Solutions to a model with Neumann boundary conditions for sea-ice growth

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Abstract: We continue to study an initial boundary value problems to a model describing the evolution in time of diffusive phase interfaces in sea-ice growth. In a previous paper global existence and the long-time of behavior of weak solutions in one space was studied under Dirichlet boundary conditions. Here we show that the global existence of weak solutions and the long-time behavior are also studied under Neumann boundary condition. In this paper we study in space dimension lower than or equal to 3.

Keywords: Nonlinear parabolic equation, Neumann boundary conditions, Existence of solutions, Maximal attractor, Inertial set

AMS subject classifications. 35K51; 74N20

1 Introduction

We have investigated a phase-field model for phase transitions in [6]. We have investigated a coupled system of two parabolic equations modeling the evolution of a phase interface in sea-ice growth and proved that in one space dimension an initial boundary value problem to this system has global solution and large-time behavior in [7]. In this paper, we will investigate the global existence and the long-time behavior of weak solutions to a phase-field model for sea-ice growth when the order parameter and the temperature satisfy homogeneous Neumann Boundary conditions.

Let $\Omega \subset \mathbb{R}^N$ $(1 \le N \le 3)$ be an bounded open domain. T_e is a positive constant, which can be chosen arbitrary large. We write $Q_{T_e} := (0, T_e) \times \Omega$, and define

$$(\upsilon, \varphi)_{\mathbb{Z}} = \int_{\mathbb{Z}} \upsilon(y)\varphi(y)dy,$$

for $\mathbb{Z} = \Omega$ or $\mathbb{Z} = Q_{T_e}$. By introducing a phase field variable (the order parameter $\phi \in \mathbb{R}$) to represent the physical state of the system in time and space, that is to distinguish the liquid phase and solid phase, such as the solid state when the variable is 1. The liquid

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phase is expressed when the variable is 0. The model reads

$$\phi_t = K\Delta\phi - \widehat{\psi}'(\phi) + 2\theta h'(\phi), \qquad (t,x) \in Q_{T_e}, \tag{1.1}$$

$$\theta_t = \nabla\cdot(D\nabla\theta) - \frac{l}{2}h'(\phi)\phi_t, \qquad (t,x) \in Q_{T_e}, \tag{1.2}$$

where K, D are constants, $\widehat{\psi}$ is a double potential function. The boundary and initial conditions are

$$\frac{\partial\phi}{\partial n} = 0, \qquad (t,x) \in [0,T_e]\times\partial\Omega, \tag{1.3}$$

$$\frac{\partial\theta}{\partial n} = 0, \qquad (t,x) \in [0,T_e]\times\partial\Omega, \tag{1.4}$$

$$\phi(0,x) = \phi_0(x), \qquad x \in \Omega, \tag{1.5}$$

$$\theta(0,x) = \theta_0(x), \qquad x \in \Omega. \tag{1.6}$$

Assumptions.

(A1) $h \in C^\infty(\mathbb{R})$ is such that $h(0) = 0$ and there exist $L \in \mathbb{R}$ and bounded Lipschitz-continuous function h', h'' such that

$$h(\xi) = \xi^2(3-2\xi), \ \xi \in \mathbb{R};$$

(A2) $\widehat{\psi} \in C^\infty(\Omega)$ with $\widehat{\psi}(0) = 0$, and there exist constants $\gamma_1 \geq 0, \gamma_2 \geq 0, \gamma_3 \geq 0$ such that

$$\gamma_1\xi^4 - \gamma_2 \leq \widehat{\psi}(\xi), \widehat{\psi}''(\xi) \geq -\gamma_3, \ \xi \in \mathbb{R}.$$

A typical case would be

$$\widehat{\psi}(\xi) = \xi^2(1-\xi)^2, \xi \in \mathbb{R}.$$

The existence of solutions and a maximal attractor have been investigated in [7]. $w(\phi) = \frac{a}{2}\phi^2 + b(\phi)$ have been proved in [4]. Here, we consider the polynomials function h of 3th degree. A main difficult of this paper is the *a priori* estimates for different dimensional spaces. Since h is no longer bounded, the proof is more difficult. Firstly, we suppose that $\|\nabla\theta\|$ and $\int_0^t \|\phi_t\|^2 dt$ are bounded. Then we prove that$\|\nabla\theta\|$ and $\int_0^t \|\phi_t\|^2 dt$ are bounded in fact. In fact, we shall prove the existence of solutions by a Faedo-Galerkin method and the well-posedness of (1.1)-(1.2) in $W^{1,4}(\Omega)$. To this end, we first rewrite (1.1)-(1.6) into the following order parameter ϕ and the energy density $e = \theta + \frac{l}{2}h(\phi)$, this gives

$$\phi_t = K\Delta\phi - \widehat{\psi}'(\phi) - lh(\phi)h'(\phi) + 2eh'(\phi), \qquad (t,x) \in Q_{T_e}, \tag{1.7}$$

$$e_t = D\Delta e - \frac{l}{2}div(h'(\phi)\nabla\phi), \qquad (t,x) \in Q_{T_e}. \tag{1.8}$$

The boundary and initial conditions are

$$\frac{\partial\phi}{\partial n} = 0, \qquad (t,x) \in [0,T_e]\times\partial\Omega, \tag{1.9}$$

$$\frac{\partial e}{\partial n} = 0, \qquad (t,x) \in [0,T_e]\times\partial\Omega, \tag{1.10}$$

$$\phi(0,x) = \phi_0(x), \qquad x \in \Omega, \tag{1.11}$$

$$e(0,x) = \theta_0 + h(\phi_0) = e_0(x), \qquad x \in \Omega. \tag{1.12}$$

Definition 1.1 *Let $\phi_0 \in H^1(\Omega), \theta_0 \in L^2(\Omega)$. A couple of functions (ϕ, θ) with*

$$\phi \in L^\infty(0, T_e; H^1(\Omega)) \cap L^2(0, T_e; H^2(\Omega)), \tag{1.13}$$
$$\theta \in L^\infty(0, T_e; L^2(\Omega)) \cap L^2(0, T_e; H^1(\Omega)), \tag{1.14}$$

is a weak solution to the problem (1.1)-(1.6), *if for all $\varphi \in C_0^\infty((-\infty, T_e) \times \Omega))$, there hold*

$$\begin{aligned} 0 &= (\phi, \varphi_t)_{Q_{T_e}} - K(\nabla\phi, \nabla\varphi)_{Q_{T_e}} - (\widehat{\psi}', \varphi)_{Q_{T_e}} \\ &\quad + 2\big(\theta h'(\phi), \varphi\big)_{Q_{T_e}} + (\phi_0, \varphi(0))_\Omega, \end{aligned} \tag{1.15}$$

$$\begin{aligned} 0 &= (\theta, \varphi_t)_{Q_{T_e}} - D(\nabla\theta, \nabla\varphi)_{Q_{T_e}} + \frac{l}{2}(h(\phi), \varphi_t)_{Q_{T_e}} \\ &\quad + (\theta_0, \varphi(0))_\Omega - \frac{l}{2}(h(\phi(0)), \varphi(0))_\Omega. \end{aligned} \tag{1.16}$$

The main results of this article are as follows.

Theorem 1.1 *For all $\phi_0 \in H^1(\Omega)$, and $\theta_0 \in L^2(\Omega)$ there exists a unique weak solution (ϕ, θ) of problem* (1.1)-(1.6), *which, in addition to* (1.13)-(1.14), *satisfies*

$$\phi_t \in L^2(Q_{T_e}) \cap L^{\frac{4}{3}}(Q_{T_e}), \ \phi \in L^4(Q_{T_e}), \ \theta_t \in L^2(0, T_e; H^{-1}(\Omega)). \tag{1.17}$$

Note that the space integral of the function e is conserved in time, namely

$$\int_\Omega e(t, x)dx = \int_\Omega e_0(x)dx, \ t \geq 0.$$

We introduce the following function spaces

$$X_\beta = (\phi, e) \in W^{1,4}(\Omega), \int_\Omega e(x)dx = |\Omega|\beta, \ \mathscr{X}_\alpha = \bigcup_{|\beta| \leq \alpha} X_\beta,$$

for any real number β and for any non-negative real number α.

Theorem 1.2 *Suppose that assumptions (A1)-(A2) hold, that the initial data $(\phi_0, e_0) \in W^{1,4}(\Omega)$, and that (ϕ_0, e_0) satisfy the compatibility conditions*

$$\begin{aligned} &\frac{\partial\phi(0,x)}{\partial n} = 0, \\ &\frac{\partial e(0,x)}{\partial n} = 0, \\ &\phi_t(0, x) = K\Delta\phi(0, x) - \widehat{\psi}'(\phi(0, x)) - h'(\phi(0, x))h(\phi(0, x)) + eh'(\phi(0, x)), \\ &e_t(0, x) = D\Delta e(0, x) - div(h'(\phi(0, x))\nabla\phi(0, x)), \end{aligned}$$

for $x \in \partial\Omega$. There exists a unique classic solution

$$(\phi, e) \in C([0, +\infty) \times \overline{\Omega}, \mathbb{R}^2) \cap C^\infty((0, +\infty) \times \overline{\Omega}, \mathbb{R}^2),$$

to the initial-boundary value problem (1.7)-(1.12). *Moreover, the mapping*

$$S(t) : (\phi_0, e_0) \mapsto (\phi(t), e(t))$$

is a strongly continuous (nonlinear) semigroup on $W^{1,4}(\Omega)$ that maps X_β into itself for $\beta > 0$.

Theorem 1.3 *For $\alpha > 0$. There exists a closed ball $\mathscr{B}_\alpha$ of $H^2(\Omega)$ such that, for any bounded subset $\mathscr{B}$ of $\mathscr{X}_\alpha$, there exists $t(\mathscr{B}) > 0$ such that*

$$S(t)(\mathscr{B}) \subset \mathscr{B}_\alpha, t \geq t(\mathscr{B}).$$

Then, the semigroup $S(t)$ possesses a maximal attractor $\mathscr{A}_\alpha$ which is bounded in $H^2(\Omega)$, compact and connected in $\mathscr{X}_\alpha$.

Proposition 1.4 *Let $\alpha > 0$ and set*

$$\mathscr{K}_\alpha = Cl_{\mathscr{X}_\alpha} \bigcup_{t \geq t(\mathscr{B}_\alpha)} S(t)\mathscr{B}_\alpha.$$

There exists an inertial set $\mathscr{M}_\alpha$ of $\mathscr{X}_\alpha$ such that
(i) $\mathscr{A}_\alpha \subset \mathscr{M}_\alpha \subset \mathscr{K}_\alpha, S(t)\mathscr{M}_\alpha \subset \mathscr{M}_\alpha$ for every $t \geq 0$;
(ii) $\mathscr{M}_\alpha$ has finite fractal dimension in $\mathscr{H}$;
(iii) there exist constants $c_0, c_1 > 0$ such that for all $t \geq 0$

$$sup_{(\phi,e)\in\mathscr{K}_\alpha} d_{\mathscr{H}}(S(t)(\phi, e), \mathscr{M}_\alpha) \leq c_0 e^{-c_1 t},$$

where $d_{\mathscr{H}}$ denotes the distance on $\mathscr{H}$ ($\mathscr{H} = L^2(\Omega) \times (H^1(\Omega))'$).
Remark The distance $d_{\mathscr{H}}$ used here is the Hausdorff distance of two sets. Throughout this paper $L^2(\Omega)$ is denoted $\|\cdot\|$ and C is different line in line.

The remaining of this article is organized as follows. In Section 2 we will prove Theorem 1.1. In Section 3 we will prove the existence of semigroup. More precisely, using abstract results of Amann [1], we will prove the Proposition 1.4. In Section 4 we will prove the long-time behavior of the semigroup $S(t)$. In Section 5 we shall prove the existence of inertial sets for the semigroup $S(t)$ on $\mathscr{H}$.

2 Existence of the solutions

In this section we will prove Theorem 1.1 to the initial-boundary value problem (1.7)-(1.12).

Theorem 2.1 *(Aubin-Lions) Let B_0 be a normed linear space imbedded compactly into another normed linear space B, which is continuously imbedded into a Hausdorff locally convex space B_1, and $1 \leq p < +\infty$. If $v, v_i \in L^p(0, t; B_0), i \in \mathbb{N}$, the sequence $\{v_i\}_{i\in\mathbb{N}}$ converges weakly to v in $L^p(0, t; B_0)$, and $\{\frac{\partial v_i}{\partial t}\}_{i\in\mathbb{N}}$ is bounded in $L^1(0, t; B_1)$, then v_i converges to v strongly in $L^p(0, t; B)$.*

A proof of Theorem 2.1 can be found in [5, p. 57].

Lemma 2.1 *Let $(0, T_e) \times \Omega$ be an open set in $\mathbb{R}^+ \times \mathbb{R}^n$. Suppose functions g_n, g are in $L^q((0, T_e) \times \Omega)$ for any given $1 < q < \infty$, which satisfy*

$$\|g_n\|_{L^q}((0, T_e) \times \Omega) \leq C, \ g_n \to g \ a.e. in (0, T_e) \times \Omega.$$

Then g_n converges to g weakly in $L^q((0, T_e) \times \Omega)$.

A proof of Lemma 2.1 can be found in [5, p. 12].

Proof of Theorem 1.1 The proof relies on the Faedo-Galerkin method. Let $\omega_{i=1}^{\infty}$ be a base of $H^1(\Omega)$. They are smooth functions and satisfy

$$-\Delta\omega_j = \lambda_j\omega_j,\ j = 1, 2, \dots, m$$
$$0 = \lambda_1 < \lambda_2 \le \dots \le \lambda_j \le \dots.$$

For each integer m we look for an approximate solution (ϕ_m, e_m) of the form

$$\phi_m(t) = \sum_{i=1}^{i=m} \phi_{im}(t)\omega_i,\ e_m = \sum_{i=1}^{i=m} e_{im}(t)\omega_i,$$

satisfying

$$(\phi_{mt}, \omega_j) + K(\nabla\phi_m, \nabla\omega_j) = -(\widehat{\psi}'(\phi_m), \omega_j)$$
$$-\frac{l}{2}(h(\phi_m)h'(\phi_m), \omega_j) + (e_m h'(\phi_m), \omega_j), \tag{2.1}$$
$$(e_{mt}, \omega_j) + D(\nabla e_m, \nabla\omega_j) = \frac{lD}{2}(h'(\phi_m)\nabla\phi_m, \nabla\omega_j), \tag{2.2}$$

for $j = 1, 2, \dots, m$ and

$$\|\phi_{0m}\| \le \|\phi_0\| \ and\ \phi_m(0) = \phi_{0m} \to \phi_0,\ in\ H^1(\Omega),\ as\ n \to \infty, \tag{2.3}$$
$$\|e_{0m}\| \le \|e_0\| \ and\ e_m(0) = e_{0m} \to e_0,\ in\ L^2(\Omega),\ as\ n \to \infty. \tag{2.4}$$

Since the nonlinear terms are Lipschitz continuous functions, and $\{\omega_j\}$ are smooth functions. According to the standard existence theory for ordinary differential equations. Problem (2.1)-(2.4) have a unique solution (ϕ_m, e_m) for a.e $0 \le t \le T_m$.

Next we derive the *a priori* estimates which is independent of m and in particular $T_m = +\infty$. Multiplying (2.1) and (2.2) by $\phi_{jmt}(t)$ and $\frac{4}{l}(e_{jm} - h(\phi_{jm}))_t$ and integrating respectively and summing on $j = 1, 2, \dots, m$ and adding, we get

$$\frac{d}{dt}\int_\Omega (\frac{K}{2}|\nabla\phi_m|^2 + \frac{2}{l}\theta_m^2 + \widehat{\psi}(\phi_m))dx. + (\frac{4D}{l}\|\nabla\theta_m\|^2 + \|\phi_{mt}\|^2) = 0. \tag{2.5}$$

Integrating (2.4) in $\tau \in (0, T_e)$, we have

$$\frac{K}{2}\|\nabla\phi_m\|^2 + \frac{2}{l}\|\theta_m\|^2 + \int_\Omega \widehat{\psi}(\phi_m)dx + \int_0^{T_e} (\frac{4D}{l}\|\nabla\theta_m\|^2 + \|\phi_{mt}\|^2)d\tau$$
$$= \frac{K}{2}\|\nabla\phi_m(0)\|^2 + \frac{2}{l}\|\theta_m(0)\|^2 + \int_\Omega \widehat{\psi}(\phi_m(0))dx \le C. \tag{2.6}$$

From this we obtain

$$\|\phi_m\|_{H^1(\Omega)} \le C, \|\theta_m\| \le C, \int_0^{T_e} \|\phi_{mt}\|^2 d\tau, \int_0^{T_e} \|\nabla\theta_m\|^2 d\tau \le C. \tag{2.7}$$

Multiplying (2.1) by ϕ_{jm}, we find

$$\|\phi_m\|_{L^4(Q_{T_e})} \le C. \tag{2.8}$$

Multiplying (2.1) by $-\lambda_j\phi_{jm}(t)$, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\phi_m\|^2+K\|\Delta\phi_m\|^2+\frac{1}{4}\int_\Omega\phi_m^2|\nabla\phi_m|^2dx\\&\leq C(\|\phi_m\|^2_{H^1(\Omega)}+\|\theta_m\|^2_{H^1(\Omega)}).\end{aligned}\tag{2.9}$$

Integrating the inequality (2.9) in $\tau\in(0,T_e)$, we have

$$\int_0^{T_e}\|\Delta\phi_m\|^2d\tau\leq C.$$

From this estimates and (2.7) we can extract a subsequence, which we still denoted by (ϕ_m,θ_m), such that as $m\to\infty$

$$\phi_m\rightharpoonup\phi,\ weakly\ in\ L^2(0,T_e;H^2(\Omega)),\tag{2.10}$$

$$\theta_m\rightharpoonup\theta,\ weakly\ in\ L^2(0,T_e;H^1(\Omega)),\tag{2.11}$$

$$\widehat{\phi}'(\phi_m)\rightharpoonup\mathcal{X},\ weakly\ in\ L^{\frac{4}{3}}(Q_{T_e}).\tag{2.12}$$

Since $L^2(0,T_e;H^2(\Omega))\Subset L^2(0,T_e;H^1(\Omega)),L^2(0,T_e;H^1(\Omega))\Subset L^2(Q_{T_e})$, we have

$$\phi_m\to\phi,\ strongly\ in\ L^2(0,T_e;H^1(\Omega)),\tag{2.13}$$

$$\phi_m\to\phi,\ a.e.\ in\ Q_{T_e},\tag{2.14}$$

$$\theta_m\to\theta,\ strongly\ in\ L^2(0,T_e;L^2(\Omega)).\tag{2.15}$$

From Theorem 2.1, Lemma 2.1 and (2.8), (2.12), (2.14) we conclude that

$$\widehat{\phi}'(\phi_m)\rightharpoonup\widehat{\psi}'(\phi),\ weakly\ in\ L^{\frac{4}{3}}(Q_{T_e}).$$

Passing to the limit in (2.1)-(2.2), we find

$$\begin{aligned}&(\phi_t,\omega_j)+K(\nabla\phi,\nabla\omega_j)=-(\widehat{\psi}'(\phi),\omega_j)\\&-(h(\phi)h'(\phi),\omega_j)+(eh'(\phi),\omega_j),\\&(e_t,\omega_j)+D(\nabla e,\nabla\omega_j)=D(h'(\phi)\nabla\phi,\nabla\omega_j).\end{aligned}$$

3 Existence of the semigroup

In this section, we shall prove Theorem 1.2. We define $y=(y_1,y_2)\in\mathbb{R}^2$ and

$$a(y)=\begin{pmatrix}K&0\\-Dh'(y_1)&D\end{pmatrix}\quad a_{jk}(x,y)=\begin{cases}a(y), & \text{if } 1\leq j=k\leq N,\\0, & otherwise.\end{cases}$$

This version, which we need here, is used in [1]. The boundary-value problem $(\mathscr{A}(y),\mathscr{B}(y))$ is defined

$$\mathscr{A}(y)\upsilon=-\sum_{j,k=1}^N\frac{\partial}{\partial x_j}(a_{jk}(\cdot,y)\frac{\partial\upsilon}{\partial x_k}),\ \upsilon=(\upsilon_1,\upsilon_2),$$

$$\mathscr{B}(y)\upsilon=-\sum_{j,k=1}^N\nu^j\gamma_\partial(a_{jk}(\cdot,y)\frac{\partial\upsilon}{\partial x_k}),\ \upsilon=(\upsilon_1,\upsilon_2),$$

which is used in [1]. Here, γ_∂ and $\nu = (\nu^1, \nu^2, \ldots, \nu^N)$ denote the trace operator and the outer unit normal vector field to $\partial\Omega$. The normal ellipticity of $(\mathscr{A}, \mathscr{B})$ is a consequence of [1, Theorem 4.4].

Finally, we define $f \in C^\infty(\mathbb{R}^2 \times \overline{\Omega}, \mathbb{R}^2)$ by

$$f(x,y) = \begin{pmatrix} -\widehat{\psi}'(y_1) - lh(y_1)h'(y_1) + 2h'(y_1)y_2 \\ 0 \end{pmatrix}.$$

With these notation, the problem (1.7)-(1.8) reads

$$y_t + \mathscr{A}(\cdot, y)y = f(\cdot, y), \tag{3.1}$$

$$\mathscr{B}(y)y = 0, \tag{3.2}$$

$$y(\cdot, 0) = (\phi_0, e_0), \tag{3.3}$$

where $y = (\phi, e)$.

Lemma 3.1 *For $(\phi_0, e_0) \in W^{1,4}(\Omega)$, (3.1)-(3.3) has a unique maximal classical solution $y = (\phi, e)$ on $[0, t^+(\phi_0, e_0))$ which is $C^\infty-$ smooth, that is*

$$(\phi, e) \in C([0, +\infty) \times \overline{\Omega}, \mathbb{R}^2) \cap C^\infty((0, +\infty) \times \overline{\Omega}, \mathbb{R}^2),$$

and $y = (\phi, e)$ satisfies (3.1)-(3.3) pointwise. Moreover, $t^+(\phi_0, e_0) = +\infty$, provided that there exist $0 < \lambda < 1$ and $c : [0, +\infty) \to [0, +\infty)$ such that

$$\|\phi\|_{C^\lambda(\overline{\Omega})} + \|e\|_{C^\lambda(\overline{\Omega})} \le c(T), \ 0 \le t \le T < +\infty, \ t < t^+(\phi_0, e_0). \tag{3.4}$$

The map $\mathscr{F} : (t, \phi_0, e_)) \mapsto (\phi(t), e(t))$ is a semiflow on $W^{1,4}(\Omega)$, that is, if we set

$$Y = \bigcup_{(\phi_0, e_0) \in W^{1,4}(\Omega)} [0, t^+(\phi_0, e_0)) \times \{(\phi_0, e_0)\},$$

then Y is open in $[0, +\infty) \times W^{1,4}(\Omega)$, $\mathscr{F}$ is continuous from Y to $W^{1,4}(\Omega)$ with

$$\mathscr{F}(0, \phi_0, e_0) = (\phi_0, e_0),$$

and if $(\phi_0, e_0) \in W^{1,4}(\Omega), t' \in [0, t^+(\phi_0, e_0))$ and $t \in [0, t^+(\mathscr{F}(t', \phi_0, e_0)))$, then

$$t' + t < t^+(\phi_0, e_0), \ \text{and} \ \mathscr{F}(t' + t, \phi_0, e_0) = (t, \mathscr{F}(t', \phi_0, e_0)).$$

The proof of Lemma 3.1 can be found in [1, Section 14, 15].

In Section 2 we have proved

$$\begin{aligned} &\|\phi\|_{L^\infty(0,T_e;H^1(\Omega))} + \|\phi_t\|_{L^2(Q_{T_e})} \\ &\quad + \|\theta\|_{L^\infty(0,T_e;L^2(\Omega))} + \|\theta\|_{L^2(0,T_e;H^1(\Omega))} \le C_{T_e}. \end{aligned} \tag{3.5}$$

Next we shall improve the *a priori* estimates.

Lemma 3.2 *There exists a constant C_{T_e} such that*

$$\begin{aligned} &\|t^{\frac{1}{2}}\phi_t\|_{L^\infty(0,T_e;L^2(\Omega))} + \|t^{\frac{1}{2}}\theta\|_{L^\infty(0,T_e;H^1(\Omega))} \\ &\quad + \|t^{\frac{1}{2}}\nabla\phi_t\|_{L^2(Q_{T_e})} + \|t^{\frac{1}{2}}\theta_t\|_{L^2(Q_{T_e})} \le C_{T_e}, \end{aligned} \tag{3.6}$$

$$\|t^{\frac{1}{2}}\phi\|_{L^\infty(0,T_e;H^2(\Omega))} \le C_{T_e}, \tag{3.7}$$

$$\|te\|_{L^\infty(0,T_e;H^2(\Omega))} \le C_{T_e}. \tag{3.8}$$

Proof. We differentiate (1.1) with respect to t, and multiply the result equation with $t\phi_t$, and multiply (1.2) by $\frac{4t}{l}\theta_t$ and integrate over Ω, and add them, which yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(t\|\phi_t\|^2+\frac{4D}{l}t\|\nabla\theta\|^2)+(Kt\|\nabla\phi_t\|^2+\frac{4}{l}t\|\theta_t\|^2)+4\int_\Omega t\phi^2\phi_t^2dx+2\int_\Omega t\phi_t^2dx\\
&\le Ct\|\phi\|_{L^6(\Omega)}\|\theta\|_{L^3(\Omega)}\|\phi_t\|^2_{L^4(\Omega)}+Ct\|\theta\|\|\phi_t\|^2_{L^4(\Omega)}\\
&+Ct\|\phi_t\|^2+C(\|\phi_t\|^2+\|\nabla\theta\|^2)\\
&\le Ct(\|\nabla\phi_t\|\|\phi_t\|+\|\phi_t\|^2)+Ct(\|\nabla\theta\|^{\frac{1}{3}}\|\theta\|^{\frac{2}{3}}+\|\theta\|)(\|\nabla\phi_t\|\|\phi_t\|+\|\phi_t\|^2)\\
&+Ct\|\phi_t\|^2+C(\|\phi_t\|^2+\|\nabla\theta\|^2)\\
&\le Ct\|\phi_t\|\|\nabla\phi_t\|+Ct\|\nabla\theta\|^{\frac{1}{3}}\|\phi_t\|\|\nabla\phi_t\|\\
&+Ct(1+\|\nabla\theta\|^{\frac{1}{3}})\|\phi_t\|^2+C(\|\phi_t\|^2+\|\nabla\theta\|^2)\\
&\le t(\frac{K}{2}+\varepsilon\|\nabla\theta\|^2)\|\nabla\phi_t\|^2+t(C+\varepsilon\|\nabla\theta\|^2)\|\phi_t\|^2\\
&+C(\|\phi_t\|^2+\|\nabla\theta\|^2),\ n=2,
\end{aligned}\tag{3.9}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(t\|\phi_t\|^2+\frac{4D}{l}t\|\nabla\theta\|^2)+(Kt\|\nabla\phi_t\|^2+\frac{4}{l}t\|\theta_t\|^2)+4\int_\Omega t\phi^2\phi_t^2dx+2\int_\Omega t\phi_t^2dx\\
&\le Ct(\|\phi\|_{L^6(\Omega)}\|\nabla\phi\|_{L^3(\Omega)}\|\phi_t\|^2_{L^4(\Omega)}\\
&+Ct\|\theta\|\|\phi_t\|^2_{L^4(\Omega)}+Ct\|\phi_t\|^2+C(\|\phi_t\|^2+\|\nabla\theta\|^2)\\
&\le Ct\|\nabla\phi_t\|^{\frac{3}{2}}\|\phi_t\|^{\frac{1}{2}}+Ct(\|\nabla\theta\|^{\frac{1}{2}}\|\|\theta\|+\|\theta\|)(\|\nabla\phi_t\|^{\frac{3}{2}}\|\phi_t\|^{\frac{1}{2}}+\|\phi_t\|^2)\\
&+Ct\|\phi_t\|^2+C(\|\phi_t\|^2+\|\nabla\theta\|^2)\\
&\le Ct\|\phi_t\|^{\frac{1}{2}}\|\nabla\phi_t\|^{\frac{3}{2}}+t\|\nabla\theta\|^{\frac{1}{2}}\|\phi_t\|^{\frac{1}{2}}\|\nabla\phi_t\|^{\frac{3}{2}}\\
&+Ct(1+\|\theta\|^{\frac{1}{2}})\|\phi_t\|^2+C(\|\phi_t\|^2+\|\nabla\theta\|^2)\\
&\le t(\frac{K}{2}+\varepsilon\|\nabla\theta\|^2)\|\nabla\phi_t\|^2+t(C+\varepsilon\|\nabla\theta\|^2)\|\phi_t\|^2\\
&+C(\|\phi_t\|^2+\|\nabla\theta\|^2),\ n=3,
\end{aligned}\tag{3.10}
$$

where we have used the estimates (3.5) and the Gagliardo-Nirenberg inequality.

Integrating the inequality (3.9) and (3.10) over $(0,t)$ for $t\in(0,T_e)$, we get

$$
\begin{aligned}
&t\|\phi_t\|^2+t(\frac{4D}{l}-\varepsilon\int_0^t\|\phi_t\|^2dt)\|\nabla\theta\|^2+\int_0^t\big(t(\frac{K}{2}-\varepsilon\|\nabla\theta\|^2)\|\nabla\phi_t\|^2+\frac{4}{l}t\|\theta_t\|^2\big)dt\\
&\le C\int_0^t t\|\phi_t\|^2dt+C.
\end{aligned}\tag{3.11}
$$

where ε is enough small.

Using Gronwall' s inequality in (3.11) yields (3.6).

Multiplying (1.1) by $t\Delta\phi$, which yields

$$
K\int_\Omega t|\Delta\phi|^2dx\le Ct\|\phi_t\|\|\Delta\phi\|+t\|\widehat{\psi}'(\phi)\|\|\Delta\phi\|+t\|\theta\|_{L^6(\Omega)}\|h'(\phi)\|_{L^3(\Omega)}\|\Delta\phi\|.
$$

From this inequality and by use of the estimates (3.5) and (3.6), we get

$$
\int_\Omega t|\Delta\phi|^2dx\le C_{T_e}.\tag{3.12}
$$

We differentiate (1.1) with respect to t, and multiply the result equation with $t\phi_t$, which yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(t\|\phi_t\|^2) + Kt\|\nabla\phi_t\|^2 + 6\int_\Omega t\phi^2\|\phi_t\|^2 dx \\
&\quad \le C(t\|\phi_t\|^2 + t\|\theta_t\|\|\Delta\phi\|^2\|\phi_t\| + t\|\phi_t\|^2_{L^4(\Omega)} + \|\phi_t\|^2) \\
&\quad \le C(t\|\phi_t\|^2 + t\|\theta_t\|^2 + \frac{K}{2}t\|\nabla\phi_t\|^2 + \|\phi_t\|^2),
\end{aligned}$$

where we have used the continuous imbedding $H^2(\Omega) \subset L^\infty(\Omega)$ and Young's inequality.

Using the Gronwall inequality yields

$$\int_0^{T_e} t\|\nabla\phi_t\|^2 dt \le C_{T_e} \tag{3.13}$$

Let $\eta = \Delta\theta$ be a solution to

$$\begin{aligned}
\eta_t - D\Delta\eta &= -div\big(h^{'}(\phi)\nabla\phi_t + h^{''}(\phi)\phi_t\nabla\phi\big), \\
\frac{\partial\eta}{\partial n} &= 0.
\end{aligned}$$

Multiplying this equation by $t^2\eta$ and integrating over Ω, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega t^2\eta^2 dx + D\|t\nabla\eta\|^2 \\
&\le C(\|t\nabla\eta\|\|t^{\frac{1}{2}}\nabla\phi_t\|\|t^{\frac{1}{2}}h^{'}(\phi)\|_{L^\infty(\Omega)} \\
&+\|h^{''}\|_{L^\infty(\Omega)}\|t\nabla\eta\|\|t^{\frac{1}{2}}\phi_t\|_{L^4(\Omega)}\|t^{\frac{1}{2}}\nabla\phi\|_{L^4(\Omega)} + t\|\eta\|^2) \\
&\le C(1+\|\phi\|^2_{H^2(\Omega)})t^2\|\phi\|^2_{H^2(\Omega)}\|\nabla\phi_t\|^2_{H^1(\Omega)} + \frac{D}{2}\|t\nabla\eta\|^2 + t\|\eta\|^2 + t\|\phi_t\|^2_{H^1(\Omega)},
\end{aligned}$$

where we have used Young's inequality and the continuous embedding $H^1 \subset L^6(\Omega)$ and $H^2 \subset L^\infty(\Omega)$. Integrating this inequality over $(0, T_e)$, when T_e is finite we conclude that

$$\int_\Omega t^2\eta^2 dx + D\int_0^{T_e}\int_\Omega t^2|\nabla\eta|^2 dxdt \le C_{T_e} + \int_0^{T_e}\int_\Omega t\eta^2 dxdt. \tag{3.14}$$

Multiplying (1.2) by $t\eta$ and integrating over Ω, we find

$$D\int_0^{T_e}\int_\Omega t\eta^2 dxd\tau \le C\int_0^{T_e}\int_\Omega t|\theta_t|^2 dxdt + C\int_0^{T_e} t\|\phi\|^2_{H^2(\Omega)}\|\phi_t\|^2_{H^1(\Omega)}dt \le C_{T_e}. \tag{3.15}$$

We have used $t\|\phi\|^2_{H^2(\Omega)} \le C_{T_e}$ in the inequalities (3.14) and (3.15) when T_e is finite. Combining (3.14) and (3.15) yields

$$\|t\theta(t)\|_{L^\infty(0,T_e;H^2(\Omega))} \le C_{T_e},\ 0 \le t \le T_e,\ t < t^+(\phi_0, \theta_0). \tag{3.16}$$

Thus, (3.8) follows from (3.7), (3.16) and from $(A2)$.

Since $H^2(\Omega) \subset W^{1,4}(\Omega)$, we infer from (3.7)-(3.8) that

$$\|\phi(t)\|_{W^{1,4}(\Omega)} + \|\theta(t)\|_{W^{1,4}(\Omega)} \le c(T_e),\ 0 \le t \le c(T_e),\ t < t^+(\phi_0, \theta_0),$$

From this we conclude the following lemma.

Lemma 3.3 *Let $(\phi_0, e_0) \in W^{1,4}(\Omega)$. Then there exists a positive constant C_{T_e} such that*

$$\|\phi\|_{C^\lambda(\Omega)} + \|e\|_{C^\lambda(\Omega)} \le C_{T_e}.$$

Due to $H^{1,4}(\Omega) \subset C^{\frac{1}{4}}(\overline{\Omega})$. Lemma 3.3 is obtained. (3.4) is proved.

We can use the standard bootstrap argument and conclude the following theorem.

Theorem 3.1 *Let $m \in \mathbb{N}^+$ and assume that $(\phi_0, e_0) \in W^{1,4}(\Omega)$. Then there exist a constant C such that*

$$\|\phi\|_{C^{m+\lambda/2,2m+\lambda}((0,+\infty)\times\Omega)} + \|e\|_{C^{m+\lambda/2,2m+\lambda}((0,+\infty)\times\Omega)} \le C.$$

Continue by using the bootstrap argument, we get the $C^\infty((0,+\infty)\times\Omega)$. Thus, we get the proof of Theorem 1.2 by use of Theorem 3.1.

4 The maximal attractor

In this section , we shall prove Theorem 1.3. To this end, we shall show the existence of absorbing sets in $H^2(\Omega)$ and the compactness in $W^{1,4}(\Omega)$ for semigroup $S(t)$.

Let $N(v)$ is unique solution and $v \in L^2(\Omega)$ satisfies $\int_\Omega v(x)dx = 0$ for

$$-\Delta N(v) = v,\ \frac{\partial N(v)}{\partial n} = 0,\ \int_\Omega N(v)dx = 0. \tag{4.1}$$

Then there exists a C_1 such that

$$\|\nabla N(v)\|_{L^2(\Omega)} \le C_1\|v\|_{L^2(\Omega)}. \tag{4.2}$$

Let $(\phi_0, e_0) \in \mathscr{X}_\alpha$, $(\phi(t), e(t)) = S(t)(\phi_0, e_0)$ and

$$m_0 = \int_\Omega (|\phi_0|^2 + |e_0|^2)dx,\ E = \frac{1}{|\Omega|}\int_\Omega e(t,x)dx = \frac{1}{|\Omega|}\int_\Omega e_0(t,x)dx,\ t \ge 0.$$

Firstly, we define a function $\sigma \in C^\infty(\mathbb{R})$ by

$$\sigma(\xi) = \frac{\gamma_3}{2}\xi^2 + \widehat{\psi}(\xi) - \widehat{\psi}'(0)\xi,\ \xi \in \mathbb{R}.$$

Let $\sigma > 0$ be a convex function such that $\sigma(0) = 0$ and $\sigma'(0) = 0$, and we have with $(A2)$,

$$\xi\sigma'(\xi) \ge \sigma(\xi) \ge 0,\ \sigma(\xi) \ge \frac{7\gamma_1}{8}\xi^6 - \gamma_4,\ \xi \in \mathbb{R}, \tag{4.3}$$

where γ_4 is a constant depending on $\gamma_1, \gamma_2, \gamma_3$ and $\widehat{\psi}(0)$. We also have the following results for h: for $\delta > 0$, there exists a constant $C_\delta > 0$ depending on $2, 3, \delta$ such that

$$\frac{1}{2}h'(\xi)\xi\big(h'(\xi)\xi - lh(\xi)\big) \le (\delta - \epsilon)\xi^6 + C_\delta, \tag{4.4}$$

$$h(\xi)^2 \le (\delta + \epsilon)\xi^6 + C_\delta, \tag{4.5}$$

where $\xi \in \mathbb{R}$. We shall show that the closed balling $\mathscr{B}_\alpha$ of $H^2(\Omega)$ exits.

Lemma 4.1 *There holds for constants $C_2, C_2^{'}$ and a monotonic function $T_1(m_0)$ such that*

$$\|\phi(t)\| + \|\nabla(N(e(t)-E)\| \leq C_2, \tag{4.6}$$

$$\int_t^{t+r}(\|\phi(\tau)\|^2_{H^1(\Omega)} + \|e(\tau)\|^2)d\tau + \int_t^{t+r}\int_\Omega \phi\sigma^{'}(\phi)dxd\tau \leq C_2^{'}, \tag{4.7}$$

where $t \geq T_1(m_0), r > 0$.

Proof. Multiplying (1.7)-(1.8) by ϕ and $\frac{2}{D}N(e-E)$ respectively, and integrating by parts over Ω and adding, which yields

$$\begin{aligned}
&\frac{d}{dt}(\frac{1}{D}\|\nabla N(e-E)\|^2 + \frac{1}{2}\|\phi\|^2) + 2\|e-E\|^2 + \int_\Omega \big(\frac{K}{2}|\nabla\phi|^2 + \phi\sigma^{'}(\phi)\big)dx \\
&\leq 2\int_\Omega |(e-E)h(\phi)|dx + C(1+\|\phi\|^2) + \|e-E\|^2 \\
&+\frac{1}{2}\int_\Omega h^{'}(\phi)\phi\big(h^{'}(\phi)\phi - lh(\phi)\big)dx \\
&\leq \frac{\epsilon+2\gamma_1}{\epsilon+\gamma_1}(\delta_0+\epsilon)\int_\Omega \phi^6 dx + \frac{\epsilon+3\gamma_1}{\epsilon+2\gamma_1}\|e-E\|^2 + (2\delta_0-\epsilon)\int_\Omega \phi^6 dx + C,
\end{aligned}$$

where $\delta_0 = \frac{\gamma_1}{4}\frac{\epsilon+\gamma_1}{3\epsilon+4\gamma_1}$. From this inequality we have

$$\begin{aligned}
&\frac{d}{dt}(\frac{1}{D}\|\nabla N(e-E)\|^2 + \frac{1}{4}\|\phi\|^2) + \frac{\gamma_1}{\epsilon+2\gamma_1}\|e-E\|^2 + \int_\Omega \big(\frac{K}{2}|\nabla\phi|^2 + \phi\sigma^{'}(\phi)\big)dx \\
&\leq \frac{3\gamma_1}{4}\int_\Omega \phi^6 dx + C.
\end{aligned} \tag{4.8}$$

From this inequality and (4.2) and (4.3) we get

$$\begin{aligned}
&\frac{d}{dt}(\frac{1}{D}\|\nabla N(e-E)\|^2 + \frac{1}{4}\|\phi\|^2) + C_3\big(\frac{1}{D}\|\nabla N(e-E)\|^2 + \frac{1}{4}\|\phi\|^2\big) \\
&+c\int_\Omega \big(\frac{K}{2}|\nabla\phi|^2 + \phi\sigma^{'}(\phi) + |e-E|^2\big)dx \leq C_4.
\end{aligned} \tag{4.9}$$

From this inequality and (4.3), and using the Gronwall Lemma we find

$$\|\nabla N(e(t)-E)\|^2 + \|\phi(t)\|^2 \leq C_5(1 + m_0 e^{-C_3 t}).$$

When $t \geq T_1(m_0)$, we have

$$\|\nabla N(e(t)-E)\|^2 + \|\phi(t)\|^2 \leq 2C_5. \tag{4.10}$$

where $T_1(\xi) = \frac{1}{C_3}\ln(\frac{\xi}{C_5})$.

Integrating (4.9) over $(t, t+r)$, when $t \geq T_1(m_0)$, we find

$$\int_t^{t+r}\int_\Omega \big(\frac{K}{2}|\nabla\phi|^2 + \phi\sigma^{'}(\phi) + |e-E|^2\big)dxd\tau \leq C_5. \tag{4.11}$$

From this inequality (4.7) obtain.

Lemma 4.2 *There exists constants C_6, C_6' and a monotonic function $T_2(m_0)$ such that*

$$\|\phi(t)\|_{H^1(\Omega)} + \|e(t)\| \le C_6, \tag{4.12}$$

$$\int_t^{t+r} (\|\phi_\tau(\tau)\|^2 + \|\theta(\tau)\|^2_{H^1(\Omega)}) d\tau \le C_6', \tag{4.13}$$

where $t \ge T_2(m_0)$ and $T_2(m_0) \ge T_1(m_0)$.

Proof. Multiplying (1.1) and (1.3) by ϕ_t and $\frac{4}{l}\theta$ respectively and integrating, using (4.3) we find

$$\begin{aligned}\frac{d}{dt}\int (\frac{K}{2}|\nabla\phi|^2 + \sigma(\phi) + \frac{2}{l}|\theta|^2)dx + \frac{1}{2}\|\phi_t\|^2 + \frac{4D}{l}\|\nabla\theta\|^2 \\ \le \int (\frac{K}{2}|\nabla\phi|^2 + \sigma(\phi) + \frac{2}{l}|\theta|^2)dx + C(1 + \|\phi\|^2). \end{aligned}\tag{4.14}$$

Using the uniform Gronwall Lemma [8, Lemma 1.1] yields

$$\begin{aligned}\int_\Omega (\frac{K}{2}|\nabla\phi(t+r)|^2 + \sigma(\phi)(t+r) + \frac{2}{l}|\theta(t+r)|^2)dx \\ \le (\frac{a_1}{r} + C)e^{C_2' r},\ t \ge 1 + T_1(m_0) = T_2(m_0). \end{aligned}\tag{4.15}$$

where a_1 is below with Lemma 4.1

$$\begin{aligned}\int_t^{t+r}\int_\Omega (\frac{K}{2}|\nabla\phi(\tau)|^2 + \sigma(\phi)(\tau) + \frac{2}{l}\|\theta(\tau)\|^2)dxd\tau \\ \le C(1 + \int_t^{t+r}\int_\Omega (|\nabla\phi|^2 + \phi\sigma'(\phi) + e^2)dxd\tau \le C. \end{aligned}\tag{4.16}$$

Integrating (4.14) over $(t, t+r)$, we obtain (4.13). (4.12) is also obtained from (4.15).

Lemma 4.3 *There holds for constants C_7, C_7' and a monotonic function $T_3(m_0)$ such that*

$$\|\phi_t\| + \|\theta\|_{H^1(\Omega)} \le C_7, \tag{4.17}$$

$$\int_t^{t+r} (\|\phi_\tau\|^2_{H^1(\Omega)} + \|\theta_\tau\|^2) d\tau \le C_7', \tag{4.18}$$

where $t \ge T_3(m_0)$ and $T_3 \ge T_2$.

Proof. Differentiating (1.1) with respect to t and multiplying these results by ϕ_t, and multiplying (1.2) by θ_t, which gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\phi_t\|^2 + \frac{4D}{l}\|\nabla\theta\|^2) + (K\|\nabla\phi_t\|^2 + \frac{4}{l}\|\theta_t\|^2) + 4\int_\Omega \phi^2\phi_t^2 dx + 2\int_\Omega \phi_t^2 dx \\ &\le C\|\theta\|\|\phi_t\|^2_{L^4(\Omega)} + C\|\phi\|_{L^6(\Omega)}\|\theta\|_{L^3(\Omega)}\|\phi_t\|^2_{L^4(\Omega)} \\ &\le C(\|\nabla\phi_t\|\|\phi_t\| + \|\phi_t\|^2) + C(\|\nabla\theta\|^{\frac{1}{3}}\|\theta\|^{\frac{2}{3}} + \|\theta\|)(\|\nabla\phi_t\|\|\phi_t\| + \|\phi_t\|^2) \\ &\le (\frac{K}{2} + \varepsilon\|\nabla\theta\|^2)\|\nabla\phi_t\|^2 + (C + \varepsilon\|\nabla\theta\|^2)\|\phi_t\|^2,\ n = 2, \end{aligned}\tag{4.19}$$

and

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\phi_t\|^2+\frac{4D}{l}\|\nabla\theta\|^2)+(K\|\nabla\phi_t\|^2+\frac{4}{l}\|\theta_t\|^2)+4\int_\Omega\phi^2\phi_t^2dx+2\int_\Omega\phi_t^2dx\\
&\quad\le C\|\theta\|\|\nabla\phi_t\|^2_{L^4(\Omega)}+C\|\phi\|_{L^6(\Omega)}\|\theta\|_{L^3(\Omega)}\|\phi_t\|^2_{L^4(\Omega)}\\
&\quad\le C(\|\nabla\phi_t\|^{\frac{3}{2}}\|\phi_t\|^{\frac{1}{2}}+\|\phi_t\|^2)+C(\|\nabla\theta\|^{\frac{1}{2}}\|\theta\|^{\frac{1}{2}}+\|\theta\|)(\|\nabla\phi_t\|^{\frac{3}{2}}\|\phi_t\|^{\frac{1}{2}}+\|\phi_t\|^2)\\
&\quad\le(\frac{K}{2}+\varepsilon\|\nabla\theta\|^2)\|\nabla\phi_t\|^2+(C+\varepsilon\|\nabla\theta\|^2)\|\phi_t\|^2,\ n=3,
\end{aligned}
\tag{4.20}
$$

where ε is sufficiently small. Here, we have used the Gagliardo-Nirenberg inequality and Young's Inequality.

Integrating the inequalities (4.19) and (4.20) over $(0,t)$ for $t\in(0,T_e)$ yield

$$
\begin{aligned}
&\frac{1}{2}\|\phi_t\|^2+(\frac{4D}{l}-\varepsilon\int_0^t\|\phi_t\|^2dt)\|\nabla\theta\|^2+\int_0^t\big((\frac{K}{2}-\varepsilon\|\nabla\theta\|^2)\|\nabla\phi_t\|^2+\frac{4}{l}\|\theta_t\|^2\big)dt\\
&\quad\le C\int_0^t\|\phi_t\|^2dt.
\end{aligned}
\tag{4.21}
$$

Using the Gronwall inequality to (4.21) yields

$$\|\phi_t\|_{L^\infty(0,T_e;L^2(\Omega))}+\|\theta\|_{L^\infty(0,T_e;H^1(\Omega))}\le C_7,\ t\ge 1+T_2=T_3. \tag{4.22}$$

Integrating the inequalities (4.19) and (4.20) over $(t,t+r)$ for $t\in(0,T_e)$ yield

$$\int_t^{t+r}(\|\phi_\tau\|^2_{H^1(\Omega)}+\|\theta_\tau\|^2)d\tau\le C_7'. \tag{4.23}$$

Lemma 4.4 *There holds for a constant C_8 and a monotonic function $T_4(m_0)$ such that*

$$\|\phi\|_{H^2(\Omega)}+\|e\|_{H^2(\Omega)}\le C_8, \tag{4.24}$$

where $t\ge T_4(m_0)$ and $T_4\ge T_3$.

Proof. Multiplying (1.1) by $\Delta\phi$ yields

$$K\int_\Omega|\Delta\phi|^2dx\le C\|\phi_t\|\|\Delta\phi\|+\|\widehat{\psi}'(\phi)\|\|\Delta\phi\|+\|\theta\|_{L^6(\Omega)}\|h'(\phi)\|_{L^3(\Omega)}\|\Delta\phi\|.$$

From this inequality we get by use of (4.17)

$$\int_\Omega|\Delta\phi|^2dx\le C. \tag{4.25}$$

Hence, by the standard elliptic theory,

$$\|\phi\|_{H^2(\Omega)}\le C. \tag{4.26}$$

Multiplying (1.2) by $-\Delta\theta_t$ and integrating, we get

$$
\begin{aligned}
&\frac{D}{2}\frac{d}{dt}\|\Delta\theta\|^2+\|\nabla\theta_t\|^2\\
&\quad\le C\big(\|h''(\phi)\|_{H^2(\Omega)}\|\phi_t\|_{H^1(\Omega)}\|\phi\|_{H^2(\Omega)}\|\nabla\theta_t\|\\
&\quad+\|h'(\phi)\|_{L^\infty(\Omega)}\|\nabla\phi_t\|\|\nabla\theta_t\|\big),
\end{aligned}
$$

where we have used $H^2(\Omega) \subset L^\infty(\Omega)$. Using Young' s inequality and the estimate (4.22) yields

$$D\frac{d}{dt}\|\Delta\theta\|^2 + \|\nabla\theta_t\|^2 \le C\|\phi_t\|^2_{H^1(\Omega)}.$$

Integrating this inequality over $(0,t)$ for $t \in (0,T_e)$ and using the estimates (4.18) yield

$$\|\Delta\theta\| \le C,\ t \ge T_4(m_0) = 1 + T_3(m_0).$$

Hence, by the standard elliptic theory,

$$\|\theta\|_{H^2(\Omega)} \le C,\ t \ge T_4(m_0). \tag{4.27}$$

Proof of Theorem 1.3. Let B_2 be the closed ball of $H^2(\Omega)$ of radius C_8, setting

$$\mathscr{B}_\alpha = B_2 \cap \mathscr{X}_\alpha. \tag{4.28}$$

Due to $H^2(\Omega) \subset H^{1,4}(\Omega)$, $\mathscr{B}_\alpha$ is an compact absorbing set for $S(t)$ in $\mathscr{X}_\alpha$. The proof of Theorem 1.3 come from an abstract result [8].

We will prove the long-time behavior of solution as $t \to \infty$ with the $\omega-$limit set of (ϕ_0, e_0). The $\omega-$limit set $\omega(\phi_0,\theta_0)$ of (ϕ_0,θ_0) in $L^2(\Omega)$ is

$$\begin{aligned}\omega(\phi_0,\theta_0) = \{&(\phi_\infty,\theta_\infty) \in L^2(\Omega), \exists t_n \to +\infty \textit{ such that}\\ &\big(\phi(t_n),\theta(t_n)\big) \to (\phi_\infty,\theta_\infty) \textit{ in } L^2(\Omega)\}.\end{aligned}$$

We put

$$M_0 = \int_\Omega \big(\theta_0 + \frac{l}{2}h(\phi_0)\big)dx.$$

Theorem 4.1 *If (ϕ_∞, e_∞) belongs to $\omega(\phi_0,\theta_0)$, it satisfies*

$$\phi_\infty \in H^1(\Omega),\ \theta_\infty \in L^2(\Omega),\ \int_\Omega \big(\theta_0 + \frac{l}{2}h(\phi_0)\big)dx = M_0,$$

such that

$$-K\Delta\phi_\infty + \widehat{\psi}'(\phi_\infty) = 2\theta_\infty h'(\phi_\infty) \textit{ in } \Omega, \tag{4.29}$$

$$\frac{\partial\phi_\infty}{\partial n} = 0 \textit{ on } \partial\Omega. \tag{4.30}$$

Proof. Considering $(\phi_\infty,\theta_\infty)$ in $\omega(\phi_0,\theta_0)$ and let $t_n > 0$ such that $t_n \to +\infty$, and

$$\big(\phi(t_n),\theta(t_n)\big) \to (\phi_\infty,\theta_\infty) \textit{ in } L^2(\Omega), \tag{4.31}$$

$$\|\phi(t_n)\| + \|\theta(t_n)\| \le C,\ n \ge 1. \tag{4.32}$$

For $t \in (0,1)$ we put

$$\phi_n(t) = \phi(t_n + t),\ \theta_n(t) = \theta(t_n + t).$$

Next we get some estimates.

Lemma 4.5

$$\|\phi_n\|_{L^\infty(0,1;H^1(\Omega))} + \|\phi_{nt}\|_{L^2(Q_1)} + \|\theta_n\|_{L^\infty(0,1;L^2(\Omega))} + \|\nabla\theta_n\|_{L^2(Q_1)} \le C, \tag{4.33}$$

$$\|\phi\|_{L^2(0,1;H^2(\Omega))} \le C, \tag{4.34}$$

$$\|\phi_t\|_{L^2(0,+\infty;L^2(\Omega))} + \|\theta_t\|_{L^2(0,+\infty;H^{-1}(\Omega))} \le C. \tag{4.35}$$

Proof It follows from (2.6) that

$$\begin{aligned}
&\frac{K}{2}\|\nabla\phi\|^2 + \frac{2}{l}\|\theta\|^2 + \int_\Omega \widehat{\psi}(\phi)dx + \int_0^{T_e}(\frac{4D}{l}\|\nabla\theta\|^2 + \|\phi_t\|^2)d\tau \\
&= \frac{K}{2}\|\nabla\phi(0)\|^2 + \frac{2}{l}\|\theta(0)\|^2 + \int_\Omega \widehat{\psi}(\phi(0))dx \le C.
\end{aligned} \tag{4.36}$$

From this inequality we get

$$\begin{aligned}
&\|\phi_t\|_{L^2(0,+\infty;L^2(\Omega))} + \|\nabla\phi\|_{L^\infty(0,+\infty;L^2(\Omega))} \\
&\quad + \|\nabla\theta\|_{L^2(0,+\infty;L^2(\Omega))} + \|\theta\|_{L^\infty(0,+\infty;L^2(\Omega))} \le C.
\end{aligned} \tag{4.37}$$

Due to the boundedness of $h^{'}$ and the estimates (4.37), and (1.2) yield

$$\|\theta_t\|_{L^2(0,+\infty;H^{-1}(\Omega))} \le C. \tag{4.38}$$

From the estimates (4.37) and (4.38) yield (4.35). It follows from (4.37) for $t \in (t_n, t_n+1)$ that

$$\|\phi(t) - \phi(t_n)\|_{L^2(\Omega)} \le (t-t_n)^{\frac{1}{2}}\|\phi_t\|_{L^2(t_n,t;L^2(\Omega))} \le C.$$

Due to (4.32), we obtain

$$\|\phi_n\|_{L^\infty(0,1;L^2(\Omega))} \le C. \tag{4.39}$$

From (4.37) and (4.39) we have (4.33). From (1.1) and (4.37) we have (4.34).

We will prove the following result.

Lemma 4.6

$$\phi_n \;\to\; \phi_\infty \; in \; L^2(Q_1), \tag{4.40}$$

$$\theta_n \;\to\; \theta_\infty \; in \; L^2(0,1;H^{-1}(\Omega)). \tag{4.41}$$

Proof For $t \in (0,1)$ we get from (4.35)

$$\begin{aligned}
\|\phi_n(t) - \phi(t_n)\| &\le t^{\frac{1}{2}}\Big(\int_{t_n}^{t_n+t}\|\phi_\tau\|^2 d\tau\Big)^{\frac{1}{2}} \\
&\le \Big(\int_{t_n}^{+\infty}\|\phi_\tau\|^2 d\tau\Big)^{\frac{1}{2}} \to 0
\end{aligned}$$

as $t_n \to +\infty$. From this and (4.31) yield

$$\|\phi_n(t) - \phi_\infty\| \to 0 \; a.e. \; in \; (0,1).$$

Thus, (4.40) follows from (4.33) and the Lebesgue dominated convergence theorem. Similarly, we prove (4.35).

From Lemma 4.5 and 4.6 we have

$$\phi_n \rightharpoonup \phi_\infty \text{ in } L^2(0,1;H^2(\Omega)) \text{ and in } W^{1,2}(0,1;L^2(\Omega)), \tag{4.42}$$

$$\theta_n \rightharpoonup \theta_\infty \text{ in } L^2(Q_1) \text{ and in } W^{1,2}(0,1;H^{-1}(\Omega)). \tag{4.43}$$

It follows from Lemma 4.6, and a compactness argument that

$$\phi_n \to \phi_\infty \text{ in } L^2(0,1;H^1(\Omega)).$$

Since $h^{'}(\phi_n)$ is a bounded Lipschitz continuous function, it follows from Lemma 4.6 that $h^{'}(\phi_n) \to h^{'}(\phi_\infty)$ in $L^p(Q_1)$ for any $p \in [1,3]$. Then, $h^{'}(\phi_n)\theta_n \rightharpoonup h^{'}(\phi_\infty)\theta(\phi_\infty)$ in $L^2(Q_1)$.

We consider $\rho \in C_0^\infty(0,1)$, $z \in C_0^\infty(\Omega)$ and take $\varphi = \rho(t-t_n)z(x)$ in (1.15), which yields

$$\int_0^1 \int_\Omega \phi_{nt}\rho(t)zdxdt + \int_0^1 \int_\Omega \widehat{\psi}'\rho(t)zdxdt + K\int_0^1 \int_\Omega \nabla\phi_n \nabla z\rho(t)dxdt$$
$$= 2\int_0^1 \int_\Omega h^{'}(\phi_n)\theta_n\rho(t)zdxdt. \tag{4.44}$$

Passing to the limit in (4.44) and getting (4.29) and (4.30). Then the proof of Theorem 4.1 is completed.

5 Exponential attractor in $\mathscr{H}$

In this section, we shall prove the existence of inertial sets for the semi-group $S(t)$ on $\mathscr{H}$. To this end, we need prove that $S(t)$ satisfies the so-called *squeezing property* in [3].

Setting

$$m(u) = \frac{1}{|\Omega|} < u, 1 >, \ u \in H^{-1}(\Omega).$$

and defining the scale product $(\cdot,\cdot)_{-1}$ by

$$(u_1,u_2)_{-1} = \frac{1}{|\Omega|} < u_1, 1 >< u_2, 1 > + \int_\Omega \nabla N(u_1 - m(u_1)) \cdot \nabla N(u_2 - m(u_2))dx,$$

where N is defined by (4.1).

Let $\omega_1, \omega_2, \ldots$ be the eigenfunctions defined in section 2 and

$$W_j = span\{\omega_1, \ldots, \omega_j\}$$

and

$$m(\omega_j) = \int_\Omega \omega_j(x)dx = 0.$$

Let p_j be the orthogonal projection from $H^{-1}(\Omega)$ onto $W_j(\Omega)$ and $q_j = Id - p_j$. Let

$$P_j(u_1,u_2) = \big(p_j(u_1), p_j(u_2)\big), \ Q_j(u_1,u_2) = (q_j(u_1), q_j(u_2)), \ (u_1,u_2) \in H^{-1}(\Omega) \times H^{-1}(\Omega),$$

and

$$\|\nabla N(u)\|^2 \le \frac{1}{\lambda_{N+1}}\|\nabla u\|^2 \le \frac{1}{\lambda_{j+1}^2}\|\nabla u\|^2,\ u \in q_j H^1(\Omega). \tag{5.1}$$

Due to $\mathscr{B}_\alpha$ which is a compact absorbing set in $\mathscr{X}_\alpha$. Thus, there exists t_α such that $S(t)\mathscr{B}_\alpha \subset \mathscr{B}_\alpha$ for any $t \ge t_\alpha$. Setting

$$\mathscr{K}_\alpha = Cl_{\mathscr{X}_\alpha}(\bigcup_{t\ge t_\alpha} S(t)\mathscr{B}_\alpha) \subset \mathscr{B}_\alpha,$$

Due to $\omega-$limit set of (ϕ_0, e_0). We have $S(t)\mathscr{K}_\alpha \subset \mathscr{K}_\alpha$ for $t \ge 0$. Which implies the Lipschitz continuity of $S(t)$. We need the consequence of [4, Lemma 4.1]. We shall prove that $S(t)$ have the squeezing property. We introduce the following Lemma before the proof of squeezing property.

Lemma 5.1 *There exists a constant $C > 0$ such that for any $(\phi_1, e_1), (\phi_2, e_2)$, the pair $(\Phi_j, E_j) := Q_j(\phi_1 - \phi_2, e_1 - e_2), (\phi_i(t), e_i(t)) = S(t)(\phi_{i,0}, e_{i,0}),\ i = 1, 2,$ satisfies*

$$\begin{aligned}&\frac{d}{dt}(\|\Phi_j\|^2 + \|E_j\|_{-1}^2) + \big(K(\lambda_{j+1} - 1) + 2\big)\|\Phi_j\|^2 + D\lambda_{j+1}\|E_j\|_{-1}^2\\ &\le C_{sq}\|(\phi_1, e_1) - (\phi_2, e_2)\|^2,\end{aligned} \tag{5.2}$$

where $j \ge 2$.

The proof of Lemma 5.1 is a similar way of [3].

Proof. Using the operator to (1.4)-(1.5) yields

$$\begin{aligned}\Phi_{jt} - K\Delta\Phi_j &= q_j\big(-\widehat{\psi}'(\phi_1) - lh(\phi_1)h'(\phi_1) + \widehat{\psi}'(\phi_2) + lh(\phi_2)h'(\phi_2)\big)\\ &\quad +2q_j\Big(e_1\big(h'(\phi_1) - h'(\phi_2)\big)\Big) + q_j\big(h'(\phi_2)(e_1 - e_2)\big),\end{aligned} \tag{5.3}$$

$$E_{jt} - D\Delta E_j = -\Delta q_j\big(h(\phi_1) - h(\phi_2)\big). \tag{5.4}$$

Multiplying (5.3) and (5.4) by Φ_j and $N(E_j)$ respectively and integrating and adding, using (4.1) we get

$$\begin{aligned}&\frac{d}{dt}(\|\Phi_j\|^2 + \|E_j\|_{-1}^2) + 2K\|\nabla\Phi_j\|^2 + 2D\|E_j\|^2\\ &\le C\|\phi_1 - \phi_2\|(\|\Phi_j\| + \|E_j\|) + 2\Big|\int_\Omega h'(\phi_2)(e_1 - e_2)\Phi_j dx\Big|\\ &\le C\|\phi_1 - \phi_2\|(\|\Phi_j\| + \|E_j\|) + 2\|h'(\phi_2)\|_{L^\infty(\Omega)}\|\nabla N(e_1 - e_2)\|\|\nabla\Phi_j\|\\ &\quad +2\|h''(\phi_2)\|_{L^\infty(\Omega)}\|\Phi_j\|_{L^4(\Omega)}\|\nabla N(e_1 - e_2)\|\|\nabla\phi_2\|_{L^4(\Omega)}\\ &\le C\|\phi_1 - \phi_2\|(\|\Phi_j\| + \|E_j\|) + C\|\Phi_j\|_{H^1(\Omega)}\|\nabla N(e_1 - e_2)\|\\ &\le C_{sq}(\|\phi_1 - \phi_2\|^2 + \|\nabla N(e_1 - e_2)\|^2) + K\|\Phi_j\|^2 + K\|\nabla\Phi_j\|^2 + D\|E_j\|^2.\end{aligned} \tag{5.5}$$

Thus, we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\Phi_j\|^2 + \|E_j\|_{-1}^2) + K\|\nabla\Phi_j\|^2 + 2\|\Phi_j\|^2 - K\|\Phi_j\|^2 + D\|E_j\|^2\\ &\le C_{sq}\|(\phi_1 - \phi_2, e_1 - e_2)\|_{\mathscr{K}}^2.\end{aligned} \tag{5.6}$$

Then, substituting (4.2) and (5.1) into (5.6) yields (5.2).

Now we shall prove the squeezing property.

Lemma 5.2 *There exist* $(\phi_{01}, e_{01}), (\phi_{02}, e_{02}) \in \mathscr{K}_\alpha$ *such that*

$$|P_{j^*}(S(t^*)(\phi_{01}, e_{01}) - S(t^*)(\phi_{02}, e_{02})|_{\mathscr{H}} \leq |Q_{j^*}(S(t^*))(\phi_{01}, e_{01}) - S(t^*)(\phi_{02}, e_{02})|_{\mathscr{H}}, \tag{5.7}$$

and

$$|S(t^*)(\phi_{01}, e_{01}) - S(t^*)(\phi_{02}, e_{02})|_{\mathscr{H}} \leq \frac{1}{16}|(\phi_{01}, e_{01}) - (\phi_{02}, e_{02})|_{\mathscr{H}]}, \tag{5.8}$$

where $j^* \geq 2$ *is an integer and* $t^* > 0$.

The proof of Lemma 5.2 is the same as that of [2, Theorem 4.2]. Now we prove the $S(t)$ satisfies Proposition 1.4.

Proof of Proposition 1.4 From Lemma 5.2 we conclude that the mapping $S(1)$ is Lipschitz continuity and maps $\mathscr{K}_\alpha$ into itself, and find

$$|S(1)(\phi, e) - S(1)(\widehat{\phi}, \widehat{e})|_{\mathscr{H}} \leq \frac{1}{16}|(\phi, e) - (\widehat{\phi}, \widehat{e})|_{\mathscr{H}},$$

or

$$|Q_{j^*}\big(S(1)(\phi, e) - S(1)(\widehat{\phi}, \widehat{e})|_{\mathscr{H}} \leq |P_{j^*}\big(S(1)(\phi, e) - S(1)(\widehat{\phi}, \widehat{e})|_{\mathscr{H}},$$

where $(\phi, e), (\widehat{\phi}, \widehat{e}) \in \mathscr{K}_\alpha$ and $j^* \geq 2$. This property is squeezing property in [3].

Due to $\mathscr{K}_\alpha \subset \mathscr{H}$, which is compact and connected subset of $\mathscr{H}$. We obtain a compact subset $\mathscr{M}_\alpha^*$ of $\mathscr{H}$ such that $\mathscr{A}_\alpha \subset \mathscr{M}_\alpha^* \subset \mathscr{K}_\alpha$, $S(1)\mathscr{M}_\alpha^* \subset \mathscr{M}_\alpha^*$ from [3, Theorem 6]. $\mathscr{M}_\alpha^*$ has finite fractal dimension in $\mathscr{H}$. Thus, there exist constant $c_0^* > 0, c_1^* > 0$ such that

$$d_{\mathscr{H}}\big(S(n)(\phi, e), \mathscr{M}_\alpha^*\big) \leq c_0^* e^{-nc_1^*},\ \forall(\phi, e) \in \mathscr{K}_\alpha,\ \forall n \geq 1. \tag{5.9}$$

Thus, $S(t)$ is well-defined on $\mathscr{M}_\alpha^*$. We obtain Proposition 1.4 (iii) from this inequality and [4, Lamma 4.1].

Setting

$$\mathscr{M}_\alpha = \bigcup_{0 \leq t \leq 1} S(t)\mathscr{M}_\alpha^*.$$

We conclude that $\mathscr{M}_\alpha$ is a compact subset of $\mathscr{H}$ from [4, Lemma 4.1]. Then, Proposition 1.1 (i) holds.

Now we find a ball of radius $0 < \epsilon < 1$, there exists the smallest integer, which cover $\mathscr{M}_\alpha$ and $\mathscr{M}_\alpha^*$. It follows from [4, Lemma 4.1] that, for any $\epsilon \in (0, 1)$,

$$n(\mathscr{M}_\alpha, e^{\widetilde{D}\epsilon}) \leq \frac{2}{\epsilon} n(\mathscr{M}_\alpha^*, \epsilon).$$

Thus, Proposition 1.4 (ii) is obtained.

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