On solutions of PDEs by using algebras

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Abstract. The components of complex differentiable functions define solutions for the Laplace's equation, and in a simply connected domain each solution of this equation is the first component of a complex analytic function. In this paper we generalize this result; for each PDE of the form $Au_{xx}+Bu_{xy}+Cu_{yy}=0$ and for each affine planar vector field φ, we give an associative and commutative 2D algebra with unit $\mathbb{A}$, with respect to which the components of all functions of the form $\mathcal{L}\circ\varphi$ define solutions for this PDE, where $\mathcal{L}$ is differentiable in the sense of Lorch with respect to $\mathbb{A}$. By using the generalized Cauchy-Riemann equations associated with $\varphi\mathbb{A}$-differentiability we show that each solution of these PDEs is a component of a $\varphi\mathbb{A}$-differentiable function. In the same way, for each PDE of the form $Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0$, the components of the exponential function e^{φ} defined with respect to A, define solutions for this PDE. Also, solutions for two independent variables 3^{th} order PDEs and 4^{th} order PDE are constructed; among these are the bi-harmonic, bi-wave, and bi-telegraph equations.

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Introduction

In this paper we consider the class of PDEs of the form

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0. \tag{1}$$

An important subclass is the PDEs having the form

$$Au_{xx}+Bu_{xy}+Cu_{yy}=0. \tag{2}$$

This includes Laplace's, wave, and heat equations between others.

When proposing a solution of the form $w = e^{ax+by}$ of (1) it is concluded that a and b must satisfy

$$Aa^2 + Bab + Cb^2 + Da + Eb + F = 0.$$

So, generically this set of solutions is parameterized by a conic. In this paper, given a PDE like (1) and a vector field

$$\varphi(x, y) = (ax + by + k, cx + dy + l) \tag{3}$$

with $Ac^2 + Bcd + Cd^2 \neq 0$, we found an algebra $\mathbb{A}$ with respect to which the components of the exponential function

$$\mathcal{E}(x, y) = e^{\varphi(x,y)} \tag{4}$$

define solutions of (1). If $D = E = 0$ in (1), similar results are obtained by using sine, cosine, hyperbolic sine, and hyperbolic cosine functions instead of the exponential function.

The components of complex analytic functions are harmonic functions, and in a simply connected domain each harmonic function is the first component of a complex analytic function. This result has been generalized in Theorems 2.2 and 2.3; for each PDE (2) and for each affine planar vector field φ with $Ac^2+Bcd+Cd^2 \neq 0$, an associative and commutative 2D algebra with unit $\mathbb{A}$ (see Section (1.1)) is given, with respect to which the components of all $\varphi\mathbb{A}$-differentiable functions (see Section (1.2)) are solutions for this PDE, and we show in each simply connected region that each solution of (2) is a component of a $\varphi\mathbb{A}$-differentiable function.

In Section 1 we introduce the definitions of algebra $\mathbb{A}$, of the pre-twisted differentiability, and the Cauchy-Rieman equations for the pre-twisted differentiability. In Section 2, given a 2^{nd} order PDE and an affine planar vector field, we give an algebra with respect to which the components of the exponential function e^{φ} define solutions of the PDE, and we give these solutions explicitly. Also used the sine, cosine, hyperbolic sine, and hyperbolic cosine functions instead the exponential function, for constructing solutions of 2^{nd} order PDEs. Moreover, for PDEs of the form (2) we construct families of pre-twisted differentiable functions whose components define solutions, and we show that each solution of these PDEs is a component of a $\varphi\mathbb{A}$-differentiable function. In the same sense in Section 3 3^{th} order PDEs and a 4^{th} order PDE are considered. In Section 2.5, given a PDEs of the form (2) and an affine planar vector fields φ, we give algebras $\mathbb{A}$ for which we show that components of the $\varphi\mathbb{A}$-differentiable functions are solutions of the given PDE.

1 Pre-twisted differentiability

1.1 Algebras $\mathbb{A}_1(p_1, p_2)$

We call to a $\mathbb{R}$-linear space $\mathbb{L}$ *an algebra* if it is endowed with a bilinear product $\mathbb{L} \times \mathbb{L} \to \mathbb{L}$ denoted by $(u, v) \mapsto uv$, which is associative and commutative $u(vw) = (uv)w$ and $uv = vu$ for all $u, v, w \in \mathbb{L}$; furthermore, there exists a unit $e \in \mathbb{L}$, which satisfies $eu = u$ for all $u \in \mathbb{L}$. An element $u \in \mathbb{L}$ is called *regular* if there exists $u^{-1} \in \mathbb{L}$ called *the inverse* of u such that $u^{-1}u = e$. We also use the notation e/u for u^{-1}. If $u \in \mathbb{L}$ is not regular, then u is called *singular*. $\mathbb{L}^*$

denotes the set of all the regular elements of $\mathbb{L}$. If $u, v \in \mathbb{L}$ and v is regular, the quotient u/v means uv^{-1}.

An *algebra* $\mathbb{A}$ will be an algebra where $\mathbb{L} = \mathbb{R}^2$ and an *algebra* $\mathbb{M}$ will be an algebra where $\mathbb{L}$ is a two dimensional matrix algebra in the space of matrices $M(2, \mathbb{R})$, where the algebra product corresponds to the matrix product. We say that two matrix algebras $\mathbb{M}_1$ and $\mathbb{M}_2$ are *conjugated* if there exists an invertible matrix T such that $\mathbb{M}_1 = T\mathbb{M}_2 T^{-1}$.

The $\mathbb{A}$-product between the elements of the canonical basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ is given by $e_i e_j = \sum_{k=1}^{2} c_{ijk} e_k$ where $c_{ijk} \in \mathbb{R}$ for $i, j, k \in \{1, 2\}$ are called *structure constants* of $\mathbb{A}$. The *first fundamental representation* of $\mathbb{A}$ is the injective linear homomorphism $R : \mathbb{A} \to M(2, \mathbb{R})$ defined by $R : e_i \mapsto R_i$, where R_i is the matrix with $[R_i]_{jk} = c_{ikj}$, for $i = 1, 2$.

The linear space $\mathbb{R}^2$ endowed with the product

$$\begin{array}{c|cc} \cdot & e_1 & e_2 \\ \hline e_1 & e_1 & e_2 \\ e_2 & e_2 & p_1 e_1 + p_2 e_2 \end{array} \tag{5}$$

is an algebra $\mathbb{A}$ which we denote by $\mathbb{A}_1(p_1, p_2)$, see [2]. These algebras are associative, commutative, and have unit $e = e_1$, see also [7]. The *first fundamental representation* of $\mathbb{A}_1(p_1, p_2)$ is defined by

$$R(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R(e_2) = \begin{pmatrix} 0 & p_1 \\ 1 & p_2 \end{pmatrix}. \tag{6}$$

This representation allows us to use the corresponding matrix algebra in order to get expressions of $\varphi\mathbb{A}$-functions, like the defined in the following section.

1.2 $\varphi\mathbb{A}$-differentiability

The pre-twisted differentiability is defined in [5], this definition is closely related with the differentiability in the sense of Lorch, see [6]. Let $\mathbb{A}$ be an algebra and φ a differentiable planar vector field in the usual sense. We say that a planar vector field $\mathcal{F}$ is $\varphi\mathbb{A}$-*differentiable (pre-twisted differentiable)* if $\mathcal{F}$ is differentiable in the usual sense and if there exists a planar vector field $\mathcal{F}'_\varphi$ which we call $\mathbb{A}$-*derivative* of $\mathcal{F}$, such that

$$d\mathcal{F}_p = \mathcal{F}'_\varphi(p) d\varphi_p, \qquad p = (x, y) \tag{7}$$

where $\mathcal{F}'_\varphi(p) d\varphi_p(v)$ denotes the $\mathbb{A}$-product of $\mathcal{F}'_\varphi(p)$ and $\varphi_p(v)$ for every vector v in $\mathbb{R}^2$. In the same way, we say that $\mathcal{F}$ has a *second order* $\varphi\mathbb{A}$-*derivative* $\mathcal{F}''_\varphi$ if $\mathcal{F}$ is $\varphi\mathbb{A}$-differentiable, $\mathcal{F}'_\varphi$ is differentiable in the usual sense, and $\mathcal{F}''_\varphi$ is a planar vector field, such that

$$d(\mathcal{F}'_\varphi)_p = \mathcal{F}''_\varphi(p) d\varphi_p, \qquad p = (x, y). \tag{8}$$

Therefore, in this way we define the *n-order* $\varphi\mathbb{A}$-*derivative* $\mathcal{F}^{(n)}_\varphi$ for $n \in \mathbb{N}$.

A $\varphi\mathbb{A}$-*polynomial function* $\mathcal{P} : \mathbb{A} \to \mathbb{A}$ is defined by

$$\mathcal{P}(\xi) = c_0 + c_1 \varphi(\xi) + c_2 (\varphi(\xi))^2 + \cdots + c_m (\varphi(\xi))^m \tag{9}$$

where $c_0, c_1, \cdots, c_m \in \mathbb{A}$ are constants, $\xi = (x, y)$ is $\mathbb{A}$-variable, and the products involved in $c_k(\varphi(\xi))^k$ for $k \in \{1, 2, \cdots, m\}$ are defined with respect to $\mathbb{A}$. In the same way *exponential*, *trigonometric*, and *hyperbolic* $\varphi\mathbb{A}$-*functions* are defined. If $\mathcal{P}$ and $\mathcal{Q}$ are $\varphi\mathbb{A}$-polynomial functions, the $\varphi\mathbb{A}$-*rational function* $\mathcal{P}/\mathcal{Q}$ is defined on the set $\mathcal{Q}^{-1}(\mathbb{A}^*)$. All these functions have n-order $\varphi\mathbb{A}$-derivatives for $n \in \mathbb{N}$ and the usual rules of differentiation are satisfied for this differentiability.

1.3 Partial derivatives of $\mathcal{F}$

We suppose that $\mathcal{F}$ has $\mathbb{A}$-derivatives of first and second orders F'_φ and F''_φ, respectively, like the cases of functions defined in last section. The first partial derivatives $\mathcal{F}_x$ and $\mathcal{F}_y$ of every $\varphi\mathbb{A}$-differentiable function $\mathcal{F}$ are expressed by

$$\mathcal{F}_x = \mathcal{F}'_\varphi \varphi_x, \qquad \mathcal{F}_y = \mathcal{F}'_\varphi \varphi_y, \tag{10}$$

the second ones $\mathcal{F}_{xx}$, $\mathcal{F}_{xy}$, and $\mathcal{F}_{yy}$ by

$$\mathcal{F}_{xx} = \mathcal{F}''_\varphi \varphi_x^2 + \mathcal{F}'_\varphi \varphi_{xx}, \qquad \mathcal{F}_{xy} = \mathcal{F}''_\varphi \varphi_x \varphi_y + \mathcal{F}'_\varphi \varphi_{xy}, \qquad \mathcal{F}_{yy} = \mathcal{F}''_\varphi \varphi_y^2 + \mathcal{F}'_\varphi \varphi_{yy}. \tag{11}$$

Since φ is affine its second partial derivatives are zero, then the second partial derivatives given in (11) become

$$\mathcal{F}_{xx} = \mathcal{F}''_\varphi \varphi_x^2, \qquad \mathcal{F}_{xy} = \mathcal{F}''_\varphi \varphi_x \varphi_y, \qquad \mathcal{F}_{yy} = \mathcal{F}''_\varphi \varphi_y^2. \tag{12}$$

By the product of $\mathbb{A} = \mathbb{A}^1(p_1, p_2)$ and the form of φ in (3) we have

$$\begin{array}{rcl} \varphi_x^2 & = & (a^2 + p_1 c^2, 2ac + p_2 c^2), \\ \varphi_x \varphi_y & = & (ab + p_1 cd, ad + bc + p_2 cd), \\ \varphi_y^2 & = & (b^2 + p_1 d^2, 2bd + p_2 d^2), \end{array} \tag{13}$$

which can be calculated easily by using the first fundamental representation of $\mathbb{A}$.

1.4 $\varphi\mathbb{A}$ Cauchy-Riemann equations

By using $\mathcal{F}_x$ and $\mathcal{F}_y$ given in (10) we obtain

$$\varphi_y \mathcal{F}_x = \varphi_x \mathcal{F}_y,$$

which for $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ and φ given by (3) define the $\varphi\mathbb{A}$ Cauchy-Riemann equations

$$\begin{array}{rcl} bf_x + p_1 dg_x & = & af_y + p_1 cg_y, \\ df_x + (b + p_2 d)g_x & = & cf_y + (a + p_2 c)g_y. \end{array} \tag{14}$$

So, if $\mathcal{F} = (f, g)$ is $\varphi\mathbb{A}$-differentiable, then their component functions f and g satisfy the set of PDEs (14). The converse affirmation is: if functions f and g satisfy (14), then $\mathcal{F} = (f, g)$ is $\varphi\mathbb{A}$-differentiable, see [5].

2 Solutions defined by $\mathcal{E} = e^\varphi$

2.1 Looking for the algebra

In the following theorem we found the algebra $\mathbb{A}$ with respect to which the components of the exponential function $\mathcal{E} = e^\varphi$ define solutions of the PDE (1).

Theorem 2.1 *Consider the PDE (1) and the vector field φ given in (3). Suppose that $Ac^2 + Bcd + Cd^2 \neq 0$. Thus, for the algebra $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ with parameters p_1 and p_2 given by*

$$p_1 = -\frac{Aa^2 + Bab + Cb^2 + Da + Eb + F}{Ac^2 + Bcd + Cd^2}, \tag{15}$$

and

$$p_2 = -\frac{2Aac + B(ad + bc) + 2Cbd + Dc + Ed}{Ac^2 + Bcd + Cd^2}, \tag{16}$$

we have that the components f and g of the exponential function (4) defined with respect to $\mathbb{A}$, are solutions of the PDE (1).

Proof. Let $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$. The equalities (15) and (16) are equivalent to

$$\begin{array}{rcl} A(a^2 + p_1c^2) + B(ab + p_1cd) + C(b^2 + p_1d^2) + Da + Eb + F & = & 0, \\ A(2ac + p_2c^2) + B(ad + bc + p_2cd) + C(2bd + p_2d^2) + Dc + Ed & = & 0, \end{array} \tag{17}$$

respectively. From (13) and (17) it can be obtained

$$A\varphi_x^2 + B\varphi_x\varphi_y + C\varphi_y^2 + D\varphi_x + E\varphi_y + F(1, 0) = 0. \tag{18}$$

As $\mathcal{E} = e^\varphi$ with respect to the product of $\mathbb{A}$, we have $\mathcal{E} = \mathcal{F}'_\varphi = F''_\varphi$. From this and the equalities (12), by multiplying $\mathcal{E}$ with respect to $\mathbb{A}$ we get

$$A\mathcal{E}_{xx} + B\mathcal{E}_{xy} + C\mathcal{E}_{yy} + D\mathcal{E}_x + E\mathcal{E}_y + F\mathcal{E} = 0. \tag{19}$$

Since $\mathcal{E} = (f, g)$ we have that f and g are solutions for (1). $\square$

By using the first fundamental representation R defined in Section 1.1, expressions for f and g can be obtained, as we see in the following example.

Example 2.1 *Consider the PDE (1) with $A = 1$, $B = 2$, $C = 3$, $D = 4$, $E = 5$, and $F = 6$. We can define φ with $c = d = 1$, and $a = b = 0$, that is, $\varphi(x, y) = (0, x + y)$. So $p_1 = -1$, $p_2 = -3/2$, and by using the first fundamental representation R we can found that f and g are given by*

$$f(x, y) = \frac{7\cos(\frac{\sqrt{7}(x+y)}{4}) + 3\sqrt{7}\sin(\frac{\sqrt{7}(x+y)}{4})}{7e^{\frac{3(x+y)}{4}}}, \tag{20}$$

and

$$g(x, y) = \frac{4\sqrt{7}\sin(\frac{\sqrt{7}(x+y)}{4})}{7e^{\frac{3(x+y)}{4}}}, \tag{21}$$

and they are solutions for (1).

If we consider the same PDE and $\varphi(x,y)=(0,x)$, so $p_1=-6$, $p_2=-4$,

$$f(x,y)=e^{-2x}\left(\cos(\sqrt{2}x)+\sqrt{2}\sin(\sqrt{2}x)\right), \tag{22}$$

and

$$g(x,y)=\frac{\sqrt{2}}{2}e^{-2x}\sin(\sqrt{2}x). \tag{23}$$

2.2 Expression for $\mathcal{E}=e^{\varphi}$

In this Section we will use Theorem 2.1 to give explicit solutions for PDEs of the type (1). By means of the first fundamental representation R defined in Section 1.1 we will work on the corresponding matrix algebra. Although calculating the exponentials of matrices is well known and commonly used in differential equations, this section will introduced so that the work is more complete and that solutions for (1) can be directly obtained.

To each matrix in its normal form a matrix algebra can be associated, see [1]; in the case of 2-by-2 real matrices, we have three types of algebras that correspond to the three types of normal canonical forms of matrices.

2.2.1 Case ${p_2}^2+4p_1<0$

When the algebra $\mathbb{A}=\mathbb{A}_1(p_1,p_2)$ is isomorphic to the algebra of the complexes $\mathbb{C}$, we have the following proposition.

Proposition 2.1 *If ${p_2}^2+4p_1<0$, then the solutions f and g of the PDE (1) given in Theorem 2.1 for $\varphi(x,y)=(ax+by,cx+dy)$ are the following*

$$f(x,y)=e^{ax+by+\frac{p_2}{2}(cx+dy)}\left(\cos\left(\frac{\gamma}{2}(cx+dy)\right)-\frac{p_2}{\gamma}\sin\left(\frac{\gamma}{2}(cx+dy)\right)\right) \tag{24}$$

and

$$g(x,y)=e^{ax+by+\frac{p_2}{2}(cx+dy)}\left(\frac{(-{p_2}^2-\gamma^2)\sin\left(\frac{\gamma}{2}(cx+dy)\right)}{2p_1\gamma}\right), \tag{25}$$

where $\gamma=\sqrt{-{p_2}^2-4p_1}$.

Proof. If ${p_2}^2+4p_1<0$, then $p_1<0$ and

$$\begin{pmatrix}0 & p_1\\ 1 & p_2\end{pmatrix}=\begin{pmatrix}0 & 2p_1\\ \gamma & p_2\end{pmatrix}\begin{pmatrix}\frac{p_2}{2} & -\frac{\gamma}{2}\\ \frac{\gamma}{2} & \frac{p_2}{2}\end{pmatrix}\begin{pmatrix}\frac{-p_2}{2p_1\gamma} & \frac{1}{\gamma}\\ \frac{1}{2p_1} & 0\end{pmatrix}.$$

We have

$$e^{R(\varphi(x,y))}=e^{ax+by}e^{\frac{p_2}{2}(cx+dy)}Me^{B(cx+dy)}M^{-1},$$

where R is the first fundamental representation of $\mathbb{A}^1(p_1, p_2)$,

$$e^{B(cx+dy)} = \begin{pmatrix} \cos\left(\frac{\gamma}{2}(cx+dy)\right) & -\sin\left(\frac{\gamma}{2}(cx+dy)\right) \\ \sin\left(\frac{\gamma}{2}(cx+dy)\right) & \cos\left(\frac{\gamma}{2}(cx+dy)\right) \end{pmatrix},$$

and

$$M = \begin{pmatrix} 0 & 2p_1 \\ \gamma & p_2 \end{pmatrix}.$$

Thus, the expressions for f and g are given by (49) and (46), respectively. $\square$

2.2.2 Case ${p_2}^2 + 4p_1 = 0$

When the algebra $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ is isomorphic to the algebra spanned by the Jordan canonical form, we have the following proposition.

Proposition 2.2 *If ${p_2}^2 + 4p_1 = 0$, then the solutions f and g of the PDE (1) given in Theorem 2.1 for $\varphi(x, y) = (ax + by, cx + dy)$ are the following*

$$f(x, y) = e^{ax+by+\frac{p_2}{2}(cx+dy)} \left(\frac{-p_2(cx + dy) + 2}{2}\right) \tag{26}$$

and

$$g(x, y) = e^{ax+by+\frac{p_2}{2}(cx+dy)}(cx + dy). \tag{27}$$

Proof. If ${p_2}^2 + 4p_1 = 0$, then $p_1 = -{p_2}^2/4$, and

$$\begin{pmatrix} 0 & p_1 \\ 1 & p_2 \end{pmatrix} = \begin{pmatrix} -\frac{p_2}{2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{p_2}{2} & 1 \\ 0 & \frac{p_2}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & \frac{p_2}{2} \end{pmatrix}.$$

We have

$$e^{R(\varphi(x,y))} = e^{ax+by+\frac{p_2}{2}(cx+dy)} \begin{pmatrix} -\frac{p_2}{2} & 1 \\ 1 & 0 \end{pmatrix} e^{(cx+dy)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ 1 & \frac{p_2}{2} \end{pmatrix},$$

then

$$e^{R(\varphi(x,y))} = e^{ax+by+\frac{p_2}{2}(cx+dy)} \begin{pmatrix} \frac{-p_2(cx+dy)+2}{2} & \frac{-{p_2}^2(cx+dy)}{4} \\ cx + dy & \frac{p_2(cx+dy)+2}{2} \end{pmatrix}.$$

Thus, in this case the expressions for f and g are given by (26) and (27), respectively. $\square$

2.2.3 Case ${p_2}^2 + 4p_1 > 0$

When the algebra $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ is the direct sum of $\mathbb{R}$ and $\mathbb{R}$, we have the following proposition.

Proposition 2.3 *Let ${p_2}^2 + 4p_1 > 0$ and $\gamma = \sqrt{{p_2}^2 + 4p_1}$. Then the solutions f and g of the PDE (1) given in Theorem 2.1 for $\varphi(x, y) = (ax + by, cx + dy)$ are the following*

1. *If* $p_1 \neq 0$,

$$f(x,y) = e^{ax+by+\frac{p_2}{2}(cx+dy)} \frac{(\gamma - p_2)e^{\frac{\gamma}{2}(cx+dy)} + (\gamma + p_2)e^{-\frac{\gamma}{2}(cx+dy)}}{2\gamma} \tag{28}$$

and

$$g(x,y) = e^{ax+by+\frac{p_2}{2}(cx+dy)} \frac{(\gamma^2 - {p_2}^2)e^{\frac{\gamma}{2}(cx+dy)} - (\gamma^2 - {p_2}^2)e^{-\frac{\gamma}{2}(cx+dy)}}{4p_1\gamma}. \tag{29}$$

2. *If* $p_1 = 0$,

$$f(x,y) = e^{ax+by} \tag{30}$$

and

$$g(x,y) = \frac{1}{p_2} e^{ax+by} \left(-1 + e^{p_2(cx+dy)}\right). \tag{31}$$

Proof. If ${p_2}^2 + 4p_1 > 0$, we have that the matrix

$$\begin{pmatrix} 0 & p_1 \\ 1 & p_2 \end{pmatrix}$$

is diagonalizable. If $p_1 \neq 0$, then

$$\begin{pmatrix} 0 & p_1 \\ 1 & p_2 \end{pmatrix} = \begin{pmatrix} 2p_1 & 2p_1 \\ p_2 + \gamma & p_2 - \gamma \end{pmatrix} \begin{pmatrix} \frac{p_2+\gamma}{2} & 0 \\ 0 & \frac{p_2-\gamma}{2} \end{pmatrix} \begin{pmatrix} \frac{-p_2+\gamma}{4p_1\gamma} & \frac{p_1}{2p_1\gamma} \\ \frac{p_2+\gamma}{4p_1\gamma} & \frac{-p_1}{2p_1\gamma} \end{pmatrix},$$

where $\gamma = \sqrt{{p_2}^2 + 4p_1}$. Thus, in this case the expressions for f and g are given by (28) and (29), respectively.

If $p_1 = 0$, then

$$\begin{pmatrix} 0 & 0 \\ 1 & p_2 \end{pmatrix} = \begin{pmatrix} p_2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & p_2 \end{pmatrix} \begin{pmatrix} \frac{1}{p_2} & 0 \\ \frac{1}{p_2} & 1 \end{pmatrix}.$$

Thus, in this case the expressions for f and g are given by (30) and (31), respectively. □

2.3 The 1D heat equation

Now we will consider the 1D heat equation.

Example 2.2 *The 1D heat equation is given by*

$$\alpha u_{xx} - u_t = 0. \tag{32}$$

In this case we change the variable y *by* t, $A = \alpha$, $E = -1$, B, C, D, *and* F *are zero. So,*

$$p_1 = -\frac{\alpha a^2 - b}{\alpha c^2}, \qquad p_2 = -\frac{2\alpha ac - d}{\alpha c^2}.$$

Suppose that $\alpha = 1/7$, $a = c = \sqrt{7}$, $b = 1$, and $d = 2$, then

$$p_1 = -\frac{\alpha a^2 - b}{\alpha c^2}, \qquad p_2 = -\frac{2\alpha ac - d}{\alpha c^2}.$$

Thus, $p_1 = 0$ and $p_2 = 0$. By Proposition 2.2

$$f(x,t) = e^{\sqrt{7}x+t}, \qquad g(x,t) = e^{\sqrt{7}x+t}(\sqrt{7}x + 2t)$$

are solutions. For the same value of α we may choose values for the constants a, b, c and d with the only condition that $c \neq 0$.

2.4 Solutions by trigonometric and hyperbolic functions of φ

In the following proposition we now consider the trigonometric sine and cosine functions instead of the exponential function.

Proposition 2.4 *Suppose that in the PDE (1) $D = 0$, $E = 0$ and $Ac^2 + Bcd + Cd^2 \neq 0$. Denote by $\mathcal{T}$ the trigonometric functions*

$$\mathcal{S}(x,y) = \sin(\varphi(x,y)), \qquad \mathcal{C}(x,y) = \cos(\varphi(x,y))$$

defined with respect to the $\mathbb{A} = \mathbb{A}_1(p_1,p_2)$ product. Let p_1 and p_2 be defined by

$$p_1 = -\frac{Aa^2 + Bab + Cb^2 - F}{Ac^2 + Bcd + Cd^2}, \tag{33}$$

and

$$p_2 = -\frac{2Aac + B(ad+bc) + 2Cbd}{Ac^2 + Bcd + Cd^2}, \tag{34}$$

then the functions f and g given by

$$(f,g) = \mathcal{T}(ax+by, cx+dy) \tag{35}$$

are solutions of the PDE (1) with $D = 0$ and $E = 0$.

Proof. Proof is similar to that of Theorem 2.1 but in this case it is used that $\mathcal{T} = -\mathcal{T}'_\varphi$. $\square$

In the following proposition are considered now the functions hyperbolic sine and hyperbolic cosine instead of the exponential function.

Proposition 2.5 *Suppose that in the PDE (1) $D = 0$, $E = 0$ and $Ac^2 + Bcd + Cd^2 \neq 0$. Denote by $\mathcal{T}$ the hyperbolic functions*

$$\mathcal{S}(x,y) = \sinh(\varphi(x,y)), \qquad \mathcal{C}(x,y) = \cosh(\varphi(x,y))$$

defined with respect to the $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ *product. Let* p_1 *and* p_2 *be defined by*

$$p_1 = -\frac{Aa^2 + Bab + Cb^2 + F}{Ac^2 + Bcd + Cd^2}, \tag{36}$$

and

$$p_2 = -\frac{2Aac + B(ad + bc) + 2Cbd}{Ac^2 + Bcd + Cd^2}, \tag{37}$$

then the functions f *and* g *defined by*

$$(f, g) = \mathcal{T}(ax + by, cx + dy) \tag{38}$$

are solutions of the PDE (1) with $D = 0$ *and* $E = 0$.

Proof. Proof is similar to that of Theorem 2.1, but in this case it is used that $\mathcal{T} = \mathcal{T}'_\varphi$. $\square$

2.5 PDEs $Au_{xx} + Bu_{xy} + Cu_{yy} = 0$

We will construct algebras $\mathbb{A}$ with respect to which the class of all the second order $\varphi\mathbb{A}$-differentiable functions have component functions f and g which are solutions of the PDE (2). Thus, the pre-twisted differentiability can be used for constructing solutions of PDEs.

Theorem 2.2 *Consider the PDE (2) and the affine planar vector field* φ *given in (3). Suppose that* $Ac^2 + Bcd + Cd^2 \neq 0$. *Thus, for the algebra* $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ *with parameters* p_1 *and* p_2 *given by*

$$p_1 = -\frac{Aa^2 + Bab + Cb^2}{Ac^2 + Bcd + Cd^2}, \tag{39}$$

and

$$p_2 = -\frac{2Aac + B(ad + bc) + 2Cbd}{Ac^2 + Bcd + Cd^2}, \tag{40}$$

we have that the components f *and* g *of each* $\varphi(\mathbb{A})$*-differentiable function* $\mathcal{F} = (f, g)$ *are solutions of the PDE (2).*

Proof. Let $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$. The equalities (39) and (40) are equivalent to

$$\begin{array}{rcl} A(a^2 + p_1c^2) + B(ab + p_1cd) + C(b^2 + p_1d^2) & = & 0, \\ A(2ac + p_2c^2) + B(ad + bc + p_2cd) + C(2bd + p_2d^2) & = & 0, \end{array} \tag{41}$$

respectively. From (13) and (41) it can be obtained

$$A\varphi_x^2 + B\varphi_x\varphi_y + C\varphi_y^2 = 0. \tag{42}$$

From this and the equalities (12), by multiplying $\mathcal{F}'_\varphi$ with respect to $\mathbb{A}$ we get

$$A\mathcal{F}_{xx} + B\mathcal{F}_{xy} + C\mathcal{F}_{yy} = 0. \tag{43}$$

Since $\mathcal{F} = (f, g)$ we have that f and g are solutions for (1). $\square$

Theorem 2.2 is a generalization of a well known and important result, as we see in the following corollary.

Corollary 2.1 *Suppose that $A = 1$, $B = 0$, $C = 1$, and $\varphi(x, y) = (x, y)$. Then, PDE (2) is the Laplace's equation $u_{xx} + u_{yy} = 0$, and $p_1 = -1$ and $p_2 = 0$. Thus, $\mathbb{A} = \mathbb{A}_1(-1, 0)$ is the algebra of the complex numbers $\mathbb{A} = \mathbb{C}$, the $\varphi\mathbb{A}$-differentiability corresponds to the usual complex differentiability, and the components of the $\varphi\mathbb{A}$-differentiable functions are solutions of the Laplace's equation.*

We consider again the case of the Laplace's equation in the following example.

Example 2.3 *Suppose that $A = 1$, $B = 0$, $C = 1$, and $\varphi(x, y) = (x + 2y, 3x + 4y)$. Then, PDE (2) is the Laplace's equation $u_{xx} + u_{yy} = 0$. By Theorem 2.2*

$$p_1 = -\frac{1}{5}, \qquad p_2 = \frac{-22}{25}.$$

Then, for $\mathbb{A} = \mathbb{A}_1(-1/5, -22/25)$ the class of all the second order $\varphi\mathbb{A}$-differentiable functions are solutions for (2). Thus, the components of $\varphi^2 = (f, g)$ are solutions for (2) for the given values of A, B, and C. Since

$$\begin{array}{rcl} (\varphi(x,y))^2 & = & \left((x+2y)^2 - \frac{1}{5}(3x+4y)^2, 2(x+2y)(3x+4y) - \frac{22}{25}(3x+4y)^2\right) \\ & = & \frac{1}{25}\left(-20x^2 + 20y^2 - 20xy, -48x^2 + 48y^2 - 28xy\right), \end{array}$$

we have

$$f(x,y) = \frac{1}{5}(-4x^2 + 4y^2 - 4xy), \qquad g(x,y) = \frac{1}{25}(-48x^2 + 48y^2 - 28xy).$$

We consider the wave equation in the following example.

Example 2.4 *Suppose that $A = 1$, $B = 0$, $C = -1$, and $\varphi(x, y) = (x + 2y, 3x + 4y)$. Then, PDE (2) is the classical wave equation $u_{xx} - u_{yy} = 0$. By Theorem 2.2*

$$p_1 = -\frac{3}{7}, \qquad p_2 = \frac{-10}{7}.$$

Then, for $\mathbb{A} = \mathbb{A}_1(-3/7, -10/7)$ the class of all the second order $\varphi\mathbb{A}$-differentiable functions are solutions for (2). Thus, the components of $\varphi^2 = (f, g)$ are solutions for (2) for the given values of A, B, and C. Since

$$\begin{array}{rcl} (\varphi(x,y))^2 & = & \left((x+2y)^2 - \frac{3}{7}(3x+4y)^2, 2(x+2y)(3x+4y) - \frac{10}{7}(3x+4y)^2\right) \\ & = & -\frac{1}{7}(20x^2 + 20y^2 + 4xy, 48x^2 + 48y^2 + 100xy), \end{array}$$

we have

$$f(x,y) = -\frac{20x^2 + 20y^2 + 4xy}{7}, \qquad g(x,y) = -\frac{48x^2 + 48y^2 + 100xy}{7}.$$

Suppose that $(k, l) \neq (0, 0)$, that is, $\varphi(x, y) = (x + 2y + k, 3x + 4y + l)$. Since

$$\begin{array}{rcl} (\varphi(x,y))^2 & = & \left((x+2y+k)^2 - \frac{3}{7}(3x+4y+l)^2, 2(x+2y+k)(3x+4y+l) - \frac{10}{7}(3x+4y+l)^2\right) \\ & = & \frac{1}{7}(7k^2 - 3l^2 - 20x^2 + 14kx - 18lx - 20y^2 + 28ky - 24ly - 44xy)e_1 \\ & & +\frac{1}{7}(-10l^2 + 14kl - 48x^2 + 42kx - 46lx - 48y^2 + 56ky - 52ly - 100xy)e_2, \end{array}$$

we have

$$f(x,y)=\frac{7k^2-3l^2-20x^2+14kx-18lx-20y^2+28ky-24ly-44xy}{7},$$

and

$$g(x,y)=\frac{-10l^2+14kl-48x^2+42kx-46lx-48y^2+56ky-52ly-100xy}{7}.$$

Example 2.5 *Consider the 1D wave equation (the PDE (2)). Then,* $A=1$*,* $B=0$*, and* $C=-1$*. Let* φ *be defined by* $\varphi(x,y)=(y,x)$*. By Theorem 2.2*

$$p_1=1, \qquad p_2=0.$$

Then, for $\mathbb{A}=\mathbb{A}_1(1,0)$ *the class of all the second order* $\varphi\mathbb{A}$*-differentiable functions are solutions for (2). Thus, the components of* $\varphi^2=(f,g)$ *are solutions for (2) for the given values of* A*,* B*, and* C*. Since*

$$(\varphi(x,y))^2=\left(x^2+y^2,2xy\right),$$

we have

$$f(x,y)=x^2+y^2, \qquad g(x,y)=2xy.$$

2.6 Solutions of PDEs and $\varphi\mathbb{A}$-differentiable functions

Each solutions of a PDEs of the form (2) is a component of a $\varphi\mathbb{A}$-differentiable function, as we see in the following theorem.

Theorem 2.3 *Consider the PDE (2) and the affine planar vector field* φ *given in (3). Suppose that* $Ac^2+Bcd+Cd^2\neq 0$*, and that equalities (39) and (40) are satisfied.*

1) If $p_1(ad-bc)\neq 0$*, then the* $\varphi\mathbb{A}$ *Cauchy-Riemann equations (14) can be expressed by*

$$\begin{array}{rcl} g_x & = & \frac{-ab+cdp_1-bcp_2}{(ad-bc)p_1}f_x+\frac{a^2-c^2p_1+acp_2}{(ad-bc)p_1}f_y, \\ g_y & = & \frac{-b^2+d^2p_1-bdp_2}{(ad-bc)p_1}f_x+\frac{ab-cdp_1+adp_2}{(ad-bc)p_1}f_y, \end{array} \tag{44}$$

from which we obtain

$$(g_x)_y-(g_y)_x=\frac{(ad-bc)}{p_1}(Af_{xx}+Bf_{xy}+Cf_{yy}). \tag{45}$$

Therefore, if f *is a solution of PDE (2) and*

$$g=\int g_x dx+\int\left[g_y-\frac{\partial}{\partial y}\int g_x dx\right]dy, \tag{46}$$

then, $\mathcal{F}=(f,g)$ *is* $\varphi\mathbb{A}$*-differentiable.*

2) If $(ad-bc)\neq 0$, then the $\varphi\mathbb{A}$ Cauchy-Riemann equations (14) can be expressed by

$$\begin{array}{rcl} f_x & = & \frac{-ab+cdp_1-adp_2}{ad-bc}g_x+\frac{a^2-c^2p_1+acp_2}{ad-bc}g_y, \\ f_y & = & \frac{-b^2+d^2p_1-bdp_2}{ad-bc}g_x+\frac{ab-cdp_1+bcp_2}{ad-bc}g_y, \end{array} \tag{47}$$

from which we obtain

$$(f_x)_y-(f_y)_x=(ad-bc)(Ag_{xx}+Bg_{xy}+Cg_{yy}). \tag{48}$$

Therefore, if g is a solution of PDE (2), and

$$f=\int f_y dy+\int\left[f_x-\frac{\partial}{\partial x}\int f_y dy\right]dx, \tag{49}$$

then, $\mathcal{F}=(f,g)$ is $\varphi\mathbb{A}$-differentiable.

Proof. The systems (44) and (47) can be obtained from system (14); in the system of 1) we use $(f_y,g_y)=(a,c)^{-1}(b,d)(f_x,g_x)$ and in 2) we use $(f_x,g_x)=(b,d)^{-1}(a,c)(f_x,g_x)$. In the first case we have

$$p_1(ad-bc)((g_x)_y-(g_y)_x)=(ad-bc)^2(Af_{xx}+Bf_{xy}+Cf_{yy}),$$

thus we obtain (45). In the first case we have

$$(ad-bc)((f_x)_y-(f_y)_x)=(ad-bc)^2(Ag_{xx}+Bg_{xy}+Cg_{yy}),$$

thus we obtain (48). Since $g_{yx}=g_{xy}$ if f is a solution of (2), we have that there exists a function $g(x,y)$ (uniquely defined under a constant additive) which satisfies (44). In the same way, since $f_{yx}=f_{xy}$ if g is a solution of (2), we have that there exists a function $f(x,y)$ (uniquely defined under a constant additive) which satisfies (47). The function $\mathcal{F}=(f,g)$ so constructed satisfy the corresponding $\varphi\mathbb{A}$ Cauchy-Riemann equations. Since φ is a linear isomorphism hypotheses of Theorem 1.2 of [5] are satisfied, so we obtain that $\mathcal{F}$ is $\varphi\mathbb{A}$-differentiable. $\Box$

This example show the relation between the solutions of the 1D wave equation and the $\varphi\mathbb{A}$-differentiable functions for the algebra $\mathbb{A}=\mathbb{A}_1(1,0)$ and for $\varphi(x,y)=(y,x)$; this is similar to the relation between 2D harmonic functions and complex analytic functions.

Example 2.6 *The $\varphi\mathbb{A}$ Cauchy-Riemann equations (14) for φ and $\mathbb{A}$ given in Example 2.5, are defined by*

$$f_x=g_y,\qquad f_y=g_x. \tag{50}$$

The function $f(x,y)=x^3+3xy^2$ is a solution of the 1D wave equation. Then, by 1) of Theorem 2.3 we have

$$g(x,y)=3x^2y+y^3+k,$$

where k is a constant.

In this case $\mathcal{F}=(f,g)$ is given by $\mathcal{F}(x,y)=(\varphi(x,y))^3+(0,k)$.

3 3^{th} order PDE

Now consider the 3^{th} order PDE

$$Gu_{xxx} + Hu_{xxy} + Ku_{xyy} + Lu_{yyy} + Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0. \tag{51}$$

In this case we also are looking for solutions which are defined by the columns the exponential functions $\mathcal{E} = e^{\varphi}$. The third partial derivatives of $\mathcal{F} = (f, g)$ for affine planar vector filed φ are given by

$$\mathcal{F}_{xxx} = \mathcal{F}'''_{\varphi}\varphi_x^3, \quad \mathcal{F}_{xxy} = \mathcal{F}'''_{\varphi}\varphi_x^2\varphi_y, \quad \mathcal{F}_{xyy} = \mathcal{F}'''_{\varphi}\varphi_x\varphi_y^2, \quad \mathcal{F}_{yyy} = \mathcal{F}'''_{\varphi}\varphi_y^3. \tag{52}$$

From the $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ product and the proposed form of φ in (3) we have

$$\begin{array}{rcc} \varphi_x^3 & = & (a^3, 3a^2c + c^3p_1) + (3ac^2 + c^3p_2)(p_1, p_2), \\ \varphi_x^2\varphi_y & = & (a^2b, 2abc + a^2d + c^2dp_1) + (bc^2 + 2acd + c^2dp_2)(p_1, p_2), \\ \varphi_x\varphi_y^2 & = & (ab^2, 2abd + b^2c + cd^2p_1) + (ad^2 + 2bcd + c^2dp_2)(p_1, p_2), \\ \varphi_y^3 & = & (b^3, 3b^2d + d^3p_1) + (3bd^2 + d^3p_2)(p_1, p_2). \end{array} \tag{53}$$

Theorem 3.1 *Consider the PDE (51), the affine planar vector field φ given in (3), and the quadratic system of equations*

$$\begin{array}{cc} & (Gc^3 + Hc^2d + Kcd^2 + Ld^3)xy \\ + & (3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2 + Ac^2 + Bcd + Cd^2)x \\ + & Ga^3 + Ha^2b + Kab^2 + Lb^3 + Aa^2 + Bab + Cb^2 + Da + Eb + F = 0, \\ & \\ & (Gc^3 + Hc^2d + Kcd^2 + Ld^3)y^2 + (Gc^3 + H(c^2d + bc^2) + Kcd^2 + Ld^3)x \\ + & (3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2 + Ac^2 + Bcd + Cd^2)y + 3Ga^2c \\ + & 2H(abc + a^2d) + K(b^2c + 2abd) + 3Lb^2d + 2Aac + B(ad + bc) + 2Cbd + Dc + Ed = 0. \end{array} \tag{54}$$

If (p_1, p_2) is a solution of the quadratic system (54), for the algebra $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ the components functions f and g of the exponential function (4) defined with respect to $\mathbb{A}$, are solutions of the PDE (51).

Proof. Let $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$, where p_1 and p_2 is a solution of the quadratic system (54). Using the equalities (12), (13), (52), (53), and the obtained by substituting p_1, p_2 in (54), we obtain that columns of $\mathcal{E}$ are solutions for (51). $\square$

The following corollary could help us to construct solutions for (51).

Corollary 3.1 *If φ is satisfies*

$$c^2(Gc + Hd) + d^2(Kc + Ld) = 0 \tag{55}$$

and

$$3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2 + Ac^2 + Bcd + Cd^2 \neq 0, \tag{56}$$

then the quadratic system (54) given in Theorem 3.1 reduces to a the linear system with solutions

$$p_1 = -\frac{Ga^3 + Ha^2b + Kab^2 + Lb^3 + Aa^2 + Bab + Cb^2 + Da + Eb + F}{3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2 + Ac^2 + Bcd + Cd^2}, \tag{57}$$

and

$$p_2 = -\frac{3Ga^2c + 2H(abc + a^2d) + K(b^2c + 2abd) + 3Lb^2d + 2Aac + B(ad + bc) + 2Cbd + Dc + Ed}{3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2 + Ac^2 + Bcd + Cd^2} - Hbc^2p_1. \tag{58}$$

Proof. It follows from the Theorem 3.1 since

$$c^2(Gc + Hd) + d^2(Kc + Ld) = Gc^3 + Hc^2d + Kcd^2 + Ld^3.$$

□

We have the following example.

Example 3.1 *Consider the PDE*

$$u_{xxx} + 2u_{xxy} + 2u_{xyy} + 4u_{yyy} + 5u_{xx} + 6u_{xy} + 7u_{yy} + 8u_x + 9u_y + 10u = 0. \tag{59}$$

For $c = 2$ and $d = -1$ we have $c + 2d = 0$, $2c + 4d = 0$, then from Corollary 3 is satisfied. Thus,

$$p_1 = -\frac{a^3 + 2a^2b + 2ab^2 + 4b^3 + 5a^2 + 6ab + 7b^2 + 8a + 9b + 10}{12a + 2(4b - 4a) + 2(a - 4b) + 12b + 20 - 12 + 7},$$

and

$$p_2 = -\frac{6a^2 + 4(2ab - a^2) + 2(2b^2 - 2ab) - 12b^2 + 20a + 6(-a + 2b) - 14b + 16 - 9}{12a + 2(4b - 4a) + 2(a - 4b) + 12b + 20 - 12 + 7} + 8p_1.$$

If we take $a = b = 0$, then $\alpha = -2/3$ and $\beta = -87/5$. In this case $p_2^2 + 4p_1 > 0$, so functions f and g given in Proposition 2.3 are solutions for the PDE (59).

Consider the 3^{th} order PDE

$$Gu_{xxx} + Hu_{xxy} + Ku_{xyy} + Lu_{yyy} = 0. \tag{60}$$

Theorem 3.2 *Consider the PDE (60), the affine planar vector field φ given in (3), and the quadratic system of equations*

$$\begin{array}{rc} & (Gc^3 + Hc^2d + Kcd^2 + Ld^3)xy \\ + & (3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2)x \\ + & Ga^3 + Ha^2b + Kab^2 + Lb^3 = 0, \\ \\ & (Gc^3 + Hc^2d + Kcd^2 + Ld^3)y^2 + (Gc^3 + H(c^2d + bc^2) + Kcd^2 + Ld^3)x \\ + & (3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2)y + 3Ga^2c \\ + & 2H(abc + a^2d) + K(b^2c + 2abd) + 3Lb^2d = 0. \end{array} \tag{61}$$

If (p_1, p_2) is a solution of the quadratic system (61), for the algebra $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ the components of all the third order $\varphi\mathbb{A}$-differentiable function, are solutions of the PDE (51).

Proof. Let $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$, where p_1 and p_2 is a solution of the quadratic system (61). Using the equalities (12), (13), (52), (53), and the obtained by substituting p_1, p_2 in (61), we obtain that columns of the $\varphi\mathbb{A}$-differentiable functions are solutions for (51). □

4 4^{th} order PDEs

The *bi-harmonic equation* is the 4^{th} order PDE

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0, \tag{62}$$

the *biwave equation* is the 4^{th} order PDE

$$u_{xxxx} - 2u_{xxyy} + u_{yyyy} = 0, \tag{63}$$

and the *bi-telegraph equation* is the 4^{th} order PDE

$$u_{xxxx} - 2u_{xxyy} + u_{yyyy} - \lambda^4 u = 0, \tag{64}$$

see [8]. For φ being an affine planar vector field, the fourth partial derivatives $\mathcal{F}_{xxxx}$, $\mathcal{F}_{xxyy}$, and $\mathcal{F}_{yyyy}$ of every fourth order $\varphi\mathbb{A}$-differentiable function $\mathcal{F}$, are given by

$$\mathcal{F}_{xxxx} = \mathcal{F}''''_{\varphi}\varphi_x^4, \qquad \mathcal{F}_{xxyy} = \mathcal{F}''''_{\varphi}\varphi_x^2\varphi_y^2, \qquad \mathcal{F}_{yyyy} = \mathcal{F}''''_{\varphi}\varphi_y^4. \tag{65}$$

From the $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ product and the proposed form of φ in (3) we have

$$\begin{array}{rcl} \varphi_x^4 & = & c^4(p_1p_2^2 + p_1^2, p_2^3 + 2p_1p_2) + 4ac^3(p_1p_2, p_2^2 + p_1) + 6a^2c^2(p_1, p_2) + a^3(a, 4c), \\ \varphi_x^2\varphi_y^2 & = & c^2d^2(p_1p_2^2 + p_1^2, p_2^3 + 2p_1p_2) + (2cd(ad + bc)p_2 + a^2d^2 + 4abcd + b^2c^2)(p_1, p_2) \\ & & +(a^2b^2, 2(ad + bc)(cdp_1 + ab)), \\ \varphi_y^4 & = & d^4(p_1p_2^2 + p_1^2, p_2^3 + 2p_1p_2) + 4bd^3(p_1p_2, p_2^2 + p_1) + 6b^2d^2(p_1, p_2) + b^3(b, 4d). \end{array} \tag{66}$$

Theorem 4.1 *Consider the PDE (62), the affine planar vector field φ given in (3), and the cubic system of two equations*

$$\begin{array}{c} (c^2 + d^2)^2(xy^2 + x^2) + 4(ac + bd)(c^2 + d^2)xy \\ +2(3a^2c^2 + a^2d^2 + 4abcd + b^2c^2 + 3b^2d^2)x + (a^2 + b^2)^2 = 0, \\ \\ (c^2 + d^2)^2(y^3 + 2xy) + 4(ac + bd)(c^2 + d^2)(y^2 + x) \\ +2(3a^2c^2 + a^2d^2 + 4abcd + b^2c^2 + 3b^2d^2)y + 4(a^2 + b^2)(ac + bd) = 0. \end{array} \tag{67}$$

If (p_1, p_2) is a solution of the cubic system (67), then for $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ the components of all the fourth order $\varphi\mathbb{A}$-differentiable functions are solutions of the PDE (62).

Proof. Let $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$, where p_1 and p_2 is a solution of the cubic system (67). Using the equalities (65), (66), and (67), we obtain that components of all the $\varphi\mathbb{A}$-differentiable functions are solutions for the bi-harmonic equation (62). $\square$

Example 4.1 *If $\varphi(x, y) = (x + y + k, x - y + l)$, then $p_1 = -1$ and $p_2 = 0$, satisfy conditions (67). Thus, for $\mathbb{A} = \mathbb{C}$ the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the bi-harmonic equation (62). Function*

$$(x + y + k, x - y + l)^{-1} = \left(\frac{x + y + k}{(x + y + k)^2 + (x - y + l)^2}, \frac{x - y + l}{(x + y + k)^2 + (x - y + l)^2}\right),$$

has components

$$f(x,y) = \frac{x+y+k}{(x+y+k)^2+(x-y+l)^2}, \qquad g(x,y) = \frac{x-y+l}{(x+y+k)^2+(x-y+l)^2},$$

which are bi-harmonic functions. Also function $(x+y+k, x-y+l)^4$, has components

$$\begin{aligned}
f(x,y) &= k^4+l^4-6k^2l^2-4x^4-8kx^3-8lx^3-24klx^2+4k^3x+4l^3x-12kl^2x-12k^2lx \\
&\quad -4y^4-8ky^3+8ly^3+24kly^2+24x^2y^2+24kxy^2+24lxy^2+4k^3y-4l^3y \\
&\quad -12kl^2y+12k^2ly+24kx^2y-24lx^2y+24k^2xy-24l^2xy, \\
g(x,y) &= -4kl^3+4k^3l+8kx^3-8lx^3+12k^2x^2-12l^2x^2+4k^3x-4l^3x-12kl^2x+12k^2lx \\
&\quad -8ky^3-8ly^3-16xy^3-12k^2y^2+12l^2y^2-24kxy^2+24lxy^2-4k^3y-4l^3y \\
&\quad +12kl^2y+12k^2ly+16x^3y+24kx^2y+24lx^2y+48klxy,
\end{aligned}$$

which are bi-harmonic functions.

Example 4.2 *If $\varphi(x,y) = (y+k, -x+l)$, then $p_1 = -1$ and $p_2 = 0$ satisfy conditions (67). Thus, for $\mathbb{A} = \mathbb{C}$ the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the bi-harmonic equation (62).*

Theorem 4.2 *Consider the PDE (63), the affine planar vector field φ given in (3), and the cubic system of two equations*

$$\begin{gathered}
(c^2-d^2)^2(xy^2+x^2)+4(ac-bd)(c^2-d^2)xy \\
+2(3a^2c^2-a^2d^2-4abcd-b^2c^2+3b^2d^2)x+(a^2-b^2)^2=0, \\
(c^2-d^2)^2(y^3+2xy)+4(ac-bd)(c^2-d^2)(y^2+x) \\
+2(3a^2c^2-a^2d^2-4abcd-b^2c^2+3b^2d^2)y+4(a^2-b^2)(ac-bd)=0.
\end{gathered} \tag{68}$$

If (p_1,p_2) is a solution of the cubic system (68), then for $\mathbb{A} = \mathbb{A}_1(p_1,p_2)$ the components of all the fourth order $\varphi\mathbb{A}$-differentiable functions are solutions of the biwave equation (63).

Proof. Let $\mathbb{A} = \mathbb{A}_1(p_1,p_2)$, where p_1 and p_2 is a solution of the cubic system (68). Using the equalities (65), (66), and (68), we obtain that components of all the $\varphi\mathbb{A}$-differentiable functions are solutions for (63). $\square$

Example 4.3 *If $\varphi(x,y) = (y+k, x-y+l)$, then $p_1 = -1/4$ and $p_2 = 1$ satisfy conditions (68). Thus, for $\mathbb{A} = \mathbb{A}_1^2(1/4,1)$ the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the biwave equation (63). Function φ^{-1} is $\varphi\mathbb{A}$-differentiable, and*

$$(y+k, x-y+l)^{-1} = (f(x,y), g(x,y)),$$

where

$$\begin{aligned}
f(x,y) &= \frac{4x+4k+4l}{(x+y)^2+(4k+2l)(x+y)+(2k+l)^2}, \\
g(x,y) &= \frac{-4x+4y-4l}{(x+y)^2+(4k+2l)(x+y)+(2k+l)^2},
\end{aligned}$$

which are solutions of the biwave equation. In the same way for φ^4 we have

$$(y+k, x-y+l)^4 = (f(x,y), g(x,y)),$$

where

$$\begin{aligned} f(x,y) \;&=\; \frac{-3x^4-16kx^3-12lx^3-24k^2x^2-18l^2x^2-48klx^2-12l^3x-48kl^2x-48k^2lx}{16} \\ &\quad \frac{+5y^4+32ky^3+12ly^3+12xy^3+72k^2y^2+6l^2y^2+48kly^2+6x^2y^2+48kxy^2}{16} \\ &\quad \frac{+12lxy^2+64k^3y-4l^3y+48k^2ly-4x^3y-12lx^2y+48k^2xy-12l^2xy}{16} \\ &\quad \frac{+16k^4-3l^4-16kl^3-24k^2l^2}{16}, \end{aligned}$$

$$\begin{aligned} g(x,y) \;&=\; \frac{x^4+6kx^3+4lx^3+12k^2x^2+6l^2x^2+18klx^2+18kl^2x+24k^2lx-y^4-6ky^3}{2} \\ &\quad \frac{-2ly^3-2xy^3+8k^3x+4l^3x-12k^2y^2-6kly^2-6kxy^2-8k^3y}{2} \\ &\quad \frac{+2l^3y+6kl^2y+2x^3y+6kx^2y+6lx^2y+6l^2xy+12klxy}{2} \\ &\quad \frac{+l^4+6kl^3+12k^2l^2+8k^3l}{2}, \end{aligned}$$

which are solutions of the biwave equation.

Example 4.4 *If $\varphi(x,y) = (x+y+k, x+l)$, then $p_1 = 0$ and $p_2 = -2$ satisfy conditions (68). Thus, for $\mathbb{A} = \mathbb{A}_1^2(0,-2)$ the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the biwave equation (63). Function φ^{-1} is $\varphi\mathbb{A}$-differentiable, and*

$$(x+y+k, x+l)^{-1} = (f(x,y), g(x,y)),$$

where

$$f(x,y) = \frac{1}{x+y+k}, \qquad g(x,y) = \frac{-x-l}{-x^2+y^2-2lx+2(k-l)y+k^2-2kl},$$

which are solutions of the biwave equation.

Theorem 4.3 *Consider the PDE (64), the affine planar vector field φ given in (3), and the cubic system of two equations*

$$\begin{gathered} (c^2-d^2)^2(xy^2+x^2)+4(ac-bd)(c^2-d^2)xy \\ +2(3a^2c^2-a^2d^2-4abcd-b^2c^2+3b^2d^2)x+(a^2-b^2)^2-\lambda^4=0, \\ \\ (c^2-d^2)^2(y^3+2xy)+4(ac-bd)(c^2-d^2)(y^2+x) \\ +2(3a^2c^2-a^2d^2-4abcd-b^2c^2+3b^2d^2)y+4(a^2-b^2)(ac-bd)=0. \end{gathered} \tag{69}$$

If (p_1, p_2) is a solution of the cubic system (69), then for $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ the components of the functions

$$e^{\varphi(x,y)}, \quad \sin(\varphi(x,y)), \quad \cos(\varphi(x,y)), \quad \sinh(\varphi(x,y)), \quad \cosh(\varphi(x,y)),$$

are solutions of the bi-telegraphic equation (64).

Proof. Let $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$, where p_1 and p_2 is a solution of the cubic system (69). Using the equalities (65), (66), and (69), we obtain that components of all the $\varphi\mathbb{A}$-differentiable functions are solutions for (64). $\square$

Example 4.5 *If $\varphi(x,y) = (\lambda x+k, \lambda x+\lambda y+l)$, then $p_1 = 0$ and $p_2 = -1$ satisfy conditions (69). Thus, for $\mathbb{A} = \mathbb{A}_1^2(0,-2)$ the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the bi-telegraphic equation (64). By Proposition 2.3 components of function $e^{\varphi(x,y)}$ are given by*

$$f(x,y) = e^{\lambda x}, \qquad g(x,y) = e^{\lambda x} - e^{-\lambda y},$$

which by Theorem 4.3 are solutions of the bi-telegraphic equation.

Example 4.6 *If $\varphi(x,y) = (x-y+k, x+y+l)$, then $p_1 = (\lambda/2)^4$ and $p_2 = 0$ satisfy conditions (69). By Proposition 2.3, for $\mathbb{A} = \mathbb{A}_1^2((\lambda/2)^4, 0)$ the components of function $e^{\varphi(x,y)}$ are given by*

$$f(x,y) = \frac{e^{x-y}}{2}(e^{\frac{\lambda^2}{4}(x+y)} + e^{-\frac{\lambda^2}{4}(x+y)}), \qquad g(x,y) = \frac{2e^{x-y}}{\lambda^2}(e^{\frac{\lambda^2}{4}(x+y)} - e^{-\frac{\lambda^2}{4}(x+y)}),$$

which by Theorem 4.3 are solutions of the bi-telegraphic equation. Components of $\sin(\varphi(x,y))$ are given by

$$f(x,y) = \sin(x-y+k)\cos\left(\frac{\lambda^2}{4}(x+y+k)\right), \quad g(x,y) = \frac{4}{\lambda^2}\cos(x-y+k)\sin\left(\frac{\lambda^2}{4}(x+y+k)\right).$$

So, by proof of Theorem 4.3 they are solutions of the bi-telegraphic equation.

Example 4.7 *If $\varphi(x,y) = (0, (\sqrt{\lambda^2+d^2}\,)x+dy+l)$, then $p_1 = 1$ and $p_2 = 0$ satisfy conditions (69). Thus, for $\mathbb{A} = \mathbb{A}_1^2(0,-2)$ the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the bi-telegraphic equation (64). Following proof of Proposition 2.3 components of functions $\sin(\varphi(x,y))$ are given by*

$$f(x,y) = 0, \qquad g(x,y) = \sin\left((\sqrt{\lambda^2+d^2}\,)x + dy + l\right),$$

which by Theorem 4.3 are solutions of the bi-telegraphic equation. In the same way components of functions $\cos(\varphi(x,y))$ are given by

$$f(x,y) = \cos\left((\sqrt{\lambda^2+d^2}\,)x + dy + l\right), \qquad g(x,y) = 0,$$

and they are solutions of the bi-telegraphic equation.

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