Sector stability criteria for a nonlinear axial motion string system

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Abstract

The paper investigates the exponential stability criterion for an axially moving string system driven by a nonlinear partial differential equation with nonlinear boundary feedback. The control criterion based on a sector condition contains a large class of nonlinearities, which is a negative feedback of the velocity at the right boundary of the moving string. By invoking nonlinear semigroup theory, the well-posedness result of the closed-loop system is verified under the sector criteria. Furthermore, a novel energy like function is constructed to establish the exponential stability of the closed-loop system by using a integral-type multiplier method and the generalized Gronwall-type integral inequality.

Keywords: Axially moving string; boundary control; exponential stability; sector stability criteria.

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1 Introduction and Main results

In the past ten years research of axial motion systems has received much attention, such as in paper sheets, conveyor belts, fiber, magnetism tapes and robotic arms, etc. However, these systems are all affected by transverse vibration. This encouraged researchers to explore different control methods to suppress vibrations in these systems. One of the most powerful mechanical devices used in this area is feedback control at the boundary due to the ease of implementation, for example, see [1, 2, 3, 4]. For a recent review of axially moving systems, we refer the reader to [5]. Most controllers are designed for linear models of axially moving strings [6, 7, 8, 9], including linear discrete systems, passive control laws with linear damping, or different active control laws.

To the best of our knowledge, there are few papers which are relaxed to the design criteria of the controller for axially moving strings described by nonlinear partial differential equations. It is worth mentioning that, [10] and [11] provided two kinds of nonlinear models to reflect the dynamic behavior of axially moving strings, based on the different tension structures of string. The motivation of this paper is to propose a more general nonlinear mathematical model of string and make a design criteria of the controller. In this paper, we consider a nonlinear axially moving string, which can be expressed by the following nonlinear partial differential equation (PDE)

$$\begin{cases} z_{tt}(x,t) + 2vz_{xt}(x,t) = [z_x(x,t)h(z_x^2(x,t))]_x, \quad x \in (0,L), \quad t > 0, \\ z_x(L,t)h(z_x(L,t)) - vz_t(L,t) = U(t), \\ z(0,t) = 0, \\ z(x,0) = f(x), z_t(x,0) = g(x), x \in [0,L], \end{cases} \tag{1.1}$$

where $z(x,t)$ stands for the transversal deflection at the position x and at time t, v is the speed of the moving string, L denotes the length of string, h is a continuous function related to the tension of string, $[\cdot]_x$ represents $\frac{\partial[\cdot]}{\partial x}$, U is the boundary control input, f, g denote the initial displacement and velocity of the string.

When $h(z_x^2(x,t)) \equiv const$ in (1.1), the exponential stability of the linear control system where the damping term $z_t + z_x$ is added in the first equation of (1.1) is established in [12] with the linear boundary feedback using the Lyapunov method combined with semigroup theory. When $h(z_x^2(x,t)) = T_0 + \frac{EA}{2}z_x^2(x,t)$ with initial tension $T_0 > 0$ and U

is a mass-damper-spring controller, applying semigroup theory the asymptotical stability of the axially moving control system (1.1) is analyzed in [13]. In this case, based on the direct Lyapunov method, the exponential stabilization of this control system (1.1) is established in [14] by means of the linear speed feedback ($U = -kz_t(L,t)$ with $k > 0$). On the other hand, when $h(z_x(x,t)) = b - v^2 + (1-b)/\sqrt{1+z_x^2(x,t)}$ with $b \geq 1 > |v|$, via linear negative velocity feedback at the right boundary of the string, the exponential stabilization of the nonlinear axially moving string (1.1) is investigated by the direct Lyapunov method in [10], where the well-posedness of the system is not provided due to the fact that the nonlinearity leads to the invalidation of some commonly used approaches like Faedo-Galerkin method and frequency domain methods. In addition, for related research of another kind of nonlinear strings (Kirchhoff strings), we refer the reader to Shahruz and Krishna [15], Shahruz [16], to name a few.

In this paper, the exponential stabilization of the nonlinear axially moving string (1.1) is considered under the nonlinear controller $U(t) = -\mathcal{G}(z_t(L,t))$ where the nonlinear function $\mathcal{G}$ is nondecreasing continuous and satisfies the following sector condition [17]

$$\mathcal{G}(0) = 0, \quad b_1 \leq \frac{\mathcal{G}(s)}{s} \leq b_2, \quad \forall\, s \in R \setminus \{0\}, \tag{1.2}$$

for any given constants $b_1, b_2 > 0$. Here the constants b_1 and b_2 are regarded as the lower and upper bounds of the sector, respectively. Actually, it is easy to find that a large class of functions satisfy the sector condition, such as $\mathcal{G}(s) = 3s + \cos(s) + \ln(1+s^2)$, $\forall s \in R$, or

$$\mathcal{G}(s) = \begin{cases} 2s + e^2 - e + 4 + \ln(s-1), \quad s > 2, \\ e^s - s - e + 8, \quad 1 < s \leq 2, \\ 5s + 2s^2, \quad 0 < s \leq 1, \\ 5s - 2s^2, \quad -1 < s \leq 0 \\ -e^{-s} + s + e - 8, \quad -2 < s \leq -1, \\ 2s - e^{-s} + e - 6 + \ln(-s-1), \quad s \leq -2, \end{cases} \tag{1.3}$$

with the lower bound $b_1 = 1$ and upper bound $b_2 = 10$, which therefore presents more flexible choices of actuators in real dynamic systems [17, 20]. The stability of dynamic system when the nonlinear controller satisfies the sector condition is referred to as absolute stability, and it was proposed in [18], which mainly concentrated on finite dimensional linear systems, for example, [19, 20]. It is worth emphasizing that, to the authors' best knowledge, there are few results on the absolute stability of infinite dimensional systems [21, 22, 23]. The main difficulty in extending the absolute stability of lumped-parameter systems to that of PDE systems is that the classical methods, the circle criterion and the Popov criterion, are to some extent ineffective. Moreover, it is another challenge to complete the well-posedness of nonlinear axially moving strings, including the literature [10].

Motivated by [27], we apply nonlinear semigroup theory to prove the well-posedness of the resulting closed-loop system. Furthermore, using the generalized Gronwall-type integral inequality instead of the Lyapunov direct method, the absolute stability for an axially moving string system is established under the sector criterion.

We now introduce the following spaces. Let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norm and the inner product of $\mathcal{L}^2(0,L)$. Set $\mathcal{H}_e^1 = \{z \in \mathcal{H}^1(0,L) : z(0) = 0\}$. It is clear that $\mathcal{H}_e^1$ is a closed subspaces of $\mathcal{H}^1(0,L)$. Then substituting (1.2) into (1.1) yields the closed-loop system:

$$\begin{cases} z_{tt}(x,t) + 2vz_{xt}(x,t) = [z_x(x,t)h(z_x^2(x,t))]_x, \quad x \in (0,L), \quad t > 0, \\ z_x(L,t)h(z_x^2(L,t)) - vz_t(L,t) = -\mathcal{G}(z_t(L,t)), \\ z(0,t) = 0, \\ z(x,0) = f(x), z_t(x,0) = g(x), x \in [0,L]. \end{cases} \tag{1.4}$$

To achieve our objective, we still assume that h is a non-decreasing continuous function such that

$$\begin{aligned} &\bullet\ 0 < \alpha \leq h(s), \quad \forall\, s \in R_+, \\ &\bullet\ |s_1 h(s_1^2) - s_2 h(s_2^2)| \leq \mathcal{T}_0 |s_1 - s_2|, \quad \forall\, s_1, s_2 \in R, \end{aligned} \tag{1.5}$$

where the constant α is referred to the initial tension of the string, and $\mathcal{T}_0$ is a constant. It is worth noting here that when $h(z_x^2(x,t)) = b - v^2 + (1-b)/\sqrt{1 + z_x^2(x,t)}$ with $b \geq 1 > |v|$ in [10], assumption (1.5) on h is obviously satisfied.

Let

$$\mathcal{E}(t) = \frac{1}{2}\int_0^L z_t^2(x,t)dx + \frac{1}{2}\int_0^L \hat{h}(z_x(x,t))dx \tag{1.6}$$

denote the string energy where $\hat{h}(s) = \int_0^{s^2} h(\tau)d\tau$. The following lemma is given to show that the energy function $\mathcal{E}(t)$ is non-increasing.

Lemma 1.1 *Suppose that assumption* (1.5) *and the sector condition* (1.2) *hold. Then, the derivative of the energy function along the solution of* (1.4) *satisfies*

$$\frac{d}{dt}\mathcal{E}(t) \leq -b_1 z_t^2(L,t). \tag{1.7}$$

As a result, $\mathcal{E}(t) \leq \mathcal{E}(0)$ for all $t \geq 0$.

By using nonlinear semigroup theory (see Barbu [24] or Komornik [25]), the well-posedness result of the closed-loop system (1.4) is provided by the following theorem.

Theorem 1 *If the assumptions of Lemma* 1.1 *and $z_0, z_1 \in H_e^1$ are satisfied, the system* (1.4) *admit a unique weak solution such that $z \in C([0,\infty), \mathcal{H}_e^1)$.*

To obtain the absolute stability for the closed-loop system (1.4), a generalized Gronwall-type integral inequality [26, p. 24] is presented in the following.

Lemma 1.2 *Assume that there exists a constant $\rho > 0$ such that*

$$\int_T^\infty \mathcal{Y}^{p+1}(s)ds \leq \frac{1}{\rho}\mathcal{Y}(T) \text{ for all } T \geq 0,$$

where $\mathcal{Y} : [0,+\infty) \to [0,+\infty)$ is a non-increasing real valued function. Then, the following inequality holds:

$$\begin{cases} \mathcal{Y}(t) \leq \mathcal{Y}(0)e^{1-\rho t}, & \text{if } p = 0, \\ \mathcal{Y}(t) \leq \mathcal{Y}(0)(\frac{1+p}{p\rho t})^{1/p}, & \text{if } p > 0. \end{cases} \tag{1.8}$$

Now, we illustrate the absolute stability of the closed-loop system (1.4).

Theorem 2 *Under the assumptions of Theorem* 1.1*, the energy function $\mathcal{E}(t)$ along the solution of the closed-loop system* (1.4) *decays exponentially:*

$$\mathcal{E}(t) \leq \mu e^{-\rho t}, \tag{1.9}$$

for all $t > 0$, where

$$\mu = \mathcal{E}(0)e, \quad \rho^{-1} = \delta + \tfrac{L^2(v+b_2)^2+\alpha L}{2b_1\alpha}, \tag{1.10}$$

with $\delta = \max\left\{\frac{L^2+2vL}{\alpha}, 1\right\}$ and $b_1, b_2 > 0$ given by (1.2).

Finally, the exponential stability of displacement response z of string and the uniform boundedness of the control input U for the closed-loop system (1.4) are established.

Corollary 1 *Under the assumptions of Theorem* 1.1, *the displacement response* z *of the closed-loop system* (1.4) *decays exponentially and*

$$\int_0^{+\infty} \mathcal{G}^2(z_t(L,t))dt \leq \frac{b_2^2}{b_1}\mathcal{E}(0), \tag{1.11}$$

where b_1, b_2 *are given by* (1.2).

2 Proof of main results

Proof of Lemma 1.1. From (1.6), the derivative rule gives,

$$\frac{d}{dt}\mathcal{E}(t) = \int_0^L z_{tt}(x,t)z_t(x,t)dx + \int_0^L z_x(x,t)h(z_x^2(x,t))z_{xt}(x,t)dx. \tag{2.1}$$

Now, we introduce the variational structure associated with (1.4), provided by

$$\int_0^L z_{tt}udx + 2v\int_0^L z_{xt}udx + \int_0^L z_x h(z_x^2)u_x dx = [vz_t(L,t) - \mathcal{G}(z_t(L,t))]u(L), \tag{2.2}$$

for any $u \in \mathcal{H}_e^1$. Taking $u = z_t$ in (2.2) together with (2.1) leads to

$$\frac{d}{dt}\mathcal{E}(t) = v[z_t(L,t)]^2 - 2v\int_0^L z_{xt}(x,t)z_t(x,t)dx - \mathcal{G}(z_t(L,t))z_t(L,t). \tag{2.3}$$

Note $z(0,t) = 0$, and thus $z_t(0,t) = 0$, and it is easy to see that

$$\int_0^L z_{xt}(x,t)z_t(x,t)dx = \frac{1}{2}z_t^2(L,t). \tag{2.4}$$

Hence, the equation (2.3) becomes

$$\frac{d}{dt}\mathcal{E}(t) = -\mathcal{G}(z_t(L,t))z_t(L,t). \tag{2.5}$$

Taking account of $\mathcal{G}(z_t(L,t))z_t(L,t) \geq b_1[z_t(L,t)]^2$, it follows from (2.5) that

$$\frac{d}{dt}\mathcal{E}(t) \leq -b_1[z_t(L,t)]^2, \tag{2.6}$$

which means $\mathcal{E}(t) \leq \mathcal{E}(0)$ for any $t > 0$. This completes the proof.

Proof of Theorem 1. Here, the idea from [27] will be used. First, set $H := \mathcal{L}^2(0, L)$, and $V := \mathcal{H}_e^1$ with dual space V' where the norm on V is given by

$$\|z\|_V^2 = \int_0^L |z_x(x)|^2 dx, \quad \forall z \in V.$$

The variational structure shown by (2.2) is equivalent to the following equation

$$\langle z_{tt} + Az + B_1 z_t - B_2 z_t, u\rangle_{V',V} = 0, \quad \forall u \in V, \tag{2.7}$$

where the mappings $A, B_1, B_2 : V \to V'$ are defined by

$$\begin{aligned}
&\langle Az, u\rangle_{V',V} = \int_0^L z_x h(z_x^2) u_x dx, \\
&\langle B_1 z_t, u\rangle_{V',V} = 2v \int_0^L z_{xt} u dx, \\
&\langle B_2 z_t, u\rangle_{V',V} = [v z_t(L, t) - \mathcal{G}(z_t(L, t))] u(L).
\end{aligned} \tag{2.8}$$

Then the existence of weak solutions of the system (1.4) is equivalent to

$$z_{tt} + Az + B_1 z_t - B_2 z_t = 0, \quad in\ V'. \tag{2.9}$$

Set $\mathcal{H} = V \times V$. Let $W = (z, y) := (z, z_t)^\top$, and then $\mathcal{A}W := (-y, Az + B_1 y - B_2 y)^\top$ with domain

$$D(\mathcal{A}) = \{(z, y) \in V \times V; Az + B_1 y - B_2 y \in H\}, \tag{2.10}$$

which is dense in $V \times H$ (Lemma 7.7, [25]). Hence, Eq. (2.9) can be be written as

$$W_t = \mathcal{A}W, \quad t \geq 0, \tag{2.11}$$

with the initial data $W(0) = (f, g)^\top$. For any $W_i = (z_i, y_i) \in D(\mathcal{A})$, $i = 1, 2$, from (2.8), one has

$$\begin{aligned}
&\langle \mathcal{A}W_1 - \mathcal{A}W_2, W_1 - W_2\rangle \\
&= -\int_0^L (z_{1x} - z_{2x})(y_{1x} - y_{2x})dx + \int_0^L [z_{1x}h(z_{1x}^2) - z_{2x}h(z_{2x}^2)](y_{1x} - y_{2x})dx \\
&\quad - [v(y_1(L) - y_2(L)) - \mathcal{G}(y_1(L)) + \mathcal{G}(y_2(L))](y_1(L) - y_2(L)) \\
&\quad + 2v \int_0^L (y_{1x} - y_{2x})(y_1 - y_2)dx.
\end{aligned} \tag{2.12}$$

Due to the sector condition (1.2), it follows that

$$[\mathcal{G}(y_1(L)) - \mathcal{G}(y_2(L))](y_1(L) - y_2(L)) \geq 0. \tag{2.13}$$

Notice $y_1(0) = y_2(0) = 0$, then

$$2v\int_0^L (y_{1x} - y_{2x})(y_1 - y_2)dx = v(y_1(L) - y_2(L))^2. \tag{2.14}$$

The assumption (1.5) implies

$$\int_0^L [z_{1x}h(z_{1x}^2) - z_{2x}h(z_{2x}^2)](y_{1x} - y_{2x})dx \leq \frac{\mathcal{T}_0}{2}[\|z_{1x} - z_{2x}\|_2^2 + \|y_{1x} - y_{2x}\|_2^2]. \tag{2.15}$$

The substitution of (2.13) – (2.15) into (2.12) yields

$$\begin{aligned}\langle \mathcal{A}W_1 - \mathcal{A}W_2, W_1 - W_2\rangle &\leq \frac{\mathcal{T}_0 + 1}{2}(\|z_1 - z_2\|_V^2 + \|y_1 - y_2\|_V^2) \\ &= \frac{\mathcal{T}_0 + 1}{2}\|W_1 - W_2\|_{\mathcal{H}}^2,\end{aligned} \tag{2.16}$$

where $\mathcal{T}_0$ is the constant stated in (1.5). Hence, $\frac{\mathcal{T}_0+1}{2}\mathcal{I} - \mathcal{A}$ is monotone where $\mathcal{I}$ is the identity mapping on $\mathcal{H}$. Now, we prove that $\lambda\mathcal{I} - \mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ is surjective for $\lambda > \frac{\mathcal{T}_0+1}{2}$, which implies that for arbitrary given $X = (x_1, x_2)^\top \in \mathcal{H}$ there exists $W = (z, y)^\top \in D(\mathcal{A})$ such that $(\lambda\mathcal{I} - \mathcal{A})W = X$ with $\lambda > \frac{\mathcal{T}_0+1}{2}$. It suffices to show that the mapping $\lambda I - A - B_1 + B_2 : V \to V'$ is onto where I is the identity mapping on V. For this, let $z = \frac{1}{\lambda}(x_1 - y)$, and clearly $(z, y)^\top \in V \times V$, then $\tilde{A}y + B_1y - B_2y = \lambda y - x_2 \in H$ where $\tilde{A}y := A(\frac{x_1-y}{\lambda})$.

In order to prove the subjectivity of the mapping $\lambda I - A - B_1 + B_2 : V \to V'$, fix $x_2 \in V', x_1 \in V$ arbitrarily, and we define the functional $\mathcal{F} : V \to R$ by

$$\mathcal{F}(y) = \frac{\lambda}{2}\|y\|^2 + \frac{1}{2}\hat{A}(y) - \frac{1}{2}\langle B_1y, y\rangle + G(y) - \langle x_2, y\rangle, \ \forall y \in V, \tag{2.17}$$

where the nonlinear functional $\hat{A} : V \to R$ is defined by

$$\hat{A}(y) = \int_0^L \int_0^{(\frac{y(x)-x_1}{\lambda})^2} h(s)dsdx$$

and the map $G : V \to R$ is given by

$$G(y) = \frac{v}{2}y^2(L) - \int_0^{y(L)} \mathcal{G}(\rho)d\rho.$$

Using assumption (1.5) and the sector condition (1.2), we can check easily that the map $\mathcal{F}$ is well-defined, continuously differentiable and that

$$\langle \mathcal{F}'(y), v\rangle_{V',V} = \langle (\lambda I - \tilde{A} - B_1 + B_2)y - x_2, v\rangle_{V',V}, \quad \forall y, v \in V. \tag{2.18}$$

Furthermore, it is easy to deduce that

$$\mathcal{F}(y) \geq C[\frac{\lambda}{2}\|y\|_V - \|x_2\|_{V'}]\|y\|_V, \tag{2.19}$$

where $C > 0$ is a constant, which leads to that $\mathcal{F}(y) \to +\infty$ as $\|y\|_V \to \infty$, i.e. $\mathcal{F}$ is coercive. It follows that the infimum of $\mathcal{F}$ is attained at some point $y \in V$. Therefore $\mathcal{F}'(y) = 0$, i.e. $(\lambda I - \tilde{A} - B_1 + B_2)y = x_2$. The proof of Theorem 1 is completed.

Proof of Theorem 2. Take the inner product with xz_x on both sides of the system equation (1.4) to obtain

$$\mathcal{W}_1(t) + \mathcal{W}_2(t) = \mathcal{W}_3(t) \tag{2.20}$$

where $\mathcal{W}_1(t) := \langle xz_x, z_{tt}\rangle$, $\mathcal{W}_2(t) := 2v\langle xz_x, z_{xt}\rangle$, $\mathcal{W}_3(t) := \langle xz_x, [z_x h(z_x^2)]_x\rangle$. Based on the derivative rules and the boundary value condition $z_t(0,t) = 0$, for $\mathcal{W}_1$ we have

$$\begin{aligned}
\mathcal{W}_1(t) &= \int_0^L [xz_x(x,t)z_t(x,t)]_t dx - \int_0^L xz_{xt}(x,t)z_t(x,t)dx \\
&= \int_0^L [xz_x(x,t)z_t(x,t)]_t dx - \frac{1}{2}\int_0^L [xz_t^2(x,t)]_x dx + \frac{1}{2}\int_0^L z_t^2(x,t)dx \\
&= \int_0^L \left(\frac{1}{2}z_t^2(x,t) + [xz_x(x,t)z_t(x,t)]_t\right) dx - \frac{L}{2}z_t^2(L,t).
\end{aligned} \tag{2.21}$$

Likewise as in (2.21), we can deduce

$$\mathcal{W}_2(t) = 2v\int_0^L xz_x(x,t)z_{xt}(x,t)dx = v\int_0^L [xz_x^2(x,t)]_t dx. \tag{2.22}$$

Due to the boundary value condition $z(0,t) = 0$ for any $t \in [0,\infty)$, we have from integration by parts that

$$\begin{aligned}
\mathcal{W}_3(t) &= \int_0^L xz_x(x,t)[z_x(x,t)h(z_x^2(x,t))]_x dx \\
&= Lz_x^2(L,t)h(z_x^2(L,t)) - \int_0^L [z_x^2(x,t)h(z_x^2(x,t)) + xz_x(x,t)z_{xx}(x,t)h(z_x^2(x,t))]dx \\
&= Lz_x^2(L,t)h(z_x^2(L,t)) - \int_0^L z_x^2(x,t)h(z_x^2(x,t))dx \\
&\quad -\frac{1}{2}\int_0^L [x\hat{h}(z_x(x,t))]_x dx + \frac{1}{2}\int_0^L \hat{h}(z_x(x,t))dx \\
&= Lz_x^2(L,t)h(z_x^2(L,t)) - \int_0^L z_x^2(x,t)h(z_x^2(x,t))dx \\
&\quad -\frac{L}{2}\hat{h}(z_x(L,t)) + \frac{1}{2}\int_0^L \hat{h}(z_x(x,t))dx,
\end{aligned} \tag{2.23}$$

where $\hat{h}$ is defined in (1.6). Since h is a non-decreasing function, due to assumption (1.5), one has $\int_0^L z_x^2(x,t)h(z_x^2(x,t))dx \geq \int_0^L \hat{h}(z_x(x,t))dx$ for any $t > 0$. Then it follows from (2.23) that

$$\mathcal{W}_3(t) \leq Lz_x^2(L,t)h(z_x^2(L,t)) - \frac{L}{2}\hat{h}(z_x(L,t)) - \frac{1}{2}\int_0^L \hat{h}(z_x(x,t))dx. \tag{2.24}$$

The substitution of (2.21), (2.22), (2.24) into (2.20) yields

$$\begin{aligned}
&\frac{1}{2}\int_0^L z_t^2(x,t)dx + \frac{1}{2}\int_0^L \hat{h}(z_x(x,t))dx \\
&\leq -\int_0^L [xz_x(x,t)z_t(x,t) + vxz_x^2(x,t)]_t dx + Lz_x^2(L,t)h(z_x^2(L,t)) \\
&\quad + \frac{L}{2}z_t^2(L,t) - \frac{1}{2}\hat{h}(z_x(L,t)).
\end{aligned} \tag{2.25}$$

Following $z_x(L,t)h(z_x^2(L,t)) - vz_t(L,t) = -\mathcal{G}(z_t(L,t))$, the sector condition (1.2) together with Young's inequality implies that

$$\begin{aligned}
Lz_x^2(L,t)h(z_x^2(L,t)) &\leq |Lz_x(L,t)[vz_t(L,t) - \mathcal{G}(z_t(L,t))]| \\
&\leq L(v+b_2)|z_x(L,t)z_t(L,t)| \\
&\leq L\eta z_x^2(L,t) + \frac{L(v+b_2)^2}{4\eta}z_t^2(L,t),
\end{aligned} \tag{2.26}$$

where $\eta > 0$ is the Young's parameter. The insertion of (2.26) in (2.25) gives

$$\begin{aligned}
\mathcal{E}(t) &= \frac{1}{2}\int_0^L z_t^2(x,t)dx + \frac{1}{2}\int_0^L \hat{h}(z_x(x,t))dx \\
&\leq -\int_0^L [xz_x(x,t)z_t(x,t) + vxz_x^2(x,t)]_t dx + \frac{L[(v+b_2)^2 + 2\eta]}{4\eta}z_t^2(L,t) \\
&\quad + L\eta z_x^2(L,t) - \frac{1}{2}\hat{h}(z_x(L,t)).
\end{aligned} \tag{2.27}$$

Note $\hat{h}(z_x(x,t)) \geq \alpha z_x^2(x,t)$ by assumption (1.5), then inserting this into (2.27) yields

$$\begin{aligned}
\mathcal{E}(t) &\leq -\int_0^L [xz_x(x,t)z_t(x,t) + vxz_x^2(x,t)]_t dx + \frac{L[(v+b_2)^2 + 2\eta]}{4\eta}z_t^2(L,t) \\
&\quad - \frac{\alpha}{2}z_x^2(L,t) + L\eta z_x^2(L,t).
\end{aligned} \tag{2.28}$$

Since $\eta > 0$ can be arbitrary, we can choose it so that $\eta = \frac{\alpha}{2L}$. Then, using Lemma 1.1 we have $z_t^2(L,t) \leq -\frac{1}{b_1}\frac{d}{dt}\mathcal{E}(t)$. It then follows from (2.28) that

$$\mathcal{E}(t) \leq -\int_0^L [xz_x(x,t)z_t(x,t) + vxz_x^2(x,t)]_t dx - \frac{L^2(v+b_2)^2 + \alpha L}{2b_1\alpha}\frac{d}{dt}\mathcal{E}(t). \tag{2.29}$$

On the other hand, in light of (1.5) and (1.6), we have that

$$
\begin{aligned}
&\int_0^L xz_t(x,t)z_x(x,t)dx + v\int_0^L xz_x^2(x,t)dx \\
&\le \frac{1}{2}\int_0^L z_t^2(x,t)dx + \frac{L^2+2vL}{2\alpha}\int_0^L \alpha z_x^2(x,t)dx \\
&\le \delta\left(\frac{1}{2}\int_0^L z_t^2(x,t)dx + \frac{1}{2}\int_0^L \hat{h}(z_x(x,t))dx\right) \\
&\le \delta\mathcal{E}(t), \qquad (2.30)
\end{aligned}
$$

for all $t \ge 0$, where $\delta = \max\{\frac{L^2+2vL}{\alpha}, 1\}$. With the help of Lemma 1.1, integrate (2.30) over (T,S) $(S \ge T)$ to obtain

$$
\begin{aligned}
-\int_T^S\int_0^L [xz_t(x,t)z_x(x,t) + vxz_x^2(x,t)]_t dxdt \;&\le\; \delta(\mathcal{E}(\mathcal{T}) - \mathcal{E}(\mathcal{S})) \\
&\le\; \delta\mathcal{E}(T). \qquad (2.31)
\end{aligned}
$$

Integrating (2.30) from T to S $(S \ge T)$, and it follows from Lemma 1.1 that

$$
\int_T^S \mathcal{E}(t)dt \le \delta\mathcal{E}(T) + \frac{L^2(v+b_2)^2 + \alpha L}{2b_1\alpha}(\mathcal{E}(T) - \mathcal{E}(S)) \le \frac{1}{\rho}\mathcal{E}(T), \qquad (2.32)
$$

where the parameters δ, ρ are given by (1.10). Passing to the limit as $S \to +\infty$ shows

$$
\int_T^{+\infty} \mathcal{E}(t)dt \le \frac{1}{\rho}\mathcal{E}(T).
$$

Lemma 1.2 then can be applied to give inequality (1.9), which completes the proof of theorem 2.

Proof of Corollary 1.1. Due to $z(0,t) = 0$, for all $t \ge 0$, it holds that

$$
|z(x,t)| = \left|\int_0^x z_x(s,t)ds\right| \le \int_0^L |z_x(x,t)|dx \le \sqrt{L}\|z_x(\cdot,t)\| \le \sqrt{\frac{2L}{\alpha}\mathcal{E}(t)}, \qquad (2.33)
$$

where we have used the fact that $\frac{\alpha}{2}\|z_x\|^2 \le \frac{1}{2}\int_0^L \hat{h}(z_x(x,t))dx$, for all $t \ge 0$ and $x \in [0,L]$. Invoking Theorem 2 to (2.33) we conclude our desired result.

For any $S > 0$, integrating over $(0,S)$ on (1.7) of Lemma 1.1 gives

$$
b_1\int_0^S z_t^2(L,s)ds \le \mathcal{E}(0) - \mathcal{E}(S) \le \mathcal{E}(0),
$$

which shows that

$$
\int_0^{+\infty} \mathcal{G}^2(z_t(L,t))dt \le b_2^2\int_0^{+\infty} z_t^2(L,s)ds \le \frac{b_2^2}{b_1}\mathcal{E}(0),
$$

using the sector condition (1.2).

3 Concluding remarks

First a general mathematical model of axially moving string which covers the model of [10] is proposed. Second, under the sector criteria, nonlinear semigroup theory is applied to establish the well-posedness result of the closed-loop system. Finally, the absolute exponential stability of the closed-loop system is established by applying the generalized Gronwall-type integral inequality (Lemma 1.2) instead of the Lyapunov approach ([10, 11]), and less regularity of the integrand is required as one of the merits of the approach adopted in this paper. In addition, when $v = 0$ in (1.1) for the nonlinear non-moving string, all the results obtained in this paper are still valid along this line. It is an interesting question whether this stability standard can be obtained by the current method when internal interference occurs in the system, which is the focus of our future work.

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Conflicts of interest

The authors declare that there are no conflict of interest.

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