Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

RESEARCH ARICLE

Mathematical modelling of dynamic characteristics of repair process for system operating under maintenance contracts

Nataša Kontrec*[1] | Jelena Vujaković[1] | Marina Tošić[1] | Stefan Panić[2] | Biljana Panić[3]

[1]Department of Mathematics, Faculty of Sciences, University of Priština - Kosovska Mitrovica, Lole Ribara St. 29, 38220 Kosovska Mitrovica, Serbia
[2]Department of Informatics, Faculty of Sciences, University of Priština - Kosovska Mitrovica, Lole Ribara St. 29, 38220 Kosovska Mitrovica, Serbia
[3]Department of Operation Research, Faculty of Organizational Sciences, University of Belgrade, Jove Ilica 154, 11000 Belgrade, Serbia

Correspondence
*Nataša Kontrec, Faculty of Sciences, University in Priština, Lole Ribara St 29, 38220 Kosovska Mitrovica, Serbia. Email: natasa.kontrec@pr.ac.rs

Summary

Repair rate is very important parameter in a system maintainability and it can be defined as frequency of the successfully performed repair actions on failed component per unit of time. This paper analyses the integral characteristics of a stochastic repair rate for corresponding values of availability in the system operating under maintenance contracts. The equation for the envelope line of the probability density function (PDF) maximums of the repair rate has been provided. This new expression can be used for planning of base stock levels and capacities of repair facilities. Namely, in that way instead of repair rate PDF equation, for some calculations we can use envelope line parameters, which are expressed in simpler mathematical form, to reduce the time required for calculations and prediction and enhance reactions in failure events. For analytical and numerical evaluation of system performance, the annual repair rate PDFs are analyzed like particular solutions of corresponding differential equation, while the existence of singular solution is considered and analyzed under different conditions. Moreover, we have derived optimal values of availability for which the PDF maximums have been obtained. Finally, in order to generalize behavior of the repair process, a partial differential equation, as a function of the repair rate process and availability parameter, has been formed.

KEYWORDS:
repair rate, dynamic characteristic, availability, maintenance contracts

1 | INTRODUCTION

The maintenance contracts are of particular interest to defence and air force industry, which has a goal to reduce the overall operating costs of maintaining a complex system with simultaneous improvement of its performance[1,2]. The major criteria for signing of these contracts are availability[3], the numbers of spare parts and reliability. Kang et al. have examined the systems for supplies management according to the maintenance contracts in detail[4]. The methodology that they have developed helps determining the system's availability based on the reliability of its components and possibilities for their maintenance. In their research, they reached the conclusions that mean time between failure (MTBF) and mean time to repair (MTTR) have the greatest impact on availability. Papers[5,6,7,8] provided major contribution concerning the issue of determining the availability of repairable systems and components under the maintenance contracts. In[9] this problem was examined with the implementation of several assumptions such as fixed frequency of failures, fixed time required to repair those failures and unlimited capacities required for repairs. Based on that authors concluded that service at maintenance and repair organization (MRO) should be as short and as cost-effective as possible. Some interesting conclusions regarding the minimal repair rate are given in[10] and[11]. Actually, according to[12] repair data are often used to predict repair time or maintenability performance of item. Time series can also be used for modeling services at MRO. For instance, ARMA models presented in[13] and[14] can be adapted for the prupose. Furher, in[15] a model for analysis of repair time has been presented. Due to importance of repair time in MRO, in[16] the authors have recommended a novel stochastic model for determining the annual repair rate for critical aircraft components in order to achieve the desired level of availability. Those results can be used for planning of required numbers of spare parts and the repair facilities' overall capacities. The described model can also be used to assess other vital parameters related to system maintenance. Moreover, setting the availability parameters to the required values and assuming that the basic level of supplies is fixed to a constant value, a repair rate can be determined not only for critical components presented in the paper, but any other system that can be repaired and fulfils the previously set assumptions. Using an expression for the probability density function (PDF) can precisely model the repair rate by generating the exact values of repair rate samples for the related values of availability and the MTBF. Such simulations can also serve for dynamic forecasting of system's characteristics, as well as for planning and implementation of new repair stations and increase of any other repair capacities in order to improve efficiency of the maintenance systems. The previous analysis of those probabilistic models for describing the repair process, depending on the individual relevant parameters for specific availability conditions, resulted in the idea to focus the subject of research at the request of the integral description of phenomena that affect the complex system renewal process. We tried to determine the analytical expressions required to describe the boundary conditions that could be useful in planning of the repair process.

This paper has been structured in the following way: In Section 2 we have discussed the preliminaries about the random repair process. The integral characteristics of the repair rate function have been presented in Section 3. In this section, we have provided an equation of an envelope curve of repair rate process maximums. Moreover, by observing this envelope equation as a singular solution, we have derived a differential equation which can describe the dynamics of this repair process, where the repair rate PDFs denote the partical solutions of such differential equation. Also, we have derived the optimal values of availability for which the PDF maximums have been obtained. The results obtained in this section have been graphically presented and analysed in the function of system parameters in Section 4. This section includes yet another contribution by this paper - a partial differential equation as a function of the repair rate and availability. Finally, the concluding remarks have been outlined in Section 5.

2 | REPAIR RATE MODEL

With regard to the system maintenance, the repairable and non-repairable systems should be clearly distinguished. In this paper we deal exclusively with the repairable systems. The repairable systems are those that can be restored to their operational state after the failure. The system can only exist in one of the two states i.e. it alternates between operative and non-operative state. The sequence of operational time of the system can be denoted by $U = \left(U_1, U_2, \dots, U_n\right)$, while the failure states can be denoted by $D = (D_1, D_2, \dots, D_n)$. We have assumed that $W = \left((U_1, D_1), (U_2, D_2), ..., (U_n, D_n)\right)$ is a set of independent alternating cycles. We have also assumed that the perfect repair had been carried out at the constant rate after which system behaves exactly the same as the new one. The renewal process corresponding to this model is called the alternating renewal process. Since is actually MTBF and is MTTR, based on the renewal theorem[17] we can determine the system's availability as:

$$A = \frac{E(U)}{E(U) + E(D)} \tag{1}$$

which is, based on the previous, equal to:

$$A = \frac{MTBF}{MTBF + MTTR}. \tag{2}$$

In[16], the authors observed such system - a system with an alternating renewal process, which, after repairs, returns to its original state and they determined the annual repair rate in the case when the MTBF has a Rayleigh distribution i.e.:

$$MTBF = \int_0^\infty t p_T(t) = \int_0^\infty \frac{2t^2}{x} \exp\left(-\frac{t^2}{x}\right) dt, \tag{3}$$

given with:

$$p_T(t) = \frac{2t}{x} \exp\left(-\frac{t^2}{x}\right) \tag{4}$$

where x is the distribution parameter determined by the relation $x = \frac{4(MTBF)^2}{\pi}$. Based on these assumptions the authors determined the PDF function of the repair rate μ as

$$p_\mu(\mu) = \frac{8A^2}{(1-A)^2 \mu^3 \pi x_0} \exp\left(\frac{-4A^2}{(1-A)^2 \mu^2 \pi x_0}\right) \tag{5}$$

where $x_0 = E(x)$ i.e. expected value of x for the observed part of the failed system. As shown in[6] this PDF is not symmetric so its mean value does not correspond to its maximal value. Since this value is an important performance indicator of repair process under maintenance contracts, it would be interesting to observe the behaviour of its maximum value.

Here, we will observe this PDF as a function of two parameters: the repair rate, μ and the availability, A, while x_0 from eq. (5) will be observed as a constant, with its value related to the MTBF. In the process of determination and analysis of the repair rate distribution integral characteristics, one of the PDF parameters will be treated as a variable, while the other is set to certain constant values of interest in practice. In this way, the PDF curves family is obtained. The analysis of the position of the maximums for this curves family can be performed analytically, using the first derivative of function, and also numerically. The same procedure is repeated for the case when the second parameter is treated as a variable, and the others are set to constant values. This procedure gives a new PDF curves family, while the maximum position is determined by a new envelope. For each of the PDF curves families, the equation of the envelope of curves maximums is considered. The direction coefficients and values on ordinate- axis are determined analytically and numerically, for both envelopes. Also, in both cases, the differential equations describing the process dynamics are determined, whereby the envelopes of the PDF curves families represent their singular solutions and PDFs represent their particular solution.

3 | INTEGRAL CHARACTERISTICS OF REPAIR PROCESS

We have observed an unmanned aerial vehicle (UAV). This concept is not new but due to insufficient data on UAV's reliability it has been poorly implemented in civil sector[18]. In[16] UAV's engine has been considered in the case when failure rate λ is known ($\lambda = 1.92$ as calculated in[5]). The repair rate has been considered as a stochastic process and the engine repair time was calculated on an annual basis for the pre-defined availability values set as:

$$A = 0.85,\ A = 0.90,\ A = 0.95.$$

The graphic presentation of the dependence of repair rate PDF versus the repair rate, for a fixed value of the MTBF parameter x_0 and different values of availability parameter A, in logarithmic scale, is shown in Figure 1. The analysis of this dependence shows that the increase of parameter A results in smaller values of the PDF maximums reached for higher values of repair rate.

It can be concluded that all the maximums lie along a single curve, so the envelope of maximums is traceable. In order to determine the values of the repair rate level when the maximums are reached, as well as the values of maximums and a direction coefficient of the repair rate, the first derivative of relative repair rate level is determined.

$$\begin{aligned} &\left.\frac{dp(\mu,A)}{d\mu}\right|_{\mu=\mu_{\max}} = 0 \\ &\mu_{\max} = \frac{2\sqrt{2}A}{\sqrt{3\pi x_0(1-A)^2}}. \end{aligned} \tag{6}$$

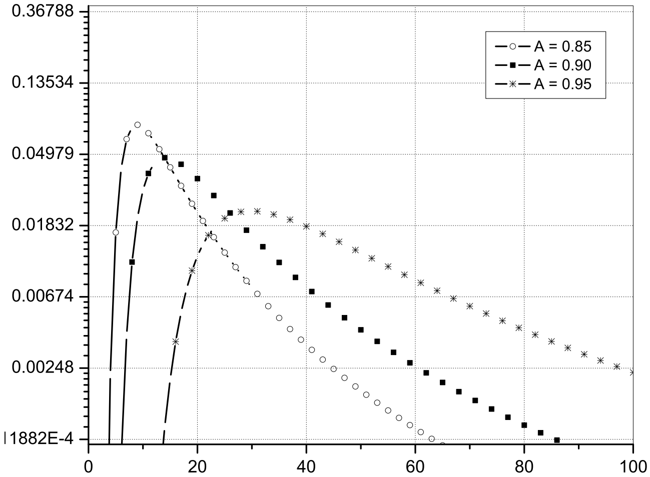

FIGURE 1 PDF of engine repair rate

Substituting (6) into (5) yields:

$$p_{\mu_{\max}}(A) = 3\sqrt{\frac{3\pi x_0}{8}}\exp\left(-\frac{3}{2}\right)\frac{1}{A} - 3\sqrt{\frac{3\pi x_0}{8}}\exp\left(-\frac{3}{2}\right), \tag{7}$$

which can be simplified as:

$$p_{\mu_{\max}}(A) \approx 0.72\sqrt{x_0}\frac{1}{A} - 0.72\sqrt{x_0} \tag{8}$$

or equivalently:

$$\begin{aligned} p_{\mu_{\max}}(\tilde{A}) &= 0.72\sqrt{x_0}\tilde{A} - 0.72\sqrt{x_0} = k\tilde{A} + n \\ \tilde{A} &= \tfrac{1}{A}; \end{aligned} \tag{9}$$

where

$$\begin{aligned} k &= k(x_0) = 0.72\sqrt{x_0}, \\ n &= n(x_0) = -0.72\sqrt{x_0}. \end{aligned} \tag{10}$$

The envelope determines a certain singular solution of differential equation which can describe the dynamics of this process, while the PDF is its particular solution [19]:

$$\frac{\partial p_\mu(\mu)}{\partial \mu} + p_\mu(\mu)\left(\frac{3}{\mu} - \frac{8A^2}{(1-A)^2\mu^3\pi x_0}\right) = 0. \tag{11}$$

According to the theory of differential equations, singular solutions cannot be found from general ones, and that is their significance, as well as the significance of the contribution of this paper. Using the obtained differential equations, we are able to simplify the calculations related to determining the characteristics of the repair rate PDF distribution model. Namely, PDF repair rate distribution functions can, instead of initial equations, be replaced in certain calculations by envelope parameters, which, as shown, are described by simpler functions. That will reduce the time required for numerical calculations and optimize appropriate system failure support. Now, in simmilar manner we can determine the values of the availability for which maximum of PDF are obtained as:

$$\frac{dp(\mu,A)}{dA}\Big| A = A_{\max} = 0;\qquad A_{\max} = \frac{\mu\sqrt{\pi}\sqrt{x_0}}{2+\mu\sqrt{\pi}\sqrt{x_0}} \tag{12}$$

The values of these maximums and coefficients of directions of envelope of maximums can be determined as in the previous case.

4 | RESULT ANALYSIS

The envelopes of the PDF curves families maximums, for different values of the MTBF parameter x_0, in case the observed availability is viewed as a parameter, are presented in Figures 2 and 3.

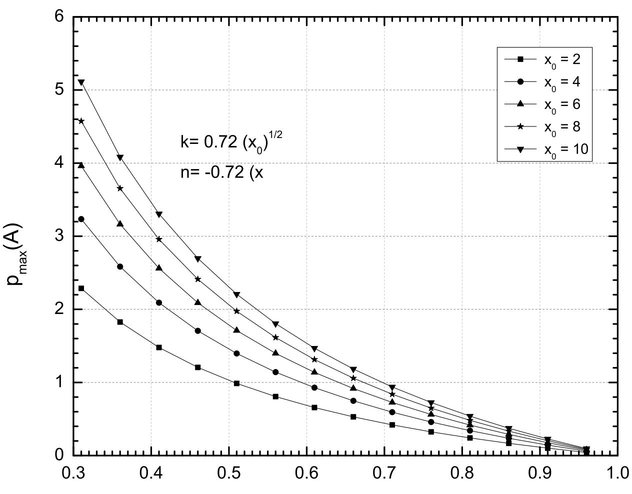

FIGURE 2 Maximums of the repair rate PDF as a function of required part availability

In that way the maximal value of the repair rate PDF can be efficiently predicted. Thus, the boundary conditions for the repair service under maintenance contract, with the predefined MTBF conditions, could be defined, while some system performance measures could also be evaluated, since they are all related to the repair rate. It can be concluded that an increase of the MTBF x_0 parameter results in increase of the repair rate maximum envelope direction coefficient and decrease of its value on the ordinate axis, when the repair rate PDF curves maximums are analyzed versus parameter x_0, as it is presented in Figure 2. Also it is interesting to observe the linearized form of the repair rate maximum envelope for the observed argument of parameter $\frac{1}{A}$, which

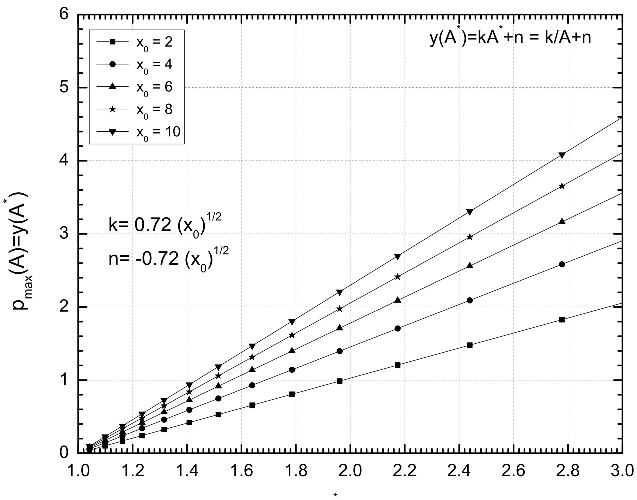

FIGURE 3 Linearized function of maximums of the repair rate PDF

is presented in Figure 3. Also, based on the aforementioned, by taking into account the influence of both the repair rate process and the availability parameter, after providing some basic mathematical transformations, the aircraft's engine repair process can be described with the following particular differential equation of first order:

$$\mu \frac{\partial p}{\partial \mu} + A\,(1 - A)\,\frac{\partial p}{\partial A} = -\frac{8A^2}{(1-A)^2 \pi x_0 \mu^3} \exp\left(-\frac{4A^2}{(1-A)^2 \mu^2 \pi x_0}\right) \tag{13}$$

where $p = p(\mu, A)$. By using this expression, these system variables can be expressed dynamically as a differential equation for the unknown repair rate of an aircraft part as a function of availability. More generally, this equation represents the law of repair rate variation of the engine repair process. Figure 4 is a graphical presentation of the partial differential equation solution for the observed engine repair process.

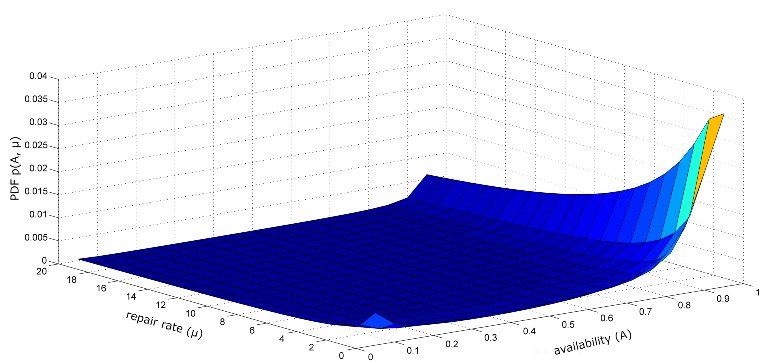

FIGURE 4 Solution for partial differential equation of repair rate process of engine

5 | CONCLUSION

Integral characteristics of repair process for corresponding availability have been analyzed. By observing repair rate PDF curve, we have concluded that all the maximums lie along a single curve, and that envelope of maximums is traceable. In that way the most common value of system repair rate can be efficiently predicted for any state of system parameters. Therefore, the boundary conditions for the repair rate process with given availability conditions could be defined, while some other performance measures could also be evaluated, since they are all related to the repair rate PDF. In addition, we have derived the optimal values of availability for which the PDF maximums have been obtained. Finally, in order to generalize the behaviour of the aircraft parts' repair rate process, a novel form of partial differential equation as a function of the repair rate and availability has been presented. Analytical considerations of repair rate PDF as particular solutions of provided partial differential equation provides the existence of singular partial differential equation solution, which could be used during the analysis of system renewal characteristics.

References

1. Phillips E. Performance based logistics: a whole new approach. *Aviation Week and Space Technology* 2005; 163(17): 52-55.

2. Randall W, Pohlen T, Hanna J. Evolving a theory of performance-based logistics using insights from service dominant logic. *Journal of Business Logistics* 2010; 31(2): 35-61. doi: https://doi.org/10.1002/j.2158-1592.2010.tb00142.x

3. Wong H, Cattrysse D, Oudheusden DV. Stocking decisions for repairable spare parts pooling in a multi-hub system. *International Journal of Production Economics* 2005; 93: 309-317. doi: https://doi.org/10.1016/j.ijpe.2004.06.029

4. Kang K, Doerr K, Boudreau M, Apte U. A decision support model for valuing proposed improvements in component reliability. tech. rep., 2005. Working paper.

5. Mirzahosseinian H, Piplani R. A study of repairable parts inventory system operating under performance-based contract. *European Journal of Operational Research* 2011; 214(2): 256-261. doi: https://doi.org/10.1016/j.ejor.2011.04.035

6. Nowicki D, Kumar U, Steudel H, Verma D. Spares provisioning under performance-based logistics contract: profit-centric approach. *Journal of the Operational Research Society* 2008; 59 (3): 342-352. doi: https://doi.org/10.1057/palgrave.jors.2602327

7. Carpitella S, Certa A, Izquierdo J, La Fata CM. k-out-of-n systems: An exact formula for the stationary availability and multi-objective configuration design based on mathematical programming and TOPSIS. *Journal of Computational and Applied Mathematics* 2018; 330: 1007 - 1015. doi: https://doi.org/10.1016/j.cam.2017.01.006

8. Jeang A, Ko C, Chung C, Chen Y, Lin I. Optimal availability for determining choice and repair policy of system components. *International Journal of Quality and Reliability Management* 2018; 36:3: 347-357. doi: https://doi.org/10.1108/IJQRM-12-2017-0255

9. Pogacnik B, Duhovnik J, Tavcar J. Aircraft fault forecasting at maintenance service on the basis of historic data and aircraft parameters. *Eksploatacja i Niezawodnosc – Maintenance and Reliability* 2017; 19:4: 624–633. doi: http://dx.doi.org/10.17531/ein.2017.4.17

10. Park M, Jung K, Park D. Optimal maintenance strategy under renewable warranty with repair time threshold. *Applied Mathematical Modelling* 2017; 43: 498-508. doi: https://doi.org/10.1016/j.apm.2016.11.015

11. Cha JH, Finkelstein M. New failure and minimal repair processes for repairable systems in a random environment. *Applied Stochastic Models in Business and Industry* 2019; 35: 522–536. doi: https://doi.org/10.1002/asmb.2331

12. Ayele YZ, Barabadi A, Fuqing Y. Mixture Lognormal Cox Regression Repair Model for Prediction of the Repair Time. In: *Proceedings of 2018 IEEE International Conference on Industrial Engineering and Engineering Management (IEEM)*.

13. Stojanovic V, Kevkic T, Ljajko E, Jelic G. Noise-Indicator ARMA Model with Application in Fitting Physically-Based Time Series. *U.P.B. Scientific Bulletin-Series A: Applied Mathematics and Physics* 2018; 81: 257-264.

14. Randjelovic M, Stojanovic V, Kevkic T. Noise-indicator autoregressive conditional heteroskedastic process with application in modeling actual time series. *U.P.B. Scientific Bulletin-Series A: Applied Mathematics and Physics* 2019; 81: 77-84.

15. Kontrec N, Panic S, Petrovic M, Milosevic H. A Stochastic Model for Achieving Required Level of Availability Based on the Repair Rate Analysis. *Tehnicki vjesnik* 2019; 26(4): 1171-1175. doi: https://doi.org/10.17559/TV-20171220201513

16. Kontrec N, Panic S, Petrovic M, Milosevic H. A stochastic model for estimation of repair rate for system operating under performance based logistics. *Eksploatacja i Niezawodnosc – Maintenance and Reliability* 2018; 20(1): 68-72. doi: http://dx.doi.org/10.17531/ein.2018.1.9

17. Ross S. *Applied probability models with optimization applications*. Courier Corporation . 2013.

18. Krawczyk M. Conditions for Unmanned Aircraft Reliability Determination.'. *Eksploatacja i Niezawodnosc – Maintenance and Reliability* 2013; 15(1): 31-36.

19. A. Polyanin VZ, Moussiaux A. *Handbook of First Order Partial Differential Equations*. Taylor & Francis, London . 2002.

□