Global Well-posedness and Asymptotics of Full Compressible Non-resistive MHD System with Large External Potential Forces

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Abstract

We consider the global well-posedness and asymptotic behavior of compressible viscous, heat-conductive, and non-resistive magnetohydrodynamics (MHD) fluid in a field of external forces over three-dimensional periodic thin domain $\Omega = \mathbb{T}^2 \times (0, \delta)$. The unique existence of the stationary solution is shown under the adhesion and the adiabatic boundary conditions. Then, it is shown that a solution to the initial boundary value problem with the same boundary and periodic conditions uniquely exists globally in time and converges to the stationary solution as time tends to infinity. Moreover, if the external forces are small or disappeared in an appropriate Sobolev space, then δ can be a general constant. Our proof relies on the two-tier energy method for the reformulated system in Lagrangian coordinates and the background magnetic field which is perpendicular to the flat layer. Compared to the work of Tan and Wang (SIAM J. Math. Anal. 50:1432–1470, 2018), we not only overcome the difficulties caused by temperature, but also consider the big external forces.

Keywords. Compressible non-resistive MHD; heat-conductive; global well-posedness; large external forces

AMS Subject Classifications (2000). 35B45, 35Q30, 35Q60, 76N10, 76W05

1 Introduction

1.1 Formulation

The magnetohydrodynamics (MHD) equations are derived from the induction equation of the magnetic field and the Navier-Stokes equations of fluid dynamics (see also [1, 2, 3, 4, 6, 10, 12]).

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In the Eulerian coordinates, the MHD equations have the following forms ($\Omega \subset \mathbb{R}^3$):

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0, \qquad\qquad (x,t) \in \Omega \times \mathbb{R}^+, \\ \rho(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \nabla P - \mu \Delta \boldsymbol{u} - (\mu + \mu')\nabla \mathrm{div} \boldsymbol{u} = \lambda \boldsymbol{H} \cdot \nabla \boldsymbol{H} - \frac{\lambda}{2} \nabla |\boldsymbol{H}|^2 + \rho \nabla f, \\ c_v \rho(\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta) + P \mathrm{div} \boldsymbol{u} - \kappa \Delta \theta = 2\mu |D(\boldsymbol{u})|^2 + \mu' |\mathrm{div} \boldsymbol{u}|^2, \\ \partial_t \boldsymbol{H} + \boldsymbol{u} \cdot \nabla \boldsymbol{H} - \boldsymbol{H} \cdot \nabla \boldsymbol{u} + \boldsymbol{H} \mathrm{div} \boldsymbol{u} = \nu \Delta \boldsymbol{H}, \ \mathrm{div} \boldsymbol{H} = 0. \end{cases} \tag{1.1}$$

Here, the mass density ρ, the velocity $\boldsymbol{u} = (u_1, u_2, u_3)$, the absolute temperature θ, and the magnetic field $\boldsymbol{H} = (H_1, H_2, H_3)$ are unknown functions. In addition, $P := R\rho\theta$ ($R > 0$) is the pressure; $f = f(x)$ denotes the known external potential force. Positive constants c_v and κ are the heat capacity and the ratio of the heat conductivity coefficient over the heat capacity respectively. Non-negative constant ν is the resistivity coefficient. μ and μ' are the viscosity coefficients, satisfying the physical conditions

$$\mu > 0, \ 2\mu + 3\mu' > 0,$$

and $\lambda > 0$ are the permeability coefficients. The operator $D(\boldsymbol{u})$ is defined by

$$D(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top).$$

The MHD equations are used to simulate the motion of a conducting fluid under the effect of the electromagnetic field and have a very wide range of applications in astrophysics, plasma, and so on. In many cosmical and geophysical problems where the conducting fluid is of extremely high conductivity (e.g., ideal conductors), it is more reasonable to ignore the resistivity (i.e., $\nu = 0$). For the highly conducting fluid, the magnetic field lines move along exactly with the fluid, rather than simply diffusing out. This type of behavior is physically described as that *the magnetic field lines are frozen into the fluid*. In effect, the fluid can flow freely along the magnetic field lines, but any motion of the conducting fluid, perpendicular to the field lines, carries them with the fluid. The "frozen-in" nature of magnetic fields plays very important roles and has a very wide range of applications in both astrophysics and nuclear fusion theory, where the magnetic Reynolds number $R_m \sim 1/\nu$ is usually very high. A typical illustration of the "frozen-in" behavior is the phenomenon of sunspots. For more details of its physical background and applications, we refer to [2, 3, 4, 6, 10, 12].

Due to its physical importance, complexity, rich phenomena, and mathematical challenges, there have been extensive studies on MHD by many physicists and mathematicians. However, similarly to those for the Navier-Stokes equations, many physically important and mathematically fundamental problems of MHD are still open. For example, to the author's knowledge, the global well-posedness of the multi-dimensional compressible non-resistive MHD equations remains unknown, even that the data is sufficiently close to the non-vacuum equilibrium state in a similar sense as that in [20] for the compressible Navier-Stokes equations. Here, we would like to refer to the recent works [15, 17, 18, 32], where the global well-posedness of the Cauchy problem of two/three-dimensional incompressible non-resistive MHD (MHD-type) equations with small data are announced. In the case that viscosity coefficient equals magnetic resistive coefficient whether they are zero or positive constant, He-Xu-Yu [11], Cai-Lei [5], and Wei-Zhang [30] they establish the global well-posedness in different methods for the incompressible MHD. For the isentropic compressible non-resistive MHD equations, Wu-Wu [31] obtain the global solutions and its asymptotic behavior in $\mathbb{R}^2$ using the special structure of equations. By means of the two-tier energy method, Tan-Wang prove the similar decay result over $\Omega := \mathbb{R}^2 \times (0, 1)$ in [27].

It is worth noting that the heating of high-temperature plasma by MHD waves is one of the most interesting and challenging problems of plasma physics, especially when the energy is injected into the system at length scales that are much larger than the dissipative ones. For compressible heat-conductive non-resistive MHD, Zhang-Zhao [33] and Li-Wang-Ye[14] obtain the global well-posedness in 1D bounded domain and whole space, respectively. Assume that there exists a magnetic background, Si-Zhao [25] establish the large time behaviors of strong solutions in $\mathbb{R}$, $\mathbb{R}^+$, and $(0,1)\subset\mathbb{R}$, respectively.

Furthermore, the large external forces can significantly affect the dynamic motion of flows and cause some serious difficulties (see [9, 13, 22, 29]). Indeed, there are lots of results for the global existence and large-time behavior of the solutions to the compressible Navier-Stokes equations when both the external forces and the initial perturbations are sufficiently small; see [7, 8, 24, 28] and the references therein. For the compressible Navier-Stokes equations, the asymptotic behavior of a solution is first considered by Matsumura and Nishida in [21], where they prove the solution converges to the corresponding stationary solution as time tends to infinity in the exterior domain of $\mathbb{R}^3$ under the smallness assumptions on the initial data and the external force. After the research of [21], Matsumura in [19] considers isothermal flow and proves that the stationary solution is time asymptotically stable for an arbitrary external force. For large external forces, literatures [9, 13, 22, 23] establish the large time behavior of weak solutions to the isentropic compressible Navier-Stokes equations. Moreover, Li-Zhang-Zhao [16] study the global existence and the large time behavior of the strong solutions.

In view of the results achieved, we consider the global well-posedness and asymptotic behavior of the compressible heat-conductive non-resistive MHD flows

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho\boldsymbol{u})=0, \qquad\qquad (x,t)\in\Omega\times\mathbb{R}^+,\\ \rho(\partial_t\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u})+\nabla P-\mu\Delta\boldsymbol{u}-(\mu+\mu')\nabla\operatorname{div}\boldsymbol{u}=\lambda\boldsymbol{H}\cdot\nabla\boldsymbol{H}-\dfrac{\lambda}{2}\nabla|\boldsymbol{H}|^2+\rho\nabla f,\\ c_v\rho(\partial_t\theta+\boldsymbol{u}\cdot\nabla\theta)+P\operatorname{div}\boldsymbol{u}-\kappa\Delta\theta=2\mu|D(\boldsymbol{u})|^2+\mu'|\operatorname{div}\boldsymbol{u}|^2,\\ \partial_t\boldsymbol{H}+\boldsymbol{u}\cdot\nabla\boldsymbol{H}-\boldsymbol{H}\cdot\nabla\boldsymbol{u}+\boldsymbol{H}\operatorname{div}\boldsymbol{u}=0,\ \operatorname{div}\boldsymbol{H}=0,\\ \boldsymbol{u}|_{x_3=0,\delta}=0, n\cdot\nabla\theta|_{x_3=0,\delta}=0,\end{cases}\tag{1.2}$$

where $\Omega:=\mathbb{T}^2\times[0,\delta]$, $\mathbb{T}^2:=[0,1]\times[0,1]$. Letting $(\rho_s,\boldsymbol{u}_s,\theta_s,\boldsymbol{H}_s)$ be the stationary solutions of the problem (1.2), multiplying $(1.2)_3$ by $1/\theta_s$, and integrating over Ω, by virtue of $n\cdot\nabla\theta_s|_{x_3=0,\delta}=0$, $\operatorname{div}(\rho_s u_s)=0$, and $u_s|_{x_3=0,\delta}=0$, one has

$$\kappa\int_\Omega\left|\frac{\nabla\theta_s}{\theta_s}\right|^2\mathrm{d}x+2\mu\int_\Omega\frac{|D(\boldsymbol{u}_s)|^2}{\theta_s}\mathrm{d}x+\mu'\int_\Omega\frac{|\operatorname{div}\boldsymbol{u}_s|^2}{\theta_s}\mathrm{d}x=0,\tag{1.3}$$

which means that θ_s and $\boldsymbol{u}_s$ are some constants. Since $\boldsymbol{u}_s|_{x_3=0,\delta}=0$, one has

$$\boldsymbol{u}_s=0.\tag{1.4}$$

For simplicity, we set

$$\theta_s=1.\tag{1.5}$$

Because we are interesting in showing that $\boldsymbol{H}\to\bar{\boldsymbol{H}}=(0,0,\bar{b})^\top$, we may assume $\boldsymbol{H}_s=\bar{\boldsymbol{H}}$. Hence, it follows from $(1.2)_2$ and (1.4)–(1.5) that

$$R\nabla\rho_s=\rho_s\nabla f,\tag{1.6}$$

which yields

$$\rho_s=\exp\{f/R+C\}.$$

To summarize, we have the following proposition.

Proposition 1.1 *Assume that $f \in H^{4N+2}$ and the integer $N \geq 1$. Then the stationary problem (1.6) has a unique solution $\rho_s = \rho_s(x)$ satisfying*

$$0 < \underline{\rho} \leq \rho_s \leq \bar{\rho} < \infty \qquad \text{and} \qquad \nabla \rho_s \in H^{4N+1}. \tag{1.7}$$

where $\underline{\rho}$ and $\bar{\rho}$ are positive constants depending only on R, $\sup_{x\in\Omega} f$ and $\inf_{x\in\Omega} f$.

Inspired of [26], letting $\boldsymbol{B} = \boldsymbol{H} - \bar{\boldsymbol{H}}$, similar as (1.3), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big(\frac{1}{2}\rho \boldsymbol{u}^2 + \frac{\kappa}{2}\boldsymbol{B}^2 + c_v\rho(\theta - \log\theta - 1) + R\rho_s\Big(\frac{\rho}{\rho_s}\log\frac{\rho}{\rho_s} - \frac{\rho}{\rho_s} + 1\Big)\Big)\mathrm{d}x \\ &\qquad + \kappa\int_\Omega \left|\frac{\nabla\theta}{\theta}\right|^2 \mathrm{d}x + 2\mu\int_\Omega \frac{|D(\boldsymbol{u})|^2}{\theta}\mathrm{d}x + \mu'\int_\Omega \frac{|\mathrm{div}\boldsymbol{u}|^2}{\theta}\mathrm{d}x = 0. \end{aligned} \tag{1.8}$$

In fact, it follows from (1.2) and $\boldsymbol{B} = \boldsymbol{H} - \bar{\boldsymbol{H}}$ that

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho\boldsymbol{u}) = 0, \qquad (x,t) \in \Omega\times\mathbb{R}^+, \\ \rho(\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \nabla P - \mu\Delta\boldsymbol{u} - (\mu+\mu')\nabla\mathrm{div}\boldsymbol{u} \\ \qquad = \lambda\boldsymbol{B}\cdot\nabla\boldsymbol{B} - \frac{\lambda}{2}\nabla|\boldsymbol{B}|^2 + \lambda\partial_3\boldsymbol{B} - \lambda\nabla\boldsymbol{B}_3 + \rho\nabla f, \\ c_v\rho(\partial_t\theta + \boldsymbol{u}\cdot\nabla\theta) + P\mathrm{div}\boldsymbol{u} - \kappa\Delta\theta = 2\mu|D(\boldsymbol{u})|^2 + \mu'|\mathrm{div}\boldsymbol{u}|^2, \\ \partial_t\boldsymbol{B} + \boldsymbol{u}\cdot\nabla\boldsymbol{B} - \boldsymbol{B}\cdot\nabla\boldsymbol{u} + \boldsymbol{B}\mathrm{div}\boldsymbol{u} = \partial_3\boldsymbol{u} - \mathrm{div}\boldsymbol{u}e_3,\ \mathrm{div}\mathbf{B} = 0, \\ \boldsymbol{u}|_{\partial\Omega} = 0, n\cdot\nabla\theta|_{\Omega} = 0. \end{cases} \tag{1.9}$$

Multiplying $(1.9)_2$, $(1.9)_4$, and $(1.9)_3$ by $\boldsymbol{u}$, $\boldsymbol{B}$, and $-\theta^{-1}$ respectively, adding them together with $(1.9)_3$, it follows from integration by parts, $(1.9)_1$, $\boldsymbol{u}|_{\partial\Omega} = 0$, and $n\cdot\nabla\theta|_{\partial\Omega} = 0$ that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\rho\boldsymbol{u}^2 + \frac{\lambda}{2}\boldsymbol{B}^2 + c_v\rho(\theta - \log\theta - 1)\right)\mathrm{d}x + \kappa\int_\Omega \left|\frac{\nabla\theta}{\theta}\right|^2 \mathrm{d}x \\ &\qquad + 2\mu\int_\Omega \frac{|D(\boldsymbol{u})|^2}{\theta}\mathrm{d}x + \mu'\int_\Omega \frac{|\mathrm{div}\boldsymbol{u}|^2}{\theta}\mathrm{d}x = \int_\Omega \rho\boldsymbol{u}\cdot\nabla f\mathrm{d}x + \int_\Omega R\rho\mathrm{div}\boldsymbol{u}\mathrm{d}x. \end{aligned} \tag{1.10}$$

By virtue of (1.6), $(1.9)_1$ and integration by parts, one can derive that

$$\begin{aligned} \int_\Omega \rho\boldsymbol{u}\cdot\nabla f\mathrm{d}x &= R\int_\Omega \rho\boldsymbol{u}\cdot\nabla\log\rho_s\mathrm{d}x = -R\int_\Omega \mathrm{div}(\rho\boldsymbol{u})\log\rho_s\mathrm{d}x \\ &= R\int_\Omega \rho_t\log\rho_s\mathrm{d}x, \end{aligned} \tag{1.11}$$

$$\begin{aligned} \int_\Omega R\rho\mathrm{div}\boldsymbol{u}\mathrm{d}x &= -R\int_\Omega \boldsymbol{u}\cdot\nabla\rho\mathrm{d}x = -R\int_\Omega \rho\boldsymbol{u}\cdot\nabla\log\rho\mathrm{d}x \\ &= R\int_\Omega \mathrm{div}(\rho\boldsymbol{u})\log\rho\mathrm{d}x = -R\int_\Omega \rho_t\log\rho\mathrm{d}x. \end{aligned} \tag{1.12}$$

Inserting (1.11)-(1.12) into (1.10), one immediately has (1.8).

The basic energy-entropy estimate (1.8) will play a very important role in our proof. For convenience to establish the high order estimate, we transform (1.2) into Lagrangian coordinate.

1.2 Reformulation

We define the Lagrangian trajectory $\mathbb{X}(t,y)$ by

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{X}(t,y) = \boldsymbol{u}(t,\mathbb{X}(t,y)), \\ \mathbb{X}(0,y) = \mathbb{X}_0(y). \end{cases} \tag{1.13}$$

For any smooth function f, we deduce from chain rule that

$$\partial_{y_j} f(\mathbb{X}(t,y),t) = \frac{\partial f}{\partial x_i}(\mathbb{X}(t,y),t)\frac{\partial \mathbb{X}_i(t,y)}{\partial y_j}, \tag{1.14}$$

where $i,j\in\{1,2,3\}$. Let us denote $A=((\nabla\mathbb{X})^{-1})^\top$, then we have

$$\nabla_x f = A_{ij}\partial_{y_j} f := \nabla_A f, \qquad \mathrm{div}_x f = A_{ij}\partial_{y_j} f_i := \mathrm{div}_A f. \tag{1.15}$$

Denote $\Delta_A f = \mathrm{div}_A \nabla_A f$, then one has

$$\begin{cases} \partial_t\rho + \rho\,\mathrm{div}_A\boldsymbol{u} = 0, \\ \rho\partial_t\boldsymbol{u} + \nabla_A P - \mu\Delta_A\boldsymbol{u} - (\mu+\mu')\nabla_A \mathrm{div}_A\boldsymbol{u} = \lambda\boldsymbol{H}\cdot\nabla_A\boldsymbol{H} - \dfrac{\lambda}{2}\nabla_A|\boldsymbol{H}|^2 + \rho\nabla_A f, \\ c_v\rho\partial_t\theta + P\,\mathrm{div}_A\boldsymbol{u} - \kappa\Delta_A\theta = 2\mu|D_A(\boldsymbol{u})|^2 + \mu'|\mathrm{div}_A\boldsymbol{u}|^2, \\ \partial_t\boldsymbol{H} - \boldsymbol{H}\cdot\nabla_A\boldsymbol{u} + \boldsymbol{H}\mathrm{div}_A\boldsymbol{u} = 0,\ \mathrm{div}_A\boldsymbol{H} = 0, \\ \boldsymbol{u}|_{\partial\Omega} = 0, n\cdot\nabla_A\theta|_{\partial\Omega} = 0, \end{cases} \tag{1.16}$$

where $D_A(\boldsymbol{u}) = \dfrac{\nabla_A\boldsymbol{u} + (\nabla_A\boldsymbol{u})^\top}{2}$. We denote that $J=\det(\nabla\mathbb{X})$, then we have $\partial_t J = J\mathrm{div}_A\boldsymbol{u}$, which combined with $(1.16)_1$ yields that

$$\partial_t(\rho J) = 0. \tag{1.17}$$

Multiplying $(1.16)_4$ by $JA^\top$, thanks to $\partial_j(JA_{ij})=0$, one can deduce that

$$\partial_t(JA^\top\boldsymbol{H}) = 0. \tag{1.18}$$

Since we are interested in showing that $(\mathbb{X},\rho,\boldsymbol{u},\theta,\boldsymbol{H}) \to (Id,\rho_s,0,1,\bar{\boldsymbol{H}})$ as $t\to\infty$ in a strong sense, due to the conservations (1.17)-(1.18), we may conclude that for any $t\geq 0$

$$\rho J = \rho_s, \quad JA^\top\boldsymbol{H} = \bar{\boldsymbol{H}} \quad \text{in } \Omega, \qquad \text{and} \qquad \mathbb{X} = Id \quad \text{on } \partial\Omega. \tag{1.19}$$

Note then that $\mathrm{div}_A\boldsymbol{H} = J^{-1}\mathrm{div}(JA^\top\boldsymbol{H}) = J^{-1}\mathrm{div}\bar{\boldsymbol{H}} = 0$. In turn, to have these we need to assume that the initial data satisfy these conditions; such conditions are necessary for our global well-posedness. We may shift $\mathbb{X}\to Id+\mathbb{X}$, and hence $J=\det(I+\nabla\mathbb{X}), A=((I+\nabla\mathbb{X})^{-1})^\top$, and we rerecord these conserved quantities in the following form:

$$\rho = \rho_s J^{-1},\ \boldsymbol{H} = J^{-1}(I+\nabla\mathbb{X})\bar{\boldsymbol{H}} = \bar{b}J^{-1}(e_3+\partial_3\mathbb{X}) \text{ in } \Omega,\ \text{and } \mathbb{X}=0 \text{ on } \partial\Omega. \tag{1.20}$$

Next, we reformulate (1.16). By virtue of $\mathbb{X}|_{\partial\Omega}=0$, one has

$$(I+\nabla\mathbb{X})|_{\partial\Omega} = \begin{pmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ \partial_3\mathbb{X}_1|_{\partial\Omega}, & \partial_3\mathbb{X}_2|_{\partial\Omega}, & 1+\partial_3\mathbb{X}_3|_{\partial\Omega} \end{pmatrix}. \tag{1.21}$$

Hence, one obtains that

$$A_{31}|_{\partial\Omega} = 0, \qquad A_{32}|_{\partial\Omega} = 0, \tag{1.22}$$

which combined with $n \cdot \nabla_A \theta|_{\partial\Omega} = 0$ yields

$$\partial_3 \theta|_{\partial\Omega} = 0. \tag{1.23}$$

According Taylor formula, one has $J^{-1} = 1 - \mathrm{div}\mathbb{X} + O(|\nabla\mathbb{X}|^2)$, which means that

$$\begin{aligned}
&\boldsymbol{H} \cdot \nabla_A \boldsymbol{H} - \nabla_A \left(\frac{|\boldsymbol{H}|^2}{2} \right) \\
={}& \bar{\boldsymbol{H}} \cdot \nabla(\boldsymbol{H} - \bar{\boldsymbol{H}}) + \nabla(\boldsymbol{H}_j - \bar{\boldsymbol{H}}_j)\bar{\boldsymbol{H}}_j \\
&+ \bar{\boldsymbol{H}} \cdot (\nabla_A - \nabla)(\boldsymbol{H} - \bar{\boldsymbol{H}}) + (\boldsymbol{H} - \bar{\boldsymbol{H}}) \cdot \nabla_A(\boldsymbol{H} - \bar{\boldsymbol{H}}) \\
&+ (\nabla_A - \nabla)(\boldsymbol{H}_j - \bar{\boldsymbol{H}}_j)\bar{\boldsymbol{H}}_j + \nabla_A(\boldsymbol{H}_j - \bar{\boldsymbol{H}}_j)(\boldsymbol{H}_j - \bar{\boldsymbol{H}}_j) \\
={}& \bar{\boldsymbol{H}} \cdot \nabla(\boldsymbol{H} - \bar{\boldsymbol{H}}) + \nabla(\boldsymbol{H}_j - \bar{\boldsymbol{H}}_j)\bar{\boldsymbol{H}}_j + O(\nabla\mathbb{X}\nabla^2\mathbb{X}) \\
={}& \bar{b}^2 \Big(\partial_3((J^{-1} - 1)e_3 + J^{-1}\partial_3\mathbb{X}) - \nabla(J^{-1} - 1 + J^{-1}\partial_3\mathbb{X}_3) \Big) + O(\nabla\mathbb{X}\nabla^2\mathbb{X}) \\
={}& \bar{b}^2 (\partial_3^2 \mathbb{X} - \partial_3 \mathrm{div}\mathbb{X} e_3 + \nabla \mathrm{div}\mathbb{X} - \nabla\partial_3\mathbb{X}_3) + O(\nabla\mathbb{X}\nabla^2\mathbb{X}).
\end{aligned} \tag{1.24}$$

About the terms of pressure and external force, we have

$$\begin{aligned}
&\nabla_A P - \rho \nabla_A f \\
={}& R\nabla_A(\rho_s J^{-1}\theta) - RJ^{-1}\nabla_A \rho_s \\
={}& R\nabla_A(\rho_s(J^{-1} - 1)\theta + \rho_s(\theta - 1)) \\
&- R(J^{-1} - 1)(\nabla_A - \nabla)\rho_s - R(J^{-1} - 1)\nabla\rho_s \\
={}& R(\nabla_A - \nabla)\Big(\rho_s(J^{-1} - 1)\theta + \rho_s(\theta - 1)\Big) - R(J^{-1} - 1)(\nabla_A - \nabla)\rho_s \\
&+ R\nabla\Big(\rho_s(J^{-1} - 1)(\theta - 1)\Big) + R\rho_s\nabla(J^{-1} - 1) + R\nabla(\rho_s(\theta - 1)) \\
={}& -G - R\rho_s\nabla\mathrm{div}\mathbb{X} + R\nabla(\rho_s(\theta - 1)) + \rho_s O(\nabla\mathbb{X}\nabla^2\mathbb{X}),
\end{aligned} \tag{1.25}$$

where

$$\begin{aligned}
G_i := {}& R(J^{-1} - 1)(A_{ij} - \delta_{ij})\partial_j\rho_s - R\partial_i\Big(\rho_s(J^{-1} - 1)(\theta - 1)\Big) \\
&- R(A_{ij} - \delta_{ij})\partial_j\Big(\rho_s(J^{-1} - 1)\theta + \rho_s(\theta - 1)\Big).
\end{aligned}$$

Similarly, one has

$$P\mathrm{div}_A u = R\rho_s \mathrm{div}\boldsymbol{u} + R\rho_s[(J^{-1} - 1)\theta + (\theta - 1)]A_{ij}\partial_j u_i + R\rho_s(A_{ij} - \delta_{ij})\partial_j u_i. \tag{1.26}$$

By means of (1.22)-(1.26), (1.16) can be rewritten as following:

$$\begin{cases}
\partial_t \mathbb{X} = \boldsymbol{u}, \qquad (\boldsymbol{u}, \mathbb{X}, \partial_3\theta, A_{31}, A_{32})|_{\partial\Omega} = 0, \qquad \Omega := \mathbb{R}^2 \times (0, \delta), \\
\rho_s \partial_t \boldsymbol{u} - \mu\Delta\boldsymbol{u} - (\mu + \mu')\nabla\mathrm{div}\boldsymbol{u} - R\rho_s\nabla\mathrm{div}\mathbb{X} + R\nabla(\rho_s(\theta - 1)) \\
\qquad = \lambda\bar{b}^2(\partial_3^2\mathbb{X} - \partial_3\mathrm{div}\mathbb{X}e_3 + \nabla\mathrm{div}\mathbb{X} - \nabla\partial_3\mathbb{X}_3) + F + G, \\
c_v \rho_s \partial_t \theta - \kappa\Delta\theta + R\rho_s\mathrm{div}\boldsymbol{u} = E,
\end{cases} \tag{1.27}$$

where

$$
\begin{aligned}
F_i :=\ & (\rho_s+1)O(\nabla\mathbb{X}\nabla^2\mathbb{X})_i + (\mu+\mu')A_{ij}\partial_j A_{kl}\partial_l u_k + \mu A_{jk}\partial_k A_{jl}\partial_l u_i \\
& + \underbrace{(\mu+\mu')(A_{ij}A_{kl}-\delta_{ij}\delta_{kl})\partial_j\partial_l u_k + \mu(A_{jk}A_{jl}-\delta_{jk}\delta_{jl})\partial_k\partial_l u_i - \rho_s(J^{-1}-1)\partial_t u_i}_{I_1}, \\
E :=\ & -R\rho_s[(J^{-1}-1)\theta + (\theta-1)]A_{ij}\partial_j u_i - R\rho_s(A_{ij}-\delta_{ij})\partial_j u_i + 2\mu|D_A(\boldsymbol{u})|^2 + \mu'(\mathrm{div}_A\boldsymbol{u})^2 \\
& + \kappa A_{jk}\partial_k A_{jl}\partial_l\theta + \underbrace{\kappa(A_{jk}A_{jl}-\delta_{jk}\delta_{jl})\partial_k\partial_l\theta - \rho_s(J^{-1}-1)\partial_t\theta}_{I_2}.
\end{aligned}
$$

Notation. Before stating our main result, we first introduce the notations and conventions used throughout this paper. The Einstein convention of summing over repeated indexes is used. We write $\int f = \int_\Omega f \mathrm{d}y$. We take $L^p(\Omega)$, $p \ge 1$ and $H^k(\Omega)$, $k \ge 0$ for the standard L^p and inhomogeneous Sobolev spaces on Ω with norms $\|\cdot\|_{L^p} := \|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_k := \|\cdot\|_{H^k(\Omega)}$. We denote $C > 0$ a generic constant that does not depend on the initial data, time and Ω, but can depend on N, $\bar{\rho}$, $\underline{\rho}$, and any of the parameters of the problem, which are allowed different from line to line. We refer to such constants as "universal". We employ the notation $A \lesssim B$ to mean that $A \le CB$ for a universal constant $C > 0$, and we write $\partial_t A + B \lesssim D$ for $\partial_t A + CB \lesssim D$. We will write $\mathbb{N} = \{0, 1, 2, \cdots\}$ for the collection of non-negative integers. When using space-time differential multi-indexes, we will write $\mathbb{N}^{1+m} = \{\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_m)\}$ to emphasize that the 0-index term is related to temporal derivatives. For just spatial derivatives we write $\mathbb{N}^m$. For $\alpha \in \mathbb{N}^{1+m}$ we write $\partial^\alpha = \partial_t^{\alpha_0}\partial_1^{\alpha_1}\cdots\partial_m^{\alpha_m}$. We define the parabolic counting of such multi-indexes by writing $|\alpha| = 2\alpha_0 + \alpha_1 + \cdots + \alpha_m$. For vector $v = (v_1, v_2, v_3)$, we write $v_h = (v_1, v_2)$ for the horizontal components. We write $\nabla_h = (\partial_1, \partial_2)$ for the horizontal gradient, $\Delta_h := \partial_1^2 + \partial_2^2$ for the horizontal Laplace operator, and div_h for the horizontal divergence, i.e. $\mathrm{div}_h v_h = \partial_1 v_1 + \partial_2 v_2$. We also have the following anisotropic Sobolev norm:

$$
\|f\|_{k,l}^2 := \sum_{\alpha_1+\alpha_2\le l} \|\partial_1^{\alpha_1}\partial_2^{\alpha_2} f\|_{H_v^k}^2,
$$

where $\|f\|_{H_v^k}^2 := \sum_{j=0}^{k} \|\partial_3^j f\|_0^2$. For a *given norm* $\|\cdot\|$ and an integer $k \ge 0$, we introduce the following notation for sums of derivatives:

$$
\|\bar{\nabla}_0^k f\|^2 := \sum_{\alpha\in\mathbb{N}^{1+3}, |\alpha|\le k} \|\partial^\alpha f\|^2.
$$

Finally, for a generic integer $n \ge 3$, we define the energy as

$$
\mathcal{E}_n := \sum_{j=0}^{n} \|\partial_t^j(\boldsymbol{u}, \theta-1)\|_{2n-2j}^2 + \|\mathbb{X}\|_{2n+1}^2 \tag{1.28}
$$

and the dissipation as

$$
\mathcal{D}_n := \sum_{j=0}^{n} \|\partial_t^j(\boldsymbol{u}, \theta-1)\|_{2n-2j+1}^2 + \|\mathrm{div}\mathbb{X}\|_{2n}^2 + \|\partial_3\mathbb{X}\|_{2n}^2 + \|\mathbb{X}\|_{2n}^2. \tag{1.29}
$$

We will consider both $n = 2N$ and $n = N+2$ for the integer $N \ge 4$. Finally, we define

$$
\mathcal{G}_{2N}(T) := \sup_{0\le t\le T} \mathcal{E}_{2N}(t) + \int_0^T \mathcal{D}_{2N}(t)\mathrm{d}t + \sup_{0\le t\le T} (1+t)^{2N-4}\mathcal{E}_{N+2}(t). \tag{1.30}
$$

Multiplying $(1.9)_{2,4}$ by $\boldsymbol{u}$ and $\boldsymbol{H}-\bar{\boldsymbol{H}}$, respectively, adding them with $(1.9)_3$, and integrating over Ω about x, we can derive

$$\frac{1}{|\Omega|}\int_\Omega \theta+\frac{\boldsymbol{u}^2+(\boldsymbol{H}-\bar{\boldsymbol{H}})^2}{2c_v}\mathrm{d}x=\frac{1}{|\Omega|}\int_\Omega \theta_0+\frac{\boldsymbol{u}_0^2+(\boldsymbol{H}_0-\bar{\boldsymbol{H}})^2}{2c_v}\mathrm{d}x.$$

For simplicity, we denote the mean value of energy at initial

$$\frac{1}{|\Omega|}\int_\Omega \theta_0+\frac{\boldsymbol{u}_0^2+(\boldsymbol{H}_0-\bar{\boldsymbol{H}})^2}{2c_v}\mathrm{d}x=1.$$

1.3 Main result.

Our main result can be stated as follows.

Theorem 1.1 *Let $N\in\mathbb{Z}_{\geq 4}$. Suppose that $u_0\in H^{4N}(\Omega)$, and $\mathbb{X}_0\in H^{4N+1}(\Omega)$, $f\in H^{4N+2}$ satisfy the appropriate compatibility conditions for the local well-posedness of (1.27), $\mathbb{X}_0=0$ on $\partial\Omega$, and δ is suitable small. There exists a constant $\varepsilon_0>0$ such that if*

$$\mathcal{E}_{2N}(0)\leq\varepsilon_0, \tag{1.31}$$

then there exists a global unique solution $(\mathbb{X},\boldsymbol{u},\theta)$ solving (1.27) on $[0,\infty)$. The solution obeys the estimate

$$\mathcal{G}_{2N}(\infty)\lesssim\mathcal{E}_{2N}(0). \tag{1.32}$$

We now comment on the analysis of this paper. Our main difficulties are from the boundary condition of θ , and that ρ_s is a function of x, not a constant. Since $\theta|_{\partial\Omega}$ is unknown, one can not use the Poincaré inequality to tackle the term of temperature. Compared to the work of Tan-Wang [27], we have the following main breakthroughs:

1. Whether in estimating the energy evolution of temporal or the horizontal derivatives, there always exists big term $\|\nabla(\theta,u)\|_0^2$ coming form the estimate of pressure which cannot be directly absorbed by the left hand of inequality. Thanks to the elementary energy estimate of (1.8) in Eulerian coordinate, we have the following estimate in Lagrangian coordinate:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(&\frac{1}{2}\rho_s\boldsymbol{u}^2+\frac{\lambda}{2}\bar{b}^2\big(J^{-1}\partial_3\mathbb{X}+e_3(J^{-1}-1)\big)^2J+c_v\rho_s(\theta-\log\theta-1)\\&+R\rho_s(J-\log J-1)\Big)+\|\nabla(\theta,\boldsymbol{u})\|_0^2\lesssim\sqrt{\mathcal{E}_n}\mathcal{D}_n,\end{aligned} \tag{1.33}$$

which means that when we estimate the higher order derivatives, the terms $\|\nabla(\theta,\boldsymbol{u})\|_0^2$ can be controlled.

2. The estimate of horizontal derivatives of $\mathbb{X}$ cannot be obtained only by the integration by parts when we tackle the pressure term, since ρ_s is a function of x and the estimate of $\|\theta\|_{0,2n}^2$ cannot be established. To overcome these difficulties, we need the Poincaré's inequality to increase the derivative order of temperature. See more details in the estimate of V_3^p.

3. For the estimates of the horizontal derivatives of velocity, we cannot use the method similar as $\mathbb{X}$. In fact, these will increase the derivative order of $\boldsymbol{u}$ and lead to appearing of the term $\|\theta-1\|_{L^\infty}^2$ which does not contained in $\mathcal{D}_n$ since the Poincaré's inequality is not suitable for temperature. Thanks to $\partial_t\mathbb{X}=\boldsymbol{u}$, we can shift the time derivative from $\mathbb{X}$ to θ for the reason that ρ_s only depends on x. However, the estimate of $\|\theta_t\|_0^2$ in $\bar{\mathcal{D}}_n^t$ cannot be established similar as $\boldsymbol{u}$. Here, we need re-estimate $\|\partial_t^r\theta\|_0^2$ in Lemma 2.2.

4. In our model, the stationary solution ρ_s depending on spatial space yields that we cannot directly use the method of Tan-Wang [27] to get the estimate of $\|\partial_3\mathrm{div}\mathbb{X}\|_{k,2n-k-1}^2$. So we have to estimate $\|\mathrm{div}\mathbb{X}\|_{k,0}^2$. Fortunately, $\bar{b}$ is a constant which makes it easier to get the form of horizontal direction of $\mathbb{X}$.

5. In the improved estimate, $\partial_3\theta$ can use Poincaré's inequality since $\partial_3\theta|_{\partial\Omega}=0$. This means that $\|\partial_3\theta\|_{k,2n-k-1}^2\lesssim\|\partial_3^2\theta\|_{k,2n-k-1}^2$. Thus, we only need to tackle the term $\|\partial_3^2\theta\|_{k,2n-k-1}^2$ when we estimate $\|\partial_3\theta\|_{k+1,2n-k-1}^2$. It follows from the conservation of energy and the Poincaré's inequality that the form of θ is similar as velocity $\boldsymbol{u}$ in $\mathcal{D}_n$ (see (2.102)).

6. Particularly, to improve the estimate of θ in Lemma 2.9, We can not directly use the elliptic regularity because the boundary value of θ is unknown. But, firstly we have $\partial_3\theta|_{\partial\Omega}=0$ which means that we can establish the estimate of $\sum_{r=1}^{n}\|\partial_3\partial_t^r\theta\|_{2n-2r}^2$ by using elliptic regularity. Secondly, due to the structure of (1.27) and Sobolev inequality, we can derive the estimate of $\|\nabla_h\partial_t^r\theta\|_{2n-2r-1}^2$. Moreover, $\sum_{r=1}^{n}\|\partial_t^r\theta\|_{2n-2r+1}^2$ can be controlled when we get the estimate of $\|\partial_3\nabla_h\partial^\beta\partial_t^r\theta\|_0^2$ and $\|\Delta_h\partial^\beta\partial_t^r\theta\|_0^2$ ($\beta\in\mathbb{N}^3$, $|\beta|=2n-2r-1$). In Lemma 2.10, besides the similar estimates as above, we also need to control $\|\partial_t^{n-1}\nabla^2\theta\|_0^2$ and $\|\partial_t^{n-1}\nabla\theta\|_0^2$ separately to obtaining the estimate of $\sum_{r=0}^{n}\|\partial_t^r\theta\|_{2n-2r}^2$.

7. To obtain the decay of $\mathcal{E}_{N+2}$, we must absorb all of terms on the right hand of inequality. However, the existence of large external forces makes the estimation of horizontal derivatives of $\mathbb{X}$ is particularly difficult, since the appearance of the terms V_i^p $(i=1,2,3)$ in (2.34). To overcome this difficulty, we need the Poincaré's inequality in Lemma 1.1 when δ is suitable small.

Remark 1.1 *Suppose the external force is small or disappeared, then the controlling of the term V_i^p $(i=1,2,3)$ would no longer be a difficulty. At this moment, the distance between two layers needn't be small. Then, we have the following Corollary.*

Corollary 1.1 *Let $N\in\mathbb{Z}_{\geq 4}$. Suppose that $u_0\in H^{4N}(\Omega)$, $\mathbb{X}_0\in H^{4N+1}(\Omega)$, and $f\in H^{4N+2}$ satisfy the appropriate compatibility conditions for the local well-posedness of (1.27), $\mathbb{X}_0=0$ on $\partial\Omega$, and δ is general constant not need small. There exists a constant $\varepsilon_0>0$ such that if*

$$\mathcal{E}_{2N}(0)+\|f\|_{4N+2}\leq\varepsilon_0, \tag{1.34}$$

then there exists a global unique solution $(\mathbb{X},\boldsymbol{u},\theta)$ solving (1.27) on $[0,\infty)$. The solution obeys the estimate

$$\mathcal{G}_{2N}(\infty)\lesssim\mathcal{E}_{2N}(0). \tag{1.35}$$

Remark 1.2 *If the external force and δ are all large, we can not derive Theorem 1.1 for the appearance of V_i^p $(i = 1, 2, 3)$. We also can not establish the global well-posedness of this MHD system if $\bar{b}$ is a function of t or y. Our next step in this program is to solve these problems.*

The following results will be used in the proof of Theorem 1.1.

Lemma 1.1 *If $f|_{\partial\Omega} = 0$, it holds*

$$\|f\|_0^2 \leq \delta^2 \|\partial_3 f\|_0^2 \tag{1.36}$$

where $\Omega := \{(y_1, y_2, y_3)|(y_1, y_2) \in \mathbb{T}^2, 0 < y_3 < \delta\}$.

Proof. Since

$$\|f\|_0^2 = \int_{\mathbb{T}^2} \mathrm{d}y_h \int_0^\delta f^2 \mathrm{d}y_3, \tag{1.37}$$

we first have the following calculation:

$$|f(y_h, y_3)|^2 = \left| \int_0^{y_3} \partial_3 f(y_h, s) \mathrm{d}s \right|^2 \leq \left| \int_0^\delta |\partial_3 f(y_h, s)| \mathrm{d}s \right|^2 \leq \delta \int_0^\delta |\partial_3 f(y_h, s)|^2 \mathrm{d}s. \tag{1.38}$$

Inserting (1.38) into (1.37), one can derive

$$\|f\|_0^2 \leq \int_{\mathbb{T}^2} \mathrm{d}y_h \left(\delta^2 \int_0^\delta |\partial_3 f(y_h, s)|^2 \mathrm{d}s \right) \leq \delta^2 \|\partial_3 f\|_0^2. \tag{1.39}$$

□

In the next section, we will only prove the Theorem 1.1. First of all, the elementary energy-entropy estimates will be established in subsection 2.1. Secondly, the estimates of temporal derivatives will be obtained in subsection 2.2. Then, we will derive the estimates of horizontal derivatives of $\mathbb{X}$, $\boldsymbol{u}$, and θ in subsection 2.3. In subsection 2.4, by means of ODE regularity and Elliptic regularity, the estimates obtained above will be improved. We finish our proof of Theorem 1.1 in subsection 2.5.

2 Proof of Theorem 1.1.

In this section, we will give a complete proof of Theorem 1.1. We assume throughout the section that the solution obeys the estimate $\mathcal{G}_{2N} \leq \eta$ for sufficiently small $0 < \eta < \varepsilon_0$.

2.1 Elementary energy-entropy estimate.

First of all, we have the following elementary energy-entropy estimate.

Lemma 2.1 *For $n \geq 3$, it holds that*

$$\frac{d}{dt} S + \|\nabla(\theta, \boldsymbol{u})\|_0^2 \lesssim \sqrt{\mathcal{E}_n} \mathcal{D}_n, \tag{2.1}$$

where

$$\begin{aligned} S := \int \Big(&\frac{1}{2} \rho_s \boldsymbol{u}^2 + \frac{\lambda}{2} \bar{b}^2 \big(J^{-1} \partial_3 \mathbb{X} + e_3 (J^{-1} - 1) \big)^2 J \\ &+ c_v \rho_s (\theta - \log \theta - 1) + R \rho_s (J - \log J - 1) \Big). \end{aligned} \tag{2.2}$$

Proof. It follows from $\dfrac{D}{Dt}g = \partial_t g + u\cdot\nabla_A g$, $\partial_j(JA_{ij}) = 0$ and $J_t = J\mathrm{div}_A\boldsymbol{u}$ that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega g(x,t)\mathrm{d}x = \int_\Omega \partial_t g(x,t)\mathrm{d}x = \int_\Omega \frac{D}{Dt}g(y,t)J\mathrm{d}y - \int_\Omega Ju\cdot\nabla_A g\mathrm{d}y \\
&= \int_\Omega \frac{D}{Dt}\Big(gJ\Big) - \Big(gJA_{ij}\partial_j u_i + Ju_iA_{ij}\partial_j g\Big)\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega gJ\mathrm{d}y,
\end{aligned} \tag{2.3}$$

which also means that

$$\int_\Omega g(x,t)\mathrm{d}x = \int_\Omega g(y,t)J\mathrm{d}y. \tag{2.4}$$

Thanks to the above observations (2.3)-(2.4) and (1.8), $\rho = \rho_s J^{-1}$, $\boldsymbol{B} = \boldsymbol{H} - \bar{\boldsymbol{H}} = \bar{b}\big(J^{-1}\partial_3\mathbb{X} + e_3(J^{-1}-1)\big)$, and (1.15), one can derive

$$\frac{\mathrm{d}}{\mathrm{d}t}S + \kappa\int\left|\frac{\nabla_A\theta}{\theta}\right|^2 J + 2\mu\int\frac{|D_A(\boldsymbol{u})|^2}{\theta}J + \mu'\int\frac{|\mathrm{div}_A\boldsymbol{u}|^2}{\theta}J = 0. \tag{2.5}$$

By virtue of $\mathcal{G}_{2N}\le\eta$, one has $|\theta-1|\le\eta$, if $\eta\le 1/2$, then

$$\begin{aligned}
&\kappa\int\left|\frac{\nabla_A\theta}{\theta}\right|^2 J + 2\mu\int\frac{|D_A(\boldsymbol{u})|^2}{\theta}J + \mu'\int\frac{|\mathrm{div}_A\boldsymbol{u}|^2}{\theta}J \\
&= \kappa\int\left|\frac{\nabla_A\theta}{\theta}\right|^2(J-1) + 2\mu\int\frac{|D_A(\boldsymbol{u})|^2}{\theta}(J-1) + \mu'\int\frac{|\mathrm{div}_A\boldsymbol{u}|^2}{\theta}(J-1) \\
&\quad + \kappa\int\frac{(A_{ij}-\delta_{ij})\partial_j\theta(\nabla_A+\nabla)\theta}{\theta^2} + \mu\int\Big((\nabla_A-\nabla)\boldsymbol{u} + (\nabla_A-\nabla)\boldsymbol{u}^\top\Big)\frac{D_A(\boldsymbol{u})+D(\boldsymbol{u})}{\theta} \\
&\quad + \mu'\int\frac{(A_{ij}-\delta_{ij})\partial_j u_i(\mathrm{div}_A+\mathrm{div})\boldsymbol{u}}{\theta} + \kappa\int\left|\frac{\nabla\theta}{\theta}\right|^2 + 2\mu\int\frac{|D(\boldsymbol{u})|^2}{\theta} + \mu'\int\frac{|\mathrm{div}\boldsymbol{u}|^2}{\theta} \\
&\ge \frac{4}{9}\kappa\int|\nabla\theta|^2 + \frac{2}{3}\left(2\mu\int|D(\boldsymbol{u})|^2 + \mu'\int|\mathrm{div}\boldsymbol{u}|^2\right) - C\sqrt{\mathcal{E}_n}\mathcal{D}_n \\
&= \frac{4}{9}\kappa\int|\nabla\theta|^2 + \frac{2}{3}\left(\mu\int|\nabla\boldsymbol{u}|^2 + (\mu+\mu')\int|\mathrm{div}\boldsymbol{u}|^2\right) - C\sqrt{\mathcal{E}_n}\mathcal{D}_n.
\end{aligned} \tag{2.6}$$

Putting (2.6) into (2.5), we can conclude (2.1). $\square$

2.2 Temporal derivatives estimates.

For $n\in\mathbb{Z}_{\ge 3}$, we define the temporal energy by

$$\begin{aligned}
\bar{\mathcal{E}}_n^t := \sum_{j=1}^n\Bigg(&\int_\Omega\rho_s\Big[J^{-1}\Big((\partial_t^j\boldsymbol{u})^2 + c_v(\partial_t^j\theta)^2\Big) + R(\partial_t^j\mathrm{div}\mathbb{X})^2\Big] \\
&+\lambda\bar{b}^2\|\partial_t^j\partial_3\mathbb{X}_h\|_0^2 + \lambda\bar{b}^2\|\partial_t^j\mathrm{div}_h\mathbb{X}_h\|_0^2 + \kappa\|\nabla\partial_t^{j-1}\theta\|_0^2\Bigg)
\end{aligned} \tag{2.7}$$

and the temporal dissipation by

$$\bar{\mathcal{D}}_n^t := \sum_{j=1}^n\|\partial_t^j(\boldsymbol{u},\theta)\|_1^2. \tag{2.8}$$

Lemma 2.2 *For $n \geq 3$, it holds that*

$$\frac{d}{dt}\bar{\mathcal{E}}_n^t + \bar{\mathcal{D}}_n^t \lesssim \sqrt{\mathcal{E}_n}\mathcal{D}_n + \|\nabla \boldsymbol{u}\|_0^2. \tag{2.9}$$

Proof. Applying the operator ∂_t^r to $(1.27)_2$ for $1 \leq r \leq n$, one has

$$\begin{aligned}
&\rho_s J^{-1}\partial_t\partial_t^r u_i - \mu\Delta\partial_t^r u_i - (\mu+\mu')\partial_i \mathrm{div}\partial_t^r \boldsymbol{u} - R\rho_s\partial_i\mathrm{div}\partial_t^r\mathbb{X} + R\partial_i(\rho_s\partial_t^r\theta)\\
&= \lambda\bar{b}^2\Big(\partial_3^2\partial_t^r\mathbb{X}_i - \partial_3\mathrm{div}(\partial_t^r\mathbb{X})\delta_{i3} + \partial_i\mathrm{div}(\partial_t^r\mathbb{X}) - \partial_i\partial_3\partial_t^r\mathbb{X}_3\Big)\\
&\quad + II_1 + II_2 + \partial_t^r(F - I_1 + G),
\end{aligned} \tag{2.10}$$

where

$$II_1 = (\mu+\mu')(A_{ij}A_{kl} - \delta_{ij}\delta_{kl})\partial_j\partial_l\partial_t^r u_k + \mu(A_{jk}A_{jl} - \delta_{jk}\delta_{jl})\partial_k\partial_l\partial_t^r u_i \tag{2.11}$$

$$\begin{aligned}
II_2 = \sum_{m=1}^{r} C_r^m \Bigg(&\rho_s\partial_t^m J^{-1}\partial_t^{r-m}\partial_t u_i + (\mu+\mu')\partial_t^m(A_{ij}A_{kl} - \delta_{ij}\delta_{kl})\partial_j\partial_l\partial_t^{r-m}u_k\\
&+\mu\partial_t^m(A_{jk}A_{jl} - \delta_{jk}\delta_{jl})\partial_k\partial_l\partial_t^{r-m}u_i\Bigg),
\end{aligned} \tag{2.12}$$

and

$$\|II_2 + \partial_t^r(F - I_1 + G)\|_0^2 \lesssim \mathcal{E}_n\mathcal{D}_n. \tag{2.13}$$

In fact, all terms in the definitions of $II_2 + \partial_t^r(F - I_1 + G)$ are at least quadratic; each term can be written in XY, where X involves fewer derivative counts than Y. We may use the usual Sobolev embeddings along with the definitions of $\mathcal{E}_n$ and $\mathcal{D}_n$ to estimate $\|X\|_{L^\infty}^2 \lesssim \mathcal{E}_n$ and $\|Y\|_0^2 \lesssim \mathcal{D}_n$. Then $\|XY\|_0^2 \leq \|X\|_{L^\infty}^2\|Y\|_0^2 \lesssim \mathcal{E}_n\mathcal{D}_n$, and the estimate (2.13) follows.

Multiplying (2.10) by $\partial_t^r u_i$, by integration by parts and $J_t = J\mathrm{div}\boldsymbol{u}$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_s J^{-1}(\partial_t^r\boldsymbol{u})^2 + \mu\|\nabla\partial_t^r\boldsymbol{u}\|_0^2 + (\mu+\mu')\|\mathrm{div}\partial_t^r\boldsymbol{u}\|_0^2\\
&\qquad\int R\rho_s\mathrm{div}\partial_t^r\mathbb{X}\mathrm{div}\partial_t^r\boldsymbol{u} + \int R\partial_i(\rho_s\partial_t^r\theta)\partial_t^r u_i\\
&\quad + \int\lambda\bar{b}^2\Bigg(\partial_3\partial_t^r\mathbb{X}\partial_3\partial_t^r\boldsymbol{u} - \mathrm{div}\partial_t^r\mathbb{X}\partial_3\partial_t^r u_3\\
&\qquad + \mathrm{div}(\partial_t^r\mathbb{X})\mathrm{div}(\partial^r\boldsymbol{u}) - \partial_3\partial_t^r\mathbb{X}_3\mathrm{div}(\partial_t^r\boldsymbol{u})\Bigg)\\
&= -R\int\partial_t^r\boldsymbol{u}\cdot\nabla\rho_s\mathrm{div}\partial_t^{r-1}\boldsymbol{u} - \frac{1}{2}\int\rho_s J^{-1}\mathrm{div}\boldsymbol{u}(\partial_t^r\boldsymbol{u})^2 + \int II_1\partial_t^r\boldsymbol{u}\\
&\quad + \int\Big(II_2 + \partial_t^r(F - I_1 + G)\Big)\partial_t^r\boldsymbol{u}.
\end{aligned} \tag{2.14}$$

By $(1.27)_1$, we get

$$\int R\rho_s\mathrm{div}\partial_t^r\mathbb{X}\mathrm{div}\partial_t^r\boldsymbol{u} = \int R\rho_s\mathrm{div}\partial_t^r\mathbb{X}\mathrm{div}\partial_t^{r+1}\mathbb{X} = \frac{R}{2}\int\rho_s|\mathrm{div}(\partial_t^r\mathbb{X})|^2 \tag{2.15}$$

and

$$\begin{aligned}
&\int \Big(\partial_3\partial_t^r\mathbb{X}\partial_3\partial_t^r\boldsymbol{u}-\mathrm{div}\partial_t^r\mathbb{X}\partial_3\partial_t^r u_3+\mathrm{div}(\partial_t^r\mathbb{X})\mathrm{div}(\partial^r\boldsymbol{u})-\partial_3\partial_t^r\mathbb{X}_3\mathrm{div}(\partial_t^r\boldsymbol{u})\Big)\\
&\quad=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(|\partial_t^r\partial_3\mathbb{X}|^2-2\mathrm{div}(\partial_t^r\mathbb{X})\partial_3(\partial_t^r\mathbb{X}_3)+|\mathrm{div}(\partial_t^r\mathbb{X})|^2\Big)\\
&\quad=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(|\partial_t^r\partial_3\mathbb{X}_h|^2+|\partial_t^r\mathrm{div}_h\mathbb{X}_h|^2\Big). \qquad (2.16)
\end{aligned}$$

It follows from Poincaré's, Hölder's, and Young's inequalities, that

$$-R\int\partial_t^r\boldsymbol{u}\cdot\nabla\rho_s\mathrm{div}\partial_t^{r-1}\boldsymbol{u}\le C\|\nabla\partial_t^r\boldsymbol{u}\|_0\|\nabla\partial_t^{r-1}\boldsymbol{u}\|_0\le\varepsilon\|\nabla\partial_t^r\boldsymbol{u}\|_0^2+C_\varepsilon\|\nabla\partial_t^{r-1}\boldsymbol{u}\|_0^2. \quad (2.17)$$

By means of integration by parts, similar as (2.15) and (2.13), it follows

$$\begin{aligned}
\int II_1\partial_t^r\boldsymbol{u}=&-(\mu+\mu')\int\Bigg((\partial_jA_{ij}A_{kl}+A_{ij}\partial_jA_{kl})\partial_l\partial_t^r u_k\partial_t^r u_i\\
&\qquad+\Big((A_{ij}-\delta_{ij})A_{kl}+\delta_{ij}(A_{kl}-\delta_{kl})\Big)\partial_l\partial_t^r u_k\partial_j\partial_t^r u_i\Bigg)\\
&-\mu\int\Bigg((\partial_lA_{jk}A_{jl}+A_{jk}\partial_lA_{jl})\partial_k\partial_t^r u_i\partial_t^r u_i\\
&\qquad+\Big((A_{jk}-\delta_{jk})A_{jl}+\delta_{jk}(A_{jl}-\delta_{jl})\Big)\partial_k\partial_t^r u_i\partial_l\partial_t^r u_i\Bigg)\\
\lesssim&\|\nabla^2\mathbb{X}\nabla\partial_t^r\boldsymbol{u}\|_0\|\partial_t^r\boldsymbol{u}\|_0+\|\nabla\mathbb{X}\nabla\partial_t^r\boldsymbol{u}\|_0\|\nabla\partial_t^r\boldsymbol{u}\|_0\lesssim\sqrt{\mathcal{E}_n}\mathcal{D}_n. \qquad (2.18)
\end{aligned}$$

By Poincaré's inequality and (2.13), one has

$$-\frac{1}{2}\int\rho_sJ^{-1}\mathrm{div}\boldsymbol{u}(\partial_t^r\boldsymbol{u})^2+\int\Big(II_2+\partial_t^r(F-I_1+G)\Big)\partial_t^r\boldsymbol{u}\lesssim\sqrt{\mathcal{E}_n}\mathcal{D}_n. \quad (2.19)$$

By virtue of integration by part, we obtain

$$\int R\partial_i(\rho_s\partial_t^r\theta)\partial_t^r u_i=-R\int\rho_s\partial_t^r\theta\mathrm{div}\partial_t^r\boldsymbol{u}. \quad (2.20)$$

Putting (2.15)-(2.20) into (2.14), choosing suitable small $\varepsilon>0$, one has

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(\rho_sJ^{-1}(\partial_t^r\boldsymbol{u})^2+R\rho_s|\mathrm{div}\partial_t^r\mathbb{X}|^2+\lambda\bar{b}^2|\partial_t^r\partial_3\mathbb{X}_h|^2+\lambda\bar{b}^2|\partial_t^r\mathrm{div}_h\mathbb{X}_h|^2\Big)\\
&\qquad+\mu\|\nabla\partial_t^r\boldsymbol{u}\|_0^2+(\mu+\mu')\|\mathrm{div}\partial_t^r\boldsymbol{u}\|_0^2-R\int\rho_s\partial_t^r\theta\mathrm{div}\partial_t^r\boldsymbol{u}\\
&\lesssim\sqrt{\mathcal{E}_n}\mathcal{D}_n+\|\nabla\partial_t^{r-1}\boldsymbol{u}\|_0^2. \qquad (2.21)
\end{aligned}$$

Applying ∂_t^r to $(1.27)_3$, $1\le r\le n$, one obtains

$$c_v\rho_sJ^{-1}\partial_t\partial_t^r\theta-\kappa\Delta\partial_t^r\theta+R\rho_s\mathrm{div}\partial_t^r\boldsymbol{u}=\partial_t^r\Big(E-I_2\Big)+III_1+III_2, \quad (2.22)$$

where

$$III_1=\kappa(A_{jk}A_{jl}-\delta_{jk}\delta_{jl})\partial_k\partial_l\partial_t^r\theta,$$

$$III_2 = \sum_{m=1}^{r} C_r^m \Big(\partial_t^m (A_{jk}A_{jl} - \delta_{jk}\delta_{jl}) \partial_k \partial_l \partial_t^{r-m} \theta - \rho_s \partial_t^m J^{-1} \partial_t \partial_t^{r-m} \theta \Big). \tag{2.23}$$

Multiplying (2.22) by $\partial_t^r \theta$, one can derive

$$\begin{aligned} \frac{c_v}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int \rho_s J^{-1} |\partial_t^r \theta|^2 + \kappa \|\nabla \partial_t^r \theta\|_0^2 + \int R\rho_s \mathrm{div} \partial_t^r \boldsymbol{u} \partial_t^r \theta = \int III_1 \partial_t^r \theta \\ + \int \Big(\partial_t^r (E - I_2) + III_2 \Big) \partial_t^r \theta - \frac{1}{2} \int \rho_s J^{-1} \mathrm{div} |\partial_t^r \theta|^2. \end{aligned} \tag{2.24}$$

First of all, similar as (2.13), we can deduce that

$$\|\partial_t^r (E - I_2) + III_2\|_0^2 \lesssim \mathcal{E}_n \mathcal{D}_n. \tag{2.25}$$

By means of (2.25) and Hölder's inequality, it follows

$$\int \Big(\partial_t^r (E - I_2) + III_2 \Big) \partial_t^r \theta - \frac{1}{2} \int \rho_s J^{-1} \mathrm{div} |\partial_t^r \theta|^2 \lesssim \sqrt{\mathcal{E}_n} \mathcal{D}_n. \tag{2.26}$$

Similar as (2.18), one obtains

$$\int III_1 \partial_t^r \theta \lesssim \sqrt{\mathcal{E}_n} \mathcal{D}_n. \tag{2.27}$$

Hence, by virtue of (2.21) and (2.24)-(2.27), one can deduce

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int \Big(\rho_s J^{-1} (\partial_t^r \boldsymbol{u})^2 + c_v \rho_s J^{-1} |\partial_t^r \theta|^2 + R\rho_s |\mathrm{div} \partial_t^r \mathbb{X}|^2 + \lambda \bar{b}^2 |\partial_t^r \partial_3 \mathbb{X}_h|^2 + \lambda \bar{b}^2 |\partial_t^r \mathrm{div}_h \mathbb{X}_h|^2 \Big) \\ + \mu \|\nabla \partial_t^r \boldsymbol{u}\|_0^2 + (\mu + \mu') \|\mathrm{div} \partial_t^r \boldsymbol{u}\|_0^2 + \kappa \|\nabla \partial_t^r \theta\|_0^2 \lesssim \sqrt{\mathcal{E}_n} \mathcal{D}_n + \|\nabla \partial_t^{r-1} \boldsymbol{u}\|_0^2. \end{aligned} \tag{2.28}$$

Since the value of θ on the boundary of Ω is unknown, we cannot use the Poincaré's inequality to get the estimate of $\|\partial_t^r \theta\|_1^2$. Similar as (2.22), applying ∂^{r-1} to $(1.27)_3$, multiplied by $\partial_t^r \theta$, and integrating over Ω we can deduce that for $1 \leq r \leq n$

$$\begin{aligned} \frac{\kappa}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \partial_t^{r-1} \theta\|_0^2 + c_v \int \rho_s J^{-1} |\partial_t^r \theta|^2 + \int R\rho_s \mathrm{div} \partial_t^{r-1} \boldsymbol{u} \partial_t^r \theta = \int IV_1 \partial_t^r \theta \\ + \int \Big(\partial_t^{r-1} (E - I_2) + IV_2 \Big) \partial_t^r \theta, \end{aligned} \tag{2.29}$$

where

$$\begin{aligned} IV_1 &= \kappa (A_{jk}A_{jl} - \delta_{jk}\delta_{jl}) \partial_k \partial_l \partial_t^{r-1} \theta, \\ IV_2 &= \sum_{m=1}^{r-1} C_{r-1}^m \Big(\partial_t^m (A_{jk}A_{jl} - \delta_{jk}\delta_{jl}) \partial_k \partial_l \partial_t^{r-1-m} \theta - \rho_s \partial_t^m J^{-1} \partial_t \partial_t^{r-1-m} \theta \Big). \end{aligned} \tag{2.30}$$

Similar as (2.25)-(2.27) and (2.6), it follows

$$\begin{aligned} \frac{\kappa}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \partial_t^{r-1} \theta\|_0^2 + c_v \underline{\rho} \|\partial_t^r \theta\|_0^2 &\lesssim \sqrt{\mathcal{E}_n} \mathcal{D}_n + \int |\mathrm{div} \partial_t^{r-1} \boldsymbol{u} \partial_t^r \theta| \\ &\lesssim \sqrt{\mathcal{E}_n} \mathcal{D}_n + C_\varepsilon \|\nabla \partial_t^{r-1} \boldsymbol{u}\|_0^2 + \varepsilon \|\partial_t^r \theta\|_0^2. \end{aligned} \tag{2.31}$$

Choosing suitable small ε, combined with (2.21), (2.28), and (2.31), the proof of this lemma can be completed by a simple induction. $\square$

2.3 Horizontal derivatives estimates.

Next, we define that

$$\bar{\mathcal{E}}_n^{h,x} := \mu\|\nabla\mathbb{X}\|_{0,2n}^2 + (\mu+\mu')\|\mathrm{div}\mathbb{X}\|_{0,2n}^2$$

and

$$\bar{\mathcal{D}}_n^{h,x} := \|\mathrm{div}\mathbb{X}\|_{0,2n}^2 + \|\partial_3\mathbb{X}\|_{0,2n}^2 + \|\mathbb{X}\|_{0,2n}^2.$$

Lemma 2.3 *For $n \geq 3$, it holds that*

$$\begin{aligned}&\frac{d}{dt}\left(\bar{\mathcal{E}}_{2N}^{h,x} + 2\sum_{\substack{\alpha\in\mathbb{N}^2\\|\alpha|\leq 4N}}\int\rho_s\partial^\alpha\boldsymbol{u}\cdot\partial^\alpha\mathbb{X}\right) + \bar{\mathcal{D}}_{2N}^{h,x}\\
&\lesssim \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N} + \sqrt{\mathcal{E}_{N+2}}\mathcal{E}_{2N} + \varepsilon^{-(4N-1)}(\|\nabla\boldsymbol{u}\|_0^2 + \|\nabla\theta\|_0^2)\\
&\quad+\varepsilon\Big(\|\nabla\partial^\alpha\boldsymbol{u}\|_0^2 + \|\boldsymbol{u}_t\|_{0,2n-1}^2 + \|\nabla\theta\|_{0,2n}^2\Big)\end{aligned}\tag{2.32}$$

and

$$\frac{d}{dt}\left(\bar{\mathcal{E}}_{N+2}^{h,x} + 2\sum_{\substack{\alpha\in\mathbb{N}^2\\|\alpha|\leq 2(N+2)}}\int\rho_s\partial^\alpha\boldsymbol{u}\cdot\partial^\alpha\mathbb{X}\right) + \bar{\mathcal{D}}_{N+2}^{h,x}\tag{2.33}$$

$$\lesssim \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{N+2} + \varepsilon^{-(2N+3)}(\|\nabla\boldsymbol{u}\|_0^2 + \|\nabla\theta\|_0^2) + \varepsilon\Big(\|\nabla\partial^\alpha\boldsymbol{u}\|_0^2 + \|\boldsymbol{u}_t\|_{0,2n-1}^2 + \|\nabla\theta\|_{0,2n}^2\Big)$$

for suitable small $0<\varepsilon<1$.

Proof. Applying the operator ∂^α to $(1.27)_2$, $0\leq|\alpha|\leq 2n$, multiplying by $\partial^\alpha\mathbb{X}$, one can derive

$$\begin{aligned}&\underbrace{\int\rho_s\partial_t\partial^\alpha\boldsymbol{u}\partial^\alpha\mathbb{X}}_{V_1^u} + \underbrace{\sum_{|l|=1}^{\alpha}C_\alpha^l\int\partial^l\rho_s\partial_t\partial^{\alpha-l}\boldsymbol{u}\partial^\alpha\mathbb{X}}_{V_2^u} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(\mu|\nabla\partial^\alpha\mathbb{X}|^2 + (\mu+\mu')|\mathrm{div}\partial^\alpha\mathbb{X}|^2\Big)\\
&+R\int\rho_s|\mathrm{div}\partial^\alpha\mathbb{X}|^2 + \underbrace{R\int\nabla\rho_s\mathrm{div}\partial^\alpha\mathbb{X}\partial^\alpha\mathbb{X}}_{V_1^p} - \underbrace{R\sum_{|l|=1}^{\alpha}C_\alpha^l\int\partial^l\rho_s\nabla\mathrm{div}\partial^{\alpha-l}\mathbb{X}\partial^\alpha\mathbb{X}}_{V_2^p}\\
&+\underbrace{R\int\partial^\alpha\nabla(\rho_s(\theta-1))\partial^\alpha\mathbb{X}}_{V_3^p} + \kappa\bar{b}^2\int\Big(|\partial^\alpha\partial_3\mathbb{X}_h|^2 + |\partial^\alpha\mathrm{div}_h\mathbb{X}_h|^2\Big) = \int\partial^\alpha(F+G)\partial^\alpha\mathbb{X}.\end{aligned}\tag{2.34}$$

By means of $(1.27)_1$, Sobolev's, Young's and Poincaré's inequality, we have

$$\begin{aligned}-V_1^u &= -\frac{\mathrm{d}}{\mathrm{d}t}\int\rho_s\partial^\alpha\boldsymbol{u}\partial^\alpha\mathbb{X} + \int\rho_s|\partial^\alpha\boldsymbol{u}|^2\\
&\lesssim -\frac{\mathrm{d}}{\mathrm{d}t}\int\rho_s\partial^\alpha\boldsymbol{u}\partial^\alpha\mathbb{X} + \varepsilon\|\nabla\partial^\alpha\boldsymbol{u}\|_0^2 + C_\varepsilon\|\nabla\boldsymbol{u}\|_0^2.\end{aligned}\tag{2.35}$$

Because that the domain studied in this paper is a thin domain, we have the following estimate

$$V_1^p \leq \varepsilon\|\mathrm{div}\mathbb{X}\|_{0,2n}^2 + C_\varepsilon\|\partial^\alpha\mathbb{X}\|_0^2 \leq \varepsilon\|\mathrm{div}\mathbb{X}\|_{0,2n}^2 + \delta^2C_\varepsilon\|\partial_3\partial^\alpha\mathbb{X}\|_0^2\tag{2.36}$$

where we have used the Poincaré inequality Lemma 1.1.

Next, the estimates of V_2^u, V_2^p, and V_3^p are divide into the following two cases:

Case 1. $\alpha > 0$. By Cauchy-Schwarz's inequality and Poincaré inequality Lemma 1.1, we have

$$
\begin{aligned}
-V_2^u + V_2^p =& -\sum_{|l|=1}^{\alpha} C_\alpha^l \int \partial^l \rho_s \partial_t \partial^{\alpha-l} \boldsymbol{u} \partial^\alpha \mathbb{X} + R \sum_{|l|=1}^{\alpha} C_\alpha^l \int \partial^l \rho_s \nabla_h \mathrm{div} \partial^{\alpha-l} \mathbb{X} \partial^\alpha \mathbb{X}_h \\
& - R \sum_{|l|=1}^{\alpha} C_\alpha^l \int \partial_3 \partial^l \rho_s \mathrm{div} \partial^{\alpha-l} \mathbb{X} \partial^\alpha \mathbb{X}_3 - R \sum_{|l|=1}^{\alpha} C_\alpha^l \int \partial^l \rho_s \mathrm{div} \partial^{\alpha-l} \mathbb{X} \partial^\alpha \partial_3 \mathbb{X}_3 \\
& \leq \varepsilon \|\boldsymbol{u}_t\|_{0,2n-1}^2 + C_\varepsilon \|\partial^\alpha \mathbb{X}\|_0^2 + \varepsilon \|\mathrm{div} \mathbb{X}\|_{0,2n}^2 + \varepsilon \|\partial_3 \partial^\alpha \mathbb{X}\|_0^2 + C_\varepsilon \|\mathrm{div} \mathbb{X}\|_{0,2n-1} \\
& \leq \varepsilon \|\boldsymbol{u}_t\|_{0,2n-1}^2 + (\varepsilon + \delta^2 C_\varepsilon) \|\partial_3 \partial^\alpha \mathbb{X}\|_0^2 + \varepsilon \|\mathrm{div} \mathbb{X}\|_{0,2n}^2 + C_\varepsilon \|\mathrm{div} \mathbb{X}\|_0.
\end{aligned} \tag{2.37}
$$

Thanks to integration by parts, Sobolev's, Poincaré's, and Young's inequalities, one can derive

$$
\begin{aligned}
|V_3^p| =& R \left| \int \partial^\alpha \rho_s (\theta - 1) \mathrm{div} \partial^\alpha \mathbb{X} + \sum_{|l|=1}^{\alpha} C_\alpha^l \int \partial^{\alpha-l} \rho_s \partial^l (\theta - 1) \mathrm{div} \partial^\alpha \mathbb{X} \right| \\
& \leq C \|\theta - 1\|_{L^6} \|\mathrm{div} \partial^\alpha \mathbb{X}\|_0 + C \|\nabla \theta\|_{0,2n-1} \|\mathrm{div} \partial^\alpha \mathbb{X}\|_0 \\
& \leq \varepsilon \|\mathrm{div} \partial^\alpha \mathbb{X}\|_0^2 + C_\varepsilon \|\nabla \theta\|_0^2 + C_\varepsilon \|\nabla \theta\|_{0,2n-1}^2 + C_\varepsilon \mathcal{E}_n \mathcal{D}_n \\
& \leq \varepsilon \|\mathrm{div} \partial^\alpha \mathbb{X}\|_0^2 + C_\varepsilon \|\nabla \theta\|_0^2 + \varepsilon \|\nabla \theta\|_{0,2n}^2 + C_\varepsilon \mathcal{E}_n \mathcal{D}_n,
\end{aligned} \tag{2.38}
$$

where the following fact has been used:

$$
\|\theta - 1\|_0^2 \lesssim \|\nabla \theta\|_0^2 + \mathcal{E}_n \mathcal{D}_n. \tag{2.39}
$$

Infect, by means of the conservation of energy, we have

$$
\begin{aligned}
\left| \bar{\theta} - 1 \right| =& \frac{1}{|\Omega|} \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\eta} \left[\int_\Omega \theta + \eta \frac{\boldsymbol{u}^2 + (\boldsymbol{H} - \bar{\boldsymbol{H}})^2}{2c_v} \mathrm{d}x \right] \mathrm{d}\eta \right| \\
=& \frac{1}{|\Omega|} \left| \int_\Omega \frac{\boldsymbol{u}^2 + (\boldsymbol{H} - \bar{\boldsymbol{H}})^2}{2c_v} \mathrm{d}x \right| \lesssim \sup_{x \in \Omega} |\boldsymbol{u}^2 + (\boldsymbol{H} - \bar{\boldsymbol{H}})^2|,
\end{aligned} \tag{2.40}
$$

where we have used the definition:

$$
\bar{\theta} = \frac{1}{|\Omega|} \int_\Omega \theta \mathrm{d}x.
$$

It follows from (2.4), (2.40), the boundary condition, Poincaré's inequality, and $(1.20)_2$ that

$$
\begin{aligned}
\|\theta - 1\|_0^2 &\lesssim \int_\Omega (\theta - \bar{\theta})^2 + (\bar{\theta} - 1)^2 dx \\
&\lesssim \int_\Omega |\nabla \theta|^2 dx + \sup_{y \in \Omega} |\boldsymbol{u}^4 + (\boldsymbol{H} - \bar{\boldsymbol{H}})^4| \\
&\lesssim \|\nabla \theta\|_0^2 + \mathcal{E}_n \mathcal{D}_n.
\end{aligned} \tag{2.41}
$$

Case 2. $\alpha = 0$. In this case, the terms V_2^u and V_2^p in (2.34) will disappear, that is

$$
V_2^u = V_2^p = 0. \tag{2.42}
$$

By $f \in H^{4N+2}$, Sobolev's and Poincaré's inequalities and (2.39), one has

$$
\begin{aligned}
-V_3^p &= -R\int \nabla\rho_s(\theta-1)\mathbb{X} - R\int \rho_s\nabla\theta\mathbb{X} \\
&\le C(\|\theta-1\|_{L^6} + \|\nabla\theta\|_0)\|\mathbb{X}\|_0 \\
&\lesssim \varepsilon\|\partial_3\mathbb{X}\|_0^2 + C_\varepsilon\|\nabla\theta\|_0^2 + C_\varepsilon\mathcal{E}_n\mathcal{D}_n.
\end{aligned}
\tag{2.43}
$$

For the right hand side of (2.34), we first consider the case $n = 2N$. If $|\alpha| \le 4N-1$, then we to have

$$
\int \partial^\alpha(F+G)\partial^\alpha\mathbb{X} \le \|\partial^\alpha\mathbb{X}\|_0\|G+F\|_{4N-1} \le \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}}, \tag{2.44}
$$

where we have used

$$
\|\bar{\nabla}_0^{2n-1}(F,G)\|_0^2 \lesssim \mathcal{E}_n\mathcal{D}_n. \tag{2.45}
$$

In fact, all terms in definitions of F and G are at least quadratic. We apply these space-time differential operators to F and G, then expand using the Leibniz rule; each product in the resulting sum is also at least quadratic. We then write each term in the form XY, where X involves fewer derivative counts than Y. Then the estimate (2.45) follows similarly as (2.13) with a slight modification when $X = \nabla^{2n+1}\mathbb{X}$; in such cases, we estimate $\|\nabla^{2n+1}\mathbb{X}\|_0^2 \lesssim \mathcal{E}_n$ and $\|Y\|_{L^\infty}^2 \lesssim \mathcal{D}_n$. Moreover, it follows more easily that

$$
\|\bar{\nabla}_0^{2n-2}(F,G,E)\|_0^2 \lesssim (\mathcal{E}_n)^2, \tag{2.46}
$$

$$
\|\bar{\nabla}_0^{2n-1}E\|_0^2 \lesssim \mathcal{E}_n\mathcal{D}_n. \tag{2.47}
$$

If $|\alpha| = 4N$, we may write $\alpha = \gamma + (\alpha-\gamma)$ for some $\gamma \in \mathbb{N}^2$ with $|\gamma| = 1$. We can then integrate by parts and use (2.45) to have

$$
\begin{aligned}
\int \partial^\alpha(G+F)\partial^\alpha\mathbb{X} &= -\int \partial^{\alpha-\gamma}(F+G)\partial^{\alpha+\gamma}\mathbb{X} \le \|F+G\|_{4N-1}\|\mathbb{X}\|_{4N+1} \\
&\lesssim \left(\mathcal{D}_{2N} + \sqrt{\mathcal{E}_{N+2}\mathcal{E}_{2N}}\right)\sqrt{\mathcal{E}_{2N}},
\end{aligned}
\tag{2.48}
$$

where we have used

$$
\|F+G\|_{4N-1}^2 \lesssim (\mathcal{D}_{2N})^2 + \mathcal{E}_{N+2}\mathcal{E}_{2N}. \tag{2.49}
$$

In fact, we again write each term of $\partial^\alpha(F+G)$ in the form XY, where X involves fewer derivative counts than Y. To derive (2.48), we estimate both $\|X\|_0^2 \lesssim \mathcal{D}_{2N}$ and $\|Y\|_{L^\infty}^2 \lesssim \mathcal{D}_{2N}$ except the cases when $X = \nabla^{2n+1}\mathbb{X}$; in such cases, we estimate $\|\nabla^{2n+1}\mathbb{X}\|_0^2 \lesssim \mathcal{E}_{2N}$ and $\|Y\|_{L^\infty}^2 \lesssim \mathcal{E}_{N+2}$, then (2.48) follows. Moreover, by estimating $\|X\|_{L^\infty}^2 \lesssim \mathcal{E}_{2N}$ and $\|Y\|_0^2 \lesssim \mathcal{D}_{N+2}$, it follows

$$
\|F+G\|_{2(N+2)}^2 \lesssim \mathcal{E}_{2N}\mathcal{D}_{N+2}. \tag{2.50}
$$

Now for the case $n = N+2$, we use (2.50) to estimate

$$
\int \partial^\alpha(F+G)\partial^\alpha\mathbb{X} \le \|F+G\|_{2(N+2)}\|\partial^\alpha\mathbb{X}\|_0 \lesssim \sqrt{\mathcal{E}_{2N}\mathcal{D}_{N+2}}\sqrt{\mathcal{D}_{N+2}}. \tag{2.51}
$$

Hence, choosing suitable small $\delta > 0$, summing up such α, and using Poincaré's inequality, we can conclude (2.32) and (2.33). □

In the following, we define

$$\bar{\mathcal{E}}_n^{h,u} := \sum_{|\alpha|=0}^{2n} \int \left(\rho_s |\partial^\alpha \boldsymbol{u}|^2 + R\rho_s |\partial^\alpha \mathrm{div}\mathbb{X}|^2 + \lambda \bar{b}^2 |\partial^\alpha \partial_3 \mathbb{X}_h|^2 + \lambda \bar{b}^2 |\partial^\alpha \mathrm{div}_h \mathbb{X}_h|^2 \right),$$

and

$$\bar{\mathcal{D}}_n^{h,u} := \sum_{|\alpha|=1}^{2n} \|\nabla \partial^\alpha \boldsymbol{u}\|_0^2.$$

Lemma 2.4 *For $n \geq 3$, it holds that*

$$\begin{aligned}
&\frac{d}{dt}\left(\bar{\mathcal{E}}_n^{h,u} + 2R \int \partial^\alpha \nabla \rho_s (\theta - 1) \partial^\alpha \mathbb{X} \right) + \bar{\mathcal{D}}_n^{h,u} \\
&\lesssim \sqrt{\mathcal{E}_n}\mathcal{D}_n + \varepsilon \|\nabla\theta\|_{0,2n}^2 + \varepsilon \bar{\mathcal{D}}_n^{h,x} + \varepsilon \|\boldsymbol{u}_t\|_{0,2n-1}^2 \\
&\quad + C_\varepsilon \|\nabla \boldsymbol{u}\|_0^2 + C_\varepsilon \bar{\mathcal{D}}_n^t + C_\varepsilon \bar{\mathcal{D}}_0^{h,x}.
\end{aligned} \tag{2.52}$$

for $\alpha \in \mathbb{N}^2$ and suitable small $0 < \varepsilon < 1$.

Proof. Applying the operator ∂^α, $\alpha \in \mathbb{N}^2$, and $1 \leq |\alpha| \leq 2n$ to $(1.27)_2$, and doing inner product with $\partial^\alpha \boldsymbol{u}$ over Ω, similar as (2.15) and (2.13), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\rho_s |\partial^\alpha \boldsymbol{u}|^2 + R\rho_s |\partial^\alpha \mathrm{div}\mathbb{X}|^2 + \lambda \bar{b}^2 |\partial^\alpha \partial_3 \mathbb{X}_h|^2 + \lambda \bar{b}^2 |\partial^\alpha \mathrm{div}_h \mathbb{X}_h|^2 \right) \\
&\qquad + \underbrace{R\int \nabla \rho_s \partial^\alpha \mathrm{div}\mathbb{X} \partial^\alpha \boldsymbol{u}}_{V_1^{h,p}} + R\int \partial^\alpha \nabla(\rho_s(\theta-1))\partial^\alpha \boldsymbol{u} + \mu \|\nabla \partial^\alpha \boldsymbol{u}\|_0^2 \\
&\qquad + (\mu+\mu')\|\mathrm{div}\partial^\alpha \boldsymbol{u}\|_0^2 + \underbrace{\sum_{|l|=1}^{\alpha} \int C_\alpha^l \partial^l \rho_s \partial_t \partial^{\alpha-l}\boldsymbol{u} \partial^\alpha \boldsymbol{u}}_{V^{h,u}} + \underbrace{\sum_{|l|=1}^{\alpha} \int C_\alpha^l R \partial^l \rho_s \nabla \mathrm{div} \partial^{\alpha-l}\mathbb{X} \partial^\alpha \boldsymbol{u}}_{V_2^{h,p}} \\
&= \int \partial^\alpha (F+G) \partial^\alpha \boldsymbol{u} = -\int \partial^{\alpha-\gamma}(F+G)\partial^{\alpha+\gamma}\boldsymbol{u} \lesssim \sqrt{\mathcal{E}_n}\mathcal{D}_n,
\end{aligned} \tag{2.53}$$

where $l, \gamma \in \mathbb{N}^2$ and $|\gamma| = 1$. Thanks to $(1.27)_1$, Sobolev's and Young's inequalities, we can derive that

$$\begin{aligned}
-R\int \partial^\alpha \nabla(\rho_s(\theta-1))\partial^\alpha \boldsymbol{u} &\lesssim R\int \partial^\alpha \nabla \rho_s (\theta-1)\partial^\alpha \boldsymbol{u} + \varepsilon \|\nabla\theta\|_{0,2n}^2 + C_\varepsilon \|\partial^\alpha \boldsymbol{u}\|_0^2 \\
&\lesssim -R\frac{\mathrm{d}}{\mathrm{d}t}\int \partial^\alpha \nabla \rho_s (\theta-1)\partial^\alpha \mathbb{X} + R\int \partial^\alpha \nabla \rho_s \theta_t \partial^\alpha \mathbb{X} \\
&\qquad + \varepsilon \|\nabla(\theta, \boldsymbol{u})\|_{0,2n}^2 + C_\varepsilon \|\nabla \boldsymbol{u}\|_0^2 \\
&\lesssim -R\frac{\mathrm{d}}{\mathrm{d}t}\int \partial^\alpha \nabla \rho_s (\theta-1)\partial^\alpha \mathbb{X} + \varepsilon \|\nabla(\theta, \boldsymbol{u})\|_{0,2n}^2 + \varepsilon \|\mathbb{X}\|_{0,2n}^2 \\
&\qquad + C_\varepsilon \|(\nabla \boldsymbol{u}, \theta_t)\|_0^2.
\end{aligned} \tag{2.54}$$

By virtue of Sobolev's, Hölder's and Young's inequalities and integration by parts, one has

$$-V^{h,u} \lesssim \varepsilon \|\nabla \partial^\alpha \boldsymbol{u}\|_0^2 + \varepsilon \|\partial_t \partial^{\alpha-\gamma}\boldsymbol{u}\|_0^2 + C_\varepsilon \|\nabla \boldsymbol{u}\|_0^2, \tag{2.55}$$

$$-V_1^{h,p} \lesssim C_\varepsilon \|\partial^\alpha \boldsymbol{u}\|_0^2 + \varepsilon \|\mathrm{div}\mathbb{X}\|_{0,2n}^2 \lesssim \varepsilon \|\nabla \partial^\alpha \boldsymbol{u}\|_0^2 + C_\varepsilon \|\nabla \boldsymbol{u}\|_0^2 + \varepsilon \|\mathrm{div}\mathbb{X}\|_{0,2n}^2, \tag{2.56}$$

$$\begin{aligned} -V_2^{h,p} &= \sum_{|l|=1}^{\alpha} \int C_\alpha^l R \nabla \partial^l \rho_s \mathrm{div} \partial^{\alpha-l} \mathbb{X} \partial^\alpha \boldsymbol{u} + \sum_{|l|=1}^{\alpha} \int C_\alpha^l R \partial^l \rho_s \mathrm{div} \partial^{\alpha-l} \mathbb{X} \partial^\alpha \mathrm{div} \boldsymbol{u} \\ &\lesssim \varepsilon \|\nabla \boldsymbol{u}\|_{0,2n}^2 + C_\varepsilon \|\mathrm{div}\mathbb{X}\|_{0,2n-1}^2 \lesssim \varepsilon \|\nabla \boldsymbol{u}\|_{0,2n}^2 + \varepsilon \|\mathrm{div}\mathbb{X}\|_{0,2n-1}^2 + C_\varepsilon \|\mathrm{div}\mathbb{X}\|_0^2. \end{aligned} \tag{2.57}$$

Putting (2.54)-(2.57) into (2.53) and summing up from $\alpha = 1$ to $\alpha = 2n$, we can find

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \sum_{|\alpha|=1}^{2n} \int \bigg(\rho_s |\partial^\alpha \boldsymbol{u}|^2 + R\rho_s |\partial^\alpha \mathrm{div}\mathbb{X}|^2 + 2R \partial^\alpha \nabla \rho_s (\theta - 1) \partial^\alpha \mathbb{X} \\ &\qquad + \lambda \bar{b}^2 |\partial^\alpha \partial_3 \mathbb{X}_h|^2 + \lambda \bar{b}^2 |\partial^\alpha \mathrm{div}_h \mathbb{X}_h|^2 \bigg) + \sum_{|\alpha|=1}^{2n} \|\nabla \partial^\alpha \boldsymbol{u}\|_0^2 \\ &\lesssim \sqrt{\mathcal{E}_n} \mathcal{D}_n + \varepsilon \|(\nabla\theta, \mathbb{X}, \mathrm{div}\mathbb{X})\|_{0,2n}^2 + \varepsilon \|\boldsymbol{u}_t\|_{0,2n-1}^2 + C_\varepsilon \|\nabla(\theta_t, \boldsymbol{u})\|_0^2 + C_\varepsilon \|\mathrm{div}\mathbb{X}\|_0^2. \end{aligned} \tag{2.58}$$

Hence, (2.52) is verified.

□

Next, we define that

$$\bar{\mathcal{E}}_n^{h,\theta} := \sum_{|\alpha|=1}^{2n} \int \rho_s |\partial^\alpha \theta|^2$$

and

$$\bar{\mathcal{D}}_n^{h,\theta} := \sum_{|\alpha|=1}^{2n} \|\nabla \partial^\alpha \theta\|_0^2$$

Lemma 2.5 *For $n \geq 3$, it holds that*

$$\frac{d}{dt} \bar{\mathcal{E}}_n^{h,\theta} + \bar{\mathcal{D}}_n^{h,\theta} \lesssim \sqrt{\mathcal{E}_n} \mathcal{D}_n + \varepsilon \bar{\mathcal{D}}_n^{h,u} + C_\varepsilon \|\nabla(\boldsymbol{u}, \theta)\|_0^2 + \varepsilon \|\theta_t\|_{0,2n-1}^2. \tag{2.59}$$

Proof. Applying ∂^α, $\alpha \in \mathbb{N}^2$ and $1 \leq \alpha \leq 2n$ to $(1.27)_3$, multiplying by $\partial^\alpha \theta$, and integrating by parts over Ω, similar as (2.15) and (2.13), one has

$$\begin{aligned} &\frac{c_v}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int \rho_s |\partial^\alpha \theta|^2 + \kappa \|\nabla \partial^\alpha \theta\|_0^2 + R \int \partial^\alpha (\rho_s \mathrm{div} \boldsymbol{u}) \partial^\alpha \theta \\ &\qquad + \sum_{l=1}^{\alpha} \int C_\alpha^l \partial^l \rho_s \partial^{\alpha-l} \theta_t \partial^\alpha \theta = - \int \partial^{\alpha-1} E \partial^{\alpha+1} \theta \lesssim \sqrt{\mathcal{E}_n} \mathcal{D}_n. \end{aligned} \tag{2.60}$$

By integration by parts, Sobolev's and Young's inequalities, we have

$$\begin{aligned} -R \int \partial^\alpha (\rho_s \mathrm{div} \boldsymbol{u}) \partial^\alpha \theta &= R \int \partial^{\alpha-\gamma} (\rho_s \mathrm{div} \boldsymbol{u}) \partial^{\alpha+\gamma} \theta \lesssim \varepsilon \|\nabla \partial^\alpha \theta\|_0^2 + C_\varepsilon \|\nabla \boldsymbol{u}\|_{0,2n-1}^2 \\ &\lesssim \varepsilon \|\nabla \partial^\alpha \theta\|_0^2 + \varepsilon \|\nabla \boldsymbol{u}\|_{0,2n}^2 + C_\varepsilon \|\nabla \boldsymbol{u}\|_0^2. \end{aligned} \tag{2.61}$$

$$\begin{aligned} \sum_{l=1}^{\alpha} \int C_\alpha^l \partial^l \rho_s \partial^{\alpha-l} \theta_t \partial^\alpha \theta &\lesssim C_\varepsilon \|\partial^\alpha \theta\|_0^2 + \varepsilon \|\theta_t\|_{0,2n-1}^2 \\ &\lesssim \varepsilon \|\nabla \partial^\alpha \theta\|_0^2 + C_\varepsilon \|\nabla \theta\|_0^2 + \varepsilon \|\theta_t\|_{0,2n-1}^2. \end{aligned} \tag{2.62}$$

Hence, putting (2.61) into (2.60), and sum them up from $|\alpha| = 1$ to $2n$, we can deduce (2.59).

□

To control the term $\|\boldsymbol{u}_t\|_{0,2n-1}$ in Lemma 2.3-2.5, we need the following estimate of horizontal derivatives of $\boldsymbol{u}_t$.

Lemma 2.6 *For $n \geq 3$, it holds that*

$$\frac{d}{dt}\bar{\mathcal{E}}_n^{h,u_t} + \bar{\mathcal{D}}_n^{h,u_t} \lesssim \sqrt{\mathcal{E}_n}\mathcal{D}_n + \varepsilon\bar{\mathcal{D}}_n^{h,u} + \varepsilon\bar{\mathcal{D}}_n^{t} + C_\varepsilon(\bar{\mathcal{D}}_n^{h,x} + \bar{\mathcal{D}}^{s} + \bar{\mathcal{D}}_n^{h,\theta} + \bar{\mathcal{D}}_n^{t}). \tag{2.63}$$

Proof. Applying ∂^α, $\alpha \in \mathbb{N}^2$, and $0 \leq |\alpha| \leq 2n-1$, to $(1.27)_2$, one can derive

$$\begin{aligned}
&\rho_s\partial_t\partial^\alpha \boldsymbol{u} + \sum_{|l|=1}^{\alpha} C_\alpha^l \partial^l\rho_s\partial_t\partial^{\alpha-l}\boldsymbol{u} - \mu\Delta\partial^\alpha\boldsymbol{u} - (\mu+\mu')\nabla\mathrm{div}\partial^\alpha\boldsymbol{u} \\
&- R\rho_s\nabla\mathrm{div}\partial^\alpha\mathbb{X} - R\sum_{|l|=1}^{\alpha} C_\alpha^l\partial^l\rho_s\nabla\mathrm{div}\partial^{\alpha-l}\mathbb{X} + R\partial^\alpha\nabla(\rho_s(\theta-1)) \\
&\quad = \lambda\bar{b}^2\Big(\partial_3^2\partial^\alpha\mathbb{X} - \partial_3\mathrm{div}\partial^\alpha\mathbb{X}e_3 + \nabla\mathrm{div}\partial^\alpha\mathbb{X} - \nabla\partial_3\partial^\alpha\mathbb{X}_3\Big) + \partial^\alpha(F+G),
\end{aligned} \tag{2.64}$$

which multiplied by $\partial_t\partial^\alpha\boldsymbol{u}$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mu\|\nabla\partial^\alpha\boldsymbol{u}\|_0^2 + (\mu+\mu')\|\mathrm{div}\partial^\alpha\boldsymbol{u}\|_0^2\Big) + \|\sqrt{\rho_s}\partial^\alpha\boldsymbol{u}_t\|_0^2 + \underbrace{\sum_{|l|=1}^{\alpha} C_\alpha^l\int\partial^l\rho_s\partial_t\partial^{\alpha-l}\boldsymbol{u}\partial^\alpha\boldsymbol{u}_t}_{V_5} \\
&\quad \underbrace{+R\int\Big(\nabla\rho_s\mathrm{div}\partial^\alpha\mathbb{X} + \sum_{|l|=1}^{\alpha} C_\alpha^l\nabla\partial^l\rho_s\mathrm{div}\partial^{\alpha-l}\mathbb{X}\Big)\partial^\alpha\boldsymbol{u}_t}_{V_1} \\
&\quad \underbrace{+R\int\Big(\rho_s\mathrm{div}\partial^\alpha\mathbb{X} + \sum_{|l|=1}^{\alpha} C_\alpha^l\partial^l\rho_s\mathrm{div}\partial^{\alpha-l}\mathbb{X}\Big)\partial^\alpha\mathrm{div}\boldsymbol{u}_t}_{V_2} + \underbrace{\int R\partial^\alpha\nabla(\rho_s(\theta-1))\partial^\alpha\boldsymbol{u}_t}_{V_3} \\
&= \underbrace{\lambda\bar{b}^2\int\Big(\partial_3^2\partial^\alpha\mathbb{X} - \partial_3\mathrm{div}\partial^\alpha\mathbb{X}e_3 + \nabla\mathrm{div}\partial^\alpha\mathbb{X} - \nabla\partial_3\partial^\alpha\mathbb{X}_3\Big)\partial^\alpha\boldsymbol{u}_t}_{V_4} + \int\partial^\alpha(F+G)\partial^\alpha\boldsymbol{u}_t,
\end{aligned} \tag{2.65}$$

where we have used the integration by parts. Apparently, one has

$$-V_1 \leq C\|\mathrm{div}\mathbb{X}\|_{0,2n}\|\partial^\alpha\boldsymbol{u}_t\|_0 \leq \varepsilon\|\partial^\alpha\boldsymbol{u}_t\|_0^2 + C_\varepsilon\|\mathrm{div}\mathbb{X}\|_{0,2n}. \tag{2.66}$$

By means of $(1.27)_1$, Sobolev's and Young's inequality, we have

$$\begin{aligned}
-V_2 &= -R\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(\rho_s\mathrm{div}\partial^\alpha\mathbb{X} + \sum_{|l|=1}^{\alpha} C_\alpha^l\partial^l\rho_s\mathrm{div}\partial^{\alpha-l}\mathbb{X}\Big)\partial^\alpha\mathrm{div}\boldsymbol{u} \\
&\qquad + R\int\Big(\rho_s\mathrm{div}\partial^\alpha\boldsymbol{u} + \sum_{|l|=1}^{\alpha} C_\alpha^l\partial^l\rho_s\mathrm{div}\partial^{\alpha-l}\boldsymbol{u}\Big)\partial^\alpha\mathrm{div}\boldsymbol{u} \\
&\leq -R\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(\rho_s\mathrm{div}\partial^\alpha\mathbb{X} + \sum_{|l|=1}^{\alpha} C_\alpha^l\partial^l\rho_s\mathrm{div}\partial^{\alpha-l}\mathbb{X}\Big)\partial^\alpha\mathrm{div}\boldsymbol{u} + C\|\mathrm{div}\boldsymbol{u}\|_{0,2n-1}^2 \\
&\leq -R\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(\rho_s\mathrm{div}\partial^\alpha\mathbb{X} + \sum_{|l|=1}^{\alpha} C_\alpha^l\partial^l\rho_s\mathrm{div}\partial^{\alpha-l}\mathbb{X}\Big)\partial^\alpha\mathrm{div}\boldsymbol{u} + \varepsilon\|\nabla\boldsymbol{u}\|_{0,2n}^2 + C_\varepsilon\|\nabla\boldsymbol{u}\|_0^2.
\end{aligned} \tag{2.67}$$

Similar as Lemma 2.2, one can derive that

$$
\begin{aligned}
V_4 = \lambda\bar{b}^2\frac{\mathrm{d}}{\mathrm{d}t}\int &\Big(\partial_3^2\partial^\alpha\mathbb{X}-\partial_3\mathrm{div}\partial^\alpha\mathbb{X}e_3+\nabla\mathrm{div}\partial^\alpha\mathbb{X}-\nabla\partial_3\partial^\alpha\mathbb{X}_3\Big)\partial^\alpha\boldsymbol{u}\\
&-\lambda\bar{b}^2\int\Big(\partial_3^2\partial^\alpha\boldsymbol{u}-\partial_3\mathrm{div}\partial^\alpha\boldsymbol{u}e_3+\nabla\mathrm{div}\partial^\alpha\boldsymbol{u}-\nabla\partial_3\partial^\alpha u_3\Big)\partial^\alpha\boldsymbol{u}\\
=\lambda\bar{b}^2\frac{\mathrm{d}}{\mathrm{d}t}\int &\Big(\partial_3^2\partial^\alpha\mathbb{X}-\partial_3\mathrm{div}\partial^\alpha\mathbb{X}e_3+\nabla\mathrm{div}\partial^\alpha\mathbb{X}-\nabla\partial_3\partial^\alpha\mathbb{X}_3\Big)\partial^\alpha\boldsymbol{u}-\|(\partial^\alpha\partial_3\boldsymbol{u}_h,\partial^\alpha\mathrm{div}_h\boldsymbol{u}_h)\|_0^2.
\end{aligned}
\tag{2.68}
$$

However, the term V_3 cannot be tackled as in $\mathbb{X}$ using the integration by parts, since the derivative order of $\boldsymbol{u}$ cannot be increased. Our method is translate the time derivative to temperature. In fact,

$$
\begin{aligned}
-V_3 &\le -\frac{\mathrm{d}}{\mathrm{d}t}\int R\partial^\alpha\nabla\rho_s(\theta-1)\partial^\alpha\boldsymbol{u}+\int R\partial^\alpha\nabla\rho_s\theta_t\partial^\alpha\boldsymbol{u}+\varepsilon\|\partial^\alpha u_t\|_0^2+C_\varepsilon\|\nabla\theta\|_{0,2n}^2\\
&\le -\frac{\mathrm{d}}{\mathrm{d}t}\int R\partial^\alpha\nabla\rho_s(\theta-1)\partial^\alpha\boldsymbol{u}+\varepsilon\|(\partial^\alpha u_t,\theta_t)\|_0^2+C_\varepsilon\|\partial^\alpha\boldsymbol{u}\|_0^2+C_\varepsilon\|\nabla\theta\|_{0,2n}^2\\
&\le -\frac{\mathrm{d}}{\mathrm{d}t}\int R\partial^\alpha\nabla\rho_s(\theta-1)\partial^\alpha\boldsymbol{u}+\varepsilon\|(\partial^\alpha u_t,\theta_t)\|_0^2+\varepsilon\|\nabla\boldsymbol{u}\|_{0,2n}^2+C_\varepsilon\|\nabla\boldsymbol{u}\|_0^2+C_\varepsilon\|\nabla\theta\|_{0,2n}^2.
\end{aligned}
\tag{2.69}
$$

By Sobolev's and Young's inequalities, for $\alpha>0$, one has

$$
-V_5\le\varepsilon\|\partial^\alpha\boldsymbol{u}_t\|_0^2+C_\varepsilon\|\partial^{\alpha-l}\boldsymbol{u}_t\|_0^2\le\varepsilon\|\partial^\alpha\boldsymbol{u}_t\|_0^2+C_\varepsilon\|\partial_t\boldsymbol{u}\|_0^2,\tag{2.70}
$$

and for $\alpha=0$, the term V_5 will disappear, that is

$$
V_5=0.\tag{2.71}
$$

Combined with (2.65)-(2.71), by means of (2.45), one has

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mu\|\nabla\boldsymbol{u}\|_{0,2n-1}^2+(\mu+\mu')\|\mathrm{div}\boldsymbol{u}\|_{0,2n-1}^2\Big)+\frac{\mathrm{d}}{\mathrm{d}t}\int\Bigg(R\Big[\partial^\alpha\nabla\rho_s(\theta-1)\partial^\alpha\boldsymbol{u}\\
&\quad+\Big(\rho_s\mathrm{div}\partial^\alpha\mathbb{X}+\sum_{|l|=1}^{\alpha}C_\alpha^l\partial^l\rho_s\mathrm{div}\partial^{\alpha-l}\mathbb{X}\Big)\partial^\alpha\mathrm{div}\boldsymbol{u}\Big]-\lambda\bar{b}^2\Big(\partial_3^2\partial^\alpha\mathbb{X}-\partial_3\mathrm{div}\partial^\alpha\mathbb{X}e_3\\
&\quad+\nabla\mathrm{div}\partial^\alpha\mathbb{X}-\nabla\partial_3\partial^\alpha\mathbb{X}_3\Big)\partial^\alpha\boldsymbol{u}\Bigg)+\|\boldsymbol{u}_t\|_{0,2n-1}^2+\|\partial_3\boldsymbol{u}_h,\mathrm{div}_h\boldsymbol{u}_h\|_{0,2n-1}^2\\
&\le C\sqrt{\mathcal{E}_n}\mathcal{D}_n+\varepsilon\|\nabla\boldsymbol{u}\|_{0,2n}^2+\varepsilon\|\theta_t\|_0^2+C_\varepsilon\big(\|\mathrm{div}\mathbb{X}\|_{0,2n}^2+\|\nabla\boldsymbol{u}\|_0^2+\|\nabla\theta\|_{0,2n}^2+\|\partial_t\boldsymbol{u}\|_0^2\big),
\end{aligned}
\tag{2.72}
$$

which means that (2.63) has been obtained when we have the following definition:

$$
\begin{aligned}
\bar{\mathcal{E}}_n^{h,u_t}:=&\Big(\mu\|\nabla\boldsymbol{u}\|_{0,2n-1}^2+(\mu+\mu')\|\mathrm{div}\boldsymbol{u}\|_{0,2n-1}^2\Big)\\
&+\int\Bigg(R\Big[\partial^\alpha\nabla\rho_s(\theta-1)\partial^\alpha\boldsymbol{u}+\Big(\rho_s\mathrm{div}\partial^\alpha\mathbb{X}+\sum_{|l|=1}^{\alpha}C_\alpha^l\partial^l\rho_s\mathrm{div}\partial^{\alpha-l}\mathbb{X}\Big)\partial^\alpha\mathrm{div}\boldsymbol{u}\Big]\\
&-\lambda\bar{b}^2\Big(\partial_3^2\partial^\alpha\mathbb{X}-\partial_3\mathrm{div}\partial^\alpha\mathbb{X}e_3+\nabla\mathrm{div}\partial^\alpha\mathbb{X}-\nabla\partial_3\partial^\alpha\mathbb{X}_3\Big)\partial^\alpha\boldsymbol{u}\Bigg),
\end{aligned}
\tag{2.73}
$$

$$
\bar{\mathcal{D}}_n^{h,u_t}:=\|\boldsymbol{u}_t\|_{0,2n-1}^2+\|(\partial_3\boldsymbol{u}_h,\mathrm{div}_h\boldsymbol{u}_h)\|_{0,2n-1}^2,\qquad\bar{\mathcal{D}}^s:=\|\nabla(\boldsymbol{u},\theta)\|_0^2.\tag{2.74}
$$

$\square$

For $n \in \mathbb{Z}_{\geq 3}$, we define

$$\bar{\mathcal{E}}_n^{h,\theta_t} := \|\nabla\theta\|_{0,2n-1}^2 \tag{2.75}$$

and

$$\bar{\mathcal{D}}_n^{h,\theta_t} := \|\theta_t\|_{0,2n-1}^2. \tag{2.76}$$

Lemma 2.7 *For $n \geq 3$, it holds that*

$$\frac{d}{dt}\bar{\mathcal{E}}_n^{h,\theta_t} + \bar{\mathcal{D}}_n^{h,\theta_t} \lesssim \sqrt{\mathcal{E}_n}\mathcal{D}_n + \varepsilon\bar{\mathcal{D}}_n^{h,u} + C_\varepsilon\bar{\mathcal{D}}^s + C_\varepsilon\bar{\mathcal{D}}_n^t. \tag{2.77}$$

Proof. Applying ∂^α, $\alpha \in \mathbb{N}^2$, and $0 \leq |\alpha| \leq 2n-1$ to $(1.27)_3$, one has

$$c_v\rho_s\partial_t\partial^\alpha\theta + c_v\sum_{|l|=1}^{|\alpha|} C_\alpha^l\partial^l\rho_s\partial_t\partial^{\alpha-l}\theta - \kappa\Delta\partial^\alpha\theta + R\partial^\alpha(\rho_s\mathrm{div}\boldsymbol{u}) = \partial^\alpha E, \tag{2.78}$$

which multiplied by $\partial^\alpha\theta_t$ yields

$$\begin{aligned}&\frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\partial^\alpha\theta\|_0^2 + c_v\|\sqrt{\rho_s}\partial^\alpha\theta_t\|_0^2 + \underbrace{c_v\int\sum_{|l|=1}^{|\alpha|} C_\alpha^l\partial^l\rho_s\partial_t\partial^{\alpha-l}\theta\partial^\alpha\theta_t + R\int\partial^\alpha(\rho_s\mathrm{div}\boldsymbol{u})\partial^\alpha\theta_t}_{VI}\\ &= \int\partial^\alpha E\partial^\alpha\theta_t \lesssim \sqrt{\mathcal{E}_n}\mathcal{D}_n.\end{aligned} \tag{2.79}$$

By virtue of Hölder, Young and Sobolev's inequality, for $1 \leq \alpha \leq 2n-1$ one has

$$\begin{aligned}-VI &\leq \varepsilon\|\partial^\alpha\theta_t\|_0^2 + C_\varepsilon\|\theta_t\|_{0,2n-2}^2 + C_\varepsilon\|\nabla\boldsymbol{u}\|_{0,2n-1}^2\\ &\leq \varepsilon\Big(\|\theta_t\|_{0,2n-1}^2 + \|\nabla\boldsymbol{u}\|_{0,2n}^2\Big) + C_\varepsilon\Big(\|\theta_t\|_0^2 + \|\nabla\boldsymbol{u}\|_0^2\Big).\end{aligned} \tag{2.80}$$

Similarly, for $\alpha = 0$, one has

$$-VI = R\int\rho_s\mathrm{div}\boldsymbol{u}\theta_t \leq \varepsilon\|\theta_t\|_0^2 + C_\varepsilon\|\nabla\boldsymbol{u}\|_0^2. \tag{2.81}$$

Combined with (2.79)-(2.81), it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{0,2n-1}^2 + \|\theta_t\|_{0,2n-1}^2 \lesssim \sqrt{\mathcal{E}_n}\mathcal{D}_n + \varepsilon\|\nabla\boldsymbol{u}\|_{0,2n}^2 + C_\varepsilon\|(\nabla\boldsymbol{u},\theta_t)\|_0^2, \tag{2.82}$$

which means that (2.77) has been obtained. $\square$

2.4 Improved estimates.

Lemma 2.8 *For $n \geq 3$, it holds that*

$$\begin{aligned}&\frac{d}{dt}\mathfrak{E}_n + \|(div\mathbb{X},\partial_3\mathbb{X},\mathbb{X})\|_{2n}^2 + \|(\boldsymbol{u},\theta-1)\|_{2n+1}^2\\ &\lesssim \|(u_t,\theta_t)\|_{2n-1}^2 + \bar{\mathcal{D}}_n^{h,u} + \bar{\mathcal{D}}_n^{h,\theta} + \bar{\mathcal{D}}_n^{h,x} + \bar{\mathcal{D}}^s + \mathcal{E}_n\mathcal{D}_n.\end{aligned} \tag{2.83}$$

Proof. Note that

$$\begin{aligned}-\mu\Delta u_3-(\mu+\mu')\partial_3\mathrm{div}\boldsymbol{u}&=-\mu\Delta_h u_3-(2\mu+\mu')\partial_3^3 u_3-(\mu+\mu')\partial_3\mathrm{div}_h u_h\\&=-\mu\Delta_h u_3-(2\mu+\mu')\partial_3\mathrm{div}\boldsymbol{u}+\mu\partial_3\mathrm{div}_h u_h.\end{aligned}\tag{2.84}$$

Hence, let $q:=-\mathrm{div}\mathbb{X}$, the third component of $(1.27)_2$ can be rewritten as

$$\begin{aligned}(2\mu+\mu')\partial_t\partial_3 q+R\rho_s\partial_3 q=&-\rho_s\partial_t u_3+\mu\Delta_h u_3-\mu\partial_3\mathrm{div}_h u_h\\&-R\partial_3(\rho_s(\theta-1))+F_3+G_3.\end{aligned}\tag{2.85}$$

Applying $\partial_3^r\partial_h^m$, $0\le r\le k$, $0\le|m|\le 2n-k-1$, one has

$$\begin{aligned}(2\mu+\mu')\partial_t\partial_3(\partial_3^r\partial_h^m q)+R\rho_s\partial_3(\partial_3^r\partial_h^m q)=-R\sum_{l_1=1}^{r}\sum_{|l_2|=0}^{|m|}C_r^{l_1}C_m^{l_2}\partial_3^{l_1}\partial_h^{l_2}\rho_s\partial_3(\partial_3^{r-l_1}\partial_h^{m-l_2}q)\\-R\sum_{|l_2|=1}^{|m|}C_m^{l_2}\partial_h^{l_2}\rho_s\partial_3\partial_3^r\partial_h^{m-l_2}q+\partial_3^r\partial_h^m\Big(-\rho_s\partial_t u_3+\mu\Delta_h u_3-\mu\partial_3\mathrm{div}_h u_h\\-R\partial_3(\rho_s(\theta-1))+F_3+G_3\Big),\end{aligned}\tag{2.86}$$

which yields

$$\begin{aligned}&R(2\mu+\mu')\frac{\mathrm{d}}{\mathrm{d}t}\sum_{r=0}^{k}\sum_{|m|=0}^{2n-k-1}\int\rho_s|\partial_3\partial_3^r\partial_h^m q|^2+(2\mu+\mu')^2\|\partial_3 q_t\|_{k,2n-k-1}^2+R^2\underline{\rho}^2\|\partial_3 q\|_{k,2n-k-1}^2\\&\le C\|\partial_3 q\|_{k-1,2n-k-1}^2+C\|\partial_3 q\|_{k,2n-k-2}^2+\Big\|\Big(-\rho_s\partial_t u_3+\mu\Delta_h u_3\\&\qquad-\mu\partial_3\mathrm{div}_h u_h-R\partial_3(\rho_s(\theta-1))+F_3+G_3\Big)\Big\|_{k,2n-k-1}^2\\&\le\varepsilon\|\partial_3 q\|_{k,2n-k-1}^2+C_\varepsilon\|\partial_3 q\|_{k,0}^2+C_\varepsilon\|q\|_{0,2n-k-1}^2+C\|\partial_t u_3\|_{2n-1}^2+C\|\boldsymbol{u}\|_{k+1,2n-k}^2\\&\qquad+C\|\boldsymbol{u}\|_{k,2n-k+1}^2+C\|\nabla\theta\|_{k,2n-k-1}^2+C\|F_3+G_3\|_{2n-1}^2.\end{aligned}\tag{2.87}$$

To estimate the term $\|\partial_3 q\|_{k,0}^2$, we need the following calculation. Applying the operator ∂_3^r to (2.85), $0\le r\le k$, one has

$$\begin{aligned}(2\mu+\mu')\partial_t\partial_3(\partial_3^r q)+R\rho_s\partial_3(\partial_3^r q)=-R\sum_{l=1}^{r}C_r^l\partial_3^l\rho_s\partial_3(\partial_3^{r-l}q)+\partial_3^r\Big(-\rho_s\partial_t u_3+\mu\Delta_h u_3\\-\mu\partial_3\mathrm{div}_h u_h-R\partial_3(\rho_s(\theta-1))+F_3+G_3\Big),\end{aligned}\tag{2.88}$$

which yields

$$\begin{aligned}&R(2\mu+\mu')\frac{\mathrm{d}}{\mathrm{d}t}\sum_{r=0}^{k}\int\rho_s|\partial_3\partial_3^r q|^2+(2\mu+\mu')^2\|\partial_t\partial_3 q\|_{k,0}^2+R^2\underline{\rho}^2\|\partial_3 q\|_{k,0}^2\qquad(2.89)\\&\lesssim\|\partial_3 q\|_{k-1,0}^2+\Big\|\Big(-\rho_s\partial_t u_3+\mu\Delta_h u_3-\mu\partial_3\mathrm{div}_h u_h-R\partial_3(\rho_s(\theta-1))+F_3+G_3\Big)\Big\|_{k,0}^2\\&\lesssim\varepsilon\|\partial_3 q\|_{k,0}^2+C_\varepsilon\|q\|_0^2+\|\partial_t u_3\|_{k,0}^2+\|\boldsymbol{u}\|_{k+1,2n-k}^2+\|\boldsymbol{u}\|_{k,2n-k+1}^2+\|F_3+G_3\|_{k,0}^2+\|\nabla\theta\|_{k,0}^2.\end{aligned}$$

Combined with (2.87) and (2.89), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{r=0}^{k}\left(\int\rho_s|\partial_3\partial_3^r q|^2+\sum_{|m|=0}^{2n-k-1}\int\rho_s|\partial_3\partial_3^r\partial_h^m q|^2\right)+\left(\|\partial_3 q_t\|_{k,2n-k-1}^2+\|\partial_3 q\|_{k,2n-k-1}^2\right) \\
&\lesssim \|q\|_{0,2n}^2+\|\partial_t u_3\|_{2n-1}^2+\|\boldsymbol{u}\|_{k+1,2n-k}^2+\|\boldsymbol{u}\|_{k,2n-k+1}^2+\|(G_3,F_3)\|_{2n-1}^2+\|\nabla\theta\|_{k,2n-k-1}^2.
\end{aligned} \tag{2.90}$$

The first two components of $(1.27)_2$ can be rewritten as

$$\begin{aligned}
&-\mu\partial_3^2 u_h-\lambda\bar{b}^2\partial_3^2\mathbb{X}_h=-\rho_s\partial_t u_h+\mu\Delta_h u_h-(\mu+\mu')\nabla_h\partial_t q \\
&-R\rho_s\nabla_h q-R\nabla_h(\rho_s(\theta-1))-\lambda\bar{b}^2(\nabla_h q+\nabla_h\partial_3\mathbb{X}_3)+G_h+F_h.
\end{aligned} \tag{2.91}$$

Taking the norm $\|\cdot\|_{k,2n-k-1}^2$ of (2.91), one has

$$\begin{aligned}
&\mu\lambda\bar{b}^2\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_3^2\mathbb{X}_h\|_{k,2n-k-1}^2+\lambda^2\bar{b}^4\|\partial_3^2\mathbb{X}_h\|_{k,2n-k-1}^2+\mu^2\|\partial_3^2 u_h\|_{k,2n-k-1}^2 \\
&=\|-\rho_s\partial_t u_h+\mu\Delta_h u_h-(\mu+\mu')\nabla_h\partial_t q-R\rho_s\nabla_h q-R\nabla_h(\rho_s(\theta-1)) \\
&\qquad -\lambda\bar{b}^2(\nabla_h q+\nabla_h\partial_3\mathbb{X}_3)+G_h+F_h\|_{k,2n-k-1}^2 \\
&\lesssim \|\partial_t u_h\|_{2n-1}^2+\|u_h\|_{k,2n-k+1}^2+\|\partial_t q\|_{k,2n-k}^2+\|q\|_{k,2n-k}^2+\|\partial_3\mathbb{X}_3\|_{k,2n-k}^2 \\
&\qquad +\|\nabla\theta\|_{k,2n-k-1}^2+\|(F_h,G_h)\|_{k,2n-k-1}^2.
\end{aligned} \tag{2.92}$$

Noting that

$$\|q\|_{0,2n}^2=\|\mathrm{div}\mathbb{X}\|_{0,2n}^2\lesssim\bar{\mathcal{D}}_n^{h,x}, \tag{2.93}$$

$$\|\partial_3^3\mathbb{X}_3\|_{k,2n-k-1}^2=\|\partial_3(q+\mathrm{div}_h\mathbb{X}_h)\|_{k,2n-k-1}^2\le\|q\|_{k+1,2n-k-1}^2+\|\partial_3\mathbb{X}_h\|_{k,2n-k}^2, \tag{2.94}$$

$$\|\partial_3^2 u_3\|_{k,2n-k-1}^2=\|\partial_3(\partial_t q+\mathrm{div}_h\mathbb{X}_h)\|_{k,2n-k-1}^2\le\|\partial_t q\|_{k+1,2n-k-1}^2+\|u_h\|_{k+1,2n-k}^2, \tag{2.95}$$

combined with (2.90) and (2.92), we derive

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{r=0}^{k}\left(\int\rho_s|\partial_3\partial_3^r q|^2+\sum_{|m|=0}^{2n-k-1}\int\rho_s|\partial_3\partial_3^r\partial_h^m q|^2\right)+\|\partial_3^2\mathbb{X}_h\|_{k,2n-k-1}^2\right] \\
&+\left[\|\partial_t q\|_{k+1,2n-k-1}^2+\|q\|_{k+1,2n-k-1}^2+\|\partial_3\mathbb{X}\|_{k+1,2n-k-1}^2+\|\boldsymbol{u}\|_{k+2,2n-k-1}^2\right] \\
&\lesssim \|\partial_t q\|_{k,2n-k}^2+\|q\|_{k,2n-k}^2+\|\partial_3\mathbb{X}\|_{k,2n-k}^2+\|\boldsymbol{u}\|_{k+1,2n-k}^2+\|\boldsymbol{u}\|_{k,2n-k+1}^2+\|\partial_t\boldsymbol{u}\|_{2n-1}^2 \\
&\qquad +\|(F,G)\|_{k,2n-k-1}^2+\bar{\mathcal{D}}_n^{h,x}+\|\nabla\theta\|_{k,2n-k-1}^2.
\end{aligned} \tag{2.96}$$

By means of $(1.27)_3$, one has

$$\kappa\partial_3^2\theta=-\kappa\Delta_h\theta+c_v\rho_s\theta_t-E+R\rho_s\mathrm{div}\boldsymbol{u}.$$

Taking the $\|\cdot\|_{k,2n-k-1}^2$, by Poincaré's inequality and $\partial_3\theta|_{\partial\Omega}=0$, we can derive

$$\begin{aligned}
\|\partial_3\theta\|_{k+1,2n-k-1}^2&\lesssim\|\partial_3^2\theta\|_{k,2n-k-1}^2+\|\partial_3\theta\|_{k,2n-k-1}^2 \\
&\lesssim\|\partial_3^2\theta\|_{k,2n-k-1}^2+\|\partial_3\theta\|_{0,2n-k-1}^2+\|\partial_3\partial_3^k\theta\|_{0,2n-k-1}^2 \\
&\lesssim\|\partial_3^2\theta\|_{k,2n-k-1}^2 \\
&\lesssim\|\theta_t\|_{2n-1}^2+\|\nabla_h^2\theta\|_{k,2n-k-1}^2+\|q_t\|_{k,2n-k-1}^2+\|E\|_{2n-1}^2.
\end{aligned} \tag{2.97}$$

Noting that

$$\|\nabla_h\theta\|_{k+1,2n-k-1}^2 = \|\nabla_h\theta\|_{k,2n-k-1}^2 + \|\partial_3\nabla_h\theta\|_{k,2n-k-1}^2, \tag{2.98}$$

one can deduce

$$\|\nabla\theta\|_{k+1,2n-k-1}^2 \lesssim \|\nabla\theta\|_{k,2n-k}^2 + \|\theta_t\|_{2n-1}^2 + \|q_t\|_{k,2n-k}^2 + \|E\|_{2n-1}^2. \tag{2.99}$$

Adding (2.99) with (2.96), it follows

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{r=0}^{k}\Big(\int\rho_s|\partial_3\partial_3^r q|^2 + \sum_{|m|=0}^{2n-k-1}\int\rho_s|\partial_3\partial_3^r\partial_h^m q|^2\Big) + \|\partial_3^2\mathbb{X}_h\|_{k,2n-k-1}^2\right]\\
&\qquad + \left[\|(\partial_t q, q, \partial_3\mathbb{X}, \nabla\theta)\|_{k+1,2n-k-1}^2 + \|\boldsymbol{u}\|_{k+2,2n-k-1}^2\right]\\
&\lesssim \|(\partial_t q, q, \partial_3\mathbb{X}, \nabla\theta)\|_{k,2n-k}^2 + \|\boldsymbol{u}\|_{k+1,2n-k}^2 + \|\boldsymbol{u}\|_{k,2n-k+1}^2\\
&\quad + \|(u_t,\theta_t)\|_{2n-1}^2 + \|(F,G,E)\|_{2n-1}^2 + \bar{\mathcal{D}}_n^{h,x}.
\end{aligned} \tag{2.100}$$

By means of (2.93),

$$\|\partial_t q\|_{0,2n} = \|\mathrm{div}\boldsymbol{u}\|_{0,2n} \le \|\boldsymbol{u}\|_{1,2n} \lesssim \bar{\mathcal{D}}_n^{h,u},$$

and the recursive inequality of (2.100) on k, we conclude that there exist constants $\lambda_k > 0$, $k = 0,\cdots,2n-1$ such that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{2n-1}\lambda_k\left[\sum_{r=0}^{k}\Big(\int\rho_s|\partial_3\partial_3^r q|^2 + \sum_{|m|=0}^{2n-k-1}\int\rho_s|\partial_3\partial_3^r\partial_h^m q|^2\Big) + \frac{\mu\lambda\bar{b}^2}{2}\|\partial_3^2\mathbb{X}_h\|_{k,2n-k-1}^2\right]\\
&\qquad + C_8\sum_{k=0}^{2n-1}\left[\|\partial_t q, q, \partial_3\mathbb{X}, \nabla\theta\|_{k+1,2n-k-1}^2 + \|\boldsymbol{u}\|_{k+2,2n-k-1}^2\right]\\
&\lesssim \|\partial_t q, q, \partial_3\mathbb{X}, \nabla\theta\|_{0,2n}^2 + \|\boldsymbol{u}\|_{1,2n}^2 + \|u_t,\theta_t\|_{2n-1}^2 + \|(F,G,E)\|_{2n-1}^2 + \bar{\mathcal{D}}_n^{h,x}\\
&\lesssim \|u_t,\theta_t\|_{2n-1}^2 + \|(F,G,E)\|_{2n-1}^2 + \bar{\mathcal{D}}_n^{h,u} + \bar{\mathcal{D}}_n^{h,\theta} + \bar{\mathcal{D}}_n^{h,x} + \bar{\mathcal{D}}^s.
\end{aligned} \tag{2.101}$$

Hence, if we define

$$\mathfrak{E}_n := \sum_{k=0}^{2n-1}\lambda_k\left[\sum_{r=0}^{k}\Big(\int\rho_s|\partial_3\partial_3^r q|^2 + \sum_{|m|=0}^{2n-k-1}\int\rho_s|\partial_3\partial_3^r\partial_h^m q|^2\Big) + \frac{\mu\lambda\bar{b}^2}{2}\|\partial_3^2\mathbb{X}_h\|_{k,2n-k-1}^2\right],$$

then $\mathfrak{E}_n$ is equivalent to $\|\partial_3 q\|_{2n-1}^2 + \|\partial_3^2\mathbb{X}_h\|_{2n-1}^2$.

Thanks to (2.39) and (2.101), we can deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}_n + \|(q_t, q, \partial_3\mathbb{X})\|_{2n}^2 + \|(\boldsymbol{u},\theta-1)\|_{2n+1}^2\\
&\quad \lesssim \|(u_t,\theta_t,F,G,E)\|_{2n-1}^2 + \bar{\mathcal{D}}_n^{h,u} + \bar{\mathcal{D}}_n^{h,\theta} + \bar{\mathcal{D}}_n^{h,x} + \bar{\mathcal{D}}^s + \mathcal{E}_n\mathcal{D}_n.
\end{aligned} \tag{2.102}$$

Using (2.45) and (2.47) to estimate $\|(F,G,E)\|_{2n-1}^2 \lesssim \mathcal{E}_n\mathcal{D}_n$ we conclude the estimate (2.83) by recalling that $q = -\mathrm{div}\mathbb{X}$ and using Poincaré's inequality. □

Lemma 2.9 *For $n \ge 3$, it holds that*

$$\sum_{r=1}^{n}\Big(\|\partial_t^r u\|_{2n-2r+1}^2 + \|\partial_t^r\theta\|_{2n-2r+1}^2\Big) \lesssim \mathcal{E}_n\mathcal{D}_n + \bar{\mathcal{D}}_n^t + \|\boldsymbol{u}\|_{2n-1}^2. \tag{2.103}$$

Proof. $(1.27)_2$ can be rewrite as

$$-\mu\Delta\boldsymbol{u}+(\mu+\mu')\nabla\mathrm{div}\boldsymbol{u}=-\rho_s\partial_t\boldsymbol{u}+R\rho_s\nabla\mathrm{div}\mathbb{X}-R\nabla(\rho_s(\theta-1))$$
$$+\lambda\bar{b}^2(\partial_3^2\mathbb{X}-\partial_3\mathrm{div}\mathbb{X}e_3+\nabla\mathrm{div}\mathbb{X}-\nabla\partial_3\mathbb{X}_3)+F+G. \tag{2.104}$$

Applying ∂_t^r to (2.104), employing elliptic estimates for $2n-2r+1\geq 3$, one can obtain

$$\begin{aligned}
&\|\partial_t^r\boldsymbol{u}\|_{2n-2r+1}^2\\
\lesssim&\|\partial_t^{r+1}\boldsymbol{u}\|_{2n-2r-1}^2+\|\nabla^2\partial_t^r\mathbb{X}\|_{2n-2r-1}^2+\|\nabla(\rho_s\partial_t^r\theta)\|_{2n-2r-1}^2+\|\partial_t^r(F+G)\|_{2n-2r-1}^2\\
\lesssim&\|\partial_t^{r+1}\boldsymbol{u}\|_{2n-2(r+1)+1}^2+\|\partial_t^r\mathbb{X}\|_{2n-2r+1}^2+\|\partial_t^r\theta\|_{2n-2r}^2+Y_n,
\end{aligned}\tag{2.105}$$

where

$$Y_n:=\|\bar{\nabla}_0^{2n-1}(F+G)\|_0^2.$$

By means of $(1.27)_1$, $\|\partial_t^n\boldsymbol{u}\|_1^2\lesssim\bar{\mathcal{D}}_n^t$, and Sobolev's and Young's inequalities, a simple induction on (2.105) yields

$$\begin{aligned}
\sum_{r=1}^{n}\|\partial_t^r\boldsymbol{u}\|_{2n-2r+1}^2\lesssim&\|\partial_t^n\boldsymbol{u}\|_1^2+\sum_{r=1}^{n-1}\|\partial_t^r\mathbb{X}\|_{2n-2r+1}^2+\sum_{r=1}^{n-1}\|\partial_t^r\theta\|_{2n-2r}^2+Y_n\\
\lesssim&\bar{\mathcal{D}}_n^t+\sum_{r=1}^{n-1}\|\partial_t^{r-1}\boldsymbol{u}\|_{2n-2(r-1)-1}^2+\sum_{r=1}^{n-1}\|\partial_t^r\theta\|_{2n-2r}^2+Y_n\\
\lesssim&\bar{\mathcal{D}}_n^t+\|\boldsymbol{u}\|_{2n-1}^2+\sum_{r=1}^{n-2}\|\partial_t^r\boldsymbol{u}\|_{2n-2r-1}^2+\sum_{r=1}^{n-1}\|\partial_t^r\theta\|_{2n-2r}^2+Y_n\\
\lesssim&\bar{\mathcal{D}}_n^t+\|\boldsymbol{u}\|_{2n-1}^2+\varepsilon\sum_{r=1}^{n-2}\|\partial_t^r\boldsymbol{u}\|_{2n-2r+1}^2+\varepsilon\sum_{r=1}^{n-1}\|\partial_t^r\theta\|_{2n-2r+1}^2\\
&+C_\varepsilon\sum_{r=1}^{n-2}\|\partial_t^r\boldsymbol{u}\|_0^2+C_\varepsilon\sum_{r=1}^{n-1}\|\partial_t^r\theta\|_0^2+Y_n\\
\lesssim&\bar{\mathcal{D}}_n^t+\|\boldsymbol{u}\|_{2n-1}^2+\varepsilon\sum_{r=1}^{n-2}\|\partial_t^r\boldsymbol{u}\|_{2n-2r+1}^2+\varepsilon\sum_{r=1}^{n-1}\|\partial_t^r\theta\|_{2n-2r+1}^2\\
&+C_\varepsilon\bar{\mathcal{D}}_n^t+Y_n.
\end{aligned}\tag{2.106}$$

By virtue of $(1.27)_3$, one has

$$-\kappa\Delta\theta=-c_v\rho_s\partial_t\theta-R\rho_s\mathrm{div}\boldsymbol{u}+E. \tag{2.107}$$

Applying $\partial_3\partial_t^r$ to (2.107), employing elliptic estimates for $2n-2r\geq 2$, one has

$$\|\partial_t^r\partial_3\theta\|_{2n-2r}^2\lesssim\|\partial_t^{r+1}\partial_3\theta\|_{2n-2(r+1)}^2+\|\partial_t^{r+1}\theta\|_{2n-2(r+1)}^2+\|\partial_t^r\boldsymbol{u}\|_{2n-2r}^2+Z_n, \tag{2.108}$$

where

$$Z_n:=\|\bar{\nabla}_0^{2n-1}E\|_0^2.$$

By means of induction on (2.108), Sobolev's and Young's inequalities, and $\|\partial_t^n\theta\|_1^2\lesssim\bar{\mathcal{D}}_n^t$, we deduce

$$\sum_{r=1}^{n}\|\partial_3\partial_t^r\theta\|_{2n-2r}^2\lesssim\|\partial_t^n\theta\|_1^2+\sum_{r=1}^{n-1}\|\partial_t^r\boldsymbol{u}\|_{2n-2r}^2+\sum_{r=1}^{n-1}\|\partial_t^{r+1}\theta\|_{2n-2(r+1)}^2+Z_n$$

$$\lesssim \varepsilon \sum_{r=1}^{n-1} \|\partial_t^r \boldsymbol{u}\|_{2n-2r+1}^2 + C_\varepsilon \sum_{r=1}^{n-1} \|\partial_t^r \boldsymbol{u}\|_0^2 + \|\partial_t^n \theta\|_1^2 + \sum_{r=2}^{n} \|\partial_t^r \theta\|_{2n-2r}^2 + Z_n$$
$$\lesssim \varepsilon \sum_{r=1}^{n-1} \|\partial_t^r \boldsymbol{u}\|_{2n-2r+1}^2 + \varepsilon \sum_{r=2}^{n} \|\partial_t^r \theta\|_{2n-2r+1}^2 + C_\varepsilon \bar{\mathcal{D}}_n^t + Z_n. \tag{2.109}$$

Next, we rewrite (2.107) as

$$-k\Delta_h \theta = k\partial_3^2 \theta - c_v \rho_s \theta_t - R\rho_s \mathrm{div}\boldsymbol{u} + E. \tag{2.110}$$

Applying $\nabla^\beta \partial_t^r$, $\beta \in \mathbb{N}^3$, and $|\beta| \le 2n-2r-1$ to above, one has

$$\begin{aligned}-\kappa\Delta_h \nabla^\beta \partial_t^r \theta = {}& \partial_3^2 \nabla^\beta \partial_t^r \theta - c_v \rho_s \nabla^\beta \partial_t^{r+1} \theta - c_v \sum_{l=1}^{\beta} C_\beta^l \nabla^l \rho_s \partial_t^{r+1} \nabla^{\beta-l} \theta \\ & - R\rho_s \nabla^\beta \partial_t^r \mathrm{div}\boldsymbol{u} - R\sum_{l=1}^{\beta} C_\beta^l \nabla^l \rho_s \partial_t^r \nabla^{\beta-l} \mathrm{div}\boldsymbol{u} + \nabla^\beta \partial_t^r E.\end{aligned} \tag{2.111}$$

Multiplying (2.111) by $\nabla^\beta \partial_t^r \theta$, by integration by parts, Cauchy-Schwarz's and Sobolev's inequalities, one has

$$\begin{aligned}\|\nabla_h \nabla^\beta \partial_t^r \theta\|_0^2 &\lesssim \|\partial_3^2 \partial_t^r \theta\|_{2n-2r-1}^2 + \|\nabla^\beta \partial_t^r \theta\|_0^2 + \|\partial_t^{r+1} \theta\|_{2n-2r-1}^2 + \|\partial_t^r \boldsymbol{u}\|_{2n-2r}^2 + Z_n \\ &\lesssim \|\partial_3 \partial_t^r \theta\|_{2n-2r}^2 + \varepsilon \|\nabla_h \nabla^\beta \partial_t^r \theta\|_0^2 + C_\varepsilon \|\partial_t^r \theta\|_0^2 + C_\varepsilon \|\partial_3 \partial_t^r \theta\|_{2n-2r}^2 \\ &\quad + \|\partial_t^{r+1} \theta\|_{2n-2(r+1)+1}^2 + \varepsilon \|\partial_t^r \boldsymbol{u}\|_{2n-2r+1}^2 + C_\varepsilon \|\partial_t^r \boldsymbol{u}\|_0^2 + Z_n.\end{aligned} \tag{2.112}$$

Summing (2.112) up from $\beta = 0$ to $2n-2r-1$, we deduce

$$\begin{aligned}\|\nabla_h \partial_t^r \theta\|_{2n-2r-1}^2 \lesssim {}& \|\partial_3 \partial_t^r \theta\|_{2n-2r}^2 + \varepsilon \|\nabla_h \partial_t^r \theta\|_{2n-2r-1}^2 + C_\varepsilon \|\partial_t^r \theta\|_0^2 + C_\varepsilon \|\partial_3 \partial_t^r \theta\|_{2n-2r}^2 \\ & \|\partial_t^{r+1} \theta\|_{2n-2(r+1)+1}^2 + \varepsilon \|\partial_t^r \boldsymbol{u}\|_{2n-2r+1}^2 + C_\varepsilon \|\partial_t^r \boldsymbol{u}\|_0^2 + Z_n.\end{aligned} \tag{2.113}$$

Because

$$\begin{aligned}\|\nabla_h \partial_t^r \theta\|_{2n-2r}^2 &\le \|\nabla_h \partial_t^r \theta\|_{2n-2r-1}^2 + \|\partial_3 \nabla_h \nabla^{2n-2r-1} \partial_t^r \theta\|_0^2 + \|\nabla_h^2 \nabla^{2n-2r-1} \partial_t^r \theta\|_0^2 \\ &\le \|\nabla_h \partial_t^r \theta\|_{2n-2r-1}^2 + \|\partial_3 \partial_t^r \theta\|_{2n-2r}^2 + \|\Delta_h \nabla^{2n-2r-1} \partial_t^r \theta\|_0^2,\end{aligned} \tag{2.114}$$

we next need to estimate $\|\Delta_h \nabla^{2n-2r-1} \partial_t^r \theta\|_0^2$. Taking $|\beta| = 2n-2r-1$ and applying $\|\cdot\|_0^2$ to (2.111), one can derive

$$\begin{aligned}\|\Delta_h \nabla^{2n-2r-1} \partial_t^r \theta\|_0^2 &\lesssim \|\partial_3^2 \nabla^{2n-2r-1} \partial_t^r \theta\|_0^2 + \|\partial_t^{r+1} \theta\|_{2n-2r-1}^2 + \|\partial_t^r \boldsymbol{u}\|_{2n-2r}^2 + Z_n \\ &\lesssim \|\partial_3 \partial_t^r \theta\|_{2n-2r}^2 + \|\partial_t^{r+1} \theta\|_{2n-2(r+1)+1}^2 + \varepsilon \|\partial_t^r \boldsymbol{u}\|_{2n-2r+1}^2 + C_\varepsilon \|\partial_t^r \boldsymbol{u}\|_0^2 + Z_n.\end{aligned} \tag{2.115}$$

Putting (2.113) and (2.115) into (2.114), one has

$$\|\nabla_h \partial_t^r \theta\|_{2n-2r}^2 \lesssim \|\partial_3 \partial_t^r \theta\|_{2n-2r}^2 + \|\partial_t^{r+1} \theta\|_{2n-2(r+1)+1}^2 + C_\varepsilon \bar{\mathcal{D}}_n^t + \varepsilon \|\partial_t^r \boldsymbol{u}\|_{2n-2r+1}^2 + Z_n \tag{2.116}$$

By virtue of (2.109) and (2.116), we have

$$\begin{aligned}\|\partial_t^r \theta\|_{2n-2r+1}^2 &= \|\nabla \partial_t^r \theta\|_{2n-2r}^2 + \|\partial_t^r \theta\|_0^2 \lesssim \|\nabla_h \partial_t^r \theta\|_{2n-2r}^2 + \|\partial_3 \partial_t^r \theta\|_{2n-2r}^2 + \bar{\mathcal{D}}_n^t \\ &\lesssim C_\varepsilon \bar{\mathcal{D}}_n^t + \varepsilon \sum_{r=1}^{n-1} \|\partial_t^r \boldsymbol{u}\|_{2n-2r+1}^2 + \varepsilon \sum_{r=2}^{n} \|\partial_t^r \theta\|_{2n-2r+1}^2 + \varepsilon \|\partial_t^r \boldsymbol{u}\|_{2n-2r+1}^2\end{aligned}$$

$$+\|\partial_t^{r+1}\theta\|^2_{2n-2(r+1)+1}+Z_n. \tag{2.117}$$

By a simple induction of (2.117), one has

$$\begin{aligned}\sum_{r=1}^{n}\|\partial_t^r\theta\|^2_{2n-2r+1} &\lesssim C_\varepsilon\bar{\mathcal{D}}_n^t+\varepsilon\sum_{r=1}^{n-1}\|\partial_t^r\boldsymbol{u}\|^2_{2n-2r+1}+\|\partial_t^n\theta\|_1^2+Z_n\\ &\lesssim C_\varepsilon\bar{\mathcal{D}}_n^t+\varepsilon\sum_{r=1}^{n-1}\|\partial_t^r\boldsymbol{u}\|^2_{2n-2r+1}+Z_n,\end{aligned} \tag{2.118}$$

which combined with (2.106) yields

$$\sum_{r=1}^{n}\left(\|\partial_t^r\boldsymbol{u}\|^2_{2n-2r+1}+\|\partial_t^r\theta\|^2_{2n-2r+1}\right)\lesssim Y_n+Z_n+\bar{\mathcal{D}}_n^t+\|\boldsymbol{u}\|^2_{2n-1}. \tag{2.119}$$

Using (2.45) and (2.47) to estimate $Y_n+Z_n\lesssim\mathcal{E}_n\mathcal{D}_n$, we then conclude (2.103). □

Lemma 2.10 *For $n\geq 3$, it holds that*

$$\mathcal{E}_n\lesssim\bar{\mathcal{E}}_n^t+\bar{\mathcal{E}}_n^{h,x}+\mathfrak{E}_n+S+\mathcal{E}_n^2. \tag{2.120}$$

Proof. Applying ∂_t^r, $0\leq r\leq n-1$, to (2.104), and then employing the elliptic estimates with $2n-2r\geq 2$, one has

$$\begin{aligned}\|\partial_t^r u\|^2_{2n-2r} &\lesssim \|\partial_t^{r+1}\boldsymbol{u}\|^2_{2n-2r-2}+\|\nabla^2\partial_t^r\mathbb{X}\|^2_{2n-2r-2}+\|\nabla(\rho_s\partial_t^r(\theta-1))\|^2_{2n-2r-2}\\ &\quad+\|\partial_t^r(F+G)\|^2_{2n-2r-2}\\ &\lesssim \|\partial_t^{r+1}\boldsymbol{u}\|^2_{2n-2(r+1)}+\|\partial_t^r\mathbb{X}\|^2_{2n-2r}+\|\partial_t^r(\theta-1)\|^2_{2n-2r-1}+U_n,\end{aligned} \tag{2.121}$$

where

$$U_n=\|\bar{\nabla}_0^{2n-2}(F+G)\|_0^2. \tag{2.122}$$

A simple induction on (2.121) yields, since $\partial_t X=\boldsymbol{u}$

$$\begin{aligned}\sum_{r=0}^{n}\|\partial_t^r\boldsymbol{u}\|^2_{2n-2r} &\lesssim \|\partial_t^n\boldsymbol{u}\|_0^2+\sum_{r=0}^{n-1}\|\partial_t^r\mathbb{X}\|^2_{2n-2r}+\sum_{r=0}^{n-1}\|\partial_t^r(\theta-1)\|^2_{2n-2r-1}+U_n\\ &\lesssim \bar{\mathcal{E}}_n^t+\|\mathbb{X}\|^2_{2n}+\sum_{r=1}^{n-1}\|\partial_t^{r-1}\boldsymbol{u}\|^2_{2n-2r}+\varepsilon\sum_{r=0}^{n-1}\|\partial_t^r(\theta-1)\|^2_{2n-2r}\\ &\quad+C_\varepsilon\sum_{r=0}^{n-1}\|\partial_t^r(\theta-1)\|_0^2+U_n\\ &\lesssim C_\varepsilon\bar{\mathcal{E}}_n^t+\bar{\mathcal{E}}_n^{h,x}+\mathfrak{E}_n+\varepsilon\sum_{r=0}^{n-2}\|\partial_t^r\boldsymbol{u}\|^2_{2n-2r}+\varepsilon\sum_{r=0}^{n-1}\|\partial_t^r(\theta-1)\|^2_{2n-2r}+U_n,\end{aligned} \tag{2.123}$$

where we have used the facts $\|\mathbb{X}\|^2_{2n+1}\lesssim\bar{\mathcal{E}}_n^{h,x}+\mathfrak{E}_n$, $\sum_{r=0}^{n-2}\|\partial_t^r\boldsymbol{u}\|_0^2\lesssim\bar{\mathcal{E}}_n^t$ and $\sum_{r=0}^{n-1}\|\partial_t^r(\theta-1)\|_0^2\lesssim\bar{\mathcal{E}}_n^t$.

Applying $\partial_t^r\partial_3$, $0\leq r\leq n-1$, to (2.107), and then employing the elliptic estimates with $2n-2r-1\geq 2$, one has

$$\|\partial_t^r\partial_3\theta\|^2_{2n-2r-1}\lesssim\|\partial_t^{r+1}\partial_3\theta\|^2_{2n-2r-3}+\|\nabla\partial_t^r\boldsymbol{u}\|^2_{2n-2r-2}+\Theta_n$$

$$\lesssim \|\partial_t^{r+1}\partial_3\theta\|^2_{2n-2(r+1)-1} + \|\partial_t^r \boldsymbol{u}\|^2_{2n-2r-1} + \Theta_n, \tag{2.124}$$

where

$$\Theta_n = \|\bar{\nabla}_0^{2n-2} E\|_0^2. \tag{2.125}$$

A simple induction on (2.124) yields

$$\begin{aligned}\sum_{r=0}^{n-1} \|\partial_t^r\partial_3\theta\|^2_{2n-2r-1} &\lesssim \|\partial_t^{n-1}\partial_3\theta\|_1^2 + \sum_{r=0}^{n-2}\|\partial_t^r\boldsymbol{u}\|^2_{2n-2r-1} + \Theta_n \\ &\lesssim \|\partial_t^{n-1}\partial_3\theta\|_1^2 + \varepsilon\sum_{r=0}^{n-2}\|\partial_t^r\boldsymbol{u}\|^2_{2n-2r} + C_\varepsilon\sum_{r=0}^{n-2}\|\partial_t^r\boldsymbol{u}\|_0^2 + \Theta_n \\ &\lesssim \|\partial_t^{n-1}\partial_3\theta\|_1^2 + C_\varepsilon\bar{\mathcal{E}}_n^t + \varepsilon\sum_{r=0}^{n-2}\|\partial_t^r\boldsymbol{u}\|^2_{2n-2r} + \Theta_n. \end{aligned} \tag{2.126}$$

Similar as (2.111)-(2.116), one can derive

$$\|\nabla_h\partial_t^r\theta\|^2_{2n-2r-1} \lesssim \|\partial_3\partial_t^r\theta\|^2_{2n-2r-1} + \|\partial_t^{r+1}\theta\|^2_{2n-2(r+1)} + C_\varepsilon\bar{\mathcal{E}}_n^t + \varepsilon\|\partial_t^r\boldsymbol{u}\|^2_{2n-2r} + \Theta_n \tag{2.127}$$

By virtue of (2.126) and (2.127), for $r \le n-1$, we have

$$\begin{aligned}&\|\partial_t^r(\theta-1)\|^2_{2n-2r} = \|\nabla\partial_t^r\theta\|^2_{2n-2r-1} + \|\partial_t^r(\theta-1)\|_0^2 \\ &\lesssim \|\nabla_h\partial_t^r\theta\|^2_{2n-2r-1} + \|\partial_3\partial_t^r\theta\|^2_{2n-2r-1} + \bar{\mathcal{E}}_n^t + S \\ &\lesssim S + C_\varepsilon\bar{\mathcal{E}}_n^t + \varepsilon\sum_{r=0}^{n-2}\|\partial_t^r\boldsymbol{u}\|^2_{2n-2r} + \varepsilon\|\partial_t^r\boldsymbol{u}\|^2_{2n-2r} \\ &\qquad + \|\partial_t^{r+1}\theta\|^2_{2n-2(r+1)} + \|\partial_t^{n-1}\nabla\partial_3\theta\|_0^2 + \Theta_n. \end{aligned} \tag{2.128}$$

By a simple induction of (2.128), one has

$$\sum_{r=0}^{n}\|\partial_t^r(\theta-1)\|^2_{2n-2r} \lesssim S + C_\varepsilon\bar{\mathcal{E}}_n^t + \varepsilon\sum_{r=0}^{n-1}\|\partial_t^r\boldsymbol{u}\|^2_{2n-2r} + \|\partial_t^{n-1}\nabla^2\theta\|_0^2 + \Theta_n. \tag{2.129}$$

Applying ∂_t^{n-1} to (2.107) and then applying $\|\cdot\|_0^2$, by integration by parts and $\partial_3\theta|_{\partial\Omega} = 0$, we can deduce

$$\begin{aligned}\|\nabla^2\partial_t^{n-1}\theta\|_0^2 = \|\Delta\partial_t^{n-1}\theta\|_0^2 &\lesssim \|\partial_t^n\theta\|_0^2 + \|\mathrm{div}\partial_t^{n-1}\boldsymbol{u}\|_0^2 + \|\partial_t^{n-1}E\|_0^2 \\ &\lesssim \bar{\mathcal{E}}_n^t + \|\partial_t^{n-1}\boldsymbol{u}\|_1^2 + \Theta_n. \end{aligned} \tag{2.130}$$

Combined with (2.123) and (2.129)-(2.130), it follows

$$\sum_{r=0}^{n}\|\partial_t^r(\boldsymbol{u},\theta-1)\|^2_{2n-2r} \lesssim \bar{\mathcal{E}}_n^t + \bar{\mathcal{E}}_n^{h,x} + \mathfrak{E} + S + U_n + \Theta_n. \tag{2.131}$$

Using (2.46) to estimate $U_n + \Theta_n \lesssim (\mathcal{E}_n)^2$, we then conclude (2.120). □

Lemma 2.11 *For $n = N+2$ or $2N$, there exists an energy $\tilde{\mathcal{E}}_n$ which is equivalent to $\mathcal{E}_n$ such that*

$$\frac{d}{dt}\tilde{\mathcal{E}}_{2N} + \mathcal{D}_{2N} \lesssim \sqrt{\mathcal{E}_{N+2}}\mathcal{E}_{2N} \tag{2.132}$$

and

$$\frac{d}{dt}\tilde{\mathcal{E}}_{N+2} + \mathcal{D}_{N+2} \le 0. \tag{2.133}$$

Proof. It follows from (2.83) of Lemma 2.8 and (2.103) of Lemma 2.9, that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}_n + \mathcal{D}_n &\lesssim \mathcal{E}_n\mathcal{D}_n + \|\boldsymbol{u}\|_{2n-1}^2 + \bar{\mathcal{D}}_n^t + \bar{\mathcal{D}}_n^{h,u} + \bar{\mathcal{D}}_n^{h,x} \\ &\lesssim (\mathcal{E}_n + \varepsilon)\mathcal{D}_n + C_\varepsilon\|\boldsymbol{u}\|_0^2 + \bar{\mathcal{D}}_n^t + \bar{\mathcal{D}}_n^{h,u} + \bar{\mathcal{D}}_n^{h,x} \\ &\lesssim (\mathcal{E}_n + \varepsilon)\mathcal{D}_n + C_\varepsilon\bar{\mathcal{D}}_n^t + \bar{\mathcal{D}}_n^{h,u} + \bar{\mathcal{D}}_n^{h,x}, \end{aligned} \tag{2.134}$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}_n + \mathcal{D}_n \lesssim \bar{\mathcal{D}}_n^t + \bar{\mathcal{D}}_n^{h,u} + \bar{\mathcal{D}}_n^{h,x}. \tag{2.135}$$

Let n denote either $2N$ or $N+2$ through the proof, and we use the compact notation Q_n with

$$Q_{2N} := \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N} + \sqrt{\mathcal{E}_{N+2}}\mathcal{E}_{2N} \quad \text{and} \quad Q_{N+2} := \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{N+2}. \tag{2.136}$$

It follows from Lemma 2.1-2.2, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}(S + \bar{\mathcal{E}}_n^t) + \bar{\mathcal{D}}^s + \bar{\mathcal{D}}_n^t \lesssim \sqrt{\mathcal{E}_n}\mathcal{D}_n. \tag{2.137}$$

By virtue of Lemma 2.3-2.4, there exists a large constant K_1, such that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathcal{E}}_n^{h,u} + K_1\left[\bar{\mathcal{E}}_0^{h,x} + 2\int \rho_s\boldsymbol{u}\cdot\mathbb{X}\right]\right) + \bar{\mathcal{D}}_n^{h,u} + (K_1 - C_\varepsilon)\bar{\mathcal{D}}_0^{h,x} \\ &\lesssim Q_n + \sqrt{\mathcal{E}_n}\mathcal{D}_n + \varepsilon(\bar{\mathcal{D}}_n^{h,\theta} + \bar{\mathcal{D}}_n^{h,u} + \bar{\mathcal{D}}_n^{h,x} + \bar{\mathcal{D}}_n^{h,u_t}) + C_\varepsilon(\bar{\mathcal{D}}^s + \bar{\mathcal{D}}_n^t). \end{aligned} \tag{2.138}$$

By Lemma 2.3 and Lemma 2.5, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathcal{E}}_n^{h,x} + 2\sum_{\substack{\alpha\in\mathbb{N}^2\\|\alpha|\le 2n}}\int \rho_s\partial^\alpha\boldsymbol{u}\cdot\partial^\alpha\mathbb{X} + \bar{\mathcal{E}}_n^{h,\theta}\right) + \bar{\mathcal{D}}_n^{h,x} + \bar{\mathcal{D}}_n^{h,\theta} \\ &\lesssim \sqrt{\mathcal{E}_n}\mathcal{D}_n + Q_n + C_\varepsilon\bar{\mathcal{D}}^s + \varepsilon(\bar{\mathcal{D}}_n^{h,u} + \bar{\mathcal{D}}_n^{h,u_t} + \bar{\mathcal{D}}_n^{h,\theta_t}). \end{aligned} \tag{2.139}$$

Hence, by means of (2.137)-(2.139), there exists a large constant $K_2 >> K_1 >> C_\varepsilon > 1$ and a small $0 < 2\varepsilon < \varepsilon_1 << 1$ such that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[K_2(S + \bar{\mathcal{E}}_n^t) + \bar{\mathcal{E}}_n^{h,u} + K_1\left[\bar{\mathcal{E}}_0^{h,x} + 2\int\rho_s\boldsymbol{u}\cdot\mathbb{X}\right] + \varepsilon_1\left(\bar{\mathcal{E}}_n^{h,x} + 2\sum_{\substack{\alpha\in\mathbb{N}^2\\|\alpha|\le 2n}}\int\rho_s\partial^\alpha\boldsymbol{u}\cdot\partial^\alpha\mathbb{X} + \bar{\mathcal{E}}_n^{h,\theta}\right)\right] \\ &+(K_2 - \varepsilon_1 C_\varepsilon - C_\varepsilon)\bar{\mathcal{D}}^s + (K_2 - C_\varepsilon)\bar{\mathcal{D}}_n^t + (1 - \varepsilon_1\varepsilon - \varepsilon)\bar{\mathcal{D}}_n^{h,u} + (\varepsilon_1 - \varepsilon)(\bar{\mathcal{D}}_n^{h,x} + \bar{\mathcal{D}}_n^{h,\theta}) \\ &\lesssim Q_n + \sqrt{\mathcal{E}_n}\mathcal{D}_n + \varepsilon(\bar{\mathcal{D}}_n^{h,\theta_t} + \bar{\mathcal{D}}_n^{h,u_t}). \end{aligned} \tag{2.140}$$

Thanks to Lemma 2.7 and Lemma 2.6, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\mathcal{E}}_n^{h,\theta_t} + \bar{\mathcal{E}}_n^{h,u_t}) + \bar{\mathcal{D}}_n^{h,\theta_t} + \bar{\mathcal{D}}_n^{h,u_t} \lesssim \sqrt{\mathcal{E}_n}\mathcal{D}_n + C_\varepsilon(\bar{\mathcal{D}}_n^t + \bar{\mathcal{D}}^s + \bar{\mathcal{D}}_n^{h,x} + \bar{\mathcal{D}}_n^{h,\theta}) + \varepsilon\bar{\mathcal{D}}_n^{h,u}. \tag{2.141}$$

Multiplying (2.141) by $0 < \varepsilon < \varepsilon_2 < \varepsilon_1 - \varepsilon << 1$ and adding with (2.140), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[K_2(S + \bar{\mathcal{E}}_n^t) + \bar{\mathcal{E}}_n^{h,u} + K_1\left(\bar{\mathcal{E}}_0^{h,x} + 2\int\rho_s\boldsymbol{u}\cdot\mathbb{X}\right) + \varepsilon_1\left(\bar{\mathcal{E}}_n^{h,x} + 2\sum_{\substack{\alpha\in\mathbb{N}^2\\|\alpha|\le 2n}}\int\rho_s\partial^\alpha\boldsymbol{u}\cdot\partial^\alpha\mathbb{X} + \bar{\mathcal{E}}_n^{h,\theta}\right)\right.$$

$$+\varepsilon_2(\bar{\mathcal{E}}_n^{h,\theta_t}+\bar{\mathcal{E}}_n^{h,u_t})\Bigg]+\bar{\mathcal{D}}^s+\bar{\mathcal{D}}_n^t+\bar{\mathcal{D}}_n^{h,u}+\bar{\mathcal{D}}_n^{h,x}+\bar{\mathcal{D}}_n^{h,\theta}+\bar{\mathcal{D}}_n^{h,u_t}+\bar{\mathcal{D}}_n^{h,\theta_t}\lesssim Q_n+\sqrt{\mathcal{E}_n}\mathcal{D}_n. \quad (2.142)$$

We then deduce from (2.142) and (2.135), for $0<\varepsilon_3<<1$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Bigg[K_2(S+\bar{\mathcal{E}}_n^t)+\bar{\mathcal{E}}_n^{h,u}+K_1\left(\bar{\mathcal{E}}_0^{h,x}+2\int\rho_s\boldsymbol{u}\cdot\mathbb{X}\right)+\varepsilon_1\Bigg(\bar{\mathcal{E}}_n^{h,x}+2\sum_{\substack{\alpha\in\mathbb{N}^2\\|\alpha|\le 2n}}\int\rho_s\partial^\alpha\boldsymbol{u}\cdot\partial^\alpha\mathbb{X}+\bar{\mathcal{E}}_n^{h,\theta}\Bigg)\\&\quad+\varepsilon_2(\bar{\mathcal{E}}_n^{h,\theta_t}+\bar{\mathcal{E}}_n^{h,u_t})+\varepsilon_3\mathfrak{E}_n\Bigg]+\bar{\mathcal{D}}^s+\bar{\mathcal{D}}_n^t+\bar{\mathcal{D}}_n^{h,u}+\bar{\mathcal{D}}_n^{h,x}+\bar{\mathcal{D}}_n^{h,\theta}+\bar{\mathcal{D}}_n^{h,u_t}+\bar{\mathcal{D}}_n^{h,\theta_t}+\mathcal{D}_n\\&\lesssim Q_n+\sqrt{\mathcal{E}_n}\mathcal{D}_n. \end{aligned}\quad (2.143)$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_n+\mathcal{D}_n\lesssim Q_n, \quad (2.144)$$

where

$$\begin{aligned}\tilde{\mathcal{E}}_n:=\Bigg[&K_2(S+\bar{\mathcal{E}}_n^t)+\bar{\mathcal{E}}_n^{h,u}+K_1\left(\bar{\mathcal{E}}_0^{h,x}+2\int\rho_s\boldsymbol{u}\cdot\mathbb{X}\right)+\varepsilon_2(\bar{\mathcal{E}}_n^{h,\theta_t}+\bar{\mathcal{E}}_n^{h,u_t})+\varepsilon_3\mathfrak{E}_n\\&+\varepsilon_1\Bigg(\bar{\mathcal{E}}_n^{h,x}+2\sum_{\substack{\alpha\in\mathbb{N}^2\\|\alpha|\le 2n}}\int\rho_s\partial^\alpha\boldsymbol{u}\cdot\partial^\alpha\mathbb{X}+\bar{\mathcal{E}}_n^{h,\theta}+\bar{\mathcal{E}}_n^{h,\theta_t}\Bigg)\Bigg].\end{aligned}$$

By virtue of (2.120), one has

$$\mathcal{E}_n\lesssim\tilde{\mathcal{E}}_n+(\mathcal{E}_n)^2, \quad (2.145)$$

that is to say $\tilde{\mathcal{E}}_n$ is equivalent to $\mathcal{E}_n$ since $\mathcal{E}_{2N}\le\eta$ is small. We thus deduce (2.132) and (2.133) from (2.144) by recalling the notation Q_n and using again that $\mathcal{E}_{2N}(T)\le\eta$ is small. □

2.5 Global energy estimates.

Lemma 2.12 *There exists a $0<\eta<1$ such that if $G_{2N}\le\eta$, then*

$$\mathcal{E}_{2N}(t)+\int_0^T\mathcal{D}_{2N}(t)\,dt\lesssim\mathcal{E}_{2N}(0) \quad (2.146)$$

for all $0\le t\le T$.

Proof. Integrating (2.132) directly in time, we find that

$$\begin{aligned}&\mathcal{E}_{2N}(t)+\int_0^T\mathcal{D}_{2N}(t)\mathrm{d}t\lesssim\mathcal{E}_{2N}(0)+\int_0^T\sqrt{\mathcal{E}_{N+2}}\mathcal{E}_{2N}\mathrm{d}t\\&\lesssim\mathcal{E}_{2N}(0)+\sup_{0\le t\le T}\mathcal{E}_{2N}(t)\int_0^T\sqrt{\eta}(1+t)^{-N+2}\mathrm{d}t\lesssim\mathcal{E}_{2N}(0)+\sqrt{\eta}\sup_{0\le t\le T}\mathcal{E}_{2N}(t),\end{aligned}\quad (2.147)$$

where we have used the fact that $N\ge 4$. This proves the estimate (2.146) since η is small. □

It remains to show the decay estimates of $\mathcal{E}_{N+2}$.

Lemma 2.13 *There exists a general constant $0 < \eta < 1$ so that if $G_{2N}(T) \leq \eta$, then*

$$(1+t)^{2N-4}\mathcal{E}_{N+2}(t) \lesssim \mathcal{E}_{2N}(0) \tag{2.148}$$

for all $0 \leq t \leq T$.

Proof. If we can obtain that $\tilde{\mathcal{E}}_{N+2} \leq M(\mathcal{D}_{N+2})^a$, $0 < a < 1$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_{N+2} + CM^{-1/a}\tilde{\mathcal{E}}_{N+2}^{1/a} \leq 0, \tag{2.149}$$

which means that

$$\tilde{\mathcal{E}}_{N+2} \lesssim \tilde{\mathcal{E}}_{N+2}(0)\left(1 + \frac{CM^{1/a}\tilde{\mathcal{E}}_{N+2}(0)^{1/a-1}}{1/a-1}t\right)^{-1/(1/a-1)}. \tag{2.150}$$

However, every term in $\mathcal{E}_{N+2}$ can be controlled by $\mathcal{D}_{N+2}$, except $\|\nabla_h^{2(N+2)+1}\mathbb{X}\|_0^2$. Using the Sobolev's inequality in horizontal, we have

$$\|\nabla_h^{2(N+2)+1}\mathbb{X}\|_0^2 \lesssim \|\nabla_h^{2(N+2)}\mathbb{X}\|_0^{2a}\|\nabla_h^{4N+1}\mathbb{X}\|_0^{2(1-a)} \lesssim (\mathcal{D}_{N+2})^a(\mathcal{E}_{2N})^{1-a}, \tag{2.151}$$

where $a = \dfrac{2N-4}{2N-3} < 1$. Thus, we may derive

$$\mathcal{E}_{N+2} \lesssim (\mathcal{D}_{N+2})^a(\mathcal{E}_{2N})^{1-a}. \tag{2.152}$$

By virtue of (2.146), one has

$$\sup_{0\leq t\leq T} \mathcal{E}_{2N}(t) \lesssim \mathcal{E}_{2N}(0), \tag{2.153}$$

which combined with (2.152) yields

$$\tilde{\mathcal{E}}_{N+2}(t) \leq C\mathcal{E}_{N+2}(t) \leq C(\mathcal{E}_{2N}(0))^{1-a}(\mathcal{D}_{N+2})^a. \tag{2.154}$$

So, one has $M = C(\mathcal{E}_{2N}(0))^{1-a}$, putting into (2.150), by $\tilde{\mathcal{E}}_{N+2}(0) \lesssim \mathcal{E}_{N+2}(0) \lesssim \mathcal{E}_{2N}(0)$, we may conclude (2.148).

The Lemmas 2.12 and 2.13 can directly get the following result.

Theorem 2.1 *There exists a general constant $0 < \eta < 1$ such that if $G_{2N} \leq \eta$, then*

$$G_{2N} \lesssim \mathcal{E}_{2N}(0) \tag{2.155}$$

for all $0 \leq t \leq T$.

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