MULTIPLICATIVE DERIVATIVE AND ITS BASIC PROPERTIES ON TIME SCALES

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Abstract: We define multiplicative derivative and its properties on time scales. Then, we restate many concepts for multiplicative analysis such as derivative, Rolle's theorem, mean value theorem and increasing decreasing property on time scales. We aim to create important fields of study by carrying this most important issue of multiplicative analysis, which has applications in economics, finance and many other fields, to time scale calculus.

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1. Introduction

Multiplicative calculus was first introduced by *Grossman and Katz* [13], [14] in 1967. This type of analysis is also called non-Newtonian analysis because of its difference from classical calculus of Newton and Leibniz. There are four important operators for this analysis, such as gradient, derivative, average and integral. Multiplicative calculus is a useful supplement to the usual calculus in that it is tailored to situations involving exponential functions in the same sense that the usual calculus is tailored to situations involving linear functions. While classical analysis works based on addition and subtraction operations, multiplicative analysis works on multiplication and division. The multiplicative calculus moves the roles of substraction and addition to division and multiplication. There are actually many reasons to study multiplication analysis. It improves the work of additive calculations indirectly. Problems that are difficult to solve in classical analysis can be solved with incredible ease in this analysis.

Multiplicative calculus has a relatively restrictive area of applications than the classical calculus. Only positive functions are concerned here. In fact, the following question may come to mind. Why is the need to develop this new analysis when there is already an existing analysis that has been developed in great detail and has many applications. This is actually the same as the answer to the question of why polar coordinates are used when there is a cartesian coordinate system. Get to know the point in the plane better. This analysis gives better results than classical analysis in many fields such as finance, economics, biology and demography. A very limited number of studies have been conducted on this analysis until the beginning of the 2000s. Recently, various studies have been carried out on this subject and quality and effective results have been obtained (see [5], [6], [11], [12], [15], [19]).

In this study, we will define the multiplicative derivative and its properties which has many applications in many fields on time scales. In this case, it will be seen that classical multiplicative analysis is a special case of time scale for $\mathbb{T} = \mathbb{R}$. Before explaining some important concepts of multiplicative analysis on time scales, let's briefly explain the concept of time scale and express its basic features.

A time scale $\mathbb{T}$ is an arbitrary, closed, non-empty subset of real numbers. This theory was first studied by *Hilger* in 1988 in his doctoral dissertation [2], [17]. *Hilger*'s aim was to gather discrete and continuous states in mathematics under the same roof. Thus, difference equations and differential equations would be combined and important results would be obtained. However, some concepts such as $\Delta-$derivative, $\Delta-$integration and their various properties on $\mathbb{T}$ are explained in detail in two important books written by *Bohner and Peterson* [7, 8]. Later, in the multivariable case, many concepts with partial delta derivatives and their properties were given by *Bohner and Svetlin* [9]. Since its first study, this theory has been studied by many mathematicians in numerous fields (see [3], [10]). We need firstly to mention on something about $\Delta-$calculus of time scale theory.

Let $a = \inf \mathbb{T}$ and $b = \sup \mathbb{T}$. Since $\mathbb{T}$ is not necessarily connected, the forward-jump and backward-jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},\ \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

respectively for $t \in \mathbb{T}$ such that $a < t < b$, $t < \sup \mathbb{T}$, $\inf \phi = \sup \mathbb{T}$, $\sup \phi = \inf \mathbb{T}$ where ϕ denotes empty set. If $\mathbb{T}$ is bounded, one can write $\sigma(b) = b$. The corresponding forward-step function μ is defined by

$$\mu : \mathbb{T}^{\kappa} \to \mathbb{R}^{+}, \mu(t) = \sigma(t) - t.$$

However, $t \in \mathbb{T}$ is left dense, left scattered, right dense, right scattered, isolated and dense iff $\rho(t) = t$, $\rho(t) < t, \sigma(t) = t, \sigma(t) > t$, $\rho(t) < t < \sigma(t)$ and $\rho(t) = t = \sigma(t)$, respectively. We also should remind $\Delta-$differentiability region $\mathbb{T}^{\kappa}$ along with the set $\mathbb{T}$ to define $\Delta-$derivative of a function. $\mathbb{T}^{\kappa} = \mathbb{T} \backslash \{b\}$ if $\mathbb{T}$ is bounded above and b is left-scattered; otherwise $\mathbb{T}^{\kappa} = \mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{R}$ be a function. f is right continuous at $t \in \mathbb{T}$ if there is some $\delta > 0$ such that $|f(t) - f(s)| < \varepsilon$ for all $s \in [t, t + \delta)$ and $\varepsilon > 0$. The set of all right continuous functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T})$.

One can define $f^{\Delta}(t)$ to be the value for $t \in \mathbb{T}^{\kappa}$, if one exists, such that for all $\varepsilon > 0$ there is a neighborhood U_1 of t such that for all $s \in U_1$

$$\left|[f^{\sigma}(t) - f(s)] - f^{\Delta}(t) (\sigma(t) - s)\right| < \varepsilon \left|\sigma(t) - s\right|.$$

Here, f is $\Delta-$differentiable on $\mathbb{T}^{\kappa}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$. We will refer to [7, 8] for detailed information on $\Delta-$derivative.

Before the details of multiplicative derivative on time scales, some definitions should be made that will provide the infrastructure of this important concept. As is known, logarithms and exponential functions have an important place in multiplicative analysis. For this reason, it is necessary to express these concepts on $\mathbb{T}$. The following important notions and conclusions are introduced by Anderson and Bohner [1].

Definition 1.1. For $h > 0$, the transformation $\zeta_h : \mathbb{C}_h \to \mathbb{C}$ by

$$\zeta_h(z) = \begin{cases} \frac{1}{h}\log(1+zh), & \text{for } h \neq 0 \\ z, & \text{for } h = 0 \end{cases},$$

is called multi-valued cylinder transformation where $\mathbb{C}$ is the set of complex numbers, $\mathbb{C}_h = \left\{ z \in \mathbb{C} : z \neq -\dfrac{1}{h} \right\}$ and log is the multi-valued complex logarithm function.

Definition 1.2. [1] Let $p : \mathbb{T} \to \mathbb{C}$ be a $\Delta-$differentiable function for $p \neq 0$. Then,

$$\ell_p(t,s) = \int_s^t \zeta_{\mu(\tau)} \left[\frac{p^\Delta(\tau)}{p(\tau)} \right] \Delta\tau : s,t \in \mathbb{T},$$

is multi-valued logarithm function on $\mathbb{T}$. Here, if $p \equiv$constant, $\ell_p(t,s) = 0$ for all $s,t \in \mathbb{T}$. Thus, this logarithm does not distinguish between either constants or constant multiples of functions. Now let's express some of the features of this function that are necessary for our proofs.

Lemma 1.3. [1] Let $f, h : \mathbb{T} \to \mathbb{C}$ be a $\Delta-$differentiable functions with $f, h \neq 0$. Then,

$$\ell_{fh}(t,s) = \ell_f(t,s) + \ell_h(t,s)$$

and

$$\ell_{\frac{f}{h}}(t,s) = \ell_f(t,s) - \ell_h(t,s).$$

The proof of these properties can be easily demonstrated by definition.

Lemma 1.4. [1] Let $\alpha \in \mathbb{R}$ and $p : \mathbb{T} \to \mathbb{C}$ be a $\Delta-$differentiable function with $p \neq 0$. Then,

$$\ell_{p^\alpha}(t,s) = \alpha \ell_p(t,s),$$

for all $s,t \in \mathbb{T}$.

The derivative of this specially defined function on $\mathbb{T}$ is one of the critical concepts for our study. This concept will have different representations as the time scale is changed. This situation will yield very important results for multiplicative analysis.

Theorem 1.5. [1] Let $p : \mathbb{T} \to \mathbb{C}$ be a $\Delta-$differentiable function with $p \neq 0$. Then,

$$\ell_p^\Delta(t,s) = \begin{cases} \frac{1}{\mu(t)} \log\left[\frac{p^\sigma(t)}{p(t)}\right], & \text{for } \mu(t) \neq 0 \\ \frac{p^\Delta(t)}{p(t)}, & \text{for } \mu(t) = 0 \end{cases},$$

for all $s,t \in \mathbb{T}$ where $\Delta-$derivative is respect to t. All this will be necessary when bringing the concept of $\Delta-$derivative to multiplicative analysis on time scale.

The rest of the work is organized as follows: In the second chapter, some basic concepts in multiplicative analysis and the concept of derivative in multiplicative analysis will be given. In the next section, the multiplicative derivative will be defined on $\mathbb{T}$ and its properties will be proved. Some basic theorems for usual calculus will be generalized to mutiplicative calculus on $\mathbb{T}$.

2. Preliminaries on Multiplicative Calculus

Now, before moving on to the main topic, let's express the notions and theorems that should be given about multiplicative analysis. First of all, it is important to express the concept of derivative in multiplicative analysis, which is the basis of our study, and to examine its properties. The operations in the difference quotient in classical derivative. By way of contrast, the derivative in multiplicative calculus for a function is based on the ratio.

Definition 2.1. [4], [18] Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be differentiable in usual case and $f(t) > 0$ for all t. If the below limit exists and positive for

$$f^*(t) = \lim_{h\to 0}\left[\frac{f(t+h)}{f(t)}\right]^{\dfrac{1}{h}}, \tag{2.1.}$$

$f^*(t)$ is called $^*-$*derivative* of f at t. Since f is positive, the quantity in square brackets is positive, and the power in (2.1.) is a well defined positive number for all arbitrary non-zero numbers h. If $f^*(t)$ exists for all t on an open set $\Omega \subset \mathbb{R}$, then $f : A \to \mathbb{R}$ is well defined.

In fact, there is a relationship we will use frequently between classical derivative and derivative in multiplicative analysis. Let's express this now.

Lemma 2.2. [4], [18] Let $f : A \to \mathbb{R}$ be positive and differentiable at t,

$$f^*(t) = e^{(\ln o f)'(t)}.$$

Repeating this procedure n times, we can obtain the relation between the $n-$th order classical derivative and $n-$th multiplicative derivative as

$$f^{*(n)}(t) = e^{(\ln o f)^{(n)}(t)}.$$

Similarly, $n-$th order classical derivative can be expressed in terms of $n-$th multiplicative derivative;

$$f^{(n)}(t) = \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-k-1)!} f^{(k)}(t)\left(\ln o f^{*(n-k)}\right)(t), n = 0, 1, ...$$

Lemma 2.3. [4], [18] If a positive function f is differentiable in usual case at t, it is also $^*-$differentiable at t. Similarly, If the positive function f is $^*-$differentiable at t and $f^*(t) \neq 0$, it is also usual differentiable at t.

Let's see the basic properties of the multiplicative derivative with the following theorem in order to better understand the main part of our study.

Theorem 2.4. [4], [18] Let f, h be $^*-$differentiable at t and p be usual differentiable at t. The following expressions are provided for multiplicative derivative.

$i.\ (cf)^{*}(t) = f^{*}(t), c \in \mathbb{R},$
$ii.\ (fh)^{*}(t) = f^{*}(t)h^{*}(t),$
$iii.\ (f/h)^{*}(t) = f^{*}(t)/h^{*}(t)$
$iv.\ (f^{p})^{*}(t) = f^{*}(t)^{p(t)} f(t)^{p'(t)},$
$v.\ (fop)^{*}(t) = f^{*}(p(t))^{p'(t)},$
$vi.\ (f+h)^{*}(t) = f^{*}(t)^{\frac{f(t)}{f(t)+h(t)}} h^{*}(t)^{\frac{h(t)}{f(t)+h(t)}}.$

The proof of each of these items can be made using the multiplicative derivative definition. Here, unlike the classical case, the multiplicative derivative of the sum or difference is more complicated. Some important theorems related to derivative, which have very important applications in classical analysis, can also be expressed in multiplicative analysis.

Theorem 2.5. (**Multiplicative Rolle's Theorem**) [4], [18] Assume that f is continuous, positive on $[a,b]$ and $^{*}-$differentiable on (a,b). If $f(a) = f(b)$, there exists $a < c < b$ such that $f^{*}(c) = 1$.

Theorem 2.6. (**Multiplicative Mean Value Theorem**) [4], [18] Assume that f is continuous, positive on $[a,b]$ and $^{*}-$differentiable on (a,b). Then, there exists $a < c < b$ such that

$$\frac{f(b)}{f(a)} = f^{*}(c)^{b-a}.$$

As can be seen, although the multiplicative derivative will have some problems especially in the case of addition and subtraction, it will have great benefits in practice. Now let's move the multiplicative derivative and its properties to the time scale, which is a more general case. There has not been any study on the time scale regarding multiplicative analysis so far. In this respect, the proof and given concepts will make important contributions to this theory.

3. Multiplicative derivative on time scales

In this section, $^{*}-$derivative and its basic properties will be defined on $\mathbb{T}$. Here, the concept of $\Delta-$derivative on $\mathbb{T}$ will be carried to multiplicative calculus and important results will be obtained. Then some theorems related to new derivative will be expressed and proved, and examples will be given.

Definition 3.1. Let $f : \mathbb{T} \to \mathbb{C}$ be a $\Delta-$differentiable and $f(t) > 0$ for all $t \in \mathbb{T}$. If the below limit exists and positive for

$$f^{\Delta^{*}}(t) = \lim_{s \to t} \left(\frac{f^{\sigma}(t)}{f(s)} \right)^{\dfrac{1}{\sigma(t) - s}}, \tag{3.1.}$$

$f^{\Delta^{*}}(t)$ is called $\Delta^{*}-$ *derivative* of f at t. To consolidate this definition and make it understandable, we need to define one sided $\Delta^{*}-$ *derivatives.*

If $f(t) > 0$ is defined on $[t_0, b) \subset \mathbb{T}$, then the right side $\Delta^{*}-$ *derivative of* f at the point t_0 is defined to be

$$f_{+}^{\Delta^{*}}(t) = \lim_{s \to t_0^{+}} \left(\frac{f^{\sigma}(t_0)}{f(s)} \right)^{\dfrac{1}{\sigma(t_0) - s}}.$$

Similarly, the left side Δ^*- *derivative of* f at the point t_0 is defined to be

$$f_-^{\Delta^*}(t) = \lim_{s \to t_0^-} \left(\frac{f^\sigma(t_0)}{f(s)} \right)^{\dfrac{1}{\sigma(t_0) - s}},$$

while $f(t) > 0$ is defined on $(a, t_0] \subset \mathbb{T}$. We can easily draw the following conclusion from here. $f(t) > 0$ is Δ^*- *differentiable at* t_0 iff $f_+^{\Delta^*}(t_0)$ and $f_-^{\Delta^*}(t_0)$ exist and

$$f^{\Delta^*}(t_0) = f_+^{\Delta^*}(t_0) = f_-^{\Delta^*}(t_0).$$

Example 3.2. Consider the function

$$f(t) = \begin{cases} t+3, & t \in [-2, 2)_{\mathbb{R}} \\ t^2 + t, & t \in \{2, 4, 8\} \end{cases}.$$

Let us evaluate $f^{\Delta^*}(2)$. Here, we will first examine the structure of one-sided Δ^*-derivatives. Since

$$f_-^{\Delta^*}(2) = 2$$

and

$$f_+^{\Delta^*}(2) = \sqrt{\frac{10}{3}},$$

f is not Δ^*-differentiable at $t = 2$.

Lemma 3.3. Suppose that $f : \mathbb{T} \to \mathbb{C}$ be $\Delta-$differentiable and $f(t) > 0$ for all $t \in \mathbb{T}$. Then

$$f^{\Delta^*}(t) = e^{\boldsymbol{\ell}_f^{\Delta}(t,s)}.$$

Proof: We use the formal definition of $\boldsymbol{\ell}_f^{\Delta}(t, s)$. Firstly, Let's make the proof for $\mu(t) \neq 0$ as

$$f^{\Delta^*}(t) = e^{\dfrac{1}{\mu(t)} \log \left(\dfrac{f^\sigma(t)}{f(s)} \right)} = e^{\boldsymbol{\ell}_f^{\Delta}(t,s)}.$$

Similarly, we get

$$f^{\Delta^*}(t) = e^{\lim\limits_{s \to t} \dfrac{\log \left(\dfrac{f^\sigma(t)}{f(s)} \right)}{\sigma(t) - s}} = e^{\boldsymbol{\ell}_f^{\Delta}(t,s)},$$

for $\mu(t) = 0$. So, from these two equations, the desired equality can be easily seen.

Similarly to the reduction of the $\Delta-$derivative to usual derivative when $\mathbb{T} = \mathbb{R}$ in particular, Δ^*-derivative is reduced to $^*-$derivative.

Theorem 3.4. If $f : \mathbb{T} \to \mathbb{C}$ is $\Delta-$differentiable and $f(t) > 0$ for all $t \in \mathbb{T}$, then it is also Δ^*-*differentiable at* t.

Theorem 3.5. If $f : \mathbb{T} \to \mathbb{C}$ is Δ^*-*differentiable*, $f(t) > 0$ and $f^{\Delta^*}(t) \neq 0$ for all $t \in \mathbb{T}$, then it is also $\Delta-$*differentiable at* t.

Proof: The following equations are obtained from the basic concepts given earlier.

$$f^{\Delta^*} = e^{\boldsymbol{\ell}_f^{\Delta}(t,s)},$$

and

$$\log f^{\Delta^*} = \boldsymbol{\ell}_f^{\Delta}(t,s) + 2k\pi i, k \in \mathbb{Z}.$$

If $\mu(t) = 0$, we get

$$2k\pi i + \frac{f^{\Delta}(t)}{f(t)} = \log f^{\Delta^*} \Rightarrow f^{\Delta}(t) = f(t)\left\{\log f^{\Delta^*} - 2k\pi i\right\}.$$

Likewise,

$$\begin{aligned} 2k\pi i + \frac{1}{\mu(t)}\log\left(\frac{f^{\sigma}(t)}{f(t)}\right) &= \log f^{\Delta^*} \Rightarrow 2k\pi i + \frac{1}{\mu(t)}\log\left(1+\mu(t)\frac{f^{\Delta}(t)}{f(t)}\right) = \log f^{\Delta^*} \\ &\Rightarrow \log\left(1+\mu(t)\frac{f^{\Delta}(t)}{f(t)}\right) = f^{\Delta^*} \\ &\Rightarrow f^{\Delta}(t) = \frac{f(t)}{\mu(t)}\left\{\left(f^{\Delta^*}\right)^{\mu(t)} - 1\right\}. \end{aligned}$$

It completes the proof.

Theorem 3.6. If $f : \mathbb{T} \to \mathbb{C}$ is Δ^*-*differentiable*, $f(t) > 0$ and $f^{\Delta^*}(t) \neq 0$ for all $t \in \mathbb{T}$, then it is continuous at t.

Proof: By the definition of Δ^*-*derivative of* $f(t) > 0$, and Theorem 3.4., it is also $\Delta-$differentiable at $t \in \mathbb{T}$. By [8], it is continuous at that point. It completes the proof.

Now let's express and prove some important properties of the multiplicative delta derivative.

Theorem 3.7. *Let* $f, h : \mathbb{T} \to \mathbb{C}$ *be* Δ^*-*differentiable functions* for all $t \in \mathbb{T}^{\kappa}$ *where* $f, h, \neq 0$. *Then,* $f.h, \frac{f}{h}, cf, f+h$ *are* Δ^*-*differentiable and*

i. $(f.h)^{\Delta^*}(t) = f^{\Delta^*}(t)\ h^{\Delta^*}(t)$.

ii. $\left(\frac{f}{h}\right)^{\Delta^*}(t) = \frac{f^{\Delta^*}(t)}{h^{\Delta^*}(t)}$ *for* $h^{\Delta^*} \neq 0$.

iii. $(cf)^{\Delta^*}(t) = f^{\Delta^*}(t)$ *for* $c \in \mathbb{R}$.

iv. $(f+h)^{\Delta^*}(t) = \left(f^{\Delta^*}(t)\right)^{\overline{\frac{f}{f+h}}}\left(h^{\Delta^*}(t)\right)^{\overline{\frac{h}{f+h}}}$.

Proof: These proofs can be obtained directly using the definition as follows.

i. $(f.h)^{\Delta^*}(t) = e^{\boldsymbol{\ell}_{fh}^{\Delta}(t,s)} = e^{\left\{\boldsymbol{\ell}_f(t,s)+\boldsymbol{\ell}_h(t,s)\right\}^{\Delta}} = e^{\boldsymbol{\ell}_f^{\Delta}(t,s)}e^{\boldsymbol{\ell}_h^{\Delta}(t,s)} = f^{\Delta^*}(t)\ h^{\Delta^*}(t)$

ii. $\left(\frac{f}{h}\right)^{\Delta^*}(t) = e^{\boldsymbol{\ell}_{\frac{f}{h}}^{\Delta}(t,s)} = e^{\left\{\boldsymbol{\ell}_f(t,s)-\boldsymbol{\ell}_h(t,s)\right\}^{\Delta}} = \frac{f^{\Delta^*}(t)}{h^{\Delta^*}(t)}$

iii. By the definition, we get $(cf)^{\Delta^*}(t) = e^{\boldsymbol{\ell}_{cf}^{\Delta}(t,s)} = e^{\boldsymbol{\ell}_c^{\Delta}(t,s)}e^{\boldsymbol{\ell}_f^{\Delta}(t,s)}$. Here we will continue the proof for two different cases of μ. Firstly, let us assume that $\mu(t) = 0$. For this case, it yields that $(cf)^{\Delta^*}(t) = f^{\Delta^*}(t)$ since $e^{\boldsymbol{\ell}_c^{\Delta}(t,s)} = 1$. Similarly, since $e^{\frac{1}{\mu(t)}\mathbf{log}\left(1+\mu(t)\frac{c^{\Delta}}{c}\right)} = 1$ for $\mu(t) \neq 0$, we get $(cf)^{\Delta^*}(t) = f^{\Delta^*}(t)$.

iv. This proof will again be done for two cases of μ. If $\mu(t)=0$,

$$(f+h)^{\Delta^*}(t)=e^{\ell^{\Delta}_{f+h}(t,s)}=e^{\frac{(f+h)^{\Delta}}{f+h}}=\left(e^{\frac{f^{\Delta}}{f}}\right)^{\frac{f}{f+h}}\left(e^{\frac{h^{\Delta}}{h}}\right)^{\frac{h}{f+h}}=\left(f^{\Delta^*}(t)\right)^{\frac{f}{f+h}}\left(h^{\Delta^*}(t)\right)^{\frac{h}{f+h}}.$$

It completes the proof.

Example 3.8. Let us consider $f:\mathbb{T}\to\mathbb{C}$, $f(t)=t$. Then,

$$f^{\Delta^*}(t)=e^{\ell^{\Delta}_{f}t}=\begin{cases} e^{\frac{1}{\mu(t)}\log\left(\frac{\sigma(t)}{t}\right)}, & \text{for } \mu(t)\neq 0\\ e^{\frac{1}{t}}, & \text{for } \mu(t)=0\end{cases}.$$

Example 3.9. If $f:\mathbb{T}\to\mathbb{C}$, $f(t)=\alpha,\alpha\in\mathbb{R}, f^{\Delta^*}(t)=1$.

Lemma 3.10. Let $f,h:\mathbb{T}\to\mathbb{C}$ be Δ^*-*differentiable functions* for all $t\in\mathbb{T}^{\kappa}$ where $f,h\neq 0$. If $f(t)=ch(t),c\in\mathbb{R}, f^{\Delta^*}(t)=h^{\Delta^*}(t)$ for all t.
Proof: We get $f^{\Delta^*}(t)=e^{\ell f^{\Delta}t}=e^{\ell^{\Delta}_{ch}t}=h^{\Delta^*}(t)$ with simple reasoning.

Lemma 3.11. If $f,h,r:\mathbb{T}\to\mathbb{C}$ are Δ^*-*differentiable functions* for all $t\in\mathbb{T}^{\kappa}$ where $f,h,r\neq 0$, then

$$(fhr)^{\Delta^*}(t)=f^{\Delta^*}(t)h^{\Delta^*}(t)r^{\Delta^*}(t).$$

Corollary 3.12. This result can be generalized as follows. Let $f:\mathbb{T}\to\mathbb{C}$ be Δ^*-*differentiable* for all $t\in\mathbb{T}^{\kappa}$. Then, $(f^n)^{\Delta^*}(t)=\left(f^{\Delta^*}(t)\right)^n$ for some $n\in\mathbb{N}$.

Theorem 3.13. Let α be a constant and $m\in\mathbb{N}$.
i. If $f(t)=(t-\alpha)^m$, then

$$f^{\Delta^*}(t)=\begin{cases} e^{\frac{m}{\mu(t)}\log\left(\frac{\sigma(t)-\alpha}{t-\alpha}\right)}, & \text{for } \mu(t)\neq 0\\ e^{\frac{m}{t-\alpha}}, & \text{for } \mu(t)=0\end{cases}.$$

ii. If h is defined by $h(t)=\dfrac{1}{(t-\alpha)^m}$,

$$h^{\Delta^*}(t)=\begin{cases} e^{\frac{-m}{\mu(t)}\log\left(\frac{\sigma(t)-\alpha}{t-\alpha}\right)}, & \text{for } \mu(t)\neq 0\\ e^{\frac{m}{\alpha-t}}, & \text{for } \mu(t)=0\end{cases}.$$

Example 3.14. Let $f(t)=\frac{t^2\sin t}{1+t}$ on $\mathbb{T}=\left(\frac{1}{2}\right)^{\mathbb{N}_0}\cup\{0\}$. Then, we have

$$f^{\Delta^*}(t)=\frac{\left(t^2\right)^{\Delta^*}(\sin t)^{\Delta^*}}{(1+t)^{\Delta^*}}.$$

If this derivative is calculated for a particular point $t=1$, $f^{\Delta^*}(1)=e^{\cot 1}$ with $\mu(t)=0$. Furthermore,

$$f^{\Delta^*}(t)=e^{\frac{1}{t}\log\left(\frac{8(t+1)\cos t}{2t+1}\right)},$$

for $\mu(t) \neq 0$.

Definition 3.15. Assume that $f : \mathbb{T} \to \mathbb{C}$ is Δ^*-*differentiable* for all $t \in \mathbb{T}^\kappa$. $\left(f^{\Delta^*}\right)^{\Delta^*} : \mathbb{T}^{\kappa^2} \to \mathbb{C}$ is second order Δ^*-*derivative of* f *provided that* f^{Δ^*} is Δ^*-*differentiable* for $t \in \mathbb{T}^{\kappa^2}$. Similarly, high order Δ^*-derivative of f is defined by $f^{(\Delta^*)^n} : \mathbb{T}^{\kappa^n} \to \mathbb{C}, n \in \mathbb{N}$. Let's now write clearer statements for the higher order Δ^*-derivative. We will do this using the definition of the first order Δ^*-derivative, $f^{\Delta^*}(t) = e^{\boldsymbol{\ell}_f^{\Delta} t}$. If the same logic continues to be used, we get

$$\left(f^{\Delta^*}\right)^{\Delta^*}(t) = e^{\boldsymbol{\ell}_f^{\Delta\Delta} t}.$$

When we generalize this situation, it yields

$$f^{(\Delta^*)^n}(t) = e^{\boldsymbol{\ell}_f^{\Delta^n} t}.$$

Now, Rolle's and mean value theorems, which have a very important place in multiplicative analysis, will be defined on time scales.

Theorem 3.16. (Δ^*-**Rolle's Theorem**) Suppose that f has Δ^*-*derivative at each point on* $[a, b]$. *If* $f(a) = f(b)$, then there exists some points $c_1, c_2 \in [a, b]$ such that

$$\begin{aligned} f^{\Delta^*}(c_1) &\leq 1 \leq f^{\Delta^*}(c_2), \text{ for } f(c_1), f(c_2) > 0, \\ f^{\Delta^*}(c_1) &\geq 1 \geq f^{\Delta^*}(c_2), \text{ for } f(c_1), f(c_2) < 0. \end{aligned}$$

Proof: Since f is Δ^*-differentiable at each point on $[a, b]$, f is also $\Delta-$differentiable on $[a, b]$. Therefore, by the assumption $f(a) = f(b)$, there exists $c_1, c_2 \in [a, b]$ such that

$$f^{\Delta}(c_1) \leq 0 \leq f^{\Delta}(c_2).$$

Let $\mu(t) = 0$.

$$\begin{aligned} &\text{If } f(c_1) < 0, \quad f^{\Delta^*}(c_1) = e^{\frac{f^{\Delta}(c_1)}{f(c_1)}} \geq 1, \\ &\text{If } f(c_1) > 0, \quad f^{\Delta^*}(c_1) = e^{\frac{f^{\Delta}(c_1)}{f(c_1)}} \leq 1, \\ &\text{If } f(c_2) < 0, \quad f^{\Delta^*}(c_2) = e^{\frac{f^{\Delta}(c_2)}{f(c_2)}} \leq 1, \\ &\text{If } f(c_2) > 0, \quad f^{\Delta^*}(c_2) = e^{\frac{f^{\Delta}(c_2)}{f(c_2)}} \geq 1. \end{aligned}$$

Consequently, we get

$$f^{\Delta^*}(c_1) \leq 1 \leq f^{\Delta^*}(c_2) \text{ or } f^{\Delta^*}(c_2) \leq 1 \leq f^{\Delta^*}(c_1),$$

if $f(c_1)f(c_2) > 0$. Let $\mu(t) \neq 0$. Then,

$$f^{\Delta^*}(t) = \left(1 + \mu(t)\frac{f^{\Delta}(t)}{f(t)}\right)^{\frac{1}{\mu(t)}}.$$

Since $\mu(t) > 0$, we get the same result. It completes the result.

Example 3.17. Let $\mathbb{T} = \mathbb{Z}$ and $f(t) = t^2$. Find $c_1, c_2 \in (-3, 3)$ such that

$$f^{\Delta^*}(c_1) \leq 1 \leq f^{\Delta^*}(c_2), \text{ for } f(c_1), f(c_2) > 0.$$

Since $\mathbb{T} = \mathbb{Z}$, it yields $\sigma(t) = t + 1, \mu(t) = 1$. Additionally, we get

$$f^{\Delta^*}(t) = \frac{(t+1)^2}{t^2}.$$

Hence,

$$f^{\Delta^*}(c_1) = \frac{(c_1+1)^2}{c_1^2} \leq 1 \text{ and } f^{\Delta^*}(c_2) = \frac{(c_2+1)^2}{c_2^2} \geq 1,$$

holds for $c_1 \in \{-2, -1\}$ and $c_2 \in \{1, 2\}$.

Theorem 3.18. (Δ^*-**Mean Value Theorem**) Suppose that f is continuous *on* $[a, b]$ and has Δ^*-*derivative at each point on* $[a, b)$. Then, there exists some $c_1, c_2 \in (a, b)$ such that

$$\left[f^{\Delta^*}(c_1)\right]^{b-a} \leq \frac{f(b)}{f(a)} \leq \left[f^{\Delta^*}(c_2)\right]^{b-a},$$

for $f(c_1), f(c_2) > 0$, and

$$\left[f^{\Delta^*}(c_1)\right]^{b-a} \geq \frac{f(b)}{f(a)} \geq \left[f^{\Delta^*}(c_2)\right]^{b-a},$$

for $f(c_1), f(c_2) < 0$.

Proof: Let us consider ϕ defined by

$$\phi(t) = \frac{f(t)}{f(a)} \left[\frac{f(b)}{f(a)}\right]^{\frac{t-a}{a-b}}.$$

Then, ϕ is continuous on $[a, b]$ and has $\Delta-$derivative at each point on $[a, b)$. Moreover, it yields $\phi(a) = \phi(b) = 1$. Then, there exists $c_1, c_2 \in [a, b)$ such that

$$\begin{aligned} \phi^{\Delta^*}(c_1) &\leq 1 \leq \phi^{\Delta^*}(c_2), \text{ for } f(c_1), f(c_2) > 0 \\ \phi^{\Delta^*}(c_1) &\geq 1 \geq \phi^{\Delta^*}(c_2), \text{ for } f(c_1), f(c_2) < 0. \end{aligned} \tag{3.1.}$$

On the other hand,

$$\phi^{\Delta^*}(c_1) = f^{\Delta^*}(c_1) \left[k^{\frac{t-a}{a-b}}\right]_{t=c_1}^{\Delta^*}, k = \frac{f(b)}{f(a)}.$$

Here, if the second multiplier on the right is calculated for different cases of μ. In both cases of $\mu(t) \neq 0, \mu(t) = 0$, we get $\left[k^{\frac{t-a}{a-b}}\right]_{t=c_1}^{\Delta^*} = k^{\frac{1}{a-b}}$. Hence we get,

$$\phi^{\Delta^*}(c_1) = f^{\Delta^*}(c_1) \left[\frac{f(b)}{f(a)}\right]^{\frac{1}{a-b}}.$$

If the same process is run similarly, we get

$$\phi^{\Delta^*}(c_2) = f^{\Delta^*}(c_2) \left[\frac{f(b)}{f(a)}\right]^{\frac{1}{a-b}}.$$

By considering these equalities and (3.1.),

$$f^{\Delta^*}(c_1)\left[\frac{f(b)}{f(a)}\right]^{\frac{1}{a-b}} \leq 1 \leq f^{\Delta^*}(c_2)\left[\frac{f(b)}{f(a)}\right]^{\frac{1}{a-b}}, \text{ for } f(c_1), f(c_2) > 0.$$

Namely,

$$\left[f^{\Delta^*}(c_1)\right]^{b-a} \leq \frac{f(b)}{f(a)} \leq \left[f^{\Delta^*}(c_2)\right]^{b-a}.$$

Similarly, it yields that

$$\left[f^{\Delta^*}(c_1)\right]^{b-a} \geq \frac{f(b)}{f(a)} \geq \left[f^{\Delta^*}(c_2)\right]^{b-a},$$

for $f(c_1), f(c_2) < 0$. This completes the proof.

Example 3.19. Consider $f(t) = t^4$ on $\mathbb{T} = \mathbb{Z}$. Let us find the values of $c_1, c_2 \in (-2, 4)$ such that

$$\left[f^{\Delta^*}(c_1)\right]^{b-a} \leq \frac{f(b)}{f(a)} \leq \left[f^{\Delta^*}(c_2)\right]^{b-a}, \text{ for } f(c_1), f(c_2) > 0,$$

on $[a, b] = [-2, 4]$. From the structure of the given time scale and the definition of the function, we get $\sigma(t) = t + 1$, $\mu(t) = 1$ on $\mathbb{T} = \mathbb{Z}$, and $f(-2) = 16, f(4) = 256$. Then,

$$f^{\Delta^*}(t) = \frac{(t+1)^4}{t^4}.$$

Hence, we have

$$\left[\frac{(c_1+1)^4}{c_1^4}\right]^6 \leq \frac{256}{16} \leq \left[\frac{(c_2+1)^4}{c_2^4}\right]^6,$$

i.e,

$$\frac{(c_1+1)^6}{c_1^6} \leq 2 \text{ and } 2 \leq \frac{(c_2+1)^6}{c_2^6}.$$

The values of c_1, c_2 which satisfy above inequalities are $c_1 = -1$ and $c_2 \in \{1, 2, 3\}$ for $\mathbb{T} = \mathbb{Z}$.

Corollary 3.20. Let f be a continuous function on $[a, b]$ that has a Δ^*-derivative at each point on $[a, b)$. If $f^{\Delta^*}(t) = 1$ for all $t \in [a, b)$, then f is a constant function.

Proof: By using Δ^*-mean value theorem, there exists some $c_1, c_2 \in [a, b)$ such that

$$1 = \left[f^{\Delta^*}(c_1)\right]^{t-a} \leq \frac{f(t)}{f(a)} \leq \left[f^{\Delta^*}(c_2)\right]^{t-a} = 1,$$

i.e., $f(t) = f(a)$ for all $t \in [a, b)$.

Theorem 3.21. Let f be continuous on $[a, b]$ and be Δ^*-differentiable at each point on $[a, b)$. Here are the following situations.

i. If $f^{\Delta^*}(t) > 1$ for every $t \in [a, b)$, then f is increasing on $[a, b]$.
ii. If $f^{\Delta^*}(t) < 1$ for every $t \in [a, b)$, then f is decreasing on $[a, b]$.
iii. If $f^{\Delta^*}(t) \geq 1$ for every $t \in [a, b)$, then f is non-decreasing on $[a, b]$.
iv. If $f^{\Delta^*}(t) \leq 1$ for every $t \in [a, b)$, then f is non-increasing on $[a, b]$.

Proof: We will only prove the provincial situation. The proof of other cases is similar.

ii. Assume that $f^{\Delta^*}(t) < 1$ for every $t \in [a,b]$. Then, for any $t_1, t_2 \in [a,b]$ and $t_1 < t_2$, there exists $c \in (t_1, t_2)$ such that

$$\frac{f(t_2)}{f(t_1)} \leq \left[f^{\Delta^*}(c)\right]^{t_2-t_1} < 1,$$

i.e. $f(t_1) > f(t_2)$. So, f is decreasing.on $[a,b]$. It completes the proof.

Example 3.22. Let us determine the intervals where $f(t) = t^3 - 2t^2 - t$ is increasing or decreasing on $\mathbb{T} = \mathbb{Z}$. By using the concept for Δ^*-derivative, we get

$$f^{\Delta^*}(t) = \frac{t^3 + t^2 - 2t - 2}{t^3 - 2t^2 - t}.$$

Let's identify the critical points by the relation $f^{\Delta^*}(t) = 1$ to determine the sign of the function. Therefore,

$$\begin{aligned} f^{\Delta^*}(t) &\geq 1 \text{ for } t \in \left[-\frac{2}{3}, 1 - \sqrt{2}\right) \cup (0,1] \cup \left(1 + \sqrt{2}, \infty\right), \\ f^{\Delta^*}(t) &\leq 1 \text{ for } t \in \left(-\infty, -\frac{2}{3}\right] \cup \left(1 - \sqrt{2}, 0\right) \cup [1, \infty). \end{aligned}$$

On $\mathbb{T} = \mathbb{Z}$, f is decreasing on $(-\infty, -1] \cup \{2\}$ and increasing on $[3, \infty)$.

4. Conclusion

In this study, the concept of $\Delta-$derivative, which is one of the basic concepts of time scale theory, has been redefined using the principles of multiplicative analysis. The basic properties and basic theorems of $\Delta-$derivative, which has many applications in many fields, are given in multiplicative analysis. We hope that this work will open up a new field for mathematicians and will be the basis for many different fields in applied mathematics.

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