Regularity, Asymptotic Solutions and Travelling Waves analysis in a porous medium system to model the interaction between invasive and invaded species.

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Abstract

This work provides an analytical approach to characterize and determine solutions to a porous medium system of equations with views in applications to invasive-invaded biological dynamics. Firstly, the existence and uniqueness of solutions are proved. Afterwards, profiles of solutions are obtained making use of the selfsimilar structure that permits to show the existence of a diffusive front. The solutions are then studied within the Travelling Waves (TW) domain showing the existence of potential and exponential profiles in the stable connection that converges to the stationary solutions in which the invasive specie predominates. The TW profiles are shown to exist based on the geometry perturbation theory together with an analytical-topological argument in the phase plane. The finding of an exponential decaying rate (related with the advection and diffusion parameters) in the invaded specie TW is not trivial in the non-linear diffusion case and reflects the existence of a TW trajectory governed by the invaded specie runaway (in the direction of the advection) and the diffusion (acting along a finite speed front or support).

Keywords: *porous medium equation, travelling waves, geometric perturbation, non-linear diffusion, advection.*

AMS Subject Classification: 35K55, 35K57, 35K59, 35K65

1 Introduction

According to the Convention of Biological Diversity (p. 1, Ch. 1 [18]) , the biological invasion is defined as "...those alien species which threaten ecosystems, habitats or species". The problem analyzed can be understood within the perspective of an invasive specie that invades a region or space previously inhabited by the invaded specie.

The invaded-invasive interaction between species has been discussed with an advection, that elucidates a non-linear diffusion, in [21]. The system derived in such reference was intended to describe the haptotactic cell invasion in a model for melanoma. In addition, [12] examines the spectral stability of travelling waves of the haptotaxis model studied in cancer invasion. The model has been analyzed making use of Evans function to a linearised operator. In these cited cases, the proposed models interpreted the advection as part of the complete random movement (no preferred direction) induced by the haptotactic evolution.

Note that even when invasive-invaded systems are not precisely speaking predator-prey models, it is worth mentioning that once and invasive occupies a domain (for example a biological organ) the invaded specie extinguishes. Such extinguishing may lead the invasive to vanish (or even die if the organ fails) as there is not further specie to invade. This a-priori dynamic constitutes a link between invasive-invaded dynamic and predator-prey. Predator-prey models have been analyzed recently with analytical and stability methods. In [5], different forms of functional responses are provided for modelling the predator-prey dynamic. The cited work considers the harvesting in the predator and the density-dependent mortality in the prey to assess the Hopf bifurcation in the proximity of the equilibrium point. In addition, the Hopf bifurcation method has been followed in [13] to study a delayed density-dependent predator-prey system with Beddington-DeAngelis functional response. The periodic dynamic in the solutions has been shown to exist as a consequences of stochastic disturbance for a Holling II functional response [28]. Similar stability and bifurcation methods have been used in [19] for a predator–prey model subject to the Allee effect with a discrete-time Holling type-IV functional response. The mentioned methods have been used as well in [14] to model an infected predator that consumes the prey according to Holling type-II response.

There exists a mathematical conection among the Hopf bifurcation method, Evans function methods and the geometric perturbation as such theories are intended to search for stable solutions, for example in the form of Travelling Waves (TW). Note that the geometric perturbation theory, as a baseline support in the search of TW profiles, is considered along this work.

As it will be shown, the non-linear diffusion drives the mathematical methods employed in this paper. Within the mathematical applications to biology, a non-linear diffusion model was proposed initially by Keller and Segel [17] to study the cells movement by chemotaxis:

$$\begin{aligned} u_t &= \nabla \cdot (d(u)u - \chi(v)u\nabla u) \quad && x \in \Omega,\ t > 0 \\ v_t &= d_v \Delta v - uv && x \in \Omega,\ t > 0, \end{aligned} \tag{1.1}$$

where u represents the cell density and v the chemical concentration. Note that $d(u)$ is the media diffusivity and $\chi(v)$ the distribution of chemical agent to which the cells are sensitive. The Keller and Segel model was progressively extended to incorporate particular kinetic dynamics in the form of reaction and absorption terms [2], [7], [26], [24] with different structures complying with a regularity criteria including the Lipschtiz condition. The problem discussed in [23] was proposed as an integro-differential system of equations with pressure effects to predict the behaviour of cancerous cells, as an invasive specie, spreading with a non-linear diffusion over healthy cells in a closed organ.

The use of non-linear diffusion is an oblique topic and has been used in other interesting applications where numerical and purely analytical approaches have been followed. Such non-linear diffusion allows to model accurately physical phenomena in which porosity is a governing parameter. The Darcy law involving nanofluids has been considered to derive numerical solutions by the Successive Local Linearization Method [22]. The authors show the different solution profiles obtained after the numerical exercise where the exponential decay can be perceived. Such an exponential decay is immediate in the linear diffusion case, nonetheless for the non-linear diffusion further analytical assessments shall be done. The effect of porosity in a partial slip for a peristaltic transport in a Jeffrey fluid has been investigated in [10]. In addition [6] employs a non-linear diffusion to improve the accuracy in simulating potential coagulation in an electromagnetic blood flow in annular vessel geometries. In all the cited cases, the finite speed of propagation defines a diffusive front. This is a common behaviour to all porous medium equations and shall be characterized for the problem discussed along the present work.

2 Model description and methods

The problem P analyzed is as follows:

$$\begin{gathered} u_t = \Delta u^m + c \cdot \nabla u + v(1-u), \\ v_t = \Delta v^m + c \cdot \nabla v - uv, \\ m > 1, \ \ u_0(x), v_0(x) > 0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \end{gathered} \tag{2.1}$$

The solving methods exposed along this work aim to obtain explicit analytical solutions in the form of maximal, minimal solutions and Travelling Waves. The spatial operator involves a non-linear diffusion with advection and is considered together with non-linear coupled reaction-absorption terms.

The problem P justification is based on studying the invasive-invaded dynamic and the mathematical properties introduced by the spatial operator. For this purpose:

Consider that the invasive specie acts in accordance with the presence of the invaded. If the quantity of invaded is high, the temporal increasing rate in the invaded (u_t) shall be high, otherwise the invasion will not succeed. As the invasive specie proliferates over time, there shall be a limit in its concentration, then the temporal ratio (considered positive) decreases because the invasive reaches progressively its maximum stationary level in the medium. In addition, the invasive temporal rate depends on the invaded concentration (v). This behavior is intended to be modelled as a reaction term of the form:

$$u_t = v(\kappa - u), \tag{2.2}$$

where κ represents the maximum invasive concentration in the given medium. Without loss of generality, assume $\kappa = 1$.

The invaded specie time vanishing rate is considered as:

$$v_t = -vu, \tag{2.3}$$

and aims to express that the decreasing rate is absolutely higher for increasing values of the invasive u and is dependant on the own invasive existence in the medium.

An advection term is proposed to account for possible forced movement in the media. The interaction of the advection, through the vector c, has the intention of modelling the desertion behaviour in the invaded specie, so that the runaway is oriented in the direction of c and toward spatial areas not previously populated (represented by $c \cdot \nabla v$). In response to the invaded specie desertion, the invasive specie will follow an advection process given by the orientation c towards the same non-populated areas (represented by $c \cdot \nabla u$).

In addition, a non-linear diffusion (Porous Medium Equation) is considered to model the random movement of both species. As an alternative to the classical linear order two diffusion, the Porous Medium Equation introduces concepts such as the finite speed of propagation in the invaded specie which is admitted to account for a further reliable model.

The study of solutions for a Porous Medium Equation with advection and no reaction has been widely analyzed in [27]. Nonetheless, the present work considers the effect of the coupled reaction and absorption for which the mixed monotone behaviour induces a different analytical treatment compared to that in [27].

This work begins by showing some regularity results together with uniqueness of solutions. The degeneracy of the diffusivity leads to consider weak solutions defined upon a test function $\phi \in C^\infty(\mathbb{R}^d)$. Afterwards, profiles of maximal and minimal solutions are shown to exist and analytically obtained based on the mixed monotone properties in the reaction-absorption terms. In addition, the problem P is studied within the geometric perturbation theory to support the construction of analytical TW profiles. Finally, the finite propagation and the exponential decay for v are shown in the proximity of the null solution, i. e. when $v \to \epsilon \to 0^+$. The finite propagation is proved with a maximal solution so that any other lower solution exhibits finite propagation as well. In addition, the finite propagation is shown to provide a compact support that is mainly dependent of the advection (invaded runaway) c. This means that the homogenizing effect of the diffusive front is lost and the advection term drives the dynamic. This behaviour can be interpreted as the invasive concentrating efforts in the runaway direction c so that the invaded specie decreases exponentially even when the diffusion is non-linear.

3 Existence and uniqueness of solutions

Let consider a test function $\phi \in C^\infty(\mathbb{R}^d)$ such tat for $0 < \tau < t < T$:

$$\int_{\mathbb{R}^d} u(t)\,\phi(t) = \int_{\mathbb{R}^d} u(\tau)\,\phi(\tau) + \int_\tau^t \int_{\mathbb{R}^d} [u\,\phi_t + u^m\,\Delta\phi + c\cdot\nabla\phi\,u + v(1-u)\,\phi]ds, \tag{3.1}$$

$$\int_{\mathbb{R}^d} v(t)\,\phi(t) = \int_{\mathbb{R}^d} v(\tau)\,\phi(\tau) + \int_\tau^t \int_{\mathbb{R}^d} [v\,\phi_t + v^m\,\Delta\phi + c\cdot\nabla\phi\,v - uv\,\phi]ds. \tag{3.2}$$

Consider $r >> r_0 > 0$, the following uniformly parabolic quasilinear set of equations is defined (named P^{ϕ}):

$$\begin{aligned} u\,\phi_t + u^m\,\Delta\phi + c\cdot\nabla\phi\,u + v(1-u)\,\phi = 0, \\ v\,\phi_t + v^m\,\Delta\phi + c\cdot\nabla\phi\,v - uv\,\phi = 0, \end{aligned} \tag{3.3}$$

in $B_r \times [0,T]$, with the following set of boundary and initial conditions:

$$\begin{aligned} (\nabla\phi + u\,c)\cdot\nu = 0, \\ (\nabla\phi + v\,c)\cdot\nu = 0. \end{aligned} \tag{3.4}$$

over $\partial B_r \times [0,T]$, where ν is the outer unitary normal vector in ∂B_r and

$$\begin{aligned} u(x,0) = u_0(x), \\ v(x,0) = v_0(x). \end{aligned} \tag{3.5}$$

initially on B_r.

The problem P^{ϕ} has existence and uniqueness of solutions based on the mixed monotone properties of the forcing terms [20]. For this purpose, note that the forcing part ϕ is a monotone increasing (with ϕ) function and $-\phi$ is a monotone decreasing function.

Theorem 1. *Given $u_0(x)$, $v_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, the set of solutions $u(x,t)$, $v(x,t)$ are bounded for all $(x,t) \in B_r \times [0,T]$ with $r >> 1$.*

Proof. Consider $\eta \in \mathbb{R}^+$ and sufficiently small, the following cut off function is defined [9]:

$$\begin{aligned} \psi_\eta \in C_0^\infty(\mathbb{R}^d), \qquad & 0 \le \psi_\eta \le 1, \\ \psi_\eta = 1 \;\; in \;\; B_{r-\eta}, \qquad & \psi_\eta = 0 \;\; in \;\; \mathbb{R}^d - B_{r-\eta}, \end{aligned} \tag{3.6}$$

so that

$$\begin{aligned} |\nabla\psi_\eta| \le \tfrac{c_a}{\eta}, \\ |\Delta\psi_\eta| \le \tfrac{c_a}{\eta^2}. \end{aligned} \tag{3.7}$$

After multiplication of (3.3) by ψ_η and integrating in $B_r \times [\tau, T]$:

$$\begin{aligned} \textstyle\int_\tau^t \int_{B_r} u\,\phi_t\psi_\eta + \int_\tau^t \int_{B_r} u^m\,\Delta\phi\psi_\eta + \int_\tau^t \int_{B_r} c\cdot\nabla\phi\,u\psi_\eta + \int_\tau^t \int_{B_r} v(1-u)\,\phi\psi_\eta = 0, \\ \textstyle\int_\tau^t \int_{B_r} v\,\phi_t\psi_\eta + \int_\tau^t \int_{B_r} v^m\,\Delta\phi\psi_\eta + \int_\tau^t \int_{B_r} c\cdot\nabla\phi\,v\psi_\eta - \int_\tau^t \int_{B_r} uv\psi_\eta\,\phi = 0. \end{aligned} \tag{3.8}$$

The integrals for the diffusion and advection terms read:

$$\int_\tau^t \int_{B_r} c\cdot\nabla\phi\,u\psi_\eta = -\int_\tau^t \int_{B_r} u\phi\,c\cdot\nabla\psi_\eta, \tag{3.9}$$

$$\int_\tau^t \int_{B_r} u^m\,\Delta\phi\psi_\eta = -\int_\tau^t \int_{B_r} u^m\,\nabla\phi\cdot\nabla\psi_\eta. \tag{3.10}$$

□

Then:

$$\begin{aligned} \textstyle\int_\tau^t \int_{B_r} u\,\phi_t\psi_\eta + \int_\tau^t \int_{B_r} v(1-u)\,\phi\psi_\eta = \int_\tau^t \int_{B_r} u\phi\,c\cdot\nabla\psi_\eta + \int_\tau^t \int_{B_r} u^m\,\nabla\phi\cdot\nabla\psi_\eta, \\ \textstyle\int_\tau^t \int_{B_r} v\,\phi_t\psi_\eta - \int_\tau^t \int_{B_r} uv\psi_\eta\,\phi = \int_\tau^t \int_{B_r} v\phi\,c\cdot\nabla\psi_\eta + \int_\tau^t \int_{B_r} v^m\,\nabla\phi\cdot\nabla\psi_\eta. \end{aligned} \tag{3.11}$$

The intention is to show that the left hand integrals in (3.11) are finite which is equivalent to search for the L^∞ norm on the right hand side terms.

For some large $r >> r_0 > 1$ [9]:

$$\int_\tau^t u^m \le c_1(\tau) r^{\frac{2m}{m-1}}, \tag{3.12}$$

and

$$\int_\tau^t u \le c_2(\tau) r^{\frac{2}{m-1}}. \tag{3.13}$$

Then:

$$\int_\tau^t \int_{B_r} u^m \nabla\phi\cdot\nabla\psi_\eta \le \int_{B_r} c_1(\tau)\, r^{\frac{2m}{m-1}}\, |\nabla\phi|\, \frac{c_a}{\eta} \le c_1(\tau)\, c_a \int_{B_r} r^{\frac{2m}{m-1}-1}|\nabla\phi|. \tag{3.14}$$

$$\int_\tau^t \int_{B_r} u\phi\, c\cdot\nabla\psi_\eta \le c\, c_2(\tau)\, c_a \int_{B_r} r^{\frac{2}{m-1}-1}\, \phi. \tag{3.15}$$

Next, consider a test function ϕ of the form:

$$\phi(x,s) = e^{g(s)}(1+r^2)^{-\beta}, \tag{3.16}$$

where $g(s) > 0$ for $0 < s < t$, $g(t) = 0$ and β shall be chosen to ensure the converge of (3.14) and (3.15) as $r \to \infty$. Then (3.14) is:

$$c_1(\tau)\, c_a \int_{B_{r\to\infty}} r^{\frac{2m}{m-1}-1} 2\beta e^{g(s)} r^{-2\beta-1}, \tag{3.17}$$

and (3.15):

$$c\, c_2(\tau)\, c_a \int_{B_r} r^{\frac{2}{m-1}-1}\, e^{g(s)} r^{-2\beta}. \tag{3.18}$$

The convergence is ensured for

$$\beta = \frac{1}{m-1}. \tag{3.19}$$

Coming to (3.11):

$$\int_\tau^t \int_{B_{r\to\infty}} u\,\phi_t\psi_\eta + \int_\tau^t \int_{B_r r\to\infty} v(1-u)\,\phi\psi_\eta \le c\, c_2(\tau)\, c_a\, e^{g(s)}, \tag{3.20}$$

$$\int_\tau^t \int_{B_{r\to\infty}} v\,\phi_t\psi_\eta - \int_\tau^t \int_{B_{r\to\infty}} uv\psi_\eta\,\phi \le c\, c_2(\tau)\, c_a\, e^{g(s)}. \tag{3.21}$$

As both integrals are finite in $\tau < s < t < T$, it is possible to conclude on the theorem postulation about the boundness of solutions in $\mathbb{R}^d \times [0,\infty]$.

The next intention is to show the uniqueness of solutions:

Theorem 2. *Let assume $(u,v) > (0,0)$ is a minimal solution for P (2.1) in $\mathbb{R}^d \times (0,T)$, then (u,v) coincides with the maximal solution, i.e., the solution is unique.*

Proof. Let $(\hat{u},\hat{v})$ be the maximal solution to P in $\mathbb{R}^d \times (0,T)$ such that:

$$(\hat{u}(x,0), \hat{v}(x,0)) = (u_0(x)+\nu, v_0(x)+\nu), \tag{3.22}$$

with $\nu > 0$ arbitrary small. In addition, let define the minimal solution:

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)), \tag{3.23}$$

such that the maximal and minimal solutions verify respectively:

$$\begin{aligned} \hat{u}_t &= \Delta\hat{u}^m + c\cdot\nabla\hat{u} + \hat{v}(1-u), \\ \hat{v}_t &= \Delta\hat{v}^m + c\cdot\nabla\hat{v} - uv. \end{aligned} \tag{3.24}$$

$$\begin{aligned} u_t &= \Delta u^m + c\cdot\nabla u + v(1-\hat{u}), \\ v_t &= \Delta v^m + c\cdot\nabla v - \hat{u}\hat{v}. \end{aligned} \tag{3.25}$$

Then for every test function $\phi \in C^\infty(\mathbb{R}^d)$, the following expressions hold:

$$0 \le \int_{\mathbb{R}^d} (\hat{u}-u)(t)\,\phi(t) = \int_0^t \int_{\mathbb{R}^d} [(\hat{u}-u)\,\phi_t + (\hat{u}^m-u^m)\,\Delta\phi + c\cdot\nabla\phi\,(\hat{u}-u) + (\hat{v}(1-u) - v(1-\hat{u})\,\phi]ds, \tag{3.26}$$

$$0 \le \int_{\mathbb{R}^d} (\hat{v}-v)(t)\,\phi(t) = \int_0^t \int_{\mathbb{R}^d} [(\hat{v}-v)\,\phi_t + (\hat{v}^m-v^m)\,\Delta\phi + c\cdot\nabla\phi\,(\hat{v}-v) + (\hat{u}\hat{v}-uv)\,\phi]ds. \tag{3.27}$$

Let define:

$$a_1(x,s) = \begin{bmatrix} \frac{\hat{u}^m - u^m}{\hat{u}-u} & for \;\; \hat{u} \not\equiv u \\ m u^{m-1} & otherwise \end{bmatrix}. \tag{3.28}$$

$$a_2(x,s) = \begin{bmatrix} \frac{\hat{v}^m - v^m}{\hat{v}-v} & for \;\; \hat{v} \not\equiv v \\ m v^{m-1} & otherwise \end{bmatrix}. \tag{3.29}$$

Given two fixed values for x and $s = t \leq T$:

$$0 \leq a_1(x,s) \leq c_1(m, \|u_0\|_\infty, T), \tag{3.30}$$

$$0 \leq a_2(x,s) \leq c_2(m, \|v_0\|_\infty, T). \tag{3.31}$$

Consider the test function:

$$\phi(|x|,s) = e^{k(T-s)}(1+|x|^2)^{-\gamma}, \tag{3.32}$$

for some constants k and γ.

The test function verifies:

$$\begin{gathered} \phi_t = -k\phi(x,s), \\ |\nabla_{|x|}\phi| \leq c_3(\gamma,d)\phi(x,s), \\ \Delta_{|x|}\phi \leq c_4(\gamma,d)\phi(x,s), \end{gathered} \tag{3.33}$$

such that:

$$\begin{aligned} (\hat{u}-u)\,\phi_t + (\hat{u}^m - u^m)\,\Delta\phi + c\cdot\nabla\phi\,(\hat{u}-u) + (\hat{v}(1-u) - v(1-\hat{u}))\,\phi \leq -(\hat{u}-u)\,k\,\phi \\ + a_1(\hat{u}-u)\,c_4\phi + c\,c_3\phi\,(\hat{u}-u) + L_1(\hat{v}-v)\,\phi, \end{aligned} \tag{3.34}$$

$$\begin{aligned} (\hat{v}-v)\,\phi_t + (\hat{v}^m - v^m)\,\Delta\phi + c\cdot\nabla\phi\,(\hat{v}-v) + (\hat{u}\hat{v} - uv)\,\phi \leq -(\hat{v}-v)\,k\,\phi \\ + a_2(\hat{v}-v)\,c_4\phi + c\,c_3\phi\,(\hat{v}-v) + L_2(\hat{u}-u)\,\phi. \end{aligned} \tag{3.35}$$

where $L_1 = \max_{\mathbb{R}^d}\{(1-u)\}$ and $L_2 = \max_{\mathbb{R}^d}\{\hat{v}\}$ are bounded as per Theorem 1.

The constant k shall be selected such that:

$$-k + a_1c_4 + c_3c \leq 0, \quad -k + a_2c_4 + c_3c \leq 0. \tag{3.36}$$

It suffices to consider:

$$k \geq \max(a_1c_4 + c_3c, a_2c_4 + c_3c), \tag{3.37}$$

so that:

$$\begin{gathered} (-k + a_1c_4 + c_3c)\phi(\hat{u}-u) \leq 0, \\ (-k + a_1c_4 + c_3c)\phi(\hat{v}-v) \leq 0. \end{gathered} \tag{3.38}$$

The expressions (3.26) and (3.27) read:

$$0 \leq \int_{\mathbb{R}^d} (\hat{u}-u)(t)\,\phi(t) \leq \int_0^t \int_{\mathbb{R}^d} L_1(\hat{v}-v)\,\phi ds, \tag{3.39}$$

$$0 \leq \int_{\mathbb{R}^d} (\hat{v}-v)(t)\,\phi(t) \leq \int_0^t \int_{\mathbb{R}^d} L_2(\hat{u}-u)\,\phi ds. \tag{3.40}$$

Making the d/dt in the previous expressions:

$$0 \leq \frac{d}{dt}\int_{\mathbb{R}^d} (\hat{u}-u)(t)\,\phi(t) \leq \int_{\mathbb{R}^d} L_1(\hat{v}-v)(t)\,\phi(t), \tag{3.41}$$

$$0 \leq \frac{d}{dt}\int_{\mathbb{R}^d} (\hat{v}-v)(t)\,\phi(t) \leq \int_{\mathbb{R}^d} L_2(\hat{u}-u)(t)\,\phi(t), \tag{3.42}$$

Let operate in the last two expressions with the equality, then:

$$\int_{\mathbb{R}^d} (\hat{u}-u)(t)\,\phi(t) = \frac{1}{L_2}\frac{d}{dt}\int_{\mathbb{R}^d} (\hat{v}-v)(t)\,\phi(t), \tag{3.43}$$

which can be replaced in (3.41):

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^d}(\hat{v}-v)(t)\,\phi(t) \leq L_1\,L_2\int_{\mathbb{R}^d}(\hat{v}-v)(t)\,\phi(t). \tag{3.44}$$

Let define:

$$g(t)=\int_{\mathbb{R}^d}(\hat{v}-v)(t)\,\phi(t), \tag{3.45}$$

then (3.44) reads as a linear standard second order equation:

$$\frac{d^2g(t)}{dt^2}=L_1\,L_2 g(t), \tag{3.46}$$

with

$$\begin{aligned} g(0)=\nu\to 0,\\ g\prime(0)=\nu\to 0. \end{aligned} \tag{3.47}$$

Then the equation (3.46) converges to the null solution. For the shake of simplicity consider $g(t)=0$, concluding that:

$$\hat{v}=v. \tag{3.48}$$

Now, returning to (3.39):

$$0\leq\int_{\mathbb{R}^d}(\hat{u}-u)(t)\,\phi(t)\leq 0, \tag{3.49}$$

to conclude:

$$\hat{u}=u, \tag{3.50}$$

showing, then, the uniqueness of any positive solution to P. □

4 Profiles of solution

This section is devoted to the searching of upper and lower solutions considering that the invasive specie $u(x,t)$ shall be non-decreasing (i.e. $u_t\geq 0$) while the invaded one $v(x,t)$ shall be non-increasing (i.e. $v_t\leq 0$).

Theorem 3. *A maximal solution $\hat{v}(x,t)$ for the invaded specie is:*

$$\hat{v}(x,t)=v_0(x)-(At)^{\frac{1}{m-1}}. \tag{4.1}$$

where

$$A\geq\frac{\delta^{m-1}}{\frac{|x|}{c}} \tag{4.2}$$

being $\delta=min\{v\}>0$, $|x|$ representing the spatial integration domain and $0<t<\frac{|x|}{c}$.

In addition, $\hat{v}$ propagates along the support in (x,t) defined by the minimal approximation:

$$|x|_{min}=\left(\frac{\delta^{m-1}c2m}{m-1}\right)^{1/3}t^{2/3}. \tag{4.3}$$

Proof. Let start with the equation in v. Considering selfsimilar solutions:

$$\hat{v}(x,t)=t^{-\alpha}f(\xi),\quad \xi=|x|t^{\beta}, \tag{4.4}$$

to the upper profile equation:

$$\hat{v}_t=\Delta\hat{v}^m+c\cdot\nabla\hat{v}. \tag{4.5}$$

Then, the following holds:

$$-\alpha t^{-\alpha-1}f+\beta\xi t^{-\alpha-1}f'=G(f'',f',f,m)t^{-\alpha m+2\beta}+ct^{-\alpha+\beta}f'. \tag{4.6}$$

Making the corresponding equalities in the time exponents:

$$\alpha=\frac{-1}{m-1};\quad \beta=-1. \tag{4.7}$$

The selfsimilar profile shall, then, be obtained as a solution to the elliptic equation:

$$-\alpha f+\beta\xi f'=(f^m)''+\frac{d-1}{\xi}(f^m)'+cf'. \tag{4.8}$$

Note that in the approximation:

$$\beta\xi>>c\,,\qquad |x|t^{-1}>>|-c|, \tag{4.9}$$

the following elliptic equation holds for $t<<\frac{|x|}{c}$.

$$\beta\xi f'=(f^m)''+\frac{d-1}{\xi}(f^m)'+\alpha f, \tag{4.10}$$

for which a solution is available [25] and [8]:

$$f(\xi)=(A-B\xi^2)^{\frac{1}{m-1}}, \tag{4.11}$$

where $A>0$ and $B=\frac{(m-1)}{2m}$. A is obtained by making $\xi=0$ in (4.11), so that:

$$\hat{v}(x,t)=A^{\frac{1}{m-1}}t^{-\alpha}. \tag{4.12}$$

Note that (4.6) provides supersolutions under the condition that each specie concentration is positive, hence $\min\{uv\}>0$. As $u_t>0$ and assuming $u_0>0$, then it is required that $min\{v\}>0$. Consider $\delta=min\{v\}>0$, then

$$\hat{v}(x,t)=A^{\frac{1}{m-1}}t^{-\alpha}>\delta, \tag{4.13}$$

so that

$$A=\frac{\delta^{m-1}}{t}\geq\frac{\delta^{m-1}}{\frac{|x|}{c}} \tag{4.14}$$

in the inner region $t<<\frac{|x|}{c}$ where $|x|$ shall be understood as the maximum spatial variable representing the integration domain.

The maximal time evolution $\hat{v}$ is then given by:

$$\hat{v}(x,t)=v_0(x)-(At)^{\frac{1}{m-1}}. \tag{4.15}$$

Finally, $\hat{v}$ is indeed an upper evolution. To show this, let consider a solution v to the original equation given by the absorption term $-uv$ and define a t_0:

$$0<t_0<\frac{|x|}{c}, \tag{4.16}$$

such that for any τ with $t_0<\tau<\frac{|x|}{c}$:

$$v(x,\tau)\leq\hat{v}(x,t_0). \tag{4.17}$$

Now, for any t with $\tau<t<\frac{|x|}{c}$:

$$v(x,\tau+t)\leq\hat{v}(x,t_0+t). \tag{4.18}$$

In the limit $\tau\to0$:

$$v(x,t)\leq\hat{v}(x,t), \tag{4.19}$$

showing the upper behaviour of $\hat{v}$ compared to v.

The propagating support is obtained making:

$$f(\xi)=0\,;\quad A=B\xi^2, \tag{4.20}$$

so that A and B can be replaced by their corresponding values. Then the support propagates as:

$$|x|\geq\left(\frac{\delta^{m-1}c2m}{m-1}\right)^{1/3}t^{2/3}=|x|_{min}. \tag{4.21}$$

$\square$

Note that solutions are of the form (4.1) in $\mathbb{R}^d \times [0, \infty]$, so that in the proximity of $\delta \to 0^+$, the following time assessment holds:

$$t_v = \frac{(\|v_0\|_\infty - \delta)^{m-1}}{A}. \tag{4.22}$$

The expression (4.22) provides an estimation in time to consider a vanishing condition for v, so that for $t > t_v$, the equation for $\tilde{u} \leq u$ reads:

$$\tilde{u}_t = \Delta \tilde{u}^m + c \cdot \nabla \tilde{u}, \tag{4.23}$$

and the following theorem holds in the search of a lower solution $\tilde{u}$:

Theorem 4. *Consider*

$$A_1 = \max_{x \in \mathbb{R}^d} \{u_0(x)\}. \tag{4.24}$$

The lower solution $\tilde{u}(x,t)$ for the invasive specie reads:

$$\tilde{u}(x,t) = u_0(x) + A_1 t^{\frac{1}{m-1}}, \tag{4.25}$$

for

$$t_v < t < \frac{|x|}{c}. \tag{4.26}$$

Proof. Again, consider selfsimilar solutions of the form:

$$\tilde{u}(x,t) = t^{-\alpha} F(\xi), \quad \xi = |x| t^{\beta}, \tag{4.27}$$

to the lower profile equation (4.23). Upon substitution of the selfsimilar solution (4.27) and operating analogously as in (4.6), the following selfsimilar exponents are determined:

$$\alpha = \frac{-1}{m-1}; \quad \beta = -1. \tag{4.28}$$

Assume $t < \frac{|x|}{c}$, the selfsimilar profile F is obtained as a solution to the elliptic equation:

$$\beta \xi F' = (F^m)'' + \frac{d-1}{\xi}(F^m)' + \alpha F, \tag{4.29}$$

for which the solution is, again, provided in [25] and [8]:

$$F(\xi) = (A_1 - B_1 \xi^2)^{\frac{1}{m-1}}, \tag{4.30}$$

for $t_v < t < \frac{|x|}{c}$ and $A_1 > 0$, $B = \frac{(m-1)}{2m}$.

Making $\xi = 0$, the solution for $\tilde{u}$ is:

$$\tilde{u}(x,t) = u_0(x) + A_1 t^{\frac{1}{m-1}}, \tag{4.31}$$

where

$$A_1 = \max_{x \in \mathbb{R}^d} \{u_0(x)\}. \tag{4.32}$$

Finally, the expression in (4.31) is shown to be a lower solution. For such purpose, let define t_0, such that $t_v < t_0 < \frac{|x|}{c}$, and τ such that $t_0 < \tau < \frac{|x|}{c}$, then:

$$\tilde{u}(x,\tau) \leq u(x, t_0). \tag{4.33}$$

This last expression is valid provided $|t_0 - \tau| << 1$. Consider t' with $\tau < t' < \frac{|x|}{c}$, then:

$$\tilde{u}(x, \tau + t') \leq u(x, t_0 + t'). \tag{4.34}$$

After a time translation $t = t_0 + t'$:

$$\tilde{u}(x,t) \leq u(x,t), \tag{4.35}$$

with $t_v < \frac{t}{2} < \frac{|x|}{c}$. $\square$

The next objective is to define a maximal solution for u and a minimal solution for v. To this end, the following theorem holds:

Theorem 5. *There exist a minimal solution $\tilde{v}(x,t)$ and a maximal solution $\hat{u}(x,t)$ in $\mathbb{R}^d \times [0,T]$. Furthermore, consider*

$$\tilde{v} = \gamma = \min_{x\in\mathbb{R}^d}\{v_0(x)\} \tag{4.36}$$

then a maximal solution for u is:

$$\hat{u}(x,t) = 1-(1-u_0(x))e^{-\gamma t} \tag{4.37}$$

Proof. For building the maximal solution $\hat{u}(x,t)$, consider the following problem in u:

$$\begin{gathered} u_t^\epsilon = \Delta(u^\epsilon)^m + c\cdot\nabla u^\epsilon + v(1-u^\epsilon),\\ u^\epsilon(x,0) = u_0(x)+\epsilon, \quad v(x,0) = v_0(x),\\ u_0(x) > 0. \;\; v_0(x) > 0 \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d). \end{gathered} \tag{4.38}$$

Solutions to problem (4.38) do exist under the Lipschitz condition in the reaction term and the positive initial conditions (refer to Theorem 1).

Let consider $\epsilon_2 < \epsilon$, so that:

$$\begin{gathered} u_t^\epsilon = \Delta(u^\epsilon)^m + c\cdot\nabla u^\epsilon + v(1-u^\epsilon),\\ u_t^{\epsilon_2} = \Delta(u^{\epsilon_2})^m + c\cdot\nabla u^{\epsilon_2} + v(1-u^\epsilon),\\ u^\epsilon(x,0) = u_0(x)+\epsilon > \;\; u^{\epsilon_2}(x,0) = u_0(x)+\epsilon_2 > 0, \end{gathered} \tag{4.39}$$

The iteration process starts with ϵ, and based on the monotony behaviour in the reaction terms of (4.39), then $u^{\epsilon_2} < u^\epsilon$ in accordance with the initial data for u^ϵ. The same argument can be repeated for $\epsilon_3 < \epsilon_2$, so that $u^{\epsilon_3} < u^{\epsilon_2}$. The sequence defined as $\{u^\epsilon\}$ is non-negative and non-increasing. Consequently the following maximal solution in the limit is defined:

$$\hat{u} = \lim_{\epsilon\to 0} u^\epsilon. \tag{4.40}$$

To construct the minimal solution for v, consider the problem:

$$\begin{gathered} v_t^\epsilon = \Delta(v^\epsilon)^m + c\cdot\nabla v^\epsilon - \left(\lim_{\epsilon\to 0} u^\epsilon\right) v^\epsilon\\ v^\epsilon(x,0) = v(x,0) = v_0(x) > 0. \end{gathered} \tag{4.41}$$

Solutions to (4.41) do exist (refer to Theorem 1).

Consider $\epsilon_2 < \epsilon$, then:

$$u^{\epsilon_2} < u^\epsilon, \;\Rightarrow\; v^{\epsilon_2} > v^\epsilon, \tag{4.42}$$

based on the monotony properties in the reaction term of (4.41). The sequence $\{v^\epsilon\}$ is non-decreasing as $\epsilon \to 0$. The minimal solutions for v is defined as per the monotone limit:

$$\tilde{v} = \lim_{\epsilon\to 0} v^\epsilon. \tag{4.43}$$

Once the maximal and minimal solutions have been constructed and shown to exist, the next intention is to search for a family of flat solutions u^ϵ, v^ϵ via the resolution of:

$$\begin{gathered} u_t^\epsilon = \gamma(1-u^\epsilon),\\ v^\epsilon = \gamma = \min_{x\in\mathbb{R}^d}\{v_0(x)\},\\ u^\epsilon(x,0) = u_0(x)+\epsilon, \quad v^\epsilon(x,0) = v_0(x),\\ u_0(x) > 0. \;\; v_0(x) > 0 \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d). \end{gathered} \tag{4.44}$$

This last system can be solved by separation of variables, so that:

$$u^\epsilon(x,t) = 1-(1-u_0(x)+\epsilon)e^{-\gamma t}, \tag{4.45}$$

then for $\epsilon \to 0$

$$\hat{u}(x,t) = 1-(1-u_0(x))e^{-\gamma t}, \tag{4.46}$$

□

On a physical sound, a value for γ shall be determined experimentally and can represent a constant homogeneous initial distribution of the invaded specie in the domain. Depending on the particular application, either the maximal or minimal solution may be considered. For instance, if the intention is to model the minimal level of invaded specie that can face an invasive, then the solution $\tilde{v}$ in (4.36) shall be taken into account together with the solution $\hat{u}$ in (4.37). Such maximal solution would make sense in this case (minimal invaded population) as the resistance to the invasion is relatively low leading the invasive to prosper quickly via a maximal solution $\hat{u}$. In addition, note that the maximal solution for the invaded specie (and hence any other solution below) propagates along the support given in (4.3) which elucidates the finite propagation feature. Note that in the inner region (inner compared to the diffusive front moving with finite propagation), the desertion will guide the invaded specie to move along the preferred direction c. If c is high then the propagating support (4.3), induced by the diffusion, will expand further.

5 Travelling Waves Existence and Regularity

The TW profiles are expressed as $u(x,t) = f(\xi)$, $\xi = x \cdot n_d - at \in \mathbb{R}$, where n_d is a unitary vector in $\mathbb{R}^d$ that defines the TW-propagation direction. a is the TW-speed and $f : \mathbb{R} \to (0, \infty)$ belongs to $L^\infty(\mathbb{R}^d)$. Note that two TW are equivalent under translation $\xi \to \xi + \xi_0$ and symmetry $\xi \to -\xi$. For the sake of simplicity, the vector n_d is $n_d = (1, 0, ..., 0)$, then $u(x,t) = f(\xi)$, $\xi = x - at \in \mathbb{R}$.

Consider $u(x,t) = f(\xi)$ and $v(x,t) = g(\xi)$, then the problem P (2.1) is transformed to the TW domain:

$$
\begin{aligned}
-af' &= (f^m)' + cf' + g(1-f), \\
-ag' &= (g^m)' + cg' - fg, \\
&f, g \in L^\infty(\mathbb{R}), \\
&f'(\xi) > 0\,,\, g'(\xi) < 0, \\
&f(\infty) = 1\,,\, g(\infty) = 0.
\end{aligned}
\tag{5.1}
$$

Working with the density and flux variables

$$
X_1 = f\,,\; Y_1 = (f^m)'\,,\; X_2 = g\,,\; Y_2 = (g^m)', \tag{5.2}
$$

the following system holds:

$$
\begin{aligned}
X_1' &= \tfrac{1}{m} X_1^{1-m} Y_1, \\
Y_1' &= -(a+c)\tfrac{1}{m} X_1^{1-m} Y_1 - X_2(1 - X_1), \\
X_2' &= \tfrac{1}{m} X_2^{1-m} Y_2, \\
Y_2' &= X_1 X_2 - (a+c)\tfrac{1}{m} X_2^{1-m} Y_2,
\end{aligned}
\tag{5.3}
$$

with the critical point $(1, 0, 0, 0)$ that represents a situation in which the invasive wins over the invaded. The analysis of the TW features in the proximity of the critical point permits to establish the following lemma:

Lemma 1. *The critical point* $(1, 0, 0, 0)$ *is a degenerate node with:*

- *One null eigenvalue,*
- *One eigenvalue approaching the null condition as the TW speed* a *increases.*
- *Two real eigenvalues, one positive and one negative.*

Proof. Firstly, in the proximity of the critical point define:

$$
\tilde{X_1} = X_1 - 1, \;\; X_2 = \epsilon << 1, \tag{5.4}
$$

so that the system (5.3) is rewritten in the compact form:

$$\begin{pmatrix} \tilde{X_1}' \\ Y_1' \\ X_2' \\ Y_2' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} & 0 & 0 \\ \epsilon & -\frac{a+c}{m} & 0 & 0 \\ 0 & 0 & 0 & \frac{\epsilon^{1-m}}{m} \\ 0 & 0 & 1 & -\frac{(a+c)\epsilon^{1-m}}{m} \end{pmatrix} \begin{pmatrix} \tilde{X_1} \\ Y_1 \\ X_2 \\ Y_2 \end{pmatrix} \tag{5.5}$$

with eigenvalues $\left(0, -\frac{\epsilon}{m}, -\frac{a+c}{2m} + \frac{1}{2}\sqrt{\frac{(a+c)^2}{m^2} + \frac{4\epsilon}{m}}, -\frac{a+c}{2m} - \frac{1}{2}\sqrt{\frac{(a+c)^2}{m^2} + \frac{4\epsilon}{m}}\right)$. In the assumption $a >> c >> \epsilon$, the eigenvalues are $\left(0, -\frac{\epsilon}{m}, 0^+, -\frac{a+c}{m}\right)$. This shows the existence of monotone stable TW solutions. The null eigenvalues corresponds to the equilibrium solution $\tilde{X_1} = 1$ $(X_1 = 2)$, $X_2 = 1$ that by a vertical translation can be interpreted as $X_1 = 1,\ X_2 = 0$.

Based on the computation of the associated monotone eigenvectors, it is possible to obtain the following leading front behaviour in the proximity of the critical point:

$$\tilde{X_1} = -c_1\, e^{-\lambda_1\, t}\,, \quad c_1 > 0\,, \quad \lambda_1 = -\frac{a+c}{2m} + \frac{1}{2}\sqrt{\frac{(a+c)^2}{m^2} + \frac{4\epsilon}{m}}, \tag{5.6}$$

or equivalently, computing the distance between the stationary $X_1 = 1$ and the solution in each time step:

$$1 - X_1 = c_1\, e^{-\lambda_1\, t}, \tag{5.7}$$

Additionally for X_2:

$$X_2 = c_2\, e^{-\lambda_2\, t}\,, \ c_2 \in \mathbb{R}\,, \ \lambda_2 = -\frac{a+c}{2m} - \frac{1}{2}\sqrt{\frac{(a+c)^2}{m^2} + \frac{4\epsilon}{m}}, \tag{5.8}$$

which proves the regularity towards convergence in the TW solutions approaching the critical point. $\square$

5.1 Geometric Perturbation Theory

The singular geometric perturbation theory is employed in this section to show the asymptotic behaviour of a two-dimensional manifold defined to make simpler the assessment of a TW analitical profile.

For this purpose, define the two-dimensional manifold as:

$$M_0 = \{X_1, Y_1, X_2, Y_2 \ / \ X_1' = \frac{1}{m} X_1^{1-m} Y_1\,;\, Y_1' = -(a+c)\frac{1}{m} X_1^{1-m} Y_1\}, \tag{5.9}$$

with critical point $(1, 0, 0, 0)$.

The perturbed manifold M_ϵ close to M_0 is defined as:

$$M_\epsilon = \{X_1, Y_1, X_2, Y_2 \ / \ X_2 = \epsilon\,;\, Y_2' = 0\,;\, X_1' = \frac{1}{m} X_1^{1-m} Y_1\,;\, Y_1' = -(a+c)\frac{1}{m} X_1^{1-m} Y_1 - \epsilon(1 - X_1)\}. \tag{5.10}$$

The intention is to use the Fenichel invariant manifold theorem [11] as formulated in [3] and [16]. For this purpose, it is required to show that M_0 is a normally hyperbolic manifold, i. e. the eigenvalues of M_0 in the linearized frame close to the critical point, and transversal to the tangent space, have non-zero real part. This is shown based on the following two dimensional equivalent flow associated to M_0:

$$\begin{pmatrix} X_1' \\ Y_1' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ 0 & -\frac{a+c}{m} \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \tag{5.11}$$

with eigenvalues $(0\,, -(a+c)/m)$. Note that for $\lambda = 0$, the associated eigenvector is $[1, 0]$ tangent to M_0. Therefore, M_0 is a hyperbolic manifold. The next intention is to show that the manifold M_ϵ is locally invariant under the flow given by the set of equations (5.3). For this purpose, it is required [16] that for all $R > 0$, for all open interval J with $(a+c) \in J$ and for any value of $i \in \mathbb{N}$, there exists a δ such that for $\epsilon \in (0, \delta)$ the manifold M_ϵ is invariant. Hence, consider $i \geq 1$ and the functions:

$$\begin{aligned}
\phi_1 &= \epsilon, \\
\phi_2 &= 0, \\
\phi_3 &= \tfrac{1}{m} X_1^{1-m} Y_1, \\
\phi_4 &= -(a+c)\tfrac{1}{m} X_1^{1-m} Y_1 - \epsilon(1 - X_1),
\end{aligned} \tag{5.12}$$

which are $C^i(\overline{B_R(0)} \times \overline{I} \times [0, \delta])$ in the proximity of the critical point $(1, 0, 0, 0)$.

A value for $R > 0$ can be chosen considering that $M_0 \cap B_R(0)$ is large enough so as to study the complete TW evolution along the domain. The determination of δ is based on assessing the distance between the flows in M_0 and M_ϵ. For this purpose, assume that the involved functions in such flows are measurable a.e. in $B_R(0)$:

$$\|\phi_4^{M_0} - \phi_4^{M_\epsilon}\| \leq \delta \|1 - X_1\|. \tag{5.13}$$

The distance between the manifolds keeps the normal hyperbolic condition for $\delta \in (0, \infty)$. For simplicity, assume $\delta = 1$.

In the same way, given the two dimensional manifold M_1:

$$M_1 = \{X_1, Y_1, X_2 \sim \epsilon, Y_2 \ / \ X_2' = \frac{1}{m} X_2^{1-m} Y_2 \, ; \, Y_2' = X_2 - (a+c)\frac{1}{m} X_2^{1-m} Y_2\}, \tag{5.14}$$

with the same critical point $(1, 0, 0, 0)$, and the perturbed M_ϵ' close to M_1:

$$M_\epsilon' = \{X_1, Y_1, X_2, Y_2 \ / \ X_1 = 1; \, Y_1' = 0 \, ; \, X_2' = \frac{1}{m} X_2^{1-m} Y_2 \, ; \, Y_2' = X_1 \, X_2 - (a+c)\frac{1}{m} X_2^{1-m} Y_2\}. \tag{5.15}$$

The Fenichel invariant manifold theorem can be applied in the same manner as for M_0. Note that the two dimensional equivalent flow associated to M_1 is:

$$\begin{pmatrix} X_2' \\ Y_2' \end{pmatrix} = \begin{pmatrix} 0 & \frac{\epsilon^{1-m}}{m} \\ 1 & -\frac{(a+c)\epsilon^{1-m}}{m} \end{pmatrix} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}. \tag{5.16}$$

The associated eigenvalues are both real $\left(-\frac{(a+c)\epsilon^{1-m}}{2m} \pm \frac{1}{2}\sqrt{\frac{((a+c)\epsilon^{1-m})^2}{m^2} + \frac{4\epsilon^{1-m}}{m}} \right)$. Hence, M_1 is a hyperbolic manifold.

In the same manner, the next intention is to show that the manifold M_ϵ' is locally invariant under the flow (5.3) so that the manifold M_1 can be represented as an asymptotic approach to M_ϵ'. For this purpose, consider the functions:

$$\begin{aligned}
\phi_1' &= 1, \\
\phi_2' &= 0, \\
\phi_3' &= \tfrac{1}{m} X_2^{1-m} Y_2, \\
\phi_4' &= Y_2' = X_1 \, X_2 - (a+c)\tfrac{1}{m} X_2^{1-m} Y_2,
\end{aligned} \tag{5.17}$$

which are $C^i(\overline{B_R(0)} \times \overline{I} \times [0, \delta])$ in the proximity of the critical point $(1, 0, 0, 0)$.

In this case, δ is determined based on the following flows that are considered to be measurable a.e. in $B_R(0)$:

$$\|\phi_4'^{M_1} - \phi_4'^{M_\epsilon'}\| \leq \delta\epsilon \|1 - X_1\| = \delta' \|1 - X_1\|. \tag{5.18}$$

The normal hyperbolic condition for both manifolds M_1 and M_ϵ' applies for $\delta' \in (0, \infty)$. In the sake of simplicity consider $\delta' = 1$.

Once we have shown that M_0 and M_1, defined as two dimensional manifolds, remain invariant with regards to the four dimensional manifolds M_ϵ and M_ϵ' respectively under the flow (5.3), the TW profiles can be obtained operating in M_0 and M_1.

6 Travelling Waves Profiles and finite propagation

Based on the normal hyperbolic condition in the manifold M_0 under the flow (5.3), asymptotic TW profiles can be obtained. For this purpose, let consider firstly:

$$X_1' = \frac{1}{m} X_1^{1-m} Y_1 \, ; \;\; Y_1' = -(a+c) \frac{1}{m} X_1^{1-m} Y_1, \tag{6.1}$$

such that the following family of trajectories in the phase plane (X_1, Y_1) holds:

$$\frac{dY_1}{dX_1} = -(a+c) - mX_2 X_1^{m-1} \frac{1-X_1}{Y_1} = H(a, c, X_1, X_2, Y_1), \tag{6.2}$$

where $0 < X_1 < 1 \, ; \, 0 < X_2 < 1 \, ; \, Y_1 > 0$.

The existence of a convergent trajectory is shown based on a comparison with subsolutions for $(a+c)$ sufficiently small and supersolutions for $(a+c)$ sufficiently large together with topological assessment and the continuity of H.

For this purpose, note that when $(a+c) \to 0$ then $dY_1/dX_1 < 0$ while when $(a+c) < 0$ with $|a+c| >> 1$, $dY_1/dX_1 > 0$. Given the continuity of H, it is possible, hence to conclude on the existence of a critical trajectory of the form:

$$-(a+c) - mX_2 X_1^{m-1} \frac{1-X_1}{Y_1} = H(a, c, X_1, X_2, Y_1) = 0. \tag{6.3}$$

Assume now that the selected TW speed satisfies $a << -c$ with $|a+c| >> 1$, then the same solution applies with minor effects due to the advection term which permits to assess a TW speed only making assumptions on the bound of solutions, indeed:

$$a \sim mX_2 X_1^{m-1} \frac{1-X_1}{Y_1}, \tag{6.4}$$

for which a maximal bound is obtained as:

$$a < \frac{m}{A}, \tag{6.5}$$

being $A = \min_{B_R(0)}\{Y_1\}$. Now, coming to (6.3) and under the infinitesimal asymptotic approach $X_2 \sim \epsilon$, the following holds:

$$Y_1 = \frac{m}{|a+c|} \epsilon X_1^{m-1} (1 - X_1), \tag{6.6}$$

in the approach $X_1 \nearrow 1$:

$$Y_1 \sim B(\epsilon)(1 - X_1), \tag{6.7}$$

for a sufficiently small $B(\epsilon)$. Equivalently in the asymptotic approximation:

$$(f^m)' + Bf = 0 \;\; \to \;\; f \left(m f^{m-2} f' + B \right) = 0 \tag{6.8}$$

where $' = \frac{d}{d\xi}$. This last equation has the solution:

$$f(\xi) = C(\alpha - B\xi)^{\frac{1}{m-1}}, \tag{6.9}$$

where $C = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}}$ and $\alpha > 0$. Note that the growing TW is obtained replacing $(-\xi)$ by the symmetric (ξ). This is a meaningful result that expresses the behaviour of the invaded specie $(\sim \xi^{\frac{1}{m-1}})$ in the proximity of the critical point.

The same process shall be repeated for the manifold M_1 under the flow (5.3):

$$X_2' = \frac{1}{m} X_2^{1-m} Y_2 \, ; \;\; Y_2' = X_2 - (a+c) \frac{1}{m} X_2^{1-m} Y_2, \tag{6.10}$$

such that in the phase plane (X_2, Y_2):

$$\frac{dY_2}{dX_2} = -\frac{a+c}{m} + mX_2^m Y_2^{-1} = G(a, c, X_2, Y_2), \tag{6.11}$$

where $0 < X_2 < 1 \, ; \, Y_2 < 0$.

In the same manner, the existence of a convergent trajectory is shown based on a topological argument and the continuity of G. For $(a+c)>0$, $dY_2/dX_2<0$ while when $(a+c)<0$ with $|a+c|>>1$, $dY_2/dX_2>0$. Given the continuity of G, a critical trajectory is given:

$$-\frac{a+c}{m}+mX_2^mY_2^{-1}=G(a,c,X_2,Y_2)=0 \tag{6.12}$$

such that

$$Y_2=-\frac{m^2}{|a+c|}X_2^m \to (g^m)'=-\frac{m^2}{|a+c|}g^m. \tag{6.13}$$

The last equation can be solved with standard means:

$$g(\xi)=D\,e^{-\frac{m}{|a+c|}\xi}, \tag{6.14}$$

for $D>0$.

The positivity condition for f (6.9) permits to conclude on some regularity results in the quasilinear parabolic operator (see Section 3). On top, the specie g verifies

$$g\to\epsilon\to 0^+ \quad in \quad B_R^T=B_R(x_0.R)\times[T-\epsilon,T+\epsilon], \tag{6.15}$$

for $T>>1$ which means the existence of a convergent tail in B_R^T towards the null condition in v. The next objective is to show the existence of finite propagation in the invaded specie v.

Theorem 6. *For $m>2$, there exists finite propagation speed when*

$$v\to\epsilon\to 0^+ \quad in \quad B_R^T=B_R(x_0.R)\times[T-\epsilon,T+\epsilon], \tag{6.16}$$

$T>>1$, where finite propagation refers to the existence of a positive convergent tail approaching the null solution.

Proof. Consider the pressure variable w:

$$w=\frac{m}{m-1}v^{m-1}, \tag{6.17}$$

so that the equation v in P reads:

$$\frac{m}{m-1}w_t=|\nabla w|^2+w\Delta w+c\cdot\nabla w\frac{m}{(m-1)^2}-(1+w)w^{\frac{m-2}{m-1}}, \tag{6.18}$$

where $w\to 0$, so that:

$$\frac{m}{m-1}w_t=|\nabla w|^2+c\cdot\nabla w\frac{m}{(m-1)^2} \tag{6.19}$$

A solution to a similar equation has been provided in [8]. Consider the following function in the search of a maximal solution:

$$W(x,t)=a\left(bt+r-\frac{1}{n}\right)_+,\ r=|x|,\ \ n\in\mathbb{N}. \tag{6.20}$$

Both a and $b>0$ are constants to be determined. In particular, for $0\le\tau\le 1$, impose:

$$b\tau=\frac{1}{2n}. \tag{6.21}$$

Under this condition:

$$W(x,t)\equiv 0 \quad for \quad r<\frac{1}{2n} \quad and \quad 0\le t\le\tau. \tag{6.22}$$

Any solution to the equation (6.19) is bounded as per Theorem 1, then:

$$v(x,t)\le K_1 \ for\ x\in\mathbb{R},\ 0\le t\le\tau\ and\ K_1(p,\|u_0\|_\infty). \tag{6.23}$$

The intention is to make $W(x,t)$ as a maximal solution:

$$W(x,t)\ge v(x,t), \tag{6.24}$$

so that

$$a\left(bt + r - \frac{1}{n}\right)_+ \geq K_1. \tag{6.25}$$

Select any $r > \frac{1}{n}$, for example $r = \frac{2}{n}$. Thus, for $t = 0$:

$$a\left(\frac{2}{n} - \frac{1}{n}\right)_+ \geq K_1, \quad a \geq nK_1. \tag{6.26}$$

Note that:

$$W(x,t) \geq v(x,t), \tag{6.27}$$

in $r = \frac{2}{n}$ and $0 \leq t \leq \tau$. The value of b shall be chosen in such a way that $W(x,t)$ is a supersolution in $0 < r < \frac{2}{n}, \quad 0 \leq t \leq \tau$:

$$W_t \geq \frac{m-1}{m}|\nabla W|^2 + c \cdot \nabla W \frac{1}{m-1}., \tag{6.28}$$

and considering that:

$$W_t = ab; \; W_r = a, \tag{6.29}$$

the following value for b is obtained:

$$b \geq \frac{m-1}{m}a + c\frac{1}{m-1} \tag{6.30}$$

For the values of a and b in expressions (6.26) and (6.30) respectively, the function $W(x,t)$ is a supersolution locally:

$$W(x,t) \geq w(x,t), \quad 0 < |x| < \frac{2}{n}, \quad 0 \leq t \leq \tau. \tag{6.31}$$

The inequality (6.31) permits to conclude that any local supersolution satisfies the null criteria in B_R^T, then, any minimal solution $w(x,t)$ satisfies such null criteria in B_R^T.

□

The expression (6.14) provides the characteristic profile in relation with the diffusion front (note the parameter m) and the advection. Once the invaded specie starts the desertion in the direction of c, it follows a curve in (x,t) given by (6.14). The invasive movement concentrates in the same trajectory to reduce the invaded population that follows an exponential decay. Furthermore, the existence of a finite propagation suggests that during the desertion, the diffusion is still relevant in the proximity of the null solution (i.e. the invaded tail). This can be interpret as the existence of a random movement of invaded specie in the tail where the invasive influence is negligible.

7 Conclusions

The proposed problem P (2.1) has been discussed stressing aspects related with existence, uniqueness, behaviour of minimal and maximal solutions and Travelling Waves supported by the geometric perturbation theory. In addition, the finite speed of propagation, induced by the porous medium diffusion, has been shown and a characterization of such property has been explored. The propagation features of the specie v when approaching the null solution have been shown. To this end, an exponential decreasing tail in the TW domain has been proved to exist. The invaded specie trajectory, to escape from the invasive, is given by the mentioned TW solution. Even when the invaded desertion is mainly governed by the advection, the non-linear diffusion still acts in the tail leading to a finite speed propagating front.

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