A note on damped wave equations with a nonlinear dissipation in non-cylindrical domains *

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Abstract

In this paper, we study the large time behavior of a class of wave equation with a nonlinear dissipation in non-cylindrical domains. The result we obtained here relaxes the conditions for the nonlinear term coefficients (in precise, that is $\beta(t)|u|^\rho u$) in [1] and [3] (which require $\beta(t)$ to be a constant or $\beta(t)$ to be decreasing with time t) and has less restriction for the defined regions.

Key words: Wave equation; stabilization; dissipation nonlinearity; non-cylindrical domain.

1 Introduction and main results

Fix $t \geq 0$. Let Ω_t be a bounded domain in $\mathbb{R}$. Given $T > 0$. Set $\widehat{Q}_T = \Omega_t \times (0,T)$ and denote by $\widehat{\Sigma}_T$ the lateral boundary of $\widehat{Q}_T$. Consider the following wave equation with a nonlinear dissipation in the non-cylindrical domain $\widehat{Q}_T$:

$$\begin{cases} u'' - \Delta u + au' + bu + \beta(t)|u|^\rho u = 0 & (x,t) \in \widehat{Q}_T, \\ u = 0 & (x,t) \in \widehat{\Sigma}_T, \\ u(x,0) = u_0(x),\ u'(x,0) = u_1(x) & x \in \Omega_0, \end{cases} \tag{1.1}$$

where (u_0, u_1) is any given initial couple, (u, u') is the state variable and $a, b > 0$.

In order to study the qualitative theory of (1.1), we need the following assumptions on the domain $\widehat{Q}_T$:

(A1) $\alpha \in C^2[0,T]$ such that $\alpha(0) = 1$, $\alpha'(t) \geq 0$ and $\sup\limits_{t\in[0,T]} \alpha'(t) < 1$.

(A2) $\beta(t), \beta'(t) \geq 0$, $t \in [0,T]$ and $\beta' \in L^\infty(0,T)$.

(A3) if $n > 2$, then $0 < \rho \leq \dfrac{2}{n-2}$; if $n = 1$ or $n = 2$, then $0 < \rho < \infty$.

The wellposedness result for (1.1) is stated as follows:

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Theorem 1.1 *Let $u_0 \in H_0^2(0,1)$ and $u_1 \in H_0^1(0,1)$. If assumptions (A1)-(A3) hold, then there exists a unique strong solution u of problem (1.1) such that $u \in L^\infty\big(0,T;H_0^1(\Omega_t)\cap H^2(\Omega_t)\big)$, $u_t \in L^\infty\big(0,T;H^1(\Omega_t)\big)$, $u_{tt} \in L^\infty\big(0,T;L^2(\Omega_t)\big)$, and*

$$\big(u'' - \Delta u + au' + bu + \beta(t)|u|^\rho u, \phi\big)(t) = 0, \ \ a.e. \ t \in (0,T),$$

where $\phi(t)$ is an arbitrary function from $L^2(\mathbb{R}^1)$. In addition, $u(0) = u_0$, $u_t(0) = u_1$.

The proof of Theorem 1.1 is quite similar to the proof of wellposedness results in [2], so we omit it (but what we need to point out is that since the assumption (A2) is different from $\beta' \le 0$, the result we obtained here just admits the solution to belong to $L^\infty\big(0,T;H_0^1(\Omega_t)\cap H^2(\Omega_t)\big)$, not to $L^\infty\big(0,\infty;H_0^1(\Omega_t)\cap H^2(\Omega_t)\big)$).

Lemma 1.1 ([4]) *Suppose that $\widehat{Q}_T$ has a regular lateral boundary $\widehat{\Sigma}_T$. If $u \in C^1\big(\mathbb{R};L^2(\Omega_t)\big)$, then we have*

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} u(x,t)dx &= \int_{\Omega_t}\frac{d}{dt}u(x,t)dx + \int_{\Gamma_t} u(x,t)\dot{x}n_x d\sigma \\ &= \int_{\Omega_t}\frac{d}{dt}u(x,t)dx - \int_{\Gamma_t} u(x,t)n_t d\sigma,\end{aligned}$$

where Γ_t is the boundary of Ω_t, $\dot{x}$ is the velocity of $x \in \Gamma_t$, and $n = (n_x, n_t)$ is the unit exterior normal to $\widehat{\Sigma}_T$. Moreover, it was observed that for $u \in H^1(\widehat{Q}_T)$ with $u = 0$ on $\widehat{\Sigma}_T$ (all tangential derivative of u also vanishes on $\widehat{\Sigma}_T$), Consequently the full gradient of u satisfies $\nabla_{x,t}u = (\partial_n u)n$ which implies that

$$u_t = (\partial_n u)n_t \quad \text{and} \quad \nabla_x u = (\partial_n u)n_x.$$

The energy of system (1.1) $\mathscr{E}(t)$ is given by

$$\mathscr{E}(t) = \int_{\Omega_t}\Big[\frac{1}{2}u_t^2(t) + \frac{1}{2}u_x^2(t) + \frac{1}{2}u^2(t) + \beta(t)\frac{1}{\rho+2}|u(t)|^{\rho+2}\Big]dx.$$

Then the main result of this paper is stated as follows.

Theorem 1.2 *One can find $\lambda > 0$ and $\beta(t)$ satisfying $\lambda(\rho+1)\beta(t) \ge \beta'(t)$, such that the inequality*

$$\mathscr{E}(t) \le C\mathscr{E}(0)\varphi^{-1}(t), \tag{1.2}$$

hold, where $\varphi(t)$ is chosen by $\varphi(t) = e^{\lambda t}$, C is some positive constant.

Proof. Firstly, let φ be a unknown continuous function. Secondly, Multiplying both sides of the first equation in (1.1) by $(u_t + \lambda u)\varphi(t)$, where $\lambda > 0$, and then integrating it on $(0,T)\times\Omega_t$, we get

$$\int_0^T\int_{\Omega_t}\big(u'' - \Delta u + au' + bu + \beta(t)|u|^\rho u\big)(u_t + \lambda u)\varphi(t)dxdt = 0.$$

Calculating the above equality, we have

$$\begin{aligned}
&\int_0^T \int_{\Omega_t} u^{''}(u_t+\lambda u)\varphi(t)dxdt \\
=& \int_0^T \int_{\Omega_t} \Big[\big(\frac{1}{2}u_t^2\varphi(t)\big)_t + \big(\lambda\varphi(t)uu_t\big)_t - \lambda\varphi(t)u_t^2 - \lambda\varphi'(t)uu_t - \frac{1}{2}\varphi'(t)u_t^2\Big]dxdt \\
=& \int_{\Omega_T} \big(\frac{1}{2}u_t^2(T)\varphi(T) + \lambda\varphi(T)u(T)u_t(T)\big)dx - \int_{\Omega_0} \big(\frac{1}{2}u_t^2(0)\varphi(0) + \lambda\varphi(0)u(0)u_t(0)\big)dx \\
&+ \int_0^T \int_{\Gamma_t} \frac{1}{2}u_t^2\varphi(t)n_t d\sigma dt - \int_0^T \int_{\Omega_t} \big[\lambda\varphi(t)u_t^2 + \lambda\varphi'(t)uu_t + \frac{1}{2}\varphi'(t)u_t^2\big]dxdt,
\end{aligned} \tag{1.3}$$

$$\begin{aligned}
&\int_0^T \int_{\Omega_t} -\Delta u(u_t+\lambda u)\varphi(t)dxdt \\
=& \int_0^T \int_{\Omega_t} \Big[\big(-u_xu_t\varphi(t)\big)_x + u_xu_{tx}\varphi(t) - \big(u_x\lambda u\varphi(t)\big)_x - \lambda\varphi(t)u_x^2 dxdt\Big] \\
=& \int_0^T \int_{\Omega_t} \Big[\big(-u_xu_t\varphi(t)\big)_x + \big(\frac{1}{2}u_x^2\varphi(t)\big)_t - \frac{1}{2}\varphi'(t)u_x^2 - \big(\lambda\varphi(t)uu_x\big)_x + \lambda\varphi(t)u_x^2\Big]dxdt \\
=& \int_0^T \int_{\Omega_t} \big(-u_xu_t\varphi(t)\big)_x dxdt + \int_{\Omega_T} \frac{1}{2}u_x^2(T)\varphi(T)dx - \int_{\Omega_0} \frac{1}{2}u_x^2(0)\varphi(0)dx \\
&+ \int_0^T \int_{\Gamma_t} \frac{1}{2}u_x^2\varphi(t)n_t d\sigma dt - \int_0^T \int_{\Omega_t} \big[\frac{1}{2}\varphi'(t)u_x^2 - \lambda\varphi(t)u_x^2\big]dxdt,
\end{aligned} \tag{1.4}$$

$$\int_0^T \int_{\Omega_t} au'(u_t+\lambda u)\varphi(t)dxdt = \int_0^T \int_{\Omega_t} \big[a\varphi(t)u_t^2 + a\lambda uu_t\varphi(t)\big]dxdt, \tag{1.5}$$

$$\begin{aligned}
&\int_0^T \int_{\Omega_t} bu(u_t+\lambda u)\varphi(t)dxdt \\
=& \int_0^T \int_{\Omega_t} \big[buu_t\varphi(t) + b\lambda\varphi(t)u^2\big]dxdt \\
=& \int_0^T \int_{\Omega_t} \Big[\big(\frac{1}{2}bu^2\varphi(t)\big)_t - \frac{b}{2}\varphi'(t)u^2 + b\lambda\varphi(t)u^2\Big]dxdt \\
=& \int_{\Omega_T} \frac{1}{2}b\varphi(T)u^2(T)dx - \int_{\Omega_0} \frac{1}{2}b\varphi(0)u^2(0)dx - \int_0^T \int_{\Omega_t} \big[\frac{b}{2}\varphi'(t)u^2 - b\lambda\varphi(t)u^2\big]dxdt,
\end{aligned} \tag{1.6}$$

$$\int_0^T \int_{\Omega_t} \beta(t)|u|^\rho u(u_t+\lambda u)\varphi(t)dxdt$$

$$
\begin{aligned}
&= \int_0^T \int_{\Omega_t} \Big[\beta(t)\big(\frac{1}{\rho+2}|u|^{\rho+2}\big)_t\varphi(t) + \lambda\beta(t)|u|^{\rho+2}\varphi(t)\Big]dxdt \\
&= \int_0^T \int_{\Omega_t} \Big(\frac{1}{\rho+2}|u|^{\rho+2}\beta(t)\varphi(t)\Big)_t - \beta'(t)\varphi(t)\frac{1}{\rho+2}|u|^{\rho+2} - \beta(t)\varphi'(t)\frac{1}{\rho+2}|u|^{\rho+2}\Big]dxdt \\
&\quad + \int_0^T \int_{\Omega_t} \lambda\beta(t)|u|^{\rho+2}\varphi(t)dxdt \qquad (1.7) \\
&= \int_{\Omega_T} \beta(T)\varphi(T)\frac{1}{\rho+2}|u(T)|^{\rho+2}dx - \int_{\Omega_0} \beta(0)\varphi(0)\frac{1}{\rho+2}|u(0)|^{\rho+2}dx \\
&\quad + \int_0^T \int_{\Omega_t} \big[\beta'(t)\varphi(t)\frac{1}{\rho+2}|u|^{\rho+2} + \beta(t)\varphi'(t)\frac{1}{\rho+2}|u|^{\rho+2} - \lambda\beta(t)|u|^{\rho+2}\varphi(t)\big]dxdt.
\end{aligned}
$$

Adding (1.3) to (1.7), we obtain

$$
\begin{aligned}
0 = &\int_{\Omega_T} \big(\frac{1}{2}u_t^2(T)\varphi(T) + \lambda\varphi(T)u(T)u_t(T)\big)dx - \int_{\Omega_0} \big(\frac{1}{2}u_t^2(0)\varphi(0) + \lambda\varphi(0)u(0)u_t(0)\big)dx \\
&+ \int_0^T \int_{\Gamma_t} \frac{1}{2}u_t^2\varphi(t)n_t d\sigma dt - \int_0^T \int_{\Omega_t} \big[\lambda\varphi(t)u_t^2 + \lambda\varphi'(t)uu_t + \frac{1}{2}\varphi'(t)u_t^2\big]dxdt \\
&- \int_0^T \int_{\Omega_t} \big(u_x u_t\varphi(t)\big)_x dxdt + \int_{\Omega_T} \frac{1}{2}u_x^2(T)\varphi(T)dx - \int_{\Omega_0} \frac{1}{2}u_x^2(0)\varphi(0)dx \\
&+ \int_0^T \int_{\Gamma_t} \frac{1}{2}u_x^2\varphi(t)n_t d\sigma dt - \int_0^T \int_{\Omega_t} \big[\frac{1}{2}\varphi'(t)u_x^2 - \lambda\varphi(t)u_x^2\big]dxdt \\
&+ \int_0^T \int_{\Omega_t} \big[a\varphi(t)u_t^2 + a\lambda uu_t\varphi(t)\big]dxdt \qquad (1.8) \\
&+ \int_{\Omega_T} \frac{1}{2}b\varphi(T)u^2(T)dx - \int_{\Omega_0} \frac{1}{2}b\varphi(0)u^2(0)dx - \int_0^T \int_{\Omega_t} \big[\frac{b}{2}\varphi'(t)u^2 - b\lambda\varphi(t)u^2\big]dxdt \\
&+ \int_{\Omega_T} \beta(T)\varphi(T)\frac{1}{\rho+2}|u(T)|^{\rho+2}dx - \int_{\Omega_0} \beta(0)\varphi(0)\frac{1}{\rho+2}|u(0)|^{\rho+2}dx \\
&+ \int_0^T \int_{\Omega_t} \big[-\beta'(t)\varphi(t)\frac{1}{\rho+2}|u|^{\rho+2} - \beta(t)\varphi'(t)\frac{1}{\rho+2}|u|^{\rho+2} + \lambda\beta(t)|u|^{\rho+2}\varphi(t)\big]dxdt.
\end{aligned}
$$

Since the assumption (A1) means that

(H1) The domain $\widehat{Q}_T$ is time-like, i.e., $|n_t| < |n_x|$.

(H2) $\widehat{Q}_T$ is monotone increasing, i.e., Ω_t is expanding with respect to t or $n_t \le 0$.

$$
\int_0^T \int_{\Gamma_t} \big[\frac{1}{2}u_t^2\varphi(t)n_t + \frac{1}{2}u_x^2\varphi(t)n_t\big]d\sigma dt - \int_0^T \int_{\Omega_t} \big(u_x u_t\varphi(t)\big)_x dxdt
$$

$$= \int_0^T \int_{\Gamma_t} \big[\frac{1}{2}u_t^2\varphi(t)n_t + \frac{1}{2}u_x^2\varphi(t)n_t\big]d\sigma dt - \int_0^T \int_{\Gamma_t} u_x u_t \varphi(t) n_x d\sigma dt$$

$$= \int_0^T \int_{\Gamma_t} \frac{1}{2}\varphi(t)|\partial_n u|^2(n_t^2 - n_x^2)n_t d\sigma dt \geq 0.$$

Furthermore, (1.8) yields

$$\int_{\Omega_T} \Big[\frac{1}{2}u_t^2(T) + \lambda u(T)u_t(T) + \frac{1}{2}u_x^2(T) + \frac{1}{2}bu^2(T) + \beta(T)\frac{1}{\rho+2}|u(T)|^{\rho+2}\Big]\varphi(T)dx$$

$$\leq \int_{\Omega_0} \Big[\frac{1}{2}u_t^2(0) + \lambda u(0)u_t(0) + \frac{1}{2}u_x^2(0) + \frac{1}{2}bu^2(0) + \beta(0)\frac{1}{\rho+2}|u(0)|^{\rho+2}\Big]\varphi(0)dx$$

$$+ \int_0^T \int_{\Omega_t} \big[\lambda\varphi(t)u_t^2 + \lambda\varphi'(t)uu_t + \frac{1}{2}\varphi'(t)u_t^2\big]dxdt + \int_0^T \int_{\Omega_t} \big[\frac{1}{2}\varphi'(t)u_x^2 - \lambda\varphi(t)u_x^2\big]dxdt$$

$$- \int_0^T \int_{\Omega_t} \big[a\varphi(t)u_t^2 + a\lambda uu_t\varphi(t)\big]dxdt + \int_0^T \int_{\Omega_t} \big[\frac{b}{2}\varphi'(t)u^2 - b\lambda\varphi(t)u^2\big]dxdt$$

$$+ \int_0^T \int_{\Omega_t} \big[\beta'(t)\varphi(t)\frac{1}{\rho+2}|u|^{\rho+2} + \beta(t)\varphi'(t)\frac{1}{\rho+2}|u|^{\rho+2} - \lambda\beta(t)|u|^{\rho+2}\varphi(t)\big]dxdt. \tag{1.9}$$

We can choose $\varphi(t) = e^{st}$, $s > 0$. In particular, let $\varphi(t) = e^{\lambda t}$ (λ be small) and

$$\lambda(\rho+1)\beta(t) \geq \beta'(t). \tag{1.10}$$

We can put

$$\beta(t) = e^{\mu t} \quad \text{with} \quad \mu \leq \lambda(\rho+1),$$

or

$$\beta(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0,$$

with $a_i > 0 (i = 0, \cdots, n)$ such that (1.10) holds.

Then the last three terms of inequality (1.9) are negative. Hence, we deduce

$$\int_{\Omega_T} \Big[\frac{1}{2}u_t^2(T) + \lambda u(T)u_t(T) + \frac{1}{2}u_x^2(T) + \frac{1}{2}bu^2(T) + \beta(T)\frac{1}{\rho+2}|u(T)|^{\rho+2}\Big]\varphi(T)dx$$

$$\leq \int_{\Omega_0} \Big[\frac{1}{2}u_t^2(0) + \lambda u(0)u_t(0) + \frac{1}{2}u_x^2(0) + \frac{1}{2}bu^2(0) + \beta(0)\frac{1}{\rho+2}|u(0)|^{\rho+2}\Big]\varphi(0)dx.$$

From the above inequality, we finally derive

$$\mathscr{E}(t) \leq C\mathscr{E}(0)\varphi^{-1}(t),$$

for some constant $C > 0$.

$\square$

Remark 1.1 *If $b = 0$ in (1.1), then use the method before, (1.9) becomes*

$$\begin{aligned}
&\int_{\Omega_T}\Big[\frac{1}{2}u_t^2(T)+\lambda u(T)u_t(T)+\frac{1}{2}u_x^2(T)+\beta(T)\frac{1}{\rho+2}|u(T)|^{\rho+2}\Big]\varphi(T)dx\\
\le\ &\int_{\Omega_0}\Big[\frac{1}{2}u_t^2(0)+\lambda u(0)u_t(0)+\frac{1}{2}u_x^2(0)+\beta(0)\frac{1}{\rho+2}|u(0)|^{\rho+2}\Big]\varphi(0)dx\\
&+\int_0^T\int_{\Omega_t}\big[\lambda\varphi(t)u_t^2+\lambda\varphi'(t)uu_t+\frac{1}{2}\varphi'(t)u_t^2\big]dxdt+\int_0^T\int_{\Omega_t}\big[\frac{1}{2}\varphi'(t)u_x^2-\lambda\varphi(t)u_x^2\big]dxdt\\
&-\int_0^T\int_{\Omega_t}\big[a\varphi(t)u_t^2+a\lambda uu_t\varphi(t)\big]dxdt\\
&+\int_0^T\int_{\Omega_t}\big[\beta'(t)\varphi(t)\frac{1}{\rho+2}|u|^{\rho+2}+\beta(t)\varphi'(t)\frac{1}{\rho+2}|u|^{\rho+2}-\lambda\beta(t)|u|^{\rho+2}\varphi(t)\big]dxdt.
\end{aligned}$$

In this case, in order to absorb the mixed term $\int_0^T\int_{\Omega_t} a\lambda uu_t\varphi(t)dxdt$, we must use poincaré inequality whose coefficients depend on geometry of the domain. That is

$$\int_{\Omega_t} u^2(x,t)dx \le |\Omega_t|^2\int_{\Omega_t} u_x^2(x,t)dx.$$

Thus

$$\begin{aligned}
&\int_0^T\int_{\Omega_t} a\lambda uu_t\varphi(t)dxdt \le \int_0^T\int_{\Omega_t}\frac{1}{2}a\lambda^2\varphi(t)u^2dxdt+\int_0^T\int_{\Omega_t}\frac{1}{2}a\varphi(t)u_t^2dxdt\\
\le &\int_0^T\int_{\Omega_t}\frac{1}{2}a\lambda^2|\Omega_t|^2\varphi(t)u_x^2dxdt+\int_0^T\int_{\Omega_t}\frac{1}{2}a\varphi(t)u_t^2dxdt.
\end{aligned}$$

When $\alpha\in L^\infty(0,\infty)$, and there exist two bounded domains $\Omega_*,\Omega^*\subset\mathbb{R}^1$ such that $\Omega_*\subset\Omega_\tau\subset\Omega_t\subset\Omega^*,\forall\tau<t$. Then we have $|\Omega_t|\le|\Omega^*|,\forall t>0$. Let $a\lambda|\Omega^*|^2<1$. With a similar argument as before, we get

$$\mathscr{E}(t)\le C\mathscr{E}(0)\varphi^{-1}(t),\quad t>0,$$

for some constant $C>0$.

If non-cylindrical domains become unbounded in some X_1-direction of space, as the time t goes to infinite, and are bounded in other X_2-direction of space. Since the projection of it in X_2-direction is a bounded open set, written as w, then the Poincaré inequality in X_2-direction turns out

$$\int_{\Omega_t} u^2(x,t)dx\le C_w^2\int_{\Omega_t}|\nabla_{X_2}u(x,t)|^2dx\le C_w^2\int_{\Omega_t}|\nabla u(x,t)|^2dx,$$

where C_w is the Poincaré constant.

Therefore, the above conclusion is still valid for this case.

Remark 1.2 *For the case of domains becoming unbounded in every spatial direction, as the time t goes to infinite, the condition $b \neq 0$ is needed to make (1.2) true. Otherwise, for any given $T > 0$, let $\lambda = \lambda(T)$ (depending on time T) be small and then it follows that*

$$\mathscr{E}(t) \leq C\mathscr{E}(0)\varphi_T^{-1}(t), \quad 0 < t < T,$$

where $\varphi_T^{-1}(t) = e^{-\lambda(T)t}$.

Since Poincaré inequality does not hold for a fixed number in any totally unbounded area, it seems difficult for us to get an estimate (1.2) without compensation ($b = 0$) and this is also an open problem that has been mentioned in some literature such as [3].

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