = 1, 2, \dots, d)$  denotes Riesz transform.

**Lemma 2.1.** [12] *For  $d \geq 2$ ,  $-d + 4 + d/p < \alpha < 1$ ,  $f \in \Theta$  and  $g \in \Theta$ . There exists a constant  $C > 0$  such that the following inequality holds*

$$\begin{aligned} & \left\| \mathcal{F} \left( \Delta_j \left( f(\tau) \nabla (-\Delta)^{-1} g(\tau) + g(\tau) \nabla (-\Delta)^{-1} f(\tau) \right) \right) \right\|_{L^p(\mathbb{R}^d)} \\ & \leq C \tau^{-\alpha/2} 2^{(3-d+d/p-\alpha)j} (\|f\|_{\mathbb{Z}_0} \|g\|_{\mathbb{Z}_\alpha} + \|g\|_{\mathbb{Z}_0} \|f\|_{\mathbb{Z}_\alpha}). \end{aligned}$$

**Lemma 2.2.** *Let  $\max\{4 - d + \frac{d}{p}, 0\} < \alpha < 1$ . Then for any  $n \in \Theta$ , there exists a constant  $C_1 > 0$  such that*

$$\|\mathbb{B}^{11}(n, n)\|_{\Theta} \leq C_1 \|n\|_{\Theta}^2.$$

*Proof.* Localizing  $\mathbb{B}^{11}(n, n)$  through the operator  $\Delta_j$  yields

$$(2.6) \quad \left\| \mathcal{F} \left( \Delta_j (\mathbb{B}^{11}(n, n)) \right) \right\|_{L^p(\mathbb{R}^d)} \lesssim \|n\|_{\Theta}^2 \int_0^t e^{-(t-s)2^{2j}} 2^{j(-d+4-\alpha+d/p)} s^{-\alpha/2} ds.$$

Therefore, multiplying both sides (2.6) by  $2^{j(-2+d-d/p)}$  and summing over  $j$ , we obtain

$$\begin{aligned} \|\mathbb{B}^{11}(n, n)\|_{\Theta} &\leq \sup_{t>0} \left\{ \sup_{j \in \mathbb{Z}} (t2^{2j})^{\frac{\alpha}{2}} \left[ 2^{j(2-\alpha)} \int_0^t e^{-(t-s)2^{2j}} s^{-\alpha/2} ds \right] \right\} \|n\|_{\Theta}^2 \\ &\quad + \sup_{t>0} \left[ 2^{j(2-\alpha)} \int_0^t e^{-(t-s)2^{2j}} s^{-\alpha/2} ds \right] \|n\|_{\Theta}^2 \\ &\leq C_1 \|n\|_{\Theta}^2, \end{aligned}$$

which finishes the proof of Lemma 2.2.  $\square$

**Lemma 2.3.** *Let  $n \in \Theta$  and  $\max\{4 - d + \frac{d}{p}, 0\} < \alpha < 1$ . Then there exists a constant  $C_2 > 0$  such that it holds that*

$$\|\mathbb{B}^{12}(n, n, n)\|_{\Theta} \leq C_2 \|n\|_{\Theta}^3.$$

*Proof.* For simplicity, we define

$$(2.7) \quad \mathbb{K}(n, n, s) := \mathbb{P}(-\Delta)^{-1}(n(s)\nabla(-\Delta)^{-1}n(s)).$$

Applying  $\Delta_j$  to (2.7), a standard Bony's decomposition (2.2) yields

$$\Delta_j(\mathbb{K}(n, n, s) \cdot \nabla n) = \sum_{k \geq j-2} \Delta_k \mathbb{K}(n, n, s) \nabla S_k n + \sum_{k \geq j-2} S_k \mathbb{K}(n, n, s) \Delta_k \nabla n.$$

Due to Bernstein's inequalities,

$$(2.8) \quad \|\mathcal{F}(S_k \nabla n)\|_{L^1(\mathbb{R}^d)} \lesssim \sum_{j' \leq k-1} 2^{3j'} \|n\|_{\mathbb{Z}_0} \lesssim 2^{3k} \|n\|_{\Theta}$$

and

$$(2.9) \quad \|\mathcal{F}(S_k \mathbb{K}(n, n, s))\|_{L^1(\mathbb{R}^d)} \lesssim \sum_{j' \leq k-1} s^{-\alpha/2} 2^{(1-\alpha)j'} \|n\|_{\mathbb{Z}_0} \|n\|_{\mathbb{Z}_\alpha} \lesssim s^{-\alpha/2} 2^{(1-\alpha)j} \|n\|_{\Theta}^2.$$

Thus, combining (2.8) and (2.9), we find for  $4 - d + d/p < \alpha < 1$  that

$$\begin{aligned} \|\mathcal{F}(\Delta_j(\mathbb{B}^{12}(n, n, n)))\|_{L^p(\mathbb{R}^d)} &\leq \int_0^t e^{-(t-s)2^{2j}} \sum_{k \geq j-2} \|\mathcal{F}(S_k \nabla n) * \mathcal{F}(\Delta_k \mathbb{K})\|_{L^p(\mathbb{R}^d)} ds \\ &\quad + \int_0^t e^{-(t-s)2^{2j}} \sum_{k \geq j-2} \|\mathcal{F}(\Delta_k \nabla n) * \mathcal{F}(S_k \mathbb{K})\|_{L^p(\mathbb{R}^d)} ds \\ (2.10) \quad &\lesssim \int_0^t e^{-(t-s)2^{2j}} \sum_{k \geq j-2} \|\mathcal{F}(S_k \nabla n)\|_{L^1(\mathbb{R}^d)} \|\mathcal{F}(\Delta_k \mathbb{K})\|_{L^p(\mathbb{R}^d)} ds \\ &\quad + \int_0^t e^{-(t-s)2^{2j}} \sum_{k \geq j-2} \|\mathcal{F}(S_k \mathbb{K})\|_{L^1(\mathbb{R}^d)} \|\mathcal{F}(\Delta_k \nabla n)\|_{L^p(\mathbb{R}^d)} ds \\ &\lesssim \|n\|_{\Theta}^3 \int_0^t e^{-(t-s)2^{2j}} \sum_{k \geq j-2} 2^{k(-d+4-\alpha+d/p)} s^{-\alpha/2} ds. \end{aligned}$$

As a consequence, thanks to (2.10), we obtain from (2.3) that

$$\begin{aligned} \|\mathbb{B}^{12}(n, n, n)\|_{\Theta} &\leq \sup_{t>0} \left\{ \sup_{j \in \mathbb{Z}} (t2^{2j})^{\frac{\alpha}{2}} \left[ 2^{j(2-\alpha)} \int_0^t e^{-(t-s)2^{2j}} s^{-\alpha/2} ds \right] \right\} \|n\|_{\mathbb{Z}_\alpha}^2 \|n\|_{\mathbb{Z}_0} \\ &\quad + \sup_{t>0} \left[ 2^{j(2-\alpha)} \int_0^t e^{-(t-s)2^{2j}} s^{-\alpha/2} ds \right] \|n\|_{\mathbb{Z}_\alpha}^2 \|n\|_{\mathbb{Z}_0} \lesssim \|n\|_{\Theta}^3, \end{aligned}$$

which completes the proof of Lemma 2.3.  $\square$

**2.2. Estimate of system (1.1).** With the aid of Duhamel formula, we construct a solution of the system (1.2) in the integral form,

$$(2.11) \quad \begin{cases} u^\epsilon = e^{\epsilon t \Delta} u_0 + \underbrace{\epsilon \int_0^t e^{\epsilon(t-s)\Delta} \mathbb{P}(n^\epsilon \nabla (-\Delta)^{-1} n^\epsilon) ds}_{F(t, n^\epsilon, n^\epsilon)}, \\ n^\epsilon = e^{t\Delta} n_0 + \int_0^t e^{(t-s)\Delta} (u^\epsilon \cdot \nabla n^\epsilon) ds + \mathbb{B}^{11}(n^\epsilon, n^\epsilon) \end{cases}$$

where

$$\mathbb{B}^{11}(n^\epsilon, n^\epsilon) := - \int_0^t e^{(t-s)\Delta} \nabla \cdot (n^\epsilon \nabla (-\Delta)^{-1} n^\epsilon) ds$$

and

$$(2.12) \quad \begin{aligned} &\int_0^t e^{(t-s)\Delta} (u^\epsilon \cdot \nabla) n^\epsilon ds \\ &= \underbrace{\int_0^t e^{(t-s)\Delta} (e^{\epsilon s \Delta} u_0 \cdot \nabla n^\epsilon) ds}_{\mathbb{L}^\epsilon(u_0, n^\epsilon)} + \underbrace{\int_0^t e^{(t-s)\Delta} (F(s, n^\epsilon, n^\epsilon) \nabla n^\epsilon) ds}_{\mathbb{B}_\epsilon^{12}(n^\epsilon, n^\epsilon, n^\epsilon)}. \end{aligned}$$

**Lemma 2.4.** *Let  $n \in \Theta$  and  $\max\{4 - d + \frac{d}{p}, 0\} < \alpha < 1$ . Then there exists a constant  $C_5 > 0$  such that it holds that*

$$\|\mathbb{B}_\epsilon^{12}(n^\epsilon, n^\epsilon, n^\epsilon)\|_{\Theta} \leq C_5 \|n\|_{\Theta}^3.$$

*Proof.* We apply  $\Delta_j$  to  $F(s, n^\epsilon, n^\epsilon)$  and take the  $L^p(\mathbb{R}^d)$  norm. By localization of the heat kernel, we thus obtain that

$$(2.13) \quad \begin{aligned} \left\| \mathcal{F} \left( \Delta_j F(s, n^\epsilon, n^\epsilon) \right) \right\|_{L^p(\mathbb{R}^d)} &\leq 2^{j(-d+1+\frac{3}{p}-\alpha)} (\epsilon 2^{2j}) \int_0^s e^{-\epsilon(\tau-s)2^{2j}} \tau^{-\alpha/2} d\tau \|n\|_{\mathbb{Z}_0} \|n\|_{\mathbb{Z}_\alpha} \\ &\lesssim 2^{j(-d+1+\frac{d}{p}-\alpha)} s^{-\alpha/2} \|n\|_{\Theta}^2. \end{aligned}$$

Exactly along the same line proof of Lemma 2.3, following (2.13), we can complete Lemma 2.4.  $\square$

**Lemma 2.5.** *Assume that  $\epsilon > 1$ ,  $u_0 \in \dot{\mathfrak{B}}_{p,\infty}^{-1+d-d/p}(\mathbb{R}^d) \cap \dot{\mathfrak{B}}_{1,1}^{-1}(\mathbb{R}^d)$ ,  $n \in \Theta$  and  $4 - d + d/p < \alpha < 1$ . There exists a constant  $C_3 > 0$  such that*

$$\|\mathbb{L}_\epsilon(u_0, n^\epsilon)\|_{\Theta} \leq C_3 \epsilon^{-\alpha/2} \|n\|_{\Theta} \|u_0\|_{\dot{\mathfrak{B}}_{p,\infty}^{-1+d-d/p}(\mathbb{R}^d) \cap \dot{\mathfrak{B}}_{1,1}^{-1}(\mathbb{R}^d)}.$$

*Proof.* By Bony's decomposition (2.2), we decompose  $\Delta_j(e^{\epsilon s \Delta} u_0 \cdot \nabla n^\epsilon)$  as

$$(2.14) \quad \Delta_j(e^{\epsilon s \Delta} u_0 \cdot \nabla n^\epsilon) = \sum_{k \geq j-2} \Delta_k e^{\epsilon s \Delta} u_0 \nabla S_k n + \sum_{k \geq j-2} S_k e^{\epsilon s \Delta} u_0 \Delta_k \nabla n.$$

By applying Fourier transform and taking  $L^p(\mathbb{R}^d)$  norm to (2.14), since  $4 - d + d/p < \ell_0 < 1, 4 - d + d/p < \alpha < 1$ , using Young's convolution inequality, (2.8), we get

$$\begin{aligned}
& \left\| \mathcal{F}(\Delta_j(e^{\epsilon s \Delta} u_0 \cdot \nabla n^\epsilon)) \right\|_{L^p(\mathbb{R}^d)} \\
& \leq \int_0^t e^{-(t-s)2^{2j}} \sum_{k \geq j-2} \|\mathcal{F}(S_k \nabla n)\|_{L^1(\mathbb{R}^d)} \|\mathcal{F}(\Delta_k e^{\epsilon s \Delta} u_0)\|_{L^p(\mathbb{R}^d)} ds \\
& \quad + \int_0^t e^{-(t-s)2^{2j}} \sum_{k \geq j-2} \|\mathcal{F}(S_k e^{\epsilon s \Delta} u_0)\|_{L^1(\mathbb{R}^d)} \|\mathcal{F}(\Delta_k \nabla n)\|_{L^p(\mathbb{R}^d)} ds \\
& \leq \int_0^t e^{-(t-s)2^{2j}} (\epsilon s)^{-\ell_0/2} \sum_{k \geq j-2} 2^{3k} \|n\|_{\mathbb{Z}_0} 2^{-k\ell_0} 2^{k(1-d+d/q)} \|u_0\|_{\mathfrak{B}_{p,\infty}^{-1+d-d/p}(\mathbb{R}^d)} ds \\
& \quad + (\epsilon s)^{-1/2} \int_0^t e^{-(t-s)2^{2j}} \sum_{k \geq j-2} \left[ \sum_{j' \leq k-1} 2^{-j'} \|\mathcal{F}(\Delta_{j'} u_0)\|_{L^1(\mathbb{R}^d)} \right] 2^{k(3-d+d/q-\alpha)} \|n\|_{\mathbb{Z}_\alpha} s^{-\alpha/2} ds,
\end{aligned}$$

which implies, by  $\epsilon > 1$ , (2.12) and (1.3), that

$$\begin{aligned}
\|\mathbb{L}_\epsilon(u_0, n^\epsilon)\|_\Theta & \leq \epsilon^{-\ell_0/2} \sup_{t>0} \left\{ \sup_{j \in \mathbb{Z}} \left( t 2^{2j} \right)^{\frac{\alpha}{2}} \left[ 2^{j(1-\alpha)} \int_0^t e^{-(t-s)2^{2j}} s^{-(1+\alpha)/2} ds \right] \right\} \|n\|_\Theta \|u_0\|_{\mathfrak{B}_{1,1}^{-1}(\mathbb{R}^d)} \\
& \quad + \epsilon^{-1/2} \sup_{t>0} \left[ 2^{j(2-\ell_0)} \int_0^t e^{-(t-s)2^{2j}} s^{-\ell_0/2} ds \right] \|u_0\|_{\mathfrak{B}_{p,\infty}^{-1+d-d/p}(\mathbb{R}^d)} \|n\|_\Theta \\
& \lesssim (\epsilon^{-1/2} + \epsilon^{-\ell_0/2}) \|n\|_\Theta \|u_0\|_{\mathfrak{B}_{p,\infty}^{-1+d-d/p}(\mathbb{R}^d) \cap \mathfrak{B}_{1,1}^{-1}(\mathbb{R}^d)} \\
& \lesssim \epsilon^{-\ell_0/2} \|n\|_\Theta \|u_0\|_{\mathfrak{B}_{p,\infty}^{-1+d-d/p}(\mathbb{R}^d) \cap \mathfrak{B}_{1,1}^{-1}(\mathbb{R}^d)}.
\end{aligned}$$

Let  $\ell_0 = \alpha$ , which thus completes the proof of Lemma 2.5.  $\square$

**2.3. Difference between system (1.1) and system (1.2).** We firstly define by

$$(2.15) \quad \begin{cases} F(\tau - \tau\sigma, n, n) := \epsilon \int_0^{\tau(1-\sigma)} e^{\epsilon(\tau-s)\Delta} \mathfrak{R}(n, n, s) ds, \\ \mathcal{H}(\tau, n, n) := \epsilon \int_{\tau(1-\sigma)}^\tau e^{\epsilon(\tau-s)\Delta} (\mathfrak{R}(n, n, s) - \mathfrak{R}(n, n, \tau)) ds, \\ Q(\tau, n, n) := \epsilon \int_{\tau(1-\sigma)}^\tau e^{\epsilon(\tau-s)\Delta} \mathfrak{R}(n, n, \tau) ds - \mathbb{K}(n, n, \tau), \end{cases}$$

where  $\mathbb{K}(n, n, \tau)$  comes from (2.7) and

$$(2.16) \quad \mathfrak{R}(n, n, s) := \mathbb{P}(n(s) \nabla (-\Delta)^{-1} n(s)).$$

Here,  $J_M$  stands for cut-off operator,

$$(2.17) \quad \mathcal{F}(J_M f(\tau)) := \chi_{\{|\xi| > \sqrt{M/\tau}\}} \mathcal{F} f(\tau), \quad \mathcal{F}(J_M^c f(\tau)) := \chi_{\{|\xi| \leq \sqrt{M/\tau}\}} \mathcal{F} f(\tau),$$

where  $0 < \sigma < 1$  and  $M > 0$  will be chosen later.

Thanks to (2.4), it holds that

$$\begin{aligned}
 & n(s) - n(\tau) \\
 &= \underbrace{\int_0^s (e^{(\tau-\varrho)\Delta} - e^{(s-\varrho)\Delta}) \mathbb{P}_1(n, n, \varrho) d\varrho + \int_0^s (e^{(\tau-\varrho)\Delta} - e^{(s-\varrho)\Delta}) \mathbb{P}_2(n, n, n, \varrho) d\varrho}_{\mathbb{V}_2} \\
 &+ \underbrace{e^{s\Delta} n_0 - e^{\tau\Delta} n_0}_{\mathbb{V}_1} + \underbrace{\int_s^\tau e^{(\tau-\varrho)\Delta} \mathbb{P}_1(n, n, \varrho) d\varrho + \int_s^\tau e^{(\tau-\varrho)\Delta} \mathbb{P}_2(n, n, n, \varrho) d\varrho}_{\mathbb{V}_3},
 \end{aligned} \tag{2.18}$$

where

$$\mathbb{P}_1(n, n, \varrho) := \nabla \cdot (n(\varrho) \nabla (-\Delta)^{-1} n(\varrho))$$

and

$$\mathbb{P}_2(n, n, n, \varrho) := [\mathbb{P}(-\Delta)^{-1} (n(\varrho) \nabla (-\Delta)^{-1} n(\varrho))] \cdot \nabla n(\varrho).$$

From (2.4), (2.16) and Fourier transform, we have

$$\begin{aligned}
 & \mathcal{F}(\mathfrak{K}(n, n, s) - \mathfrak{K}(n, n, \tau)) \\
 &= \mathcal{F}((n(s) - n(\tau)) \nabla (-\Delta)^{-1} n(s)) + \mathcal{F}(n(\tau) \nabla (-\Delta)^{-1} (n(s) - n(\tau))) \\
 &= \Pi_1 + \Pi_2,
 \end{aligned} \tag{2.19}$$

where

$$\Pi_2 := \mathcal{F}(J_M(n(s) - n(\tau))) * \mathcal{F}(\nabla(-\Delta)^{-1} n(s)) + (\mathcal{F} n(\tau)) * \mathcal{F}(J_M \nabla(-\Delta)^{-1} (n(s) - n(\tau))). \tag{2.20}$$

and

$$\begin{aligned}
 \Pi_1 &:= \sum_{i=1}^3 \left[ \mathcal{F}(J_M^c \mathbb{V}_i) * \mathcal{F}(\nabla(-\Delta)^{-1} n(s)) + (\mathcal{F} n(\tau)) * \mathcal{F}(J_M^c \nabla(-\Delta)^{-1} \mathbb{V}_i) \right] \\
 &= \sum_{i=1}^3 \Pi_{1i}
 \end{aligned} \tag{2.21}$$

Thus, it follows from (2.16), (2.19), (2.21) and (2.20) that

$$\mathcal{F} \left( \epsilon \int_{\tau(1-\sigma)}^\tau e^{\epsilon(\tau-s)\Delta} (\mathfrak{K}(n, n, s) - \mathfrak{K}(n, n, \tau)) ds \right) \leq \mathcal{H}_1(n, \tau) + \mathcal{H}_2(n, \tau), \tag{2.22}$$

where

$$\mathcal{H}_1(n, \tau) := \epsilon \int_{\tau(1-\sigma)}^\tau e^{\epsilon(\tau-s)\Delta} \mathcal{F}^{-1} \Pi_1 ds \tag{2.23}$$

and

$$\mathcal{H}_2(n, \tau) := \epsilon \int_{\tau(1-\sigma)}^\tau e^{\epsilon(\tau-s)\Delta} \mathcal{F}^{-1} \Pi_2 ds. \tag{2.24}$$

Let

$$\Gamma(n, n_0) := \left( 1 - e^{-M\sigma} + \left( \frac{1}{1-\sigma} - 1 \right)^{\alpha/2} \right) \left( \|n\|_\Theta^4 + \|n\|_\Theta^3 + \|n\|_\Theta \|n_0\|_{\mathfrak{H}_{p,\infty}^{-2+d/p}(\mathbb{R}^d)} \right), \tag{2.25}$$

where  $0 < \sigma < 1$  and  $M > 0$  will be explained below.



With these definitions (2.23), (2.24) and (2.25) in hand, we shall establish the following Lemmas in Lebesgue space.

**Lemma 2.6.** *Let  $n \in \Theta$  and  $\max\{4 - d + \frac{d}{p}, 0\} < \alpha < 1$ . There exists a absolute constant  $C$  such that*

$$\left\| \mathcal{F}(\Delta_j \mathcal{H}_1(n, \tau)) \right\|_{L^p(\mathbb{R}^d)} \leq C \Gamma(n, n_0) 2^{j(1-d+d/p)} + C(1-\sigma)^{-\alpha/2} \Gamma(n, n_0) 2^{j(1-d+d/p-\alpha)} \tau^{-\alpha/2}.$$

*Proof.* It follows from (2.21), one gets

$$\begin{aligned} \|\varphi_j \Pi_{1i}\|_{L^p(\mathbb{R}^d)} &\leq \int_{\tau(1-\sigma)}^{\tau} e^{-(t-s)2^{2j}} \sum_{k \geq j-2} \left\| \mathcal{F}(S_k J_M^c \mathbb{V}_i) \right\|_{L^1(\mathbb{R}^d)} \left\| \mathcal{F}(\Delta_k \nabla(-\Delta)^{-1} n(s)) \right\|_{L^p(\mathbb{R}^d)} ds \\ &\quad + \int_{\tau(1-\sigma)}^{\tau} e^{-(t-s)2^{2j}} \sum_{k \geq j-2} \left\| \mathcal{F}(S_k \nabla(-\Delta)^{-1} n(s)) \right\|_{L^1(\mathbb{R}^d)} \left\| \mathcal{F}(\Delta_k J_M^c \mathbb{V}_i) \right\|_{L^p(\mathbb{R}^d)} ds \\ &\quad + \int_{\tau(1-\sigma)}^{\tau} e^{-(t-s)2^{2j}} \sum_{k \geq j-2} \left\| \mathcal{F}(S_k n(s)) \right\|_{L^1(\mathbb{R}^d)} \left\| \mathcal{F}(\Delta_k J_M^c \nabla(-\Delta)^{-1} \mathbb{V}_i) \right\|_{L^p(\mathbb{R}^d)} ds \\ (2.26) \quad &\quad + \int_{\tau(1-\sigma)}^{\tau} e^{-(t-s)2^{2j}} \sum_{k \geq j-2} \left\| \mathcal{F}(S_k J_M^c \nabla(-\Delta)^{-1} \mathbb{V}_i) \right\|_{L^1(\mathbb{R}^d)} \left\| \mathcal{F}(\Delta_k n(s)) \right\|_{L^p(\mathbb{R}^d)} ds \\ &\leq \int_{\tau(1-\sigma)}^{\tau} e^{-(t-s)2^{2j}} \times \left\{ \sum_{k \geq j-2} 2^{k(1-d+d/p)} \|n\|_{\mathbb{Z}_0} \sum_{j' \leq k-1} 2^{j'd(1-1/p)} \left\| \mathcal{F}(\Delta_{j'} J_M^c \mathbb{V}_i) \right\|_{L^p(\mathbb{R}^d)} \right. \\ &\quad + \sum_{k \geq j-2} 2^{k(2-d+d/p)} \|n\|_{\mathbb{Z}_0} \sum_{j' \leq k-1} 2^{j'(d-d/p-1)} \left\| \mathcal{F}(\Delta_{j'} J_M^c \mathbb{V}_i) \right\|_{L^p(\mathbb{R}^d)} \\ &\quad \left. + \sum_{k \geq j-2} 2^k \left\| \mathcal{F}(\Delta_k J_M^c \mathbb{V}_i) \right\|_{L^p(\mathbb{R}^d)} \|n\|_{\mathbb{Z}_0} \right\} ds. \end{aligned}$$

With the definition (2.18), thanks to Lemma 2.4 and Lemma 2.5, we have

$$\begin{aligned} &\left\| \mathcal{F}(\Delta_j J_M^c \mathbb{V}_2) \right\|_{L^p(\mathbb{R}^d)} \\ (2.27) \quad &\lesssim (1 - e^{-M\sigma}) \int_0^s e^{-(s-\varrho)2^{2j}} \left\| \mathcal{F}(\Delta_j (\mathbb{P}_1(n, n, \varrho) + \mathbb{P}_2(n, n, n, \varrho))) \right\|_{L^p(\mathbb{R}^d)} d\varrho \\ &\lesssim (1 - e^{-M\sigma}) 2^{(4-d+d/p-\alpha)j} \left( \|n\|_{\mathbb{Z}_\alpha} \|n\|_{\mathbb{Z}_0}^2 + \|n\|_{\mathbb{Z}_\alpha} \|n\|_{\mathbb{Z}_0} \right) \int_0^s e^{-(s-\varrho)2^{2j}} \varrho^{-\alpha/2} d\varrho \\ &\lesssim \left( \|n\|_{\Theta}^3 + \|n\|_{\Theta}^2 \right) (1 - e^{-M\sigma}) 2^{(2-d+d/p-\alpha)j} s^{-\alpha/2} \end{aligned}$$

and

$$\begin{aligned}
(2.28) \quad & \left\| \mathcal{F}(\Delta_j J_M^c \mathbb{V}_3) \right\|_{L^p(\mathbb{R}^d)} \\
& \lesssim \left\| \mathcal{F} \left( \Delta_j \left[ \int_s^\tau e^{(\tau-\varrho)\Delta} (\mathbb{P}_1(n, n, \varrho) + \mathbb{P}_2(n, n, n, \varrho)) d\varrho \right] \right) \right\|_{L^p(\mathbb{R}^d)} \\
& \lesssim \left( \|n\|_{\mathbb{Z}_\alpha} \|n\|_{\mathbb{Z}_0}^2 + \|n\|_{\mathbb{Z}_\alpha} \|n\|_{\mathbb{Z}_0} \right) 2^{j'(4-d+d/p-\alpha)} \left( \int_s^\tau e^{-(\tau-\varrho)2^{2j}} \varrho^{-\frac{\alpha}{2}} d\varrho \right) \\
& \lesssim \left( \|n\|_\Theta^3 + \|n\|_\Theta^2 \right) 2^{j'(2-d+d/p)} \left( \frac{\tau}{s} - 1 \right)^{\alpha/2} \left( \frac{1 - e^{-2^{2j}(\tau-s)}}{(2^{2j}(\tau-s))^{\alpha/2}} \right) \\
& \lesssim \left( \|n\|_\Theta^3 + \|n\|_\Theta^2 \right) 2^{j'(2-d+d/p)} \left( \frac{1}{1-\sigma} - 1 \right)^{\alpha/2}
\end{aligned}$$

and finally

$$(2.29) \quad \left\| \mathcal{F}(\Delta_j J_M^c \mathbb{V}_1) \right\|_{L^p(\mathbb{R}^d)} \lesssim (1 - e^{-M\sigma}) 2^{j'(2-d+d/q)} \|n_0\|_{\mathfrak{B}_{p,\infty}^{-2+d-d/p}(\mathbb{R}^d)}.$$

Thanks to (2.26)-(2.29), we thus have

$$\begin{aligned}
& \left\| \mathcal{F}(\Delta_j \mathcal{H}_1(n, \tau)) \right\|_{L^p(\mathbb{R}^d)} \\
& \leq \epsilon \sum_{i=1,2,3} \int_{\tau(1-\sigma)}^\tau e^{-\epsilon(\tau-s)2^{2j}} \left\| \varphi_j \Pi_{1i} \right\|_{L^p(\mathbb{R}^d)} ds \\
& \leq \Gamma(n, n_0) \sum_{\iota=0,\alpha} \left( \int_{\tau(1-\sigma)}^\tau e^{-(\tau-s)2^{2j}} \sum_{k \geq j-2} 2^{k(1-d+d/p)} \sum_{j' \leq k-1} 2^{(2-\iota)j'} s^{-\iota/2} ds \right. \\
& \quad + \int_{\tau(1-\sigma)}^\tau e^{-(t-s)2^{2j}} \sum_{k \geq j-2} 2^{k(2-d+d/p)} \sum_{j' \leq k-1} 2^{j'(1-\iota)} s^{-\iota/2} ds \\
& \quad \left. + \int_{\tau(1-\sigma)}^\tau e^{-(\tau-s)2^{2j}} \sum_{k \geq j-2} 2^{(3-d+d/p-\iota)k} s^{-\iota/2} ds \right) \\
& \lesssim \Gamma(n, n_0) 2^{j(1-d+d/p)} + \Gamma(n, n_0) 2^{j(1-d+d/p-\alpha)} \tau^{-\alpha/2} (1-\sigma)^{-\alpha/2},
\end{aligned}$$

where  $\Gamma(n, n_0)$  is from (2.25). This completes the proof of Lemma 2.6.  $\square$

Let

$$(2.30) \quad \Lambda(\theta) := M^{-\theta} \left[ 1 + \left( \frac{1}{1-\sigma} \right)^{\frac{\alpha}{2}} \right],$$

where  $0 < \sigma < 1$  and  $M > 0$  will be chosen below and  $0 < \theta < \alpha - 2 + \frac{d}{2} - \frac{d}{2p}$ .

**Lemma 2.7.** *Assume that  $n \in \Theta$  and  $0 < \theta < \alpha - 2 + \frac{d}{2} - \frac{d}{2p}$ . Then there exists a absolute constant  $C > 0$  such that*

$$\left\| \mathcal{F}(\Delta_j J_M \mathcal{H}_2(n, \tau)) \right\|_{L^p(\mathbb{R}^d)} \leq C \Lambda(\theta) 2^{j(2\theta+1-d-\alpha+d/p)} \tau^{-\alpha/2+\theta} \|n\|_\Theta^2.$$

*Proof.* From (2.17) and (2.30), we obtain

$$(2.31) \quad \left\| \mathcal{F}(\Delta_j J_M(n(s) - n(\tau))) \right\|_{L^p(\mathbb{R}^d)} \lesssim \left( \sqrt{\frac{M}{\tau}} \right)^{-2\theta} \left[ 1 + \left( \frac{1}{1-\sigma} \right)^{\frac{\alpha}{2}} \right] \tau^{-\frac{\alpha}{2}} \|n\|_{\mathbb{Z}_\alpha} 2^{j(2\theta+2-d-\alpha+d/p)} \\ \lesssim \Lambda(\theta) \tau^{-\alpha/2+\theta} 2^{j(2\theta+2-d-\alpha+d/p)} \|n\|_{\Theta}.$$

Therefore, by (2.20), (2.31), (2.24) we obtain

$$\begin{aligned} & \left\| \mathcal{F}(\Delta_j \mathcal{H}_2(n, \tau)) \right\|_{L^q(\mathbb{R}^d)} \\ & \lesssim \epsilon \int_{\tau(1-\sigma)}^{\tau} e^{-\epsilon(\tau-s)2^{2j}} \left\| \varphi_j(\mathcal{F}(J_M(n(s) - n(\tau)))) * \mathcal{F}(\nabla(-\Delta)^{-1}n(s)) \right\|_{L^p(\mathbb{R}^d)} ds \\ & \quad + \epsilon \int_{\tau(1-\sigma)}^{\tau} e^{-\epsilon(\tau-s)2^{2j}} \left\| \varphi_j(\mathcal{F}n(\tau)) * (\mathcal{F}(J_M \nabla(-\Delta)^{-1}(n(s) - n(\tau)))) \right\|_{L^p(\mathbb{R}^d)} ds \\ & \lesssim \epsilon \|n\|_{\Theta} \int_{\tau(1-\sigma)}^{\tau} e^{-\epsilon(\tau-s)2^{2j}} \Lambda(\theta) \tau^{-\alpha/2+\theta} \times \left\{ \sum_{k \geq j-2} 2^{k(1-d+d/p)} \|n\|_{\mathbb{Z}_0} \sum_{j' \leq k-1} 2^{j'(2+2\theta-\alpha)} \right. \\ & \quad \left. + \epsilon \sum_{k \geq j-2} 2^{k(2\theta+3-d-\alpha+d/p)} + \sum_{k \geq j-2} 2^{k(2-d+d/p)} \|n\|_{\mathbb{Z}_0} \sum_{j' \leq k-1} 2^{j'(1+2\theta-\alpha)} \right\} \\ & \lesssim \epsilon \|n\|_{\Theta}^2 \Lambda(\theta) \int_{\tau(1-\sigma)}^{\tau} e^{-\epsilon(\tau-s)2^{2j}} 2^{j(2\theta+3-d-\alpha+d/p)} \tau^{-\alpha/2+\theta} ds \\ & \lesssim \|n\|_{\Theta}^2 \Lambda(\theta) 2^{j(2\theta+1-d-\alpha+d/p)} \tau^{-\alpha/2+\theta}. \end{aligned}$$

We thus conclude Lemma 2.7.  $\square$

**Lemma 2.8.** *Let  $n \in \Theta$  and  $0 \leq \varpi < 1$ . Then by (2.15), there exists a absolute constant  $C > 0$  such that*

$$\left\| \mathcal{F}(\Delta_j F(\tau - \tau\sigma, n, n)) \right\|_{L^p(\mathbb{R}^d)} \leq C \epsilon^{\frac{\varpi-1}{2}} \sigma^{\frac{\varpi-1}{2}} (1-\sigma)^{-\alpha/2} 2^{j(-d+\frac{d}{p}-\alpha+\varpi)} \|n\|_{\Theta}^2 \tau^{\frac{\varpi-1-\alpha}{2}}.$$

*Proof.* Since  $0 \leq \varpi < 1$ , according to Gaussian bound  $2^{j(1-\varpi)} e^{-\sigma\epsilon\tau 2^{2j}} \lesssim (\sigma\epsilon\tau)^{-\frac{1-\varpi}{2}}$ , it is easy to verify that

$$\begin{aligned} & \left\| \mathcal{F}(\Delta_j F(\tau - \tau\sigma, n, n)) \right\|_{L^p(\mathbb{R}^d)} \\ & \lesssim 2^{j(1-d+\frac{d}{p}-\alpha)} (\epsilon 2^{2j}) \int_0^{\tau(1-\sigma)} e^{-\epsilon(\tau(1-\sigma)-s)2^{2j}} s^{-\alpha/2} ds \|n\|_{\mathbb{Z}_0} \|n\|_{\mathbb{Z}_\alpha} e^{-\epsilon\tau\sigma 2^{2j}} \\ & \lesssim 2^{j(1-d+\frac{d}{p}-\alpha)} \|n\|_{\Theta}^2 2^{j(-1+\varpi)} \tau^{\frac{\varpi-1}{2}} (\epsilon\sigma)^{\frac{\varpi-1}{2}} \tau^{-\alpha/2} (1-\sigma)^{-\alpha/2}. \end{aligned}$$

Thus Lemma 2.8 follows.  $\square$

**Lemma 2.9.** *Let  $n \in \Theta$  and  $0 \leq \varpi < 1$ . Then there exists a absolute constant  $C > 0$  such that*

$$\|Q(\tau, n, n)\|_{L^p(\mathbb{R}^d)} \leq C 2^{j(-d+\frac{d}{p}-\alpha+\varpi)} \tau^{-(\alpha+1-\varpi)/2} \|n\|_{\Theta}^2 (\sigma\epsilon)^{-\frac{1-\varpi}{2}}.$$

*Proof.* Applying the definition of  $Q(\tau, n, n)$  ( see (2.15)), explicit calculation yields that

$$\begin{aligned}
\|Q(\tau, n, n)\|_{L^q(\mathbb{R}^d)} &\lesssim 2^{j(1-d+\frac{d}{p}-\alpha)} \tau^{-\alpha/2} \|n\|_{\mathbb{Z}_0} \|n\|_{\mathbb{Z}_\alpha} \left[ \int_{\tau(1-\sigma)}^\tau \epsilon e^{-\epsilon(\tau-s)2^{2j}} 2^{2j} ds - 1 \right] \\
&\lesssim 2^{j(-d+\frac{d}{p}-\alpha)} \left( 2^j e^{-\tau\sigma\epsilon 2^{2j}} \right) \tau^{-\alpha/2} \|n\|_{\mathbb{Z}_0} \|n\|_{\mathbb{Z}_\alpha} \\
&\lesssim 2^{j(-d+\frac{d}{p}-\alpha+\varpi)} \left( 2^{j(1-\varpi)} e^{-\tau\sigma\epsilon 2^{2j}} \right) \tau^{-\alpha/2} \|n\|_\Theta^2 \\
&\lesssim 2^{j(-d+\frac{d}{p}-\alpha+\varpi)} \tau^{-(\alpha+1-\varpi)/2} \|n\|_\Theta^2 (\sigma\epsilon)^{-\frac{1-\varpi}{2}}.
\end{aligned}$$

This ends the proof.  $\square$

By (2.5), (2.12), (2.15), (2.23) and (2.24), we arrive at

$$\begin{aligned}
&\mathbb{B}_\epsilon^{1,2}(n, n, n) - \mathbb{B}^{1,2}(n, n, n) \\
(2.32) \quad &= \int_0^t e^{(t-\tau)\Delta} \left( F(\tau - \tau\sigma, n, n) \nabla n(\tau) \right) d\tau + \int_0^t e^{(t-\tau)\Delta} \left( Q(\tau, n, n) \nabla n(\tau) \right) d\tau \\
&\quad + \int_0^t e^{(t-\tau)\Delta} \left( \mathcal{H}_1(n, \tau) \nabla n(\tau) \right) d\tau + \int_0^t e^{(t-\tau)\Delta} \left( \mathcal{H}_2(n, \tau) \nabla n(\tau) \right) d\tau.
\end{aligned}$$

Let

$$(2.33) \quad \Omega(M, \theta, \sigma) := \left( M^{-\theta} + (\epsilon\sigma)^{-\frac{1}{2}} + (\epsilon\sigma)^{\frac{\varpi-1}{2}} + 1 - e^{-M\sigma} + \left( \frac{1}{1-\sigma} - 1 \right)^{\alpha/2} \right),$$

where  $0 < \theta < \alpha - 2 + \frac{d}{2} - \frac{d}{2p}$  and  $\alpha < \varpi < 1$ .

**Proposition 2.10.** *Let  $n \in \Theta$ ,  $n_0 \in \dot{\mathfrak{B}}_{p,\infty}^{-2+d-d/p}(\mathbb{R}^d)$  and  $\alpha < \varpi < 1$ . Then There exists a absolute constant  $C > 0$  such that*

$$\left\| \mathbb{B}_\epsilon^{1,2}(n, n, n) - \mathbb{B}^{1,2}(n, n, n) \right\|_\Theta \leq C \Omega(M, \theta, \sigma) \left( \|n\|_\Theta^5 + \|n\|_\Theta^4 + \|n\|_\Theta^3 + \|n\|_\Theta^2 \|n_0\|_{\dot{\mathfrak{B}}_{p,\infty}^{-2+d-d/p}(\mathbb{R}^d)} \right).$$

*Proof.* Denote  $\mathcal{H}_3(n, \tau) := F(\tau - \tau\sigma, n, n)$  and  $\mathcal{H}_4(n, \tau) := Q(\tau, n, n)$ . We apply  $\Delta_j$  to (2.32) and take the  $L^p(\mathbb{R}^d)$  norm. By paraproduct decomposition (2.2), we obtain

$$\begin{aligned}
(2.34) \quad &\left\| \mathcal{F} \left( \Delta_j (\mathbb{B}_\epsilon^{1,2}(n, n, n) - \mathbb{B}^{1,2}(n, n, n)) \right) \right\|_{L^p(\mathbb{R}^d)} \\
&\leq \sum_{i=1}^4 \int_0^t e^{-(t-\tau)2^{2j}} \sum_{k \geq j-2} \left\| \mathcal{F} (S_k \nabla n) * \mathcal{F} (\Delta_k \mathcal{H}_i(n, \tau)) \right\|_{L^p(\mathbb{R}^d)} d\tau \\
&\quad + \sum_{i=1}^4 \int_0^t e^{-(t-s)2^{2j}} \sum_{k \geq j-2} \left\| \mathcal{F} (\Delta_k \nabla n) * \mathcal{F} (S_k \mathcal{H}_i(n, \tau)) \right\|_{L^p(\mathbb{R}^d)} d\tau \\
&\leq \sum_{i=1}^4 \left\{ \int_0^t e^{-(t-\tau)2^{2j}} \sum_{k \geq j-2} \|n\|_{\mathbb{Z}_\alpha} 2^{(3-\alpha)k} \left\| \mathcal{F} (\Delta_k \mathcal{H}_i(n, \tau)) \right\|_{L^p(\mathbb{R}^d)} \tau^{-\alpha/2} d\tau \right. \\
&\quad \left. + \int_0^t e^{-(t-\tau)2^{2j}} \sum_{k \geq j-2} \sum_{j' \leq k-1} 2^{j'd(1-1/p)} \left\| \mathcal{F} (\Delta_{j'} \mathcal{H}_i(n, \tau)) \right\|_{L^p(\mathbb{R}^d)} \right. \\
&\quad \left. \times 2^{k(3-d+d/q-\alpha)} \|n\|_{\mathbb{Z}_\alpha} \tau^{-\alpha/2} d\tau \right\} := \sum_{i=1}^4 \Psi_i.
\end{aligned}$$

We will start with the estimate of  $\Psi_1$ , due to Young's convolution inequality, Lemma 2.6, it leads to that

$$\begin{aligned}
 \Psi_1 &\leq \Gamma(n, n_0) \left[ \int_0^t e^{-(t-\tau)2^{2j}} \sum_{k \geq j-2} \|n\|_{\Theta} 2^{(3-\alpha)k} 2^{k(1-d+\frac{d}{p})} \tau^{-\alpha/2} \left[ 1 + 2^{-k\alpha} (1-\sigma)^{-\alpha/2} \tau^{-\alpha/2} \right] d\tau \right. \\
 &\quad + \int_0^t e^{-(t-\tau)2^{2j}} \sum_{k \geq j-2} \sum_{j' \leq k-1} 2^{j'd(1-1/p)} 2^{j'(1-d+\frac{d}{p})} \times 2^{k(3-d+d/q-\alpha)} \\
 &\quad \times \|n\|_{\mathbb{Z}_\alpha} \left[ 1 + 2^{-j'\alpha} (1-\sigma)^{-\alpha/2} \tau^{-\alpha/2} \right] \tau^{-\alpha/2} ds \Big] \\
 (2.35) \quad &\lesssim 2^{j(2-d+d/p)} \|n\|_{\Theta} (1-\sigma)^{-\alpha/2} \Gamma(n, n_0) 2^{j(2-2\alpha)} \int_0^t e^{-(t-\tau)2^{2j}} \tau^{-\alpha} d\tau \\
 &\quad + 2^{j(2-d+d/p)} \|n\|_{\Theta} \Gamma(n, n_0) 2^{j(2-\alpha)} \int_0^t e^{-(t-\tau)2^{2j}} \tau^{-\alpha/2} d\tau \\
 &\lesssim 2^{j(2-d+d/p)} \left[ 1 + (1-\sigma)^{-\alpha/2} \right] \|n\|_{\Theta} \Gamma(n, n_0).
 \end{aligned}$$

To handle the term  $\Psi_2$ , with the aid of Young's convolution inequality, Lemma 2.7 and  $0 < \theta < \alpha - 2 + \frac{d}{2} - \frac{d}{2p}$ , we have

$$\begin{aligned}
 \Psi_2 &\leq \Lambda(n_0) \|n\|_{\Theta}^2 \left[ \int_0^t e^{-(t-\tau)2^{2j}} \sum_{k \geq j-2} \|n\|_{\mathbb{Z}_\alpha} 2^{(3-\alpha)k} 2^{k(2\theta+1-d+\frac{d}{p}-\alpha)} \tau^{-\alpha+\theta} d\tau \right. \\
 &\quad + \int_0^t e^{-(t-\tau)2^{2j}} \sum_{k \geq j-2} \sum_{j' \leq k-1} 2^{j'd(1-1/p)} 2^{j'(2\theta+1-d+\frac{d}{p}-\alpha)} \times 2^{k(3-d+d/q-\alpha)} \|n\|_{\mathbb{Z}_\alpha} \tau^{-\alpha+\theta} d\tau \Big] \\
 (2.36) \quad &\lesssim 2^{j(2-d+d/p)} \|n\|_{\Theta}^3 \Lambda(\theta) 2^{j(2+2\theta-2\alpha)} \int_0^t e^{-(t-\tau)2^{2j}} \tau^{-\alpha+\theta} d\tau \\
 &\lesssim 2^{j(2-d+d/p)} \|n\|_{\Theta}^3 \Lambda(\theta).
 \end{aligned}$$

Next, we estimate  $\Psi_3$  and  $\Psi_4$ , in term of Young's convolution inequality, Lemma 2.8 and Lemma 2.9,  $\alpha < \varpi < 1 < 2\alpha - 3 + d - \frac{d}{p}$ , we obtain

$$\begin{aligned}
 \Psi_3 + \Psi_4 &\leq \|n\|_{\Theta}^2 \left( \epsilon^{-\frac{1}{2}} \sigma^{-\frac{1}{2}} + \epsilon^{\frac{\varpi-1}{2}} \sigma^{\frac{\varpi-1}{2}} \right) \left[ (1-\sigma)^{-\alpha/2} + 1 \right] \left[ \int_0^t e^{-(t-\tau)2^{2j}} \sum_{k \geq j-2} \|n\|_{\mathbb{Z}_0} 2^{3k} 2^{k(-d+\frac{d}{p}-\alpha)} \tau^{\frac{-1-\alpha}{2}} ds \right. \\
 &\quad + \int_0^t e^{-(t-\tau)2^{2j}} \sum_{k \geq j-2} \sum_{j' \leq k-1} 2^{j'd(1-1/p)} 2^{j'(-d+\frac{d}{p}-\alpha+\varpi)} \tau^{\frac{\varpi-1-\alpha}{2}} \times 2^{k(3-d+d/q-\alpha)} \|n\|_{\mathbb{Z}_\alpha} \tau^{-\alpha/2} ds \Big] \\
 &\lesssim 2^{j(2-d+d/p)} \|n\|_{\Theta}^3 \left( \epsilon^{-\frac{1}{2}} \sigma^{-\frac{1}{2}} + \epsilon^{\frac{\varpi-1}{2}} \sigma^{\frac{\varpi-1}{2}} \right) \left[ (1-\sigma)^{-\alpha/2} + 1 \right] \\
 &\quad \times \left( 2^{j(1-\alpha)} \int_0^t e^{-(t-\tau)2^{2j}} \tau^{\frac{-1-\alpha}{2}} d\tau + \int_0^t e^{-(t-\tau)2^{2j}} 2^{j(1-2\alpha+\varpi)} \tau^{\frac{\varpi-1-2\alpha}{2}} d\tau \right) \\
 (2.37) \quad &\lesssim 2^{j(2-d+d/p)} \|n\|_{\Theta}^3 \left( \epsilon^{-\frac{1}{2}} \sigma^{-\frac{1}{2}} + \epsilon^{\frac{\varpi-1}{2}} \sigma^{\frac{\varpi-1}{2}} \right) \left[ (1-\sigma)^{-\alpha/2} + 1 \right].
 \end{aligned}$$

Therefore, we have by (2.34), (2.35), (2.36) and (2.37) that

$$\begin{aligned}
 (2.38) \quad &\left\| \mathbb{B}_\epsilon^{1,2}(n, n, n) - \mathbb{B}^{1,2}(n, n, n) \right\|_{\Theta} \lesssim \Omega(M, \theta, \sigma) \left( \|n\|_{\Theta}^5 + \|n\|_{\Theta}^3 + \|n\|_{\Theta}^3 + \|n\|_{\Theta}^2 \|n_0\|_{\dot{\mathbb{B}}_{p,\infty}^{-2+d-d/p}(\mathbb{R}^d)} \right).
 \end{aligned}$$

This implies Proposition 2.10.  $\square$

### 3. PROOF OF THE MAIN RESULT

In this section, we shall give a complete proof of Theorem 1.1 by using some results established in previous sections. Now, we will divide two parts to finish the proof of Theorem 1.1.

**3.1. Proof of first part of Theorem 1.1. Case I: Existence of system (1.2).** Firstly, thanks to the definition of  $\Theta$  space (1.3), we easily conclude that

$$\|e^{t\Delta}n_0\|_{\Theta} \leq C_0 \|n_0\|_{\mathfrak{B}_{p,\infty}^{-2+d-d/p}(\mathbb{R}^d)},$$

where  $C_0$  is independent of  $n_0$ . If  $\|e^{t\Delta}n_0\|_{\Theta} \leq C_0 \|n_0\|_{\mathfrak{B}_{p,\infty}^{-2+d-d/p}(\mathbb{R}^d)} \leq \alpha_0$  with

$$\alpha_0 := \min \left\{ \frac{1}{72C_1}, \sqrt{\frac{1}{324C_2}} \right\},$$

where  $C_1$  and  $C_2$  come from Lemma 2.2 and Lemma 2.3, respectively. by (2.4),

$$n = e^{t\Delta}n_0 + \mathbb{B}^{11}(n, n) + \mathbb{B}^{12}(n, n, n),$$

we can know that the system (1.2) has a unique solution in the ball with center 0 and radius  $2\alpha_0$  in  $\Theta$ .

**Case II: Existence of system (1.1).** According to (2.11) and (2.12), the system (1.1) can be reduced to construct a solution  $n^\epsilon$  of the following equation:

$$n^\epsilon = e^{t\Delta}n_0 + \mathbb{L}^\epsilon(u_0, n^\epsilon) + \mathbb{B}^{11}(n^\epsilon, n^\epsilon) + \mathbb{B}_\epsilon^{12}(n^\epsilon, n^\epsilon, n^\epsilon).$$

By virtue of Lemma 2.5, for  $\epsilon > 1$  we can get when

$$\|\mathbb{L}^\epsilon\| := C_3 \epsilon^{-\alpha/2} \|u_0\|_{\mathfrak{B}_{p,\infty}^{-1+d-d/p}(\mathbb{R}^d) \cap \mathfrak{B}_{1,1}^{-1}(\mathbb{R}^d)} < 1,$$

and if

$$\|e^{t\Delta}n_0\|_{\Theta} \leq C_0 \|n_0\|_{\mathfrak{B}_{p,\infty}^{-2+d-d/p}(\mathbb{R}^d)} \leq \beta_0$$

with

$$\beta_0 := \min \left\{ \frac{1 - \|\mathbb{L}^\epsilon\|}{24AC_2}, \sqrt{\frac{1 - \|\mathbb{L}^\epsilon\|}{36A^2C_5}} \right\},$$

where

$$A = \frac{3}{1 - \|\mathbb{L}^\epsilon\|},$$

the constants  $C_4$  and  $C_5$  come from Lemma 2.2 and Lemma 2.4, respectively. Then, by above estimates, we have that a unique solution of the system (1.2) fulfills  $\|n\|_{\Theta} \leq 2\beta_0$ .

To complete first part of Theorem 1.1, we only need to let  $\ell = \frac{1}{C_3}$  and

$$\varepsilon = \min \left\{ \frac{\alpha_0}{C_0}, \frac{\beta_0}{C_0} \right\}.$$

Therefore, the proof of first part of Theorem 1.1 is complete.

**3.2. Proof of second part of Theorem 1.1.** The smallness of initial date (see the part one of Theorem 1.1) can guarantee that there exists a absolute constant  $\lambda < 1$  such that

$$(3.1) \quad \|n\|_{\Theta} + \|n^{\epsilon}\|_{\Theta} < \lambda \ll 1,$$

where  $n$  and  $n^{\epsilon}$  are the solutions of (1.2) and (1.1), respectively.

Employing (2.4), (2.11) and (2.12), it follows from Lemma 2.2, Lemma 2.3, Lemma 2.4 and Lemma 2.5 that

$$\begin{aligned} \|n - n^{\epsilon}\|_{\Theta} &= \left\| e^{t\Delta} n_0 + \mathbb{B}^{11}(n, n) + \mathbb{B}^{12}(n, n, n) \right. \\ &\quad \left. - \left[ e^{t\Delta} n_0 + \mathbb{L}^{\epsilon}(u_0, n^{\epsilon}) + \mathbb{B}^{11}(n^{\epsilon}, n^{\epsilon}) + \mathbb{B}_{\epsilon}^{12}(n^{\epsilon}, n^{\epsilon}, n^{\epsilon}) \right] \right\|_{\Theta} \\ &\leq \left\| \mathbb{B}^{1,1}(n - n^{\epsilon}, n) \right\|_{\Theta} + \left\| \mathbb{B}^{1,1}(n^{\epsilon}, n - n^{\epsilon}) \right\|_{\Theta} + \left\| \mathbb{B}_{\epsilon}^{1,2}(n, n, n) - \mathbb{B}^{1,2}(n, n, n) \right\|_{\Theta} \\ &\quad + \left\| \int_0^t e^{(t-\tau)\Delta} (F(\tau, n^{\epsilon}, n^{\epsilon}) \nabla(n^{\epsilon}(\tau) - n(\tau))) d\tau \right\|_{\Theta} + \left\| \mathbb{L}^{\epsilon}(u_0, n^{\epsilon}) \right\|_{\Theta} \\ &\quad + \left\| \int_0^t e^{(t-\tau)\Delta} ((F(\tau, n^{\epsilon}, n^{\epsilon}) - F(\tau, n, n)) \nabla n(\tau)) d\tau \right\|_{\Theta} \\ &\lesssim \left\| \mathbb{B}_{\epsilon}^{1,2}(n, n, n) - \mathbb{B}^{1,2}(n, n, n) \right\|_{\Theta} + \|n - n^{\epsilon}\|_{\Theta} (\|n\|_{\Theta} + \|n^{\epsilon}\|_{\Theta}) \\ &\quad + \|n - n^{\epsilon}\|_{\Theta} (\|n^{\epsilon}\|_{\Theta}^2 + \|n\|_{\Theta}^2 + \|n^{\epsilon}\|_{\Theta} \|n\|_{\Theta}) + \epsilon^{-\alpha/2} \|n\|_{\Theta} \|u_0\|_{\mathfrak{B}_{p,\infty}^{-1+d-d/p}(\mathbb{R}^d) \cap \mathfrak{B}_{1,1}^{-1}(\mathbb{R}^d)}, \end{aligned}$$

from which, (3.1) and Proposition 2.10, we obtain

$$\begin{aligned} \|n - n^{\epsilon}\|_{\Theta} (1 - C\lambda - C\lambda^2) &\leq C\lambda \epsilon^{-\alpha/2} \|u_0\|_{\mathfrak{B}_{p,\infty}^{-1+d-d/p}(\mathbb{R}^d) \cap \mathfrak{B}_{1,1}^{-1}(\mathbb{R}^d)} \\ &\quad + \Omega(M, \theta, \sigma) \left( \lambda^5 + \lambda^4 + \lambda^2 \|n_0\|_{\mathfrak{B}_{p,\infty}^{-2+d-d/p}(\mathbb{R}^d)} \right) := C(\epsilon). \end{aligned}$$

Due to

$$(3.2) \quad \begin{cases} d > 3, \quad p > \frac{d}{d-3}, \\ 4 - d + \frac{d}{p} < \alpha < 1, \quad 0 < \alpha < \varpi < 1, \\ 0 < \theta < \alpha - 2 + \frac{d}{2} - \frac{d}{2p}, \end{cases}$$

and let  $\sigma = \epsilon^{-\frac{1}{2}}$ ,  $M = \epsilon^{\frac{1}{4}}$ , along with (2.33) and (3.2), when  $\epsilon \rightarrow +\infty$ , we achieve

$$\Omega(M, \theta, \sigma) := \left( M^{-\theta} + (\epsilon\sigma)^{-\frac{1}{2}} + (\epsilon\sigma)^{\frac{\varpi-1}{2}} + 1 - e^{-M\sigma} + \left( \frac{1}{1-\sigma} - 1 \right)^{\alpha/2} \right) \rightarrow 0,$$

and thus  $C(\epsilon) \rightarrow 0$ . Then we have from (3.2) that

$$\|n - n^{\epsilon}\|_{\Theta} \rightarrow 0,$$

as  $\epsilon \rightarrow +\infty$ . The second part of Theorem 1.1 is finally proved.

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