Average sampling in mixed shift-invariant subspaces with generators in hybrid-norm spaces

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Abstract: This paper mainly studies the average sampling and reconstruction in shift-invariant subspaces of mixed Lebesgue spaces $L^{p,q}(\mathbb{R}^{d+1})$, under the condition that the generator φ of the shift-invariant subspace belongs to a hybrid-norm space of mixed form, which is weaker than the usual assumption of Wiener amalgam space and allows to control the orders p, q. First, the sampling stability for two kinds of average sampling functionals are established. Then, we give the corresponding iterative approximation projection algorithms with exponential convergence for recovering the time-varying shift-invariant signals from the average samples.

Keywords: average sampling; shift-invariant subspace; mixed Lebesgue space; hybrid-norm space

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1 Introduction

In the practice, some signals are time variant and mixed Lebesgue space is a suitable tool for modeling and measuring such signals. Mixed Lebesgue spaces arise for considering functions that depend on independent quantities with different properties, which were first described in detail in [1] and were furtherly studied in [2, 3, 4, 5, 6, 7, 8, 9] from the view of harmonic analysis and operator theory. The flexibility for the separate integrability of each variable plays an important role in the study of time-based partial differential equations.

The mixed Lebesgue space $L^{p,q}(\mathbb{R}^{d+1})$ consists of all measurable functions $f = f(x, y)$ defined on $\mathbb{R} \times \mathbb{R}^d$ such that

$$||f||_{L^{p,q}} = \Big\|\big\|f(x, y)\big\|_{L^q_y(\mathbb{R}^d)}\Big\|_{L^p_x(\mathbb{R})} < \infty,\ 1 \leq p, q \leq \infty. \tag{1.1}$$

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The corresponding sequence spaces are defined by

$$\ell^{p,q}(\mathbb{Z}^{d+1}) = \Big\{c : ||c||_{\ell^{p,q}} = \Big\|\big\|\{c(k_1,k_2)\}_{k_1\in\mathbb{Z},k_2\in\mathbb{Z}^d}\big\|_{\ell^q_{k_2}(\mathbb{Z}^d)}\Big\|_{L^p_{k_1}(\mathbb{Z})} < \infty\Big\},\ 1 \le p,q \le \infty. \quad (1.2)$$

Obviously, $L^{p,p}(\mathbb{R}^{d+1}) = L^p(\mathbb{R}^{d+1})$ and $\ell^{p,p}(\mathbb{Z}^{d+1}) = \ell^p(\mathbb{Z}^{d+1})$.

Sampling in mixed Lebesgue spaces will be significant for processing time-varying signals. In fact, sampling for band-limited signals in a mixed Lebesgue space had been studied in [10]. Moreover, nonuniform sampling in shift-invariant subspaces

$$V_{p,q}(\varphi) = \Big\{\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d} c(k_1,k_2)\varphi(x-k_1,y-k_2) :\ \{c(k_1,k_2)\}_{k_1\in\mathbb{Z},k_2\in\mathbb{Z}^d} \in \ell^{p,q}(\mathbb{Z}^{d+1})\Big\} \quad (1.3)$$

were discussed in [11, 12]. In [11], the generator φ was assumed to be in the classical Wiener amalgam space $W(L^1)(\mathbb{R}^{d+1})$ defined by

$$W(L^1)(\mathbb{R}^{d+1}) = \Big\{f :\ ||f||_{W(L^1)} = \sum_{n\in\mathbb{Z}^{d+1}} \sup_{x\in[0,1]^{d+1}} |f(x+n)| < \infty\Big\}.$$

The reference [12] weakened the condition in [11] and assumed that φ belongs to a mixed Wiener amalgam space

$$W(L^{1,1})(\mathbb{R}^{d+1}) = \Big\{f :\ ||f||_{W(L^{p,q})} = \sum_{n\in\mathbb{Z}} \sup_{x\in[0,1]} \sum_{\ell\in\mathbb{Z}^d} \sup_{y\in[0,1]^d} |f(x+n,y+\ell)| < \infty\Big\}.$$

In realistic applications, samples are often not exactly obtained on the sampling positions due to the physical limitation of measuring equipments, which leads to the wide interests in average sampling, such as [13, 14, 15]. For average sampling, the actual measured samples are the local mean of the signal near the sampling points, which can be realized by suitable average sampling functionals. If the generator $\varphi \in W(L^{1,1})(\mathbb{R}^{d+1})$, average sampling schemes had been studied in shift-invariant subspace $V_{p,q}(\varphi)$ [16]. In this paper, we mainly study the average sampling and reconstruction for time-varying signals in $V_{p,q}(\varphi)$ under a weaker condition that φ belongs to a mixed hybrid-norm space $W^{p,q}(\mathbb{R}^{d+1})$, which generalizes the semi-average sampling in [17] to average sampling under the same conditions.

For $1 \le p,q \le \infty$, the mixed hybrid-norm space $W^{p,q}(\mathbb{R}^{d+1})$ is defined as the linear space of all functions $f(x,y)$ for which

$$\|f\|_{W^{p,q}} := \Big\|\sum_{k_1\in\mathbb{Z}}\Big\|\sum_{k_2\in\mathbb{Z}^d}|f(x+k_1,y+k_2)|\Big\|_{L^q_y([0,1]^d)}\Big\|_{L^p_x([0,1])} < \infty. \quad (1.4)$$

In fact, it is a generalization of hybrid-norm space in [18]. It is obvious that $W^{p,q}(\mathbb{R}^{d+1}) \subset L^{p,q}(\mathbb{R}^{d+1})$. Moreover,

$$W^{p,q_2}(\mathbb{R}^{d+1}) \subset W^{p,q_1}(\mathbb{R}^{d+1}),\ 1 \le q_1 \le q_2 \le \infty$$

and

$$W^{p_2,q}(\mathbb{R}^{d+1}) \subset W^{p_1,q}(\mathbb{R}^{d+1}),\ 1 \le p_1 \le p_2 \le \infty.$$

Therefore, $W(L^{1,1})(\mathbb{R}^{d+1}) \subset W^{\infty,\infty}(\mathbb{R}^{d+1}) \subset W^{p,q}(\mathbb{R}^{d+1})$.

We always assume that the generator φ satisfies

(A1) $\varphi \in W^{\widetilde{p},\widetilde{q}}(\mathbb{R}^{d+1})$, where $\widetilde{p} = \max\{p, p'\}$ and $\widetilde{q} = \max\{q, q'\}$.

(A2) $\lim\limits_{\delta\to 0} \|\omega_\delta(\varphi)\|_{W^{p,q}} = 0$, where the modulus of continuity is defined by

$$\omega_\delta(\varphi)(x, y) = \sup_{\sqrt{s^2+|t|^2}\le\delta} |\varphi(x + s, y + t) - \varphi(x, y)|.$$

(A3) $\|\omega_1(\varphi)\|_{W^{p,q}} < \infty$.

(A4) There exist positive constants A and B such that for any $c = \{c(k_1, k_2)\}_{k_1\in\mathbb{Z},k_2\in\mathbb{Z}^d} \in \ell^{p,q}(\mathbb{Z}^{d+1})$, one has

$$A\|c\|_{\ell^{p,q}} \le \Big\| \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}^d} c(k_1, k_2)\varphi(x - k_1, y - k_2)\Big\|_{L^{p,q}} \le B\|c\|_{\ell^{p,q}}. \tag{1.5}$$

The sampling set $\Gamma = \{\gamma_{j,k} = (x_j, y_k) : x_j \in \mathbb{R}, y_k \in \mathbb{R}^d, j \in \mathbb{J}_1, k \in \mathbb{J}_2\} \subset \mathbb{R}^{d+1}$ is assumed to be relatively-separated for both variables x and y, that is,

$$B_{\Gamma,x}(\delta_1) = \sup_{x\in\mathbb{R}} \sum_{j\in\mathbb{J}_1} \chi_{B(x_j,\delta_1)}(x) < \infty$$

and

$$B_{\Gamma,y}(\delta_2) = \sup_{y\in\mathbb{R}^d} \sum_{k\in\mathbb{J}_2} \chi_{B(y_k,\delta_2)}(y) < \infty$$

for some $\delta_1 > 0$ and $\delta_2 > 0$. Furthermore, $\delta_1 > 0$ and $\delta_2 > 0$ are said to be the gaps of Γ if

$$A_{\Gamma,x}(\delta_1) = \inf_{x\in\mathbb{R}} \sum_{j\in\mathbb{J}_1} \chi_{B(x_j,\delta_1)}(x) \ge 1$$

and

$$A_{\Gamma,y}(\delta_2) = \inf_{y\in\mathbb{R}^d} \sum_{k\in\mathbb{J}_2} \chi_{B(y_k,\delta_2)}(y) \ge 1.$$

Here, $\mathbb{J}_1$ and $\mathbb{J}_2$ are countable index sets, $B(x, \delta)$ and $B(y, \delta)$ are balls in $\mathbb{R}$ and $\mathbb{R}^d$, respectively.

Given a relatively-separated sampling set Γ, the first average sampling scheme is

$$\langle f, \psi_{j,k}\rangle = \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(x, y)\psi_{j,k}(x, y)dxdy,\ j \in \mathbb{J}_1, k \in \mathbb{J}_2,$$

where the average sampling functionals $\{\psi_{j,k} : j \in \mathbb{J}_1, k \in \mathbb{J}_2\}$ satisfy

(i) $\int_{\mathbb{R}} \int_{\mathbb{R}^d} \psi_{j,k}(x, y)dxdy = 1$ for all $j \in \mathbb{J}_1$ and $k \in \mathbb{J}_2$;

(ii) There exists a $M > 0$ such that $\int_{\mathbb{R}}\int_{\mathbb{R}^d} |\psi_{j,k}(x,y)|dxdy \le M$ for all $j \in \mathbb{J}_1$ and $k \in \mathbb{J}_2$;

(iii) $supp\psi_{j,k} \subset B(\gamma_{j,k}, a)$ for some $a > 0$.

Note that the first sampling scheme requires the sampling functionals to have compact support, we also consider the second average sampling scheme which is defined as

$$\big\langle f, \psi_a(\cdot - \gamma_{j,k})\big\rangle = f * \psi_a^*(\gamma_{j,k}), j \in \mathbb{J}_1, k \in \mathbb{J}_2,$$

where $\psi \in L^1(\mathbb{R}^{d+1})$ satisfies $\int_{\mathbb{R}}\int_{\mathbb{R}^d} \psi(x,y)dxdy = 1$, $\psi_a(\cdot) = \frac{1}{a^{d+1}}\psi(\frac{\cdot}{a})$ and $\psi_a^*(\cdot) = \overline{\psi_a(-\cdot)}$.

This paper is organized as follows. In section 2, some lemmas are given. In section 3, the sampling stability for two kinds of average sampling functionals are established. Section 4 is devoted to demonstrating two iterative reconstruction algorithms for recovering the signals in $V_{p,q}(\varphi)$ from the corresponding two kinds of average samples.

2 Some lemmas

In this section, we will give some lemmas which provide theoretical basis for sampling stability and reconstruction algorithms.

Lemma 2.1 *[11] Let $1 \le p, q \le \infty$. If $\varphi \in W^{p,q}(\mathbb{R}^{d+1})$ and $c \in \ell^{p,q}(\mathbb{Z}^{d+1})$, then*

$$\Big\| \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}^d} c(k_1,k_2)\varphi(x-k_1, y-k_2)\Big\|_{L^{p,q}} \le \|c\|_{\ell^{p,q}}\|\varphi\|_{W^{p,q}}.$$

Lemma 2.2 *Suppose that $\varphi \in W^{p,q}(\mathbb{R}^{d+1})$ satisfies the assumption $(A3)$, then*

$$\|\omega_m(\varphi)\|_{W^{p,q}} \le \|\omega_1(\varphi)\|_{W^{p,q}} + 2\|\varphi\|_{W^{p,q}}$$

hold for any positive integer $m \in \mathbb{N}$.

Proof For any positive integer $m \in \mathbb{N}$ and $(s,t) \in \mathbb{R}\times\mathbb{R}^d$ satisfying $\sqrt{s^2+|t|^2} \le m$, there exist $m_1 \in \mathbb{Z}$ and $m_2 \in \mathbb{Z}^d$ such that $\sqrt{(s-m_1)^2+|t-m_2|^2} \le 1$. Then

$$\begin{aligned}
\|\omega_m(\varphi)\|_{W^{p,q}} \le &\bigg\| \sum_{k_1\in\mathbb{Z}} \Big\| \sum_{k_2\in\mathbb{Z}^d} \sup_{\sqrt{s^2+|t|^2}\le m} |\varphi(x+k_1+s, y+k_2+t)|\Big\|_{L^q_y([0,1]^d)}\bigg\|_{L^p_x([0,1])} \\
&+ \bigg\| \sum_{k_1\in\mathbb{Z}} \Big\| \sum_{k_2\in\mathbb{Z}^d} |\varphi(x+k_1, y+k_2)|\Big\|_{L^q_y([0,1]^d)}\bigg\|_{L^p_x([0,1])} \\
\le &\bigg\| \sum_{k_1\in\mathbb{Z}} \Big\| \sum_{k_2\in\mathbb{Z}^d} \sup_{\sqrt{s^2+|t|^2}\le m} |\varphi(x+k_1+m_1+s-m_1, y+k_2+m_2+t-m_2)- \\
&\varphi(x+k_1+m_1, y+k_2+m_2)|\Big\|_{L^q_y([0,1]^d)}\bigg\|_{L^p_x([0,1])} \\
&+ 2\bigg\| \sum_{k_1\in\mathbb{Z}} \Big\| \sum_{k_2\in\mathbb{Z}^d} |\varphi(x+k_1, y+k_2)|\Big\|_{L^q_y([0,1]^d)}\bigg\|_{L^p_x([0,1])} \\
\le &\|\omega_1(\varphi)\|_{W^{p,q}} + 2\|\varphi\|_{W^{p,q}}.
\end{aligned}$$

Lemma 2.3 *Suppose that $\varphi \in W^{p,q}(\mathbb{R}^{d+1})$ satisfies the assumptions (A2) and (A3). If $\psi \in L^1(\mathbb{R}^{d+1})$ satisfies*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \psi(x,y)dxdy = 1,$$

then $\lim\limits_{a\to 0} \|\varphi^a\|_{W^{p,q}} = 0$, where $\varphi^a(x,y) = \varphi(x,y) - \varphi * \psi_a^*(x,y)$.

Proof Note that

$$\begin{aligned}
|\varphi^a(x,y)| &\le \Big(\int_{\sqrt{s^2+|t|^2}\ge 1} + \int_{\sqrt{s^2+|t|^2}\le 1} \Big) \big|\varphi(x,y) - \varphi(x+s,y+t)\big|\big|\psi_a(s,t)\big|dsdt \\
&=: I_1(x,y) + I_2(x,y). \qquad (2.1)
\end{aligned}$$

Now, we estimate $\|I_1\|_{W^{p,q}}$ and $\|I_2\|_{W^{p,q}}$, respectively. It follows from Lemma 2.2 that

$$\begin{aligned}
\|I_1\|_{W^{p,q}} &\le \Big\| \sum_{k_1\in\mathbb{Z}} \Big\| \sum_{k_2\in\mathbb{Z}^d} \int_{\sqrt{s^2+|t|^2}\ge 1} \omega_{\sqrt{s^2+|t|^2}}(\varphi)(x+k_1,y+k_2)|\psi_a(s,t)|dsdt \Big\|_{L_y^q([0,1]^d)} \Big\|_{L_x^p([0,1])} \\
&\le \sum_{m=1}^{\infty} \Big\| \sum_{k_1\in\mathbb{Z}} \Big\| \sum_{k_2\in\mathbb{Z}^d} \int_{m\le\sqrt{s^2+|t|^2}\le m+1} \omega_{m+1}(\varphi)(x+k_1,y+k_2)|\psi_a(s,t)|dsdt \Big\|_{L_y^q([0,1]^d)} \Big\|_{L_x^p([0,1])} \\
&\le \sum_{m=1}^{\infty} \|\omega_{m+1}(\varphi)\|_{W^{p,q}} \int_{m\le\sqrt{s^2+|t|^2}\le m+1} |\psi_a(s,t)|dsdt \\
&\le \big(\|\omega_1(\varphi)\|_{W^{p,q}} + 2\|\varphi\|_{W^{p,q}}\big) \int_{\sqrt{s^2+|t|^2}\ge 1/a} |\psi(s,t)|dsdt \longrightarrow 0, \ \text{as } a \to 0. \qquad (2.2)
\end{aligned}$$

Since $\lim\limits_{\delta\to 0} \|\omega_\delta(\varphi)\|_{W^{p,q}} = 0$, then for any $\epsilon > 0$, there exists a $\delta_0 > 0$ such that

$$\|\omega_\delta(\varphi)\|_{W^{p,q}} < \epsilon, \ \forall\ \delta \le \delta_0.$$

Now, we begin to estimate $\|I_2\|_{W^{p,q}}$.

$$\begin{aligned}
\|I_2\|_{W^{p,q}} &\le \Big\| \int_{\sqrt{s^2+|t|^2}\le 1} \omega_{\sqrt{s^2+|t|^2}}(\varphi)(x,y)|\psi_a(s,t)|dsdt \Big\|_{W^{p,q}} \\
&\le \Big\| \int_{\sqrt{s^2+|t|^2}<\delta_0} \omega_{\sqrt{s^2+|t|^2}}(\varphi)(x,y)|\psi_a(s,t)|dsdt \Big\|_{W^{p,q}} \\
&+ \Big\| \int_{\sqrt{s^2+|t|^2}\ge\delta_0} \omega_1(\varphi)(x,y)|\psi_a(s,t)|dsdt \Big\|_{W^{p,q}} \\
&\le \epsilon\|\psi\|_1 + \|\omega_1(\varphi)\|_{W^{p,q}} \int_{\sqrt{s^2+|t|^2}\ge\delta_0/a} |\psi(s,t)|dsdt.
\end{aligned}$$

Since $\lim\limits_{a\to 0} \int_{\sqrt{s^2+|t|^2}\ge\delta_0/a} |\psi(s,t)|dsdt = 0$, then $\lim\limits_{a\to 0} \|I_2\|_{W^{p,q}} = 0$ and the final result follows from (2.1) and (2.2).

Lemma 2.4 *Let φ and ψ be as in Lemma 2.3. Then*

$$\lim_{(a,\delta)\to(0,0)} \|\omega_\delta(\varphi^a)\|_{W^{p,q}} = 0.$$

Proof Direct computation gives

$$
\begin{aligned}
\omega_\delta(\varphi^a)(x,y) &\le \omega_\delta(\varphi)(x,y) + \sup_{\sqrt{s^2+|t|^2}\le\delta}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\big|\varphi(x+s+u,y+t+v)-\varphi(x+u,y+v)\big|\big|\psi_a(u,v)\big|dudv \\
&\le \omega_\delta(\varphi)(x,y) + \int_{\mathbb{R}}\int_{\mathbb{R}^d}\omega_\delta(\varphi)(x+u,y+v)\big|\psi_a(u,v)\big|dudv \\
&\le \omega_\delta(\varphi)(x,y) + \int_{\mathbb{R}}\int_{\mathbb{R}^d}\omega_\delta(\varphi)(x,y)|\psi_a(u,v)|dudv \\
&+ \int_{\mathbb{R}}\int_{\mathbb{R}^d}\omega_{\sqrt{u^2+|v|^2}}(\omega_\delta(\varphi))(x,y)|\psi_a(u,v)|dudv. \qquad (2.3)
\end{aligned}
$$

Then, we can obtain

$$
\|\omega_\delta(\varphi^a)\|_{W^{p,q}} \le \big(1+\|\psi\|_1\big)\|\omega_\delta(\varphi)\|_{W^{p,q}} + \Big\|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\omega_{\sqrt{u^2+|v|^2}}(\omega_\delta(\varphi))(x,y)|\psi_a(u,v)|dudv\Big\|_{W^{p,q}}.
$$

Now, it is enough to prove that

$$
\lim_{(a,\delta)\to(0,0)}\Big\|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\omega_{\sqrt{u^2+|v|^2}}(\omega_\delta(\varphi))(x,y)|\psi_a(u,v)|dudv\Big\|_{W^{p,q}} = 0. \qquad (2.4)
$$

Restricting $\delta\le 1$, then we can obtain

$$
\omega_1(\omega_\delta(\varphi))(x,y) \le \omega_2(\varphi)(x,y)+\omega_1(\varphi)(x,y)+\omega_\delta(\varphi)(x,y). \qquad (2.5)
$$

By Lemma 2.2, it is similar to (2.2) that

$$
\begin{aligned}
&\Big\|\int_{\sqrt{u^2+|v|^2}\ge 1}\omega_{\sqrt{u^2+|v|^2}}(\omega_\delta(\varphi))(x,y)|\psi_a(u,v)|dudv\Big\|_{W^{p,q}} \\
&\le \big(\|\omega_1(\omega_\delta(\varphi))\|_{W^{p,q}} + 2\|\omega_\delta(\varphi)\|_{W^{p,q}}\big)\int_{\sqrt{s^2+|t|^2}\ge 1/a}|\psi(s,t)|dsdt \\
&\le \big(2\|\varphi\|_{W^{p,q}} + 2\|\omega_1(\varphi)\|_{W^{p,q}} + 3\|\omega_\delta(\varphi)\|_{W^{p,q}}\big)\int_{\sqrt{s^2+|t|^2}\ge 1/a}|\psi(s,t)|dsdt \\
&\longrightarrow 0,\ \ as\ a\to\ 0. \qquad (2.6)
\end{aligned}
$$

Since $\lim_{\delta\to 0}\|\omega_\delta(\varphi)\|_{W^{p,q}}=0$, then for any $\epsilon>0$, there exists a $\delta_0>0$ such that

$$
\|\omega_\delta(\varphi)\|_{W^{p,q}} < \epsilon,\ \forall\ \delta\le\delta_0.
$$

Restricting $\delta\le\min\left\{\frac{\delta_0}{2},1\right\}$, then we can obtain

$$
\begin{aligned}
&\Big\|\int_{\sqrt{u^2+|v|^2}\le 1}\omega_{\sqrt{u^2+|v|^2}}(\omega_\delta(\varphi))(x,y)|\psi_a(u,v)|dudv\Big\|_{W^{p,q}} \\
&\le \Big\|\int_{\sqrt{u^2+|v|^2}\le\frac{\delta_0}{2}}\Big(\omega_\delta(\varphi)(x,y)+\omega_{\sqrt{u^2+|v|^2}}(\varphi)(x,y)+\omega_{\sqrt{2(u^2+|v|^2)+2\delta^2}}(\varphi)(x,y)\Big)|\psi_a(u,v)|dudv\Big\|_{W^{p,q}} \\
&+ \Big\|\int_{\sqrt{u^2+|v|^2}\ge\frac{\delta_0}{2}}\Big(2\omega_1(\varphi)(x,y)+\omega_2(\varphi)(x,y)\Big)|\psi_a(u,v)|dudv\Big\|_{W^{p,q}} \\
&< 3\epsilon\|\psi\|_1 + \big(2\|\varphi\|_{W^{p,q}}+3\|\omega_1(\varphi)\|_{W^{p,q}}\big)\int_{\sqrt{s^2+|t|^2}\ge\frac{\delta_0}{2a}}|\psi(s,t)|dsdt.
\end{aligned}
$$

This together with (2.6) proves (2.4).

3 Sampling stability

In this section, we will establish the sampling stability for two kinds of average sampling functionals.

Theorem 3.1 *Let $1 \le p, q \le \infty$, φ satisfy the assumptions $(A1) - (A4)$. Suppose that $\Gamma = \{\gamma_{j,k} = (x_j, y_k) : x_j \in \mathbb{R}, y_k \in \mathbb{R}^d, j \in \mathbb{J}_1, k \in \mathbb{J}_2\}$ is a relatively-separated set with gaps δ_1 and δ_2 for both variables, and $\{\psi_{j,k} : j \in \mathbb{J}_1, k \in \mathbb{J}_2\}$ is the first kind of average sampling functionals with support radius a. If δ_1, δ_2 and a are chosen such that*

$$r_1 := MA^{-1}\|\omega_{a+\delta_1+\delta_2}(\varphi)\|_{W^{p,q}} < 1, \tag{3.1}$$

then any signals $f \in V_{p,q}(\varphi)$ can be stably reconstructed from the samples $\{\langle f, \psi_{j,k}\rangle\}_{j\in\mathbb{J}_1, k\in\mathbb{J}_2}$, and

$$\Big(\frac{2\delta_1}{A_{\Gamma,x}(\delta_1)}\Big)^{-1/p}\Big(\frac{V_d\delta_2^d}{A_{\Gamma,y}(\delta_2)}\Big)^{-1/q}(1-r_1)\|f\|_{L^{p,q}} \le \Big\|\{\langle f, \psi_{j,k}\rangle\}_{j\in\mathbb{J}_1,k\in\mathbb{J}_2}\Big\|_{\ell^{p,q}} \le$$
$$\Big(\frac{2\delta_1}{B_{\Gamma,x}(\delta_1)}\Big)^{-1/p}\Big(\frac{V_d\delta_2^d}{B_{\Gamma,y}(\delta_2)}\Big)^{-1/q}(1+r_1)\|f\|_{L^{p,q}}, \tag{3.2}$$

where $V_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ is the volume of d-dimensional unit sphere.

Proof For any $x \in B(x_j, \delta_1)$ and $y \in B(y_k, \delta_2)$, one has

$$\begin{aligned} |\langle f, \psi_{j,k}\rangle - f(x,y)| &\le \int_{B(\gamma_{j,k},a)} |f(s,t) - f(x,y)||\psi_{j,k}(s,t)|dsdt \\ &\le M\omega_{a+\delta_1+\delta_2}(f)(x,y) =: F_1(x,y). \end{aligned} \tag{3.3}$$

Furthermore, for any $f(x,y) = \sum\limits_{k_1\in\mathbb{Z}}\sum\limits_{k_2\in\mathbb{Z}^d} c(k_1,k_2)\varphi(x-k_1, y-k_2) \in V_{p,q}(\varphi)$, it follows from Lemma 2.1 that

$$\begin{aligned} \|F_1\|_{L^{p,q}} &\le M\Big\|\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d}|c(k_1,k_2)|\omega_{a+\delta_1+\delta_2}(\varphi)(x-k_1,y-k_2)\Big\|_{L^{p,q}} \\ &\le M\|c\|_{\ell^{p,q}}\|\omega_{a+\delta_1+\delta_2}(\varphi)\|_{W^{p,q}} \\ &\le MA^{-1}\|\omega_{a+\delta_1+\delta_2}(\varphi)\|_{W^{p,q}}\|f\|_{L^{p,q}} \\ &= r_1\|f\|_{L^{p,q}}. \end{aligned} \tag{3.4}$$

Define

$$u_{j,k}(x,y) = \alpha_j(x)\beta_k(y) = \frac{\chi_{B(x_j,\delta_1)}(x)}{\sum\limits_{j'\in\mathbb{J}_1}\chi_{B(x_{j'},\delta_1)}(x)} \cdot \frac{\chi_{B(y_k,\delta_2)}(y)}{\sum\limits_{k'\in\mathbb{J}_2}\chi_{B(y_{k'},\delta_2)}(y)}. \tag{3.5}$$

Let $1 \le p, q < \infty$. The case $p = \infty$ or $q = \infty$ can be proved similarly. It follows from (3.3) that

$$|\langle f, \psi_{j,k}\rangle|\alpha_j^{1/p}(x)\beta_k^{1/q}(y) \le |f(x,y)|\alpha_j^{1/p}(x)\beta_k^{1/q}(y) + |F_1(x,y)|\alpha_j^{1/p}(x)\beta_k^{1/q}(y). \tag{3.6}$$

Taking ℓ^q-norm for variable $k \in \mathbb{J}_2$ on both sides of (3.6), applying triangular inequality and then taking L^q-norm for variable $y \in \mathbb{R}^d$, one has

$$\Big(\sum_{k\in\mathbb{J}_2} |\langle f, \psi_{j,k}\rangle|^q \|\beta_k\|_{L^1}\Big)^{1/q} \alpha_j^{1/p}(x) \le \alpha_j^{1/p}(x)\|f(x,\cdot)\|_{L^q} + \alpha_j^{1/p}(x)\|F_1(x,\cdot)\|_{L^q}. \tag{3.7}$$

Taking ℓ^p-norm for variable $j \in \mathbb{J}_1$ on both sides of (3.7), and then taking L^p-norm for variable $x \in \mathbb{R}$, applying triangular inequality can obtain

$$\Big[\sum_{j\in\mathbb{J}_1}\Big(\sum_{k\in\mathbb{J}_2} |\langle f, \psi_{j,k}\rangle|^q \|\beta_k\|_{L^1}\Big)^{p/q} \|\alpha_j\|_{L^1}\Big]^{1/p} \le \|f\|_{L^{p,q}} + \|F_1\|_{L^{p,q}} \le (1+r_1)\|f\|_{L^{p,q}}. \tag{3.8}$$

It is easy to verify that

$$2\delta_1 B_{\Gamma,x}^{-1}(\delta_1) \le \|\alpha_j\|_{L^1} \le 2\delta_1 A_{\Gamma,x}^{-1}(\delta_1),\ j \in \mathbb{J}_1$$

and

$$V_d\delta_2^d B_{\Gamma,y}^{-1}(\delta_2) \le \|\beta_k\|_{L^1} \le V_d\delta_2^d A_{\Gamma,y}^{-1}(\delta_2),\ k \in \mathbb{J}_2.$$

Then the right hand side of (3.2) follows from (3.8). The left side can be proved by the same method from the inequality

$$|f(x,y)|\alpha_j^{1/p}(x)\beta_k^{1/q}(y) \le |\langle f, \psi_{j,k}\rangle|\alpha_j^{1/p}(x)\beta_k^{1/q}(y) + |F_1(x,y)|\alpha_j^{1/p}(x)\beta_k^{1/q}(y).$$

Theorem 3.2 *Let $1 \le p,q \le \infty$, φ satisfy the assumptions (A1) − (A4). Suppose that $\Gamma = \{\gamma_{j,k} = (x_j, y_k) : x_j \in \mathbb{R}, y_k \in \mathbb{R}^d, j \in \mathbb{J}_1, k \in \mathbb{J}_2\}$ is a relatively-separated set with gaps δ_3 and δ_4 for both variables, and $\psi \in L^1(\mathbb{R}^{d+1})$ satisfies $\int_{\mathbb{R}}\int_{\mathbb{R}^d}\psi(x,y)dxdy = 1$. If δ_3, δ_4 and a are chosen such that*

$$r_2 := A^{-1}\Big(\big\|\omega_{\sqrt{\delta_3^2+\delta_4^2}}(\varphi)\big\|_{W^{p,q}} + \big\|\omega_{\sqrt{\delta_3^2+\delta_4^2}}(\varphi^a)\big\|_{W^{p,q}} + \|\varphi^a\|_{W^{p,q}}\Big) < 1, \tag{3.9}$$

then any signals $f \in V_{p,q}(\varphi)$ can be stably reconstructed from the corresponding average samples $\{f * \psi_a^*(\gamma_{j,k})\}_{j\in\mathbb{J}_1,k\in\mathbb{J}_2}$, and

$$\begin{aligned}\Big(\frac{2\delta_3}{A_{\Gamma,x}(\delta_3)}\Big)^{-1/p}\Big(\frac{V_d\delta_4^d}{A_{\Gamma,y}(\delta_4)}\Big)^{-1/q}(1-r_2)\|f\|_{L^{p,q}} &\le \big\|\{f * \psi_a^*(\gamma_{j,k})\}_{j\in\mathbb{J}_1,k\in\mathbb{J}_2}\big\|_{\ell^{p,q}} \le \\ &\Big(\frac{2\delta_3}{B_{\Gamma,x}(\delta_3)}\Big)^{-1/p}\Big(\frac{V_d\delta_4^d}{B_{\Gamma,y}(\delta_4)}\Big)^{-1/q}(1+r_2)\|f\|_{L^{p,q}}.\end{aligned} \tag{3.10}$$

Proof For any $x \in B(x_j,\delta_1)$ and $y \in B(y_k,\delta_2)$, if $f(x,y) = \sum\limits_{k_1\in\mathbb{Z}}\sum\limits_{k_2\in\mathbb{Z}^d} c(k_1,k_2)\varphi(x-k_1, y-k_2) \in V_{p,q}(\varphi)$, then we can obtain

$$\begin{aligned}\big|f * \psi_a^*(\gamma_{j,k}) - f(x,y)\big| &\le \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d}|c(k_1,k_2)|\big|\varphi(x-k_1,y-k_2) - \varphi * \psi_a^*(x_j-k_1, y_k-k_2)\big| \\ &\le \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d}|c(k_1,k_2)|\Big(\omega_{\sqrt{\delta_3^2+\delta_4^2}}(\varphi)(x-k_1,y-k_2) \\ &+ \omega_{\sqrt{\delta_3^2+\delta_4^2}}(\varphi^a)(x-k_1,y-k_2) + \varphi^a(x-k_1,y-k_2)\Big) \\ &=: F_2(x,y).\end{aligned}$$

Furthermore, one has

$$\begin{aligned}\|F_2\|_{L^{p,q}} &\le \|c\|_{\ell^{p,q}}\Big(\big\|\omega_{\sqrt{\delta_3^2+\delta_4^2}}(\varphi)\big\|_{W^{p,q}} + \big\|\omega_{\sqrt{\delta_3^2+\delta_4^2}}(\varphi^a)\big\|_{W^{p,q}} + \|\varphi^a\|_{W^{p,q}}\Big)\\ &\le A^{-1}\Big(\big\|\omega_{\sqrt{\delta_3^2+\delta_4^2}}(\varphi)\big\|_{W^{p,q}} + \big\|\omega_{\sqrt{\delta_3^2+\delta_4^2}}(\varphi^a)\big\|_{W^{p,q}} + \|\varphi^a\|_{W^{p,q}}\Big)\|f\|_{L^{p,q}}\\ &= r_2\|f\|_{L^{p,q}}.\end{aligned}$$

The remained proof is similar to that of Theorem 3.1.

4 Iterative reconstruction algorithms

In this section, we will give the iterative approximation projection algorithms for recovering the signals in $V_{p,q}(\varphi)$ from two kinds of average samples. For two kinds of average sampling functionals, we define the pre-reconstruction operators

$$A_\Gamma f := \sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}\langle f, \psi_{j,k}\rangle u_{j,k}$$

and

$$A_{\Gamma,a} f := \sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2} f * \psi_a^*(\gamma_{j,k}) u_{j,k},$$

where $u_{j,k}$ is defined by (3.5). Let P be a bounded projection from $L^{p,q}(\mathbb{R}^{d+1})$ onto $V_{p,q}(\varphi)$. Then the corresponding approximation projection algorithms are given as

$$\begin{cases} f_0 = P\Big(\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2} c_0(j,k)u_{j,k}\Big) \\ f_n = f_0 + f_{n-1} - PA_\Gamma f_{n-1},\ n\ge 1 \end{cases} \tag{4.1}$$

and

$$\begin{cases} f_0 = P\Big(\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2} c_0(j,k)u_{j,k}\Big) \\ f_n = f_0 + f_{n-1} - PA_{\Gamma,a} f_{n-1},\ n\ge 1. \end{cases} \tag{4.2}$$

Theorem 4.1 *Let p, q, φ and $\{\psi_{j,k} : j \in \mathbb{J}_1, k \in \mathbb{J}_2\}$ be as in Theorem 3.1. Suppose that $\Gamma = \{\gamma_{j,k} = (x_j, y_k) : x_j \in \mathbb{R}, y_k \in \mathbb{R}^d, j \in \mathbb{J}_1, k \in \mathbb{J}_2\}$ is a relatively-separated set with gaps δ_5 and δ_6 for both variables x and y. If δ_5, δ_6 and a are chosen such that*

$$\begin{aligned} r_3 := A^{-1}\|P\|_{op}\Big[&2M\|\omega_a(\varphi)\|_{W^{p,q}} + (1+M)\big\|\omega_{\sqrt{\delta_5^2+\delta_6^2}}(\varphi)\big\|_{W^{p,q}}\\ &+M\big\|\omega_{\sqrt{2(\delta_5^2+\delta_6^2+a^2)}}(\varphi)\big\|_{W^{p,q}}\Big] < 1, \end{aligned} \tag{4.3}$$

then the algorithm (4.1) exponentially converges to some $f_\infty \in V_{p,q}(\varphi)$, and

$$\|f_n - f_\infty\|_{L^{p,q}} \le \frac{r_3^{n+1}}{1-r_3}\|f_0\|_{L^{p,q}}. \tag{4.4}$$

If $c_0(j,k) = \langle f, \psi_{j,k}\rangle, j \in \mathbb{J}_1, k \in \mathbb{J}_2$ for $f \in V_{p,q}(\varphi)$, then $f_\infty = f$.

Proof Note that $f_{n+1} - f_n = (I - PA_\Gamma)(f_n - f_{n-1}),\ n \geq 1$. For any $f(x,y) \in V_{p,q}(\varphi)$,

$$\begin{aligned}\|f - PA_\Gamma f\|_{L^{p,q}} &\leq \|P\|_{op}\|f - A_\Gamma f\|_{L^{p,q}} \\ &\leq \|P\|_{op}\big(\|f - Q_\Gamma f\|_{L^{p,q}} + \|Q_\Gamma f - A_\Gamma f\|_{L^{p,q}}\big),\end{aligned} \tag{4.5}$$

where $Q_\Gamma f(x,y) = \sum\limits_{j\in\mathbb{J}_1}\sum\limits_{k\in\mathbb{J}_2} f(x_j,y_k)u_{j,k}(x,y)$. Furthermore,

$$\begin{aligned}|f(x,y) - Q_\Gamma f(x,y)| &\leq \sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}|f(x,y) - f(x_j,y_k)|u_{j,k}(x,y) \\ &\leq \omega_{\sqrt{\delta_5^2+\delta_6^2}}(f)(x,y) \\ &\leq \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d}|c(k_1,k_2)|\omega_{\sqrt{\delta_5^2+\delta_6^2}}(\varphi)(x-k_1,y-k_2).\end{aligned}$$

By Lemma 2.1, we can obtain

$$\|f - Q_\Gamma f\|_{L^{p,q}} \leq A^{-1}\big\|\omega_{\sqrt{\delta_5^2+\delta_6^2}}(\varphi)\big\|_{W^{p,q}}\|f\|_{L^{p,q}}. \tag{4.6}$$

On the other hand, we have

$$\begin{aligned}|Q_\Gamma f(x,y) - A_\Gamma f(x,y)| &\leq \sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}\int_{\mathbb{R}}\int_{\mathbb{R}^d}|f(x_j,y_k) - f(s,t)||\psi_{j,k}(s,t)|u_{j,k}(x,y)dsdt \\ &\leq M\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}\omega_a(f)(x_j,y_k)u_{j,k}(x,y) \\ &\leq M\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}\Big(\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d}|c(k_1,k_2)|\omega_a(\varphi)(x_j-k_1,y_k-k_2)\Big)u_{j,k}(x,y) \\ &= MQ_\Gamma\Big(\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d}|c(k_1,k_2)|\omega_a(\varphi)(\cdot-k_1,\cdot-k_2)\Big)(x,y) \\ &\leq M\Big|\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d}|c(k_1,k_2)|\omega_a(\varphi)(x-k_1,y-k_2) \\ &\quad - Q_\Gamma\Big(\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d}|c(k_1,k_2)|\omega_a(\varphi)(\cdot-k_1,\cdot-k_2)\Big)(x,y)\Big| \\ &\quad + M\Big|\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d}|c(k_1,k_2)|\omega_a(\varphi)(x-k_1,y-k_2)\Big|.\end{aligned}$$

By the same method as (4.6), we obtain

$$\begin{aligned}\|Q_\Gamma f - A_\Gamma f\|_{L^{p,q}} &\leq MA^{-1}\Big(\big\|\omega_{\sqrt{\delta_5^2+\delta_6^2}}(\omega_a(\varphi))\big\|_{W^{p,q}} + \|\omega_a(\varphi)\|_{W^{p,q}}\Big)\|f\|_{L^{p,q}} \\ &\leq MA^{-1}\Big(\big\|\omega_{\sqrt{2(\delta_5^2+\delta_6^2+a^2)}}(\varphi)\big\|_{W^{p,q}} + \big\|\omega_{\sqrt{\delta_5^2+\delta_6^2}}(\varphi)\big\|_{W^{p,q}} \\ &\quad + 2\|\omega_a(\varphi)\|_{W^{p,q}}\Big)\|f\|_{L^{p,q}}.\end{aligned}$$

This together with (4.5) and (4.6) gives

$$\|f - PA_\Gamma f\|_{L^{p,q}} \leq r_3\|f\|_{L^{p,q}}.$$

Therefore, (4.4) is proved. Define

$$R_\Gamma := I + \sum_{n=1}^{\infty}(I - PA_\Gamma)^n.$$

Then $R_\Gamma PA_\Gamma = PA_\Gamma R_\Gamma = I$ on $V_{p,q}(\varphi)$. If $c_0(j,k) = \langle f, \psi_{j,k}\rangle$ for $f \in V_{p,q}(\varphi)$, then

$$f_\infty = R_\Gamma f_0 = R_\Gamma PA_\Gamma f = If = f.$$

Theorem 4.2 *Let p, q, φ and ψ be as in Theorem 3.2. Suppose that $\Gamma = \{\gamma_{j,k} = (x_j, y_k) : x_j \in \mathbb{R}, y_k \in \mathbb{R}^d, j \in \mathbb{J}_1, k \in \mathbb{J}_2\}$ is a relatively-separated set with gaps δ_7 and δ_8 for both variables x and y. If δ_7, δ_8 and a are chosen such that*

$$r_4 := A^{-1}\|P\|_{op}\Big[\big\|\omega_{\sqrt{\delta_7^2+\delta_8^2}}(\varphi)\big\|_{W^{p,q}} + \big\|\omega_{\sqrt{\delta_7^2+\delta_8^2}}(\varphi^a)\big\|_{W^{p,q}} + \|\varphi^a\|_{W^{p,q}}\Big] < 1, \tag{4.7}$$

then the algorithm (4.2) exponentially converges to some $f_\infty \in V_{p,q}(\varphi)$, and

$$\|f_n - f_\infty\|_{L^{p,q}} \le \frac{r_4^{n+1}}{1 - r_4}\|f_0\|_{L^{p,q}}. \tag{4.8}$$

If $c_0(j,k) = f * \psi_a^*(\gamma_{j,k}), j \in \mathbb{J}_1, k \in \mathbb{J}_2$ for $f \in V_{p,q}(\varphi)$, then $f_\infty = f$.

Proof For any $f(x,y) = \sum\limits_{k_1\in\mathbb{Z}}\sum\limits_{k_2\in\mathbb{Z}^d} c(k_1,k_2)\varphi(x-k_1, y-k_2) \in V_{p,q}(\varphi)$, one has

$$\|f - PA_{\Gamma,a}f\|_{L^{p,q}} \le \|P\|_{op}\big(\|f - Q_\Gamma f\|_{L^{p,q}} + \|Q_\Gamma f - Q_\Gamma(f * \psi_a^*)\|_{L^{p,q}}\big). \tag{4.9}$$

By (4.6), we know

$$\|f - Q_\Gamma f\|_{L^{p,q}} \le A^{-1}\big\|\omega_{\sqrt{\delta_7^2+\delta_8^2}}(\varphi)\big\|_{W^{p,q}}\|f\|_{L^{p,q}}. \tag{4.10}$$

Since $Q_\Gamma f(x,y) - Q_\Gamma(f * \psi_a^*)(x,y) = Q_\Gamma\Big(\sum\limits_{k_1\in\mathbb{Z}}\sum\limits_{k_2\in\mathbb{Z}^d} c(k_1,k_2)\varphi^a(\cdot - k_1, \cdot - k_2)\Big)(x,y)$, then

$$\begin{aligned}
\|Q_\Gamma f - Q_\Gamma(f * \psi_a^*)\|_{L^{p,q}} &\le \Big\|\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d} c(k_1,k_2)\varphi^a(x-k_1, y-k_2)\Big\|_{L^{p,q}}\\
&\quad + \Big\|\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d} c(k_1,k_2)\varphi^a(x-k_1, y-k_2)\\
&\quad - Q_\Gamma\Big(\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d} c(k_1,k_2)\varphi^a(\cdot - k_1, \cdot - k_2)\Big)(x,y)\Big\|_{L^{p,q}}\\
&\le A^{-1}\Big(\|\varphi^a\|_{W^{p,q}} + \big\|\omega_{\sqrt{\delta_7^2+\delta_8^2}}(\varphi^a)\big\|_{W^{p,q}}\Big)\|f\|_{L^{p,q}}.
\end{aligned}$$

This together with (4.9) and (4.10) obtains

$$\|f - PA_{\Gamma,a}f\|_{L^{p,q}} \le r_4\|f\|_{L^{p,q}}.$$

Therefore, (4.8) is proved. Define

$$R_{\Gamma,a} := I + \sum_{n=1}^{\infty}(I - PA_{\Gamma,a})^n.$$

Then $R_{\Gamma,a}PA_{\Gamma,a} = PA_{\Gamma,a}R_{\Gamma,a} = I$ on $V_{p,q}(\varphi)$. If $c_0(j,k) = f * \psi_a^*(\gamma_{j,k})$ for $f \in V_{p,q}(\varphi)$, then

$$f_\infty = R_{\Gamma,a}f_0 = R_{\Gamma,a}PA_{\Gamma,a}f = If = f.$$

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